We study phenomenologically the production of the neutron-rich hypernucleus $^6_\Lambda H$ in the $^6\text{Li}(\pi^-, K^+)$ reaction at 1.2 GeV/c, using a distorted-wave impulse approximation in a one-step mechanism, $\pi^- p \rightarrow K^+ \Sigma^-$ via $\Sigma^-$ by nucleon coupling. The production cross section at $^6_\Lambda H(1_{\text{exc}}^+)$ is evaluated by a coupled $(^3\text{He}-\Lambda) + (^4\text{He}-\Sigma^-)$ model with a spreading potential, in comparison with the data of the J-PARC E10 experiment. The result indicates that the $\Sigma^-$ mixing probabilities in $^6_\Lambda H(1_{\text{exc}}^+)$ are $P_{\Sigma^-} < 0.2\%$ both for $s_\Sigma$ state and for $p_\Sigma$ state in order to reproduce no significant peak in the $\Lambda$ production data, so that the cross section of $^6_\Lambda H$ is less than on the order of 0.4 nb/sr. The sensitivity of the $\Sigma\Lambda$ coupling and $\Lambda$ potentials to the near-$\Lambda$-threshold spectrum is discussed. The shape and magnitude of the spectrum provide valuable information on the $\Sigma\Lambda$ coupling in the production mechanism and also the nuclear structure of $^6_\Lambda H(1_{\text{exc}}^+)$. DOI: 10.1103/PhysRevC.95.044610

I. INTRODUCTION

Recently, the J-PARC E10 collaboration [1,2] performed experimental measurements of the double-charge-exchange (DCX) reaction ($\pi^-, K^+$) on a $^6\text{Li}$ target at $p_{\pi^-} = 1.2$ GeV/c in order to confirm a neutron-rich hypernucleus $^6_\Lambda H$ in which an unbound $^5\text{H}$ core, with neutron-proton excess ratio $(N-Z)/(N+Z) = 0.6$ is expected to be stable by $\Lambda$ stabilization or glue [3,4]. No significant peak structure below and near the $^4_\Lambda H + 2n$ threshold was observed in missing mass spectra with $K^+$ forward-direction angles of $\theta_{\text{lab}} = 2^\circ - 14^\circ$. This is inconsistent with the observation of the $^6_\Lambda H$ reaction in FINUDA experiments [5] which indicated evidence of $^6_\Lambda H$ with a binding energy of $B_A(^6_\Lambda H) = 4.5 \pm 1.2$ MeV with respect to the $^3\text{He} + \Lambda$ threshold.

Dalitz and Levi–Setti [3] first discussed the $\Lambda$ stabilization of the neutron-rich $^5_\Lambda H$ hypernucleus with the particle-unstable $^5\text{H}$ core beyond the neutron-drip line. Akashi and Myint [6] paid attention to $^5_\Lambda H$ as a test ground for an attractive three-body $\Lambda NN$ force caused by the $\Lambda N - N$ coupling which may be more coherently enhanced in such neutron-excess environments [7,8]. Thus the $^4_\Lambda H$ state of $^5_\Lambda H$ was predicted to have a large binding energy of $B_A(^5_\Lambda H) = 5.8$ MeV with respect to the $^4_\Lambda H + \Lambda$ threshold due to a large contribution of 1.4 MeV by the coherent $\Lambda \Sigma$ mixing [6]. Gal and Millener [9] showed that recent shell-model calculations including the $\Lambda\Sigma$ coupling give $B_A(^5_\Lambda H) = 3.8 \pm 0.2$ MeV which seems to be in good agreement with $B_A(^5_\Lambda H) = 4.5 \pm 1.2$ MeV reported in the FINUDA experiments [5,9]. Hiyama et al. [10] suggested a lesser binding energy of $B_A(^5_\Lambda H) = 2.47$ MeV corresponding to an unbound state with respect to the $^4_\Lambda H + 2n$ threshold in $n\Lambda n$ four-body cluster-model calculations. The value of $B_A(^5_\Lambda H)$ is often calculated by the $\Lambda$-nucleus potential which strongly depends on the structure of the nuclear core as well as $\Lambda N$ interaction involving the $\Lambda\Sigma$ coupling. Therefore, it is very important to clarify the production and structure of $^5_\Lambda H$ which is strongly related to the structure of $^3\text{He}$ in nuclear physics.

The DCX ($\pi^-, K^+$) reaction is one of the most promising ways of searching for a bound state of the neutron-rich $\Lambda$ hypernuclei with stabilized effects by $\Lambda$ added. Indeed, Saha et al. [11] performed the first measurement of a significant yield for the $^6\Lambda\text{Li}$ hypernucleus in $(\pi^-, K^+)$ reactions on a $^{10}\Lambda\text{B}$ target, whereas no clear peak has been observed with the lack of the experimental statistics. The data show that the absolute cross section for $^{10}\Lambda\text{Li}$ at 1.20 GeV/c ($d\sigma/d\Omega \sim 11 \text{ nb/sr}$) is twice larger than that at 1.05 GeV/c ($d\sigma/d\Omega \sim 6 \text{ nb/sr}$). This incident-momentum dependence of $d\sigma/d\Omega$ exhibits a trend in the opposite direction for the theoretical prediction by Tret’yakova and Landskoy [12]. This might imply that the one-step mechanism, $\pi^- p \rightarrow K^+ \Sigma^-$ via $\Sigma^-$ by nucleon coupling [13] is rather favored over the two-step mechanism, $\pi^- p \rightarrow \pi^0 n$ followed by $\pi^0 p \rightarrow K^+ \Lambda$ (or $\pi^- p \rightarrow K^0 \Lambda$ followed by $K^0 p \rightarrow K^+ n$) in the production of neutron-rich $\Lambda$ hypernuclear states, as pointed out in Ref. [11].

In this paper, we study phenomenologically the production of the neutron-rich $^5_\Lambda H$ hypernuclear states in the $^6\text{Li}(\pi^-, K^+)$ reaction at 1.2 GeV/c. We demonstrate the calculated spectrum near the $\Lambda$ threshold within a distorted-wave impulse approximation (DWIA) by using a coupled $(^3\text{He}-\Lambda) + (^4\text{He}-\Sigma^-)$ model with a spreading potential [14]. Comparing the spectrum with the data of the J-PARC E10 experiment [1,2], we discuss the $\Sigma\Lambda$ couplings related to the $\Sigma$-mixing probabilities and the strengths of the $\Lambda\Sigma$ potentials which depend on the structure of the $^3\text{He}$ nuclear core in $^5_\Lambda H$.

II. CALCULATIONS

A. Distorted wave impulse approximation

The inclusive $K^+$ double-differential laboratory cross section of $\Lambda$ production on a nuclear target in the DCX ($\pi^-, K^+$) reaction [15] is calculated by the Green’s function method.
assumingly the one-step mechanism, \( \pi^- p \rightarrow K^+ \Sigma^- \) via \( \Sigma^- p \leftrightarrow \Lambda n \) coupling, for production of \( \Lambda \) hypernuclear states by the DCX nuclear \((\pi^-, K^+)\) reactions.

Thus the s.p. wave functions for the proton in 1\( p_{3/2} \) and 1\( s_{1/2} \) are calculated by the Woods–Saxon (WS) potential with \( a = 0.67 \) fm, \( R = 1.27A^{1/3} = 2.31 \) fm [20]. The strength parameter of the potential is adjusted to be \( V_0^N = -55.5 \) MeV \((-58.0 \) MeV) for the proton in the p\( s_{1/2} \) (s\( 1/2 \)) state, and \( V_{so}^N = -0.44V_0^N \), in order to reproduce the data of proton s.p. energies in \( ^3\)Li(\( p, 2p \)) reactions [21,22]. Thus the s.p. energies for 1\( p_{3/2} \) and 1\( s_{1/2} \) amount to \(-4.61 \) MeV and \(-21.48 \) MeV, respectively. The charge radius for \( ^6\)Li \( (1^+_{\Sigma^+}) \) becomes 2.48 fm of which value is slightly smaller than that of 2.56 \( \pm 0.05 \) fm in electron elastic scatterings [23] due to the s.p. description. If we replace the s.p. wave function for the 1\( p_{3/2} \) (s\( 1/2 \)) state by a spectroscopic amplitude describing a \( p_{3/2} \) (s\( 1/2 \)) proton removal from \( ^3\)Li \( (1^+_{\Sigma^+}) \) within the \( \alpha + d \) cluster model [24], we recognize that the calculated cross sections decrease by about 5\%, in comparison with the results which will be discussed in Sec. III B. Thus our conclusion obtained in the s.p. description would be reliable.

B. Wave functions for \(^6\)H

To fully describe the one-step process, as shown in Fig. 1 and to estimate the production cross section of \(^6\)H, we perform \( \Lambda - \Sigma \) coupled-channel calculations [14] which reproduce the shape and magnitude of the data of the J-PARC E10 experiment in the \( \Lambda \) and \( \Sigma^- \) quasi-free (QF) regions [19]. Here we employ a multichannel coupled wave function of the \( \Lambda - \Sigma \) nuclear state for a total spin \( J_B \) within a weak-coupling basis. It is written as

\[
\begin{align*}
|\Psi_{j_B}^{(6)H}(\Lambda)\rangle &= \sum_{JJ',j_B} \left[ \Phi_{J,\Lambda}(5\text{He}) \phi_{j_B}(\Lambda) \right]_{J_B} \\
&+ \sum_{JJ',j_B} \left[ \Phi_{J,\Sigma}^{(6)He} \phi_{j_B}^{(\Sigma^-)}(\Sigma) \right]_{J_B},
\end{align*}
\]

with

\[
\begin{align*}
\Phi_{J,\Lambda}(5\text{He}) &= A[\Phi_{J}(s^3p)\phi_{j_B}(\rho_n)\rho_{\Lambda}(\rho_n)]_{J'He}^\Lambda, \\
\Phi_{J,\Sigma}^{(6)He} &= A[\Phi_{J}(s^3p)\phi_{j_B}^{(\Sigma^-)}(\rho_n)]_{J'He},
\end{align*}
\]

where \( \Phi_{J}(s^3p) \) is a wave function of the \( s^3p \) configuration state, \( A \) is the antisymmetrized operator for nucleons, and \( \phi_{j_B}^{(\Lambda)} \), \( \phi_{j_B}^{(\Sigma^-)} \), and \( \phi_{j_B}^{(\rho_n)} \) describe the relative wave functions of shell model states (that occupy \( j_B \), \( j_Z \), and \( j_{n,p} \) orbits) for the \( \Lambda \), \( \Sigma^- \), and neutron (proton), respectively; \( r_n \) (\( r_p \)) denotes the relative coordinate between the \( s^3p \) nucleus and the neutron or proton, and \( \rho_{\Lambda}(\rho_n) \) denotes the relative coordinate between the center of mass of the \( 3\)He \( (\text{He}) \) subsystem and the \( \Lambda \) \( \Sigma^- \). We take the \( 3\)He core-nucleus state with \( J^\pi = 1/2^+ \) [ground state (g.s.)], and the \( 5\)He core-nucleus states with \( J^\pi = 3/2^- \) (g.s.), \( 1/2^+, \) \( 3/2^+ \), and \( 1/2^0 \) that are in \((1^+ \otimes \rho_{3\Sigma^+})_{3/2^-} \) and \((1^+ \otimes \rho_{1/2} \rho_{3\Sigma^-})_{1/2^+} \) configurations formed by a proton-hole state on \(^6\)Li \( (1^+_{\Sigma^+}) \). If the \( \Lambda \) component is dominant in a bound or resonant state, we can identify it as a state of the \( \Lambda \) hypernucleus \(^6\)H, in which the \( \Sigma^- \) mixing probability can be estimated by

\[
P_{\Sigma^-} = \sum_{j_B} \int_0^\infty \rho_{j_B}(r) r^2 dr,
\]

FIG. 1. Diagrams of a one-step mechanism, \( \pi^- p \rightarrow K^+ \Sigma^- \) via \( \Sigma^- p \leftrightarrow \Lambda n \) conversion within the DWIA [13]. Figure 1 illustrates diagrams for the one-step mechanism, \( \pi^- p \rightarrow K^+ \Sigma^- \) via \( \Sigma^- p \leftrightarrow \Lambda n \) conversion within the nuclear \((\pi^-, K^+)\) reaction. The inclusive \( K^+ \) double-differential laboratory cross section on the nuclear target with a spin \( J_A \) and its \( z \) component \( M_A \) [15] is given by

\[
\begin{align*}
\frac{d^2\sigma}{d\Omega dE} &= \frac{1}{|M_A|} \sum_{M_A} S(E_B),
\end{align*}
\]

as a function of the energy \( E_B \) for hypernuclear final states, where \( F_{j_B}^{(\Sigma)} \) is the \( \Sigma \) production amplitude defined by

\[
\begin{align*}
F_{j_B}^{(\Sigma)} &= \beta^2 \langle \Psi_p | \Sigma_{-\pi^-}^{(a)} | \psi_{p_A}^{(a)} \rangle (\alpha|\bar{\psi}_p | \Psi_A),
\end{align*}
\]

and \((\alpha|\bar{\psi}_p | \Psi_A)\) is a hole-state wave function for a struck proton in the target; \( \alpha \) denotes the complete set of eigenstates for the system. The energy and momentum transfer is \( \omega = E_K - E_p \) and \( q = p_K - p_p \). The kinematical factor \( \beta \) denotes the translation from a two-body \( \pi^- p \) laboratory system to a \( \pi^- K \) system. \( \langle \pi^- K \rangle \) is a Fermi-averaged amplitude for the \( \pi^- p \rightarrow K^+ \Sigma^- \) reaction in nuclear medium [17].

Distorted waves for outgoing \( K^+ \) and incoming \( \pi^- \) mesons, \( \chi_{p_A}^{(-)} \) and \( \chi_{p_A}^{(+)} \), are estimated with the help of the eikonal approximation in which total cross sections of \( \sigma_{\pi} = 32 \) mb for \( \pi^- N \) and \( \sigma_{K^+} = 12 \) mb for \( K^+ N \), and \( \alpha_{\pi} = \alpha_K = 0 \) are used as distortion parameters [17]. The recoil effects are taken into account in our calculations because an effective momentum transfer becomes \( q_{\text{eff}} \simeq (1 - 1/A)q \simeq 0.83q \) for the light nuclear system with \( A = 6 \) due to large momentum transfer \( q = 320-600 \) MeV/c in the \((\pi^-, K^+)\) reaction. Although the \( 1^+ \) ground state of \( ^4\)Li is well described as \( \alpha + d \) clusters [18], wave functions for the \( ^6\)Li target are used in the single-particle (s.p.) description for simplicity. This s.p. description has also been used to study the \( \Sigma^- \) nucleon potential for \( A = 6 \) by the missing-mass \(^6\)Li \( (\pi^-, K^+)\) spectrum at the J-PARC E10 experiment [19].
we assume that the structure of $^5\mathrm{H}$ is still uncertain experimentally \cite{10,22,28}, with respect to $^3\mathrm{H} + \Lambda$ and $^3\mathrm{H}(1^+)+2n$ thresholds, respectively. The threshold-energy difference between $^5\mathrm{H}(1/2^+_\text{g.s.})$ and the $^3\mathrm{H}+2n$ threshold is assumed to be 4.0 MeV.

where $\rho_{j\Lambda}(r) = (\psi_{j\Lambda}^*(r))^2$ denotes a $\Sigma^-$ density distribution with the $j\Lambda$ shell under the normalization of

$$\sum_{j\Lambda} \int_0^\infty \rho_{j\Lambda}(r)^2 dr + \sum_{j\Lambda} \int_0^\infty \rho_{j\Lambda}(r)^2 dr = 1,$$

together with the $\Lambda$ density distribution $\rho_{j\Lambda}(r)$ with $j\Lambda$. Because we assume the $\Sigma^-$ doorways that are selectively produced by non-spin-flip processes in the $\pi^-p \rightarrow K^+\Sigma^-$ reaction, we consider positive-parity (negative-parity) states with $j_B^\Sigma = 1^+, 2^+, 3^+, \ldots$, $(j_B^\Lambda = 0^-, 1^-, 2^-, 3^-, \ldots)$ for final states, which are populated on the nuclear $^9\text{Li}(g.s.)$ targets; the $0^-$ ground state of $^9\text{Li}(0^+)$ is forbidden.

Several theoretical calculations \cite{22,25} predicted the $^5\text{H}$ ground state with $J^\pi = 1/2^+$(g.s.), $T = 3/2$ as a continuum or unbound state, $E_r \simeq 1.6$–4.0 MeV, $\Gamma \simeq 1.5$–4.0 MeV with respect to the $^3\text{H} + 2n$ threshold in $^3\text{H} + n + 3n$ three-body calculations \cite{25} and $E_r \simeq 3.0$–4.5 MeV in standard shell-model calculation with $sp$ shell \cite{26,27}. Since the structure of $^5\text{H}$ is still uncertain experimentally \cite{10,22,28}, we assume that the $^5\text{H}(1/2^+_\text{g.s.})$ nuclear core is a resonant state with $E_r = 4.0$ MeV \cite{25} in the viewpoint of shell-model calculations, rather than that with $E_r = 1.7$ MeV in Ref. \cite{29}. Thus the energy difference between $^3\text{He} + \Sigma^-$ and $^5\text{H} + \Lambda$ channels is $\Delta M = M(^3\text{He}) + m_{\Sigma^-} - M(^5\text{H}) - m_\Lambda = 57.6$ MeV, where $M(^3\text{He})$, $M(^5\text{H})$, $m_{\Sigma^-}$, and $m_\Lambda$ are masses of $^3\text{He}$, $^5\text{H}$, $\Sigma^-$ and $\Lambda$, respectively. Figure 2 illustrates the energy spectrum and decay threshold for the $^5\text{H}$ hypernucleus, where $B_{\Lambda}(^5\text{H})$ and $B_{2\Lambda}(^5\text{H})$ denote the binding energies with respect to $^5\text{H} + \Lambda$ and $^5\text{H}(1^+)+2n$ thresholds, respectively.

C. Multichannel Green’s functions

The Green’s function method is one of the most powerful treatments in calculations for the spectrum \cite{16}. The complete Green’s function $G(E)$ describes all information concerning $(^3\text{H} \otimes \Lambda) + (^3\text{He} \otimes \Sigma^-)$ coupled-channel dynamics. We obtain it by solving the following equation with the hyperon-nucleus potential $U$ numerically:

$$G(E) = G^{(0)}(E) + G^{(0)}(E)U G(E),$$

where

$$G(E) = \left( \begin{array}{cc} G\Lambda(E) & G\Sigma(E) \\ G\Sigma(E) & G\Sigma(E) \end{array} \right), \quad U = \left( \begin{array}{ccc} U_\Lambda & U_J & U_\Sigma \\ U_J & U_X & U_\Sigma \\ U_\Sigma & U_\Sigma & U_\Sigma \end{array} \right).$$

and the free Green’s function $G^{(0)}(E)$. The diagonal parts $U_\Lambda$ ($U_J$) for $U$ are the $\Lambda$-nucleus ($\Sigma$-nucleus) potentials, and the off-diagonal parts $U_X$ are the $\Sigma$-$\Lambda$ coupling potentials. Thus the inclusive $\Lambda$ spectrum in Eq. (2) can be decomposed into different physical processes \cite{14,16} by using the identity

$$\text{Im}(F^{(5)}_\Sigma G\Sigma(E)F\Sigma) = F^{(5)}_\Sigma \Omega^{-\delta}(\text{Im}G^{(5)}_\Lambda(E))\Omega^{-\delta}F\Sigma,$$

where $\Omega^{-\delta}$ is the Möller wave operator and $F\Sigma$ is the production amplitude for $\Sigma^-$. The remarkable production of $^8\text{H}$ arises from the term of $F^{(5)}_\Sigma G\Sigma(E)G\Sigma(E)F\Sigma$.

The $\Sigma$-$\Lambda$ (optical) potentials for $Y = \Lambda$ or $\Sigma^-$ are given by the Woods–Saxon (WS) form:

$$U_Y(r) = [V_Y + iW_Y g(E\Lambda)]f_Y(r),$$

where $f_Y(r) = \{1 + \exp [(r - R)/a]\}^{-1}$. For the $^3\text{H}$-$\Lambda$ channel, we use $a = 0.60$ fm, $r_0 = 1.080 + 0.395 A^{-2/3}$ fm and $R = r_0 A^{1/3} = 2.05$ fm \cite{30}. Considering that the $^5\text{H}$ nuclear core may be an unbound state or a broad resonant state \cite{10}, the strength parameters of $V_\Lambda$ should be adjusted appropriately to reproduce the experimental data. The spreading imaginary potential, $U_Y$, can represent complicated excited states for $^8\text{H}$; $g(E\Lambda)$ is assumed to be an energy-dependent function which linearly increases from 0 at $E_\Lambda = 0$ MeV to 1 at $E_\Lambda = 60$ MeV with respect to the $^5\text{H} + \Lambda$ threshold, as often used in nuclear optical models. For the $^5\text{He}$-$\Sigma^-$ potential, we use the WS potential with $R = 1.1A^{1/3} = 1.88$ fm and $a = 0.67$ fm, in comparison with the data of the J-PARC E10 experiment \cite{2}. We take the strengths of $(V_\Sigma, W_\Sigma) = (20\text{ MeV}, -20\text{ MeV})$ which can fully reproduce the data in $\Sigma^-$ region, leading to the reduced $\chi^2$ value of $\chi^2/N \simeq 0.97$ \cite{19}. The spreading potential $W_\Sigma$ expresses nuclear core breakup processes caused by the $\Sigma^- p \rightarrow \Lambda n$ conversion in the $^5\text{He}$ nucleus, and its effect is not involved in $U_X$ which we will mention below.

D. $\Sigma$-$\Lambda$ coupling potentials

The $\Sigma$-$\Lambda$ coupling potential $U_X$ in off-diagonal parts of $U$ can be obtained by a two-body $\Sigma-N-\Lambda$ potential $v_{\Sigma,N,\Lambda}(r,r')$ with the spin $S = 1$, 0 isospin $I = 1/2$ state. Here we use a zero-range interaction $v_{\Sigma,N,\Lambda}(r,r') = v_{\Sigma,N,\Lambda}\delta(r-r')$ in a real potential for simplicity, where $v_{\Sigma,N,\Lambda}$ is the strength parameter that should be connected with volume integral $\int v_{\Sigma,N,\Lambda}(r)dr = v_{\Sigma,N,\Lambda}$. Thus the matrix elements can be easily estimated by use of Racah
algebra [31,32]:

\[
U_\chi(r) = \left| \left[ \Phi_J (^3\text{He}) \otimes \mathcal{Y}_{J'}^{(\Sigma')} \right] \right|_{\Sigma}\n \times \frac{1}{\sqrt{3}} \sum_i v_{\Sigma,N,\Lambda}(r', r) \tau_j \cdot \phi
\times \left| \left[ \Phi_J (^5\text{He}) \otimes \mathcal{Y}_{J''}^{(\Sigma'')} \right] \right|_{\Sigma',\Lambda'},
\]

where \( \tau_j \) denotes the \( j \)-th nucleon isospin operator and \( \phi \) is defined as \( |\Sigma\rangle = |\Lambda\rangle \) in isospin space [33], and \( \mathcal{Y}_{J''}^{(\Sigma'')} \) is a spin-orbit function and \( C_{LSK}(J'-J'') \) is a purely geometrical factor [31]; \( F_{J''}^{(\Sigma'')} \) is the normal form factor including a one-body transition factor for the \( \Lambda \) = 5 shell model [26] and the center-of-mass correction of a factor \( \sqrt{A/(A-1)} \) [34].

Three parameters, \( v_{\Sigma,N,\Lambda} \), \( v_{\Sigma,N,\Lambda}^0 \), and \( V_\Lambda \), are very important for determining the \( \Sigma^- \)-mixing probability in \( ^6\text{He} \) and the production cross section of \( ^6\text{He} \) within the one-step mechanism [13]. These parameters are strongly connected each other for the shape of the spectrum and its magnitude. The effects of the \( \Sigma N N \) coupling can be evaluated by the volume integrals for \( \Sigma N N \) \( g \)-matrices based on Nijmegen potentials [35–38], in which these values are important for determining the \( \Sigma^- \)-mixing probability related to the production \( \Sigma^- \) transitions from \( ^6\text{Li}(1^-_{\text{exc}}) \) as a target, whereas the \( 1^+ \) state of \( ^6\text{He}(1^+_{\text{exc}}) \) can be produced in the reaction. The \( ^6\text{Li}(1^+_{\text{exc}}) \) in the one-step mechanism via \( \Sigma^- \) doorways, we have

\[
\begin{align*}
\text{FIG. 3. Schematic illustration of shell-model configurations for } & (a) \ p\Sigma^- \rightarrow p\Sigma_h \ 	ext{transitions from } s_{\nu'}^+ s_\Sigma \text{ components, and (b) } \ p\Sigma^- \rightarrow p\Sigma_h \text{ transitions from } p_{\nu'}^+ p\Sigma \text{ components in } ^6\text{He}(1^+_{\text{exc}}). \\
\text{A. } \Sigma^- \text{ doorways} & \\
\text{The nuclear } (\pi^-, K^+) \text{ reaction can predominantly populate spin-stretched states of } ^7\text{He} \otimes ^{-}\Sigma^- \text{ doorways with } T = 3/2 \text{ because the momentum transfer is very large (} q \simeq 359 \text{ MeV}/c \text{ around the } \Lambda \text{ threshold) in the } \pi^- p \rightarrow K^+ \Sigma^- \text{ reaction at } 1.20 \text{ GeV}/c \ [40]. \text{ It is also considered that non-spin-flip reactions are dominant near the forward direction in this reaction } [41]. \text{ Thus the } 0^+ \text{ ground state of } ^6\Lambda \text{H}(0^+_{\text{g.s.}}) \text{ that is expected to have a large contribution by the coherent } \Lambda \Sigma \text{ mixing in } [6] \text{ is forbidden by spin-parity conservation when choosing } ^6\text{Li}(1^-_{\text{exc}}) \text{ as a target, whereas the } 1^+ \text{ excited state of } ^6\text{He}(1^+_{\text{exc}}) \text{ can be produced in the reaction. For } ^6\text{He}(1^+_{\text{exc}}) \text{ in the one-step mechanism via } \Sigma^- \text{ doorways, we have} \\
\text{in the } s_{\nu'}^+ s_\Sigma \text{ configuration formed by the } \pi^- p \rightarrow K^+ \Sigma^- \text{ reaction, and} \\
\text{in the } p_{\nu'}^+ p\Sigma \text{ configuration. Figure 3 illustrates these shell-model configurations in } ^6\text{He}(1^+_{\text{exc}}) \text{ schematically. The former process indicates the coherent } \Lambda \Sigma \text{ coupling with the } p\Sigma^- \rightarrow p\Sigma_h \text{ transition } [7]. \text{ The latter process also contributes to } ^6\text{He}(1^+_{\text{exc}}) \text{ due to the } s_{\nu'}^+ p\Sigma^- \rightarrow p\Sigma_h \text{ transition which induces nucleon-hole states with nuclear core-excitaiton in the } \Lambda \text{ hypernucleus, as discussed in } ab \ initio \text{ calculations for } ^7\text{Li}(1/2^+_1) \text{ by Nemura et al. } [42]. \text{ The type of this coupling is called as “incoherent” } \Lambda \Sigma \text{ coupling. We used single-particle wave functions for a proton in } ^6\text{Li}(1^+_{\text{g.s.}}), \text{ reproducing the } s\text{-hole and } p\text{-hole energies in } ^6\text{Li}(p, 2p) \text{ reactions } [21].)
\end{align*}
\]

\[ (11) \]

\[ \text{III. RESULTS} \]

Now let us examine the dependence of the shape and magnitude of the spectrum on \( v_{\Sigma,N,\Lambda} \) and \( V_\Lambda \), comparing the calculated inclusive \( \Lambda \) spectrum for \( ^5\text{He} \) with the data of the \( ^6\text{Li}(\pi^-, K^+) \) reaction at the J-PARC E10 experiment. In our calculation, we also took the energy-dependent Fermi-averaged \( t \) matrix for the \( \pi^- p \rightarrow K^+ \Sigma^- \) reaction which is essential to explain the \( \Sigma^- \) QF spectra of the \( (\pi^-, K^+) \) data on nuclear targets [17]. Therefore, it should be noticed that the following calculated spectra have reproduced the data in the \( \Sigma^- \) and \( \Lambda \) QF regions [19].
cross section for $^6_A\Lambda H(1^{+}\text{exc})$ in one-step mechanism. In Table I, we show configurations of the $[J^e_c \otimes (\ell_j)_{\Lambda}]$ state in $^6_A\Lambda H(1^{+}\text{exc})$ composed by the $\Lambda = 5$ core nucleus with $J^e_c$ and $(\ell_j)$-shell hyperon. In Fig. 4, we display the calculated $\Sigma \Lambda$ coupling potentials $U_{\Lambda}(r)$ between $[^5\text{He}(J^e_c) \otimes (\ell_j)_{\Sigma^-}]$ and $[^3\text{H}(1/2^{+}_s) \otimes (s_{1/2})_{\Lambda}]$ as a function of the relative distance between $^5\text{He}$ and $\Lambda = (\Sigma^-)$, using the $\Sigma \Lambda$ coupling strengths of $^{\Sigma \Lambda}_{\Sigma^-}$ in $^5\text{H}(1^{+}\text{exc})$, as well as those for $^5\text{H}(1^{+}\text{exc})$ as a function of the relative distance between $^5\text{He}$ and $\Lambda = \Sigma^-$. The coupling potentials are classified by the orbital angular momentum transfers $\Delta \ell$ to the hyperon in $^5\text{H}(1^{+}\text{exc})$, where $\Delta \ell = |\ell_\Sigma^- - \ell_\Lambda|$. We find that the following coupling potentials are dominant:

i. $[1/2^+ \otimes (s_{1/2})_{\Sigma^-}] \rightarrow [1/2^+ \otimes (s_{1/2})_{\Lambda}]$ for $\Delta \ell = 0$;
ii. $[1/2^- \otimes (p_{1/2})_{\Sigma^-}] \rightarrow [1/2^+ \otimes (s_{1/2})_{\Lambda}]$ for $\Delta \ell = 1$;
iii. $[3/2^- \otimes (p_{3/2})_{\Sigma^-}] \rightarrow [1/2^+ \otimes (s_{1/2})_{\Lambda}]$ for $\Delta \ell = 1$.

This nature may originate from the fact that a significant $\pi + \rho$ meson exchange related with the SU(3) coupling constant generates a $(\sigma_N \otimes \sigma_Y \otimes \rho_N \cdot \rho_Y)$ component in $\Lambda - \Sigma$ potentials, and that the nuclear form factors $F^2_{J;K}(r)$ in Eq. (11) have the collectivity of nuclear core excitations in a $\Lambda$ shell-model calculations. We recognize that the $s_p\Sigma^- \rightarrow p_n\Sigma^- \rightarrow p_n\Sigma^+$ transitions are significant to describe $\Lambda - \Sigma$ dynamics in $^5\text{H}(1^{+})$ as well as the $p_n\Sigma^- \rightarrow p_n\Sigma^+$ transitions caused by coherent $\Delta \Sigma$ couplings, as discussed by Akaishi et al. [7].

To see the dependence of the spectrum on $\tilde{v}^2_{\Sigma^- N,NN}$, here we take $V_A = -19$ MeV for $^6_A\Lambda H(1^{+}\text{exc})$, whose potential gives the binding energy of $B_A(^6_A\Lambda H) = 1.492$ MeV when omitting $\tilde{v}^2_{\Sigma^- N,NN}$. This value of $B_A$ is moderately larger than that of $B_A(^4\text{H}) = 0.96 \pm 0.04$ MeV for the $^4\text{H}(1^{+})$ subsystem in $^6_A\Lambda H$. We consider single-particle wave functions for $\Lambda$, $n$ in $^5\text{H}(1^{+}\text{exc})$ as well as those for $^6\text{H}(1^{+}\text{exc})$ in which the $s_{1/2}$ state has the root-mean-square radius of $(\langle r^2 \rangle)^{1/2} = 3.35$ fm, in comparison with $(\langle r^2 \rangle)^{1/2} = 4.01$ fm for valence neutrons in $^6\text{H}$. Thus the $\Lambda, n$ distributions in $^6_A\Lambda H$ simulate a similar structure to the layer distributions of single-particle $t$, $\Lambda$, and $n$ densities obtained by the $tnn\Lambda$ four-body calculations [10].

### 1. Binding energies and $\Sigma^-$-mixing probabilities

In Table II, we show the results of the binding energies and $\Sigma^-$-mixing probabilities in $^6_A\Lambda H(1^{+}\text{exc})$. When we take $\tilde{v}^0_{\Sigma^- N,NN} = -450, -900, -1350$, and $-1800$ MeV ($\tilde{v}^0_{\Sigma^- N,NN} = 250, 500, 750, 1000$ MeV), we find the $\Sigma^-$-mixing probabilities of $P_{\Sigma^-}$ = 0.07%, 0.32%, 0.79%, and 1.58%, respectively. We stress that there appear not only $s_{1/2}$ components but also $p_{1/2}$ components in the $\Sigma^-$-mixing probabilities; the value of $P_{\Sigma^-}(s_{1/2}) = 0.04\%$–0.82\% is larger than that of $P_{\Sigma^-}(s_{1/2}) = 0.03\%$–0.68\%. The $d_{3/2}$ components are relatively small. The corresponding energy positions of $^6_A\Lambda H(1^{+}\text{exc})$ are shifted downward by the $\Sigma \Lambda$ coupling. We obtain the energy-level shift $\Delta E_{\Lambda}$ caused by the $p_n\Sigma^- \leftrightarrow p_n\Sigma^+$ coupling in Eq. (12), e.g., $\Delta E_{\Lambda} \approx -148$ keV when $\tilde{v}^0_{\Sigma^- N,NN} = -900$ MeV and $\tilde{v}^0_{\Sigma^- N,NN} = 500$ MeV. This value is slightly smaller than that of $^9\text{Li}$ in several microscopic shell-model calculations [43,44]. For $\Delta E_{\Lambda}$ caused by the $s_p\Sigma^- \leftrightarrow p_n\Sigma^+$ coupling in Eq. (13), we estimate $\Delta E_{\Lambda} \approx -201$ keV. This effect may be often eliminated in the model space by $g$-matrix description, and it is not taken into account explicitly in standard calculations [43,44].

In Fig. 5, we display the density distribution of $\rho_{\Sigma^-}(r)$ for $Y = \Lambda, \Sigma^-$ with $\alpha = (n,j)$ in $^6_A\Lambda H(1^{+}\text{exc})$ when we use the $\Sigma \Lambda$ coupling potential given in Fig. 4. Thus we have $P_{\Sigma^-}(s_{1/2}) = 0.13\%$ and $P_{\Sigma^-}(p_{1/2}) = 0.17\%$, as seen in Table II. We find that the $\Sigma^-$ components are located near the center of $^5\text{He}(1^{+}\text{exc})$, e.g., the renormalized root-mean-square radius of $(\langle r^2 \rangle)^{1/2} = 1.47\%$ fm for $s_{3/2}$ ($s_{1/2}$) states, respectively, in comparison with those of $(\langle r^2 \rangle)^{1/2} = 1.98\%$ (3.03) fm for $p_{1/2}$ ($p_{3/2}$) states in $^6\text{Li}(1^{+}\text{exc})$. This compactness of these $\Sigma^-$ distributions

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**TABLE I. Configurations for $^6_A\Lambda H(1^{+}\text{exc})$.**

<table>
<thead>
<tr>
<th>$^5\text{H}(T = 3/2) \otimes \Lambda$</th>
<th>$^4\text{He}(T = 1/2) \otimes \Sigma^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J^e_c \otimes (\ell_j)_{\Lambda}$</td>
<td>$J^e_c \otimes (\ell_j)_{\Sigma^-}$</td>
</tr>
<tr>
<td>$1/2^+$</td>
<td>$3/2^+; 1/2^+$</td>
</tr>
<tr>
<td>$s_{1/2}$</td>
<td>$p_{1/2}, p_{1/2}, f_{3/2}$</td>
</tr>
<tr>
<td>$1/2^+$</td>
<td>$3/2^+; 3/2^+$</td>
</tr>
<tr>
<td>$s_{1/2}, d_{5/2}, d_{3/2}$</td>
<td>$s_{1/2}, d_{5/2}, d_{3/2}$</td>
</tr>
</tbody>
</table>

---

**FIG. 4.** Calculated $\Sigma \Lambda$ coupling potentials $U_{\Lambda}(r)$ between $[^5\text{He}(J^e_c) \otimes (\ell_j)_{\Sigma^-}]$ and $[^3\text{H}(1/2^{+}_s) \otimes (s_{1/2})_{\Lambda}]$, with $\Delta \ell = |\ell_\Sigma^- - \ell_\Lambda| = 0, 1, 2$ in $^5\text{H}(1^{+}\text{exc})$ at $E_{\Lambda} = 0$ MeV in Eq. (11), as a function of the relative distance between $^5\text{He}$ and $\Lambda = (\Sigma^-)$. $\tilde{v}^0_{\Sigma^- N,NN} = -900$ MeV and $\tilde{v}^0_{\Sigma^- N,NN} = 500$ MeV are used. The dot-dashed curve denote the $^4\text{H}-\Lambda$ potential as a guide.
TABLE II. Calculated production cross sections of $d\sigma/d\Omega$ for $^6\text{Li}(^{1}\text{H}$, ) by one-step mechanism in the $^6\text{Li}(\pi^-, K^+)$ reaction at $1.2 \text{ GeV/c}$ ($^7$), depending on the $\Sigma\Lambda$ coupling parameters of $\vec{v}_{\Sigma\Lambda}^s$. $P_{\Sigma^-}$ is the $\Sigma^-$-mixing probability, and $B_{\Lambda}(\Lambda^0\text{H})$ and $B_{2\Lambda}(\Lambda^0\text{H})$ are binding energies of $\Lambda$ and $2\Lambda$, respectively. $V_\Lambda = -19 \text{ MeV}$ is used.

<table>
<thead>
<tr>
<th>$\vec{v}_{\Sigma\Lambda}^s$ (MeV)</th>
<th>$B_{\Lambda}(\Lambda^0\text{H})$ (MeV)</th>
<th>$B_{2\Lambda}(\Lambda^0\text{H})$ (MeV)</th>
<th>$P_{\Sigma^-}$ (%)</th>
<th>$d\sigma/d\Omega$ (nb/sr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S = 1$</td>
<td>$S = 0$</td>
<td>$S = 1$</td>
<td>$S = 0$</td>
<td>$S = 1$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1.492</td>
<td>-3.508</td>
<td>0.00</td>
</tr>
<tr>
<td>-450</td>
<td>250</td>
<td>1.576</td>
<td>-3.424</td>
<td>0.03</td>
</tr>
<tr>
<td>-900</td>
<td>500</td>
<td>1.84</td>
<td>-3.159</td>
<td>0.13</td>
</tr>
<tr>
<td>-1350</td>
<td>750</td>
<td>2.328</td>
<td>-2.672</td>
<td>0.34</td>
</tr>
<tr>
<td>-1800</td>
<td>1000</td>
<td>3.100</td>
<td>-1.900</td>
<td>0.68</td>
</tr>
</tbody>
</table>

may originate from the short-range nature of the $\Sigma\Lambda$ coupling potentials obtained in Eq. (11), and this nature is already seen in the $ab\text{initio}$ calculation by Ref. [42].

2. Inclusive $\Lambda$ spectra and cross sections

In Fig. 6, we show the calculated inclusive $\Lambda$ spectrum of the $^6\text{Li}(\pi^-, K^+)$ reaction at $p_{\pi^-} = 1.20 \text{ GeV/c}$ and $\theta_{ab} = 7^o$, together with the data for the average cross section $\bar{\sigma}_{7^o}$, taken into account a detector resolution of 3.2 MeV FWHM. We find that the calculated spectrum below the $^5\text{H}$ + $2\Lambda$ threshold is rather sensitive to $\vec{v}_{\Sigma\Lambda}^s$ in the one-step mechanism, where $^5\Lambda\text{H}(1_{\text{exc}}^+)$ is particle unstable above the $^5\Lambda\text{H} + 2\Lambda$ threshold. The integrated cross sections for $^6\Lambda\text{H}(1_{\text{exc}}^+)$ account for $d\sigma/d\Omega = 0.04$–1.2 nb/sr for $\vec{v}_{\Sigma\Lambda}^s$ as a bin with a finite width of 1 MeV for particle decay channels at $M_{\pi} \approx 5806.16$–5804.63 MeV/c$^2$, as also shown in Fig. 6. It is remarkable that the $\Lambda$ production spectra are composed of proton-hole states, $p_{\Lambda}^+$ and $p_{\Lambda}^-$, populated by the $(\pi^-, K^+)$ reactions. The value of $d\sigma(p_{\Lambda}^-)/d\Omega = 0.01$–0.32 nb/sr is considerably smaller than that of $d\sigma(p_{\Lambda}^+)/d\Omega = 0.03$–1.00 nb/sr, whereas $P_{\Sigma^-}(p_{\Lambda}^+)$ is $0.04$–$0.82\%$ larger than $P_{\Sigma^-}(p_{\Lambda}^-) = 0.03$–$0.68\%$, as mentioned above.

IV. DISCUSSION

A. $s$-hole proton vs $p$-hole proton

To see the feasibility of producing the neutron-rich $\Lambda$ hypernucleus in the one-step mechanism, we consider the contribution of the inclusive spectra via $\Sigma^-\Lambda$ through the proton $p_{\Lambda}^-$ ($s_{\Lambda}^-$) state on the $^6\text{Li}$ target. The integrated laboratory cross section may be roughly written as

$$\frac{d\sigma(j_p^\Lambda)}{d\Omega} = \beta |F_{p^-\to K^-\Sigma^-}|^2 \times S_p(p_{\Lambda}^-)|F^{(j_p^\Lambda-j_{\Sigma^-})}_{\Delta L=0}(q)|^2 P_{\Sigma^-}(j_{\Sigma^-}),$$

(14)

where $S_p(j_p)$ is a spectroscopic factor for $j_p$-shell proton, and $F_{p^-\to K^-\Sigma^-}$ is a Fermi-averaged amplitude for the $\pi^- p \to K^- \Sigma^-$ reactions. Thus we recognize the behavior of the form factor $F^{(j^\Lambda-j_{\Sigma^-})}_{\Delta L=0}(q)$ for the $j_p \to j_{\Sigma^-}$ transition with angular-momentum transfer $\Delta L$, depending on the momentum transfer $q$ in the $(\pi^-, K^+)$ reactions. Using a harmonic-oscillator model in the plane-wave approximation [40], we can estimate

$$\frac{S_p(p_{\Lambda}^-)}{S_p(s_p)}|F^{(j_{\Sigma^-})}_{\Delta L=0}(q)|^2 \approx \frac{1}{2} \left[ 1 - \frac{1}{3} (bq)^2 + \frac{7}{180} (bq)^4 \right] \approx 0.20,$$

(15)

for $q \approx 360 \text{ MeV/c}$ corresponding to the $\Lambda$ threshold at 1.2 GeV/c. Here we adopted $S_p(p_{\Lambda}^-)/S_p(s_p) \approx 1/2$ for $^6\text{Li}$ and the oscillator radius parameter $b \approx 1.38$ fm. This $b$ value indicates that the $\Sigma^-$ components are distributed near the center of $^6\Lambda\text{H}(1_{\text{exc}}^+)$. As a result, we confirm that the value of $d\sigma(p_{\Lambda}^-)/d\Omega$ is considerably smaller than that of $d\sigma(s_{\Lambda}^-)/d\Omega$, whereas $P_{\Sigma^-}(s_{\Lambda}^-)$ and $P_{\Sigma^-}(s_{\Lambda}^-)$ have almost the same value.

B. $\vec{v}_{\Sigma\Lambda}^s$ strengths

As far as $\vec{v}_{\Sigma\Lambda}^s = (0.0)\rightarrow(-900) \text{ MeV}$ and $\vec{v}_{\Sigma\Lambda}^s = 0.0$–$500 \text{ MeV}$ leading to $P_{\Sigma^-}(s_{\Lambda}^-) = 0.0$–$0.13\%$ and $P_{\Sigma^-}(s_{\Lambda}^-) = 0.0$–$0.17\%$, therefore, the calculated spectra can fairly explain the data of the J-PARC E10 experiment. No peak structure of $^6\Lambda\text{H}$ originates from the small $\Sigma\Lambda$ coupling
FIG. 6. Calculated missing mass spectra of the $^6$Li($\pi^-, K^+$) reactions near the $\Lambda$ threshold at 1.2 GeV/c and $\theta_{lab} = 7^\circ$, with a detector resolution of 3.2 MeV FWHM. The $\Sigma\Lambda$ coupling strengths of $\hat{v}_{\Sigma\Lambda} = (a) -1800$, (b) $-1350$, (c) $-900$, and (d) $-450$ MeV are used, together with $V_{\Lambda} = -19$ MeV for the $\Lambda$-$^5$H potential. Solid, dashed and dot-dashed curves denote contribution of total, $p$-hole, and $s$-hole spectra, respectively. The data are taken from Ref. [1]. The bins with a finite width of 1 MeV denote the cross sections for $^6\Lambda$H(1$^+_\text{exc}$) which is located in particle decay channels. The $\Sigma^-\Lambda$ coupling strength and also the loosely resonant $\Lambda$ state in the $^5$H nuclear core. Although the $\Sigma^-\Lambda$-mixing probabilities for $^6\Lambda$H are very small, the sensitivity of the spectrum below the $^5$H+$\Lambda$ threshold on $\hat{v}_{\Sigma\Lambda} = (a) -1800$, (b) $-1350$, (c) $-900$, and (d) $-450$ MeV indicates the possibility to extract the precise $\Sigma^-\Lambda$ components in wave functions for $^6\Lambda$HHint the nuclear ($\pi^-, K^+$) reactions. We confirm that the $\Sigma\Lambda$ coupling potential plays an essential role in the formation of the $\Lambda$ hypernuclear state near the $\Lambda$ threshold. Consequently, the calculated spectrum seems to be in good agreement with that of the $^6\Lambda$H(1$^+_\text{exc}$) data when we use the $\Sigma\Lambda$ coupling strengths of $\hat{v}_{\Sigma\Lambda} = (a) -1800$, (b) $-1350$, (c) $-900$, and (d) $-450$ MeV , whose values correspond to those of the volume integrals for the D2$^g$ potential [39].

C. $V_{\Lambda}$ strengths

On the other hand, another important parameter $V_{\Lambda}$ for the $^5$H-$\Lambda$ potential also affects the binding energies and the production cross sections for $^6\Lambda$H(1$^+_\text{exc}$). The energy position of $^6\Lambda$H(1$^+_\text{exc}$) is shifted downward by the attraction of $V_{\Lambda}$. We find that, when $\hat{v}_{\Sigma\Lambda} = -900$ MeV and $\hat{v}_{\Sigma\Lambda} = 500$ MeV, the binding energies are $B_{\Lambda}(^6\Lambda) = 0.050, 1.841, 3.726$ and 5.493 MeV for $V_{\Lambda} = -11$, $-19$, $-24$, and $-28$ MeV, respectively, so that the $\Sigma^-\Lambda$-mixing probabilities amount to $P_{\Sigma^-} = 0.07\%, 0.32\%, 0.38\%$, and $0.40\%$. In Fig. 7, we show the dependence of the inclusive $\Lambda$ spectrum for the $^6\Lambda$H(1$^+_\text{exc}$) production on these values of $V_{\Lambda}$ when $\hat{v}_{\Sigma\Lambda} = -900$ MeV and $\hat{v}_{\Sigma\Lambda} = 500$ MeV. We show that the calculated spectrum for $^6\Lambda$H(1$^+_\text{exc}$) is considerably changed by the value of $V_{\Lambda}$, where the integrated cross sections of $^6\Lambda$H(1$^+_\text{exc}$) become $d\sigma/d\Omega = 0.04, 0.22, 0.34$

FIG. 7. Dependence of the calculated inclusive $\Lambda$ spectrum in the $^6\Lambda$Li($\pi^-, K^+$) reaction at $p_{\pi^-} = 1.2$ GeV/c ($7^\circ$) on various strengths of $V_{\Lambda}$, together with the experimental data [1]. Solid curves denote the spectra by $V_{\Lambda} = -28, -24, -19, and -11$ MeV when $\hat{v}_{\Sigma\Lambda} = -900$ MeV and $\hat{v}_{\Sigma\Lambda} = 500$ MeV with a detector resolution of 3.2 MeV FWHM.
and 0.41 nb/sr for \( V_{\Lambda} = -11, -19, -24, \) and \(-28 \) MeV, respectively. The calculated spectra with \( V_{\Lambda} = (-24)\rightarrow(-28) \) MeV seem to disagree with the data of no peak structure below the \( ^5\text{H} + \Lambda \) threshold. This fact may indicate that the \( ^5\text{H}-\Lambda \) potential is quite shallow in comparison with the \( \Lambda \)-nucleus potentials which are well known as \( V_{\Lambda} \approx -28 \) MeV in ordinary nuclei [30], and the neutron-rich nuclear core \( ^3\text{H} \) should be an unbound or broad resonant state.

### D. \( V_{\Sigma} \) and \( W_{\Sigma} \) strengths

As discussed above, we recognize that the calculated spectrum is in good agreement with that of the \( ^6\text{Li}(\pi^-, K^+) \) data when we use the \( \Sigma \Lambda \) coupling strengths of \( \vec{v}_{\Sigma,N,\Lambda} \approx 900 \) MeV and \( \vec{v}_{\Sigma,N,\Lambda}^0 \approx 500 \) MeV, together with \( V_{\Sigma} \approx (+20)\rightarrow(+30) \) MeV and \( W_{\Sigma} \approx -20 \) MeV for the \( ^3\text{He}-\Sigma^- \) potential [19]. The nature of the repulsive component in this potential is consistent with that in the \( \Sigma \)-nucleus potential obtained on heavier targets [17]. The calculated spectrum fully explains the data in the \( \Sigma^- \) and \( \Lambda \) QF regions by the one-step mechanism, \( \pi^- p \rightarrow K^+ \Sigma^- \) via \( \Sigma^- \) doorways caused by the \( \Sigma^- p \leftrightarrow \Lambda n \) coupling.

### E. \( ^5\text{H}(1/2^+_{ex}) \) resonant state

Current experiments have reported that the \( ^5\text{H} \) ground state is located at \( E_r = 1.7 \pm 0.3 \) MeV with \( \Gamma = 1.9 \pm 0.4 \) MeV above the \( ^3\text{He} + n \) threshold [29], or at \( E_r = 5.5 \pm 0.2 \) MeV with \( \Gamma = 5.4 \pm 0.6 \) MeV [45]. The problem of whether the \( ^5\text{H}(1/2^+_{ex}) \) ground state exists as a narrow resonant state with \( E_r = 1.7 \) MeV and \( \Gamma = 1.9 \) MeV may still be unsettled [22,28]. Several theoretical investigations [22,25] suggest the energy of the \( ^5\text{H} \) ground state with \( E_r \approx 1.6\rightarrow3.0 \) MeV, \( \Gamma \approx 1.5\rightarrow4.0 \) MeV in \( tnn \) three-body calculations [25] and \( E_r \approx 3.0\rightarrow4.5 \) MeV in the shell-model calculations with \( spsd \) space [26,27]. It is expected that the \( \Sigma \Lambda \) coupling matrix elements work reasonably within the shell-model description. In the viewpoint of shell-model calculations, we assume that the \( ^5\text{H}(1/2^+_{ex}) \) nuclear core is a resonant state with \( E_r = 4.0 \) MeV, rather than that with \( E_r = 1.7 \) MeV if we have \( E_r = 1.7 \) MeV in the shell models, we would need to artificially add an extreme attraction to the \( ^5\text{H} \) system, e.g., through nuclei forces [10]. To see effects of the energy of the \( ^5\text{H} + \Lambda \) threshold on \( ^5\text{H}(1^+_\text{exc}) \), we calculate the inclusive spectrum near the \( \Lambda \) threshold, changing the energy of the \( ^5\text{H}(1/2^+_{ex}) \) resonant state. In Fig. 8, we show the dependence of the inclusive \( \Lambda \) spectrum for \( ^5\text{H}(1^+_\text{exc}) \) near the \( \Lambda \) threshold, using \( E_r = 4.0 \) MeV and \( 1.7 \) MeV which determine the position of the \( ^3\text{He} + \Lambda \) threshold. We recognize that the shape of the calculated spectrum for \( ^5\text{H}(1^+_\text{exc}) \) is considerably changed by the value of \( E_r \), which depends on whether \( ^5\text{H} \) is a narrow resonant state. The structure of \( ^5\text{H} \) may influence the scenario of production of \( ^5\text{H} \) at FINUDA [5]. Thus the spectrum near the \( \Lambda \) threshold provides the ability to study the structure of the \( ^5\text{H} \) core nucleus in detailed comparison with the precise data, as well as the structure of \( ^5\text{H}(1^+_\text{exc}) \).

### F. Finite range

To clarify the one-step mechanism for production of the neutron-rich \( \Lambda \) hypernucleus, we obtained the \( \Sigma \Lambda \) coupling potential constructed by the zero-range two-body interaction for simplicity, using the WS form for diagonal potentials in \( ^1\text{H} + \Lambda \) and \( ^3\text{He} + \Sigma^- \) channels. On the other hand, it is known that a finite range of the two-body interaction provides modified nuclear potentials [31]. To see effects of the finite range of the interaction, we have a Gaussian shape, \( v_{\Sigma,N,\Lambda}^0(r',r) = v_{\Sigma,N,\Lambda}^0(FR) \exp(-|r'-r|^2/\beta^2) \), where \( \beta \) is a range parameter. Here we choose \( v_{\Sigma,N,\Lambda}^0(FR) = -369.4 \) MeV and \( v_{\Sigma,N,\Lambda}^0(FR) = 205.2 \) MeV for \( \beta = 0.8 \) fm; these strength parameters correspond to a spin-averaged \( \Lambda N \) strength of \( \vec{v}_{\Lambda N}(FR) = -105.9 \) MeV with \( \beta = 0.8 \) fm, which reproduce the \( \Lambda \) binding energies for light \( p\)-shell nuclei. In the folding potential model, we realize that the radial shape of the \( \Sigma \Lambda \) coupling potential \( U_{\Sigma}(r) \) is more smoothly behaved and the range of \( U_{\Sigma}(r) \) becomes slightly extended. Thus we find that the \( \Sigma^- \) mixing probabilities for \( ^5\text{H}(1^+_\text{exc}) \) account for \( P_{\Sigma^-}(s\Sigma) = 0.13\% \) and \( P_{\Sigma^-}(p\Sigma) = 0.11\% \) in comparison with 0.13\% and 0.17\% shown in Table II. The integrated cross section for \( ^5\text{H}(1^+_\text{exc}) \) is \( d\sigma/d\Omega = 0.17 \) nb/sr and the \( (\pi^-, K^+) \) spectrum is not so modified. It seems that a value of \( P_{\Sigma^-}(p\Sigma) \) is relatively reduced whereas \( P_{\Sigma^-}(s\Sigma) \) is not changed. This modification may depend on nuclear structures of the \( ^5\text{H} \) and \( ^3\text{He} \) core states as well as properties of the two-body \( \Lambda N, \Sigma N \) and \( \Lambda N-\Sigma N \) effective interactions. Therefore, more investigation is needed to qualitatively clarify nuclear dynamics by sophisticated microscopic calculations.

### G. Two-step processes of \( \pi^- p \rightarrow K^0\Lambda \) followed by \( K^0 p \rightarrow K^+ n \)

Finally we discuss the integrated laboratory cross sections of \( d\sigma/d\Omega \) for \( ^5\text{H}(1^+_\text{exc}) \) by the two-step mechanism, \( \pi^- p \rightarrow K^0\Lambda \) followed by \( K^0 p \rightarrow K^+ n \) or \( \pi^- p \rightarrow \pi^0 n \) followed by \( \pi^0 p \rightarrow K^+\Lambda \) in the DCX \( ^6\text{Li}(\pi^-, K^+) \) reaction for production of the neutron-rich \( \Lambda \) hypernuclei [13]. We

![Fig. 8. Comparison between the calculated inclusive \( \Lambda \) spectrum of \( E_r = 4.0 \) MeV and that of \( E_r = 1.7 \) MeV for the energy of \( ^5\text{H}(1/2^+_{ex}) \) in the \( ^6\text{Li}(\pi^-, K^+) \) reaction at \( p_T = 1.2 \) GeV/c (7\(^a\)). \( V_{\Lambda} = -19 \) MeV is used. See also the caption in Fig. 7.](image-url)
roughly estimate the contribution of the two-step processes for \( \pi^- p \to K^+ p \) followed by \( K^0 p \to K^+ \Lambda \), which are expected to be a main component, rather than those for \( \pi^- p \to \pi^0 n \) followed by \( \pi^0 p \to K^+ \Lambda \). The sum of the cross sections by a harmonic oscillator model [46] for \(^6\text{Li}\) targets at \( p_\pi = 1.2 \) GeV/c (0°) is given as

\[
\sum_j \left( \frac{d\sigma_j}{d\Omega_L} \right)_{0^+} \approx 2\pi \xi \left( \frac{1}{r^2} \right) \left( \frac{\alpha}{d\sigma/d\Omega_L} \right)_{0^+} \phi_{\text{lab}}(K^0) \times \left( \frac{\alpha}{d\sigma/d\Omega_L} \right)_{0^+} \phi_{\text{lab}}(K^0)_{0^+} N_{pp}^{\Lambda},
\]

where \( \xi \approx 0.0370 \) mb\(^{-1}\) is the factor integrated over angle \( \phi_{\text{lab}}(K^0) \) for \( \pi^- p \to K^0\Lambda \) with \( \phi_{\text{lab}}(K^0) \) for \( K^0 p \to K^+ n \) to restore \( \phi_{\text{lab}} = 0^+ \) in the angular distributions of the two elementary processes, \( p_K \approx 0.842 \) GeV/c is the intermediate kaon momentum, and \( \langle 1/r^2 \rangle \approx 0.0280 \) mb\(^{-1}\) is the mean inverse-square radial separation of the proton pair. \( N_{pp}^{\Lambda} \approx 1 \) is the effective number of proton pairs including the nuclear distortion effects. The elementary laboratory cross section \( (d\sigma/d\Omega_L)_{0^+} \) is estimated to be \( \sim 0.55 \) mb/sr for \( \pi^- p \to K^0\Lambda \) or \( \sim 1.96 \) mb/sr for \( K^0 p \to K^+ n \), depending on the nuclear medium corrections. The results show \( \sum_j (d\sigma_j/d\Omega_L)_{0^+} \approx 1.4-1.9 \) mb/sr for \( \pi^- p \to K^0 n \) followed by \( K^0 p \to K^+\Lambda \), and also \( 0.20-0.34 \) mb/sr for \( \pi^- p \to \pi^0 n \) followed by \( \pi^0 p \to K^+\Lambda \). Considering the large momentum transfer \( q \approx 360 \) MeV/c in the \( (\pi^-, K^+) \) reactions, we expect that the production probabilities for loosely bound or resonant \( \Lambda \) states do not exceed \( 10^{-3} \)% in the quasielastic \( \Lambda n \) production, so that the cross section of \( \Lambda \) in the two-step mechanism may be on the order of \( 10^{-2} \) nb/sr at \( \phi_{\text{lab}} = 7^\circ \). This result suggests that the one-step mechanism, \( \pi^- p \to K^+ \Sigma^- \) via \( \Sigma^- \) doorways caused by the \( \Sigma^- p \to \Lambda n \) coupling is rather favored than the two-step mechanism.

### V. SUMMARY AND CONCLUSION

We studied phenomenologically the production of a neutron-rich hypernucleus \(^6\Lambda\) in the \(^6\text{Li}(\pi^-, K^+) \) reaction at 1.2 GeV/c, considering the DIWA in the one-step mechanism, \( \pi^- p \to K^+ \Sigma^- \) via \( \Sigma^- \) doorways caused by the \( \Sigma^- p \to \Lambda n \) coupling. We evaluated the production cross section of \(^6\Lambda\) \( (\Sigma^-) \) by using the coupled \((\Sigma^-) + (\Sigma^-) + 5\text{He}\) model with a spreading potential and compared it with the data of the missing mass spectrum at the J-PARC E10 experiment. The results are summarized as follows:

(i) The \( \Sigma^- \)-mixing probabilities in \(^6\Lambda\) \( (\Sigma^-) \) are \( P_{\Sigma^-} \lesssim 0.2 \)% both for the \( s_\Sigma \) state and for the \( P_{\Sigma^-} \) state in order to reproduce no significant peak in the \( \Lambda \) production data, so that the cross section of \(^6\Lambda\) in the two-step mechanism is less than on the order of \( 0.4 \) nb/sr.

(ii) The shape and magnitude of the near-\( \Lambda \)-threshold spectrum significantly depend on the \( \Sigma \Lambda \) coupling and \( \Lambda \) potentials.

(iii) The cross section of \(^6\Lambda\) \( (\Sigma^-) \) also sensitive to the structure of the \(^3\Lambda\) core nucleus independent of whether the \(^5\Lambda\) \( (\Sigma^-) \) ground state exists as a resonant state bound with a narrow width.

(iv) The one-step mechanism via \( \Sigma^- \) doorways seems to be rather favored over the two-step mechanism because the cross section of \(^6\Lambda\) in the two-step mechanism may be on the order of \( 10^{-2} \) nb/sr at \( \phi_{\text{lab}} = 7^\circ \) by the harmonic-oscillator model.

In conclusion, the calculated spectrum of the \(^6\Lambda\) hypernucleus by the one-step mechanism via \( \Sigma^- \) doorways can evaluate the near-\( \Lambda \)-threshold data of the DCX \(^6\text{Li}(\pi^-, K^+) \) reaction at 1.2 GeV/c. The result shows that the \( \Sigma^- \)-mixing probabilities in \(^6\Lambda\) \( (\Sigma^-) \) are \( P_{\Sigma^-} \lesssim 0.2 \)% both for the \( s_\Sigma \) state and for the \( P_{\Sigma^-} \) state in order to explain no significant peak in the \( \Lambda \) production spectrum obtained at the J-PARC E10 experiment. The sensitivity to the potential parameters implies that the nuclear \( (\pi^-, K^+) \) reactions with much less background experimentally provide the high ability to study precise wave functions for \( \Lambda \), \( \Sigma^- \), and \(^5\Lambda\) nuclear core in the neutron-rich \(^6\Lambda\) hypernucleus. Systematic analysis based on microscopic calculations is required for the extended J-PARC E10 experiment [47]. This investigation is in progress.

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