Incisional cyclic steps of permanent form in mixed bedrock–alluvial rivers

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Abstract Most bedrock river channels have a relatively thin, discontinuous cover of alluvium and are thus termed mixed bedrock–alluvial channels. Such channels often show a series of steps formed at relatively regular intervals. This bed form is the bedrock equivalent of cyclic steps formed on beds composed of cohesive soil in gullies. In this paper, we perform a full nonlinear analysis for the case of cyclic steps in mixed bedrock–alluvial channels to explain the formation of these steps. We employ the shallow water equations in conjunction with equations describing the process of bedrock incision. As a model of bedrock incision, we employ the recently introduced Macro-Roughness Saltation Abrasion Alluviation model, which allows direct interaction between alluvial and bedrock morphodynamics. The analysis is greatly simplified by making the quasi-steady assumption that alluvial processes occur much faster than bedrock erosional processes. From our analysis, we obtain the conditions for the formation of cyclic steps in bedrock, as well as the longitudinal profiles of bed elevation, water surface elevation, and areal fraction of alluvial cover. It is found from the analysis that when the sediment supply is small relative to the transport capacity, cyclic steps form only on slopes with very high gradients. The analysis indicates that the shape of a step formed on bedrock is characterized by a relatively short upstream portion with an adverse slope and a long, almost planar downstream portion with a constant slope.

1. Introduction

Where sediment transport capacity is dominant over sediment supply, as is commonly the case in mountain river reaches, it is often observed that bedrock is locally exposed in parts of the riverbed. If the riverbed is completely alluviated, even if the cover is thin, the channel may be able to reach a dynamic equilibrium associated with a constant sediment supply rate, without any bed degradation. A change in conditions (e.g., flow acceleration) is required to initiate bed degradation and eventually expose bedrock. If, however, the supply rate is insufficient to keep the bed covered with alluvium when the river is morphologically active, the bedrock can be very slowly, but repeatedly, eroded during floods, so that the cumulative bed incision over time can be substantial.

In recent years in particular, bedrock has been increasingly exposed in rivers throughout Japan, even in moderate-gradient reaches [Yamamoto, 2010]. This might be due to the decrease in sediment supply from mountain regions associated with dam construction and excessive erosion control. Bedrock exposure and erosion and the resulting bed degradation have caused the destabilization of river structures such as bridge piers and bank revetments [Yonezawa et al., 2007; Tadatsu et al., 2009; Mori et al., 2010; Inoue et al., 2011]. In addition, bedrock rivers that are devoid of sediment can provide very poor habitat for aquatic animals, as compared to mixed bedrock–alluvial or purely alluvial rivers [Inoue et al., 2011]. Therefore, the problem of bedrock exposure is important from an environmental as well as engineering point of view.

In such rivers with exposed bedrock, a series of steps is commonly observed to form at relatively uniform intervals. These steps are bedrock equivalents to cyclic steps formed on beds composed of cohesive soil [Parker and Izumi, 2000]. Kostic et al. [2010] provide an overview of cyclic steps in diverse settings, including those formed by flowing water in different stream types (alluvial, bedrock, and cohesive sediment) and by turbidity currents on the seafloor. Cyclic steps can also form on ice, for example, due to katabatic winds on Mars [Smith et al., 2013] or flowing water in supraglacial streams (Figure 1a), a process that has recently been
Figure 1. Cyclic steps observed in a variety of environments. (a) Cyclic steps in a supraglacial meltwater stream on the Llewellyn Glacier, British Columbia, Canada. An ice ax on the right bank with a length of about 0.7 m provides scale. Photo courtesy of Leif Karlstrom; more information about the site is provided in Karlstrom et al. [2014]. (b) Cyclic steps in the Nanatsugama Reach in Nishizawa Canyon, Yamanashi, Japan. The reach displays five successive steps, four of which are shown in the image. Step height is 2–3 m. Both exposed bedrock and alluvium can be seen just upstream of the lip of the step farthest downstream in the image. Image courtesy Yamanashi City Tourist Association. (c and d) Two steps of a train of cyclic steps in cohesive material in a discontinuous gully on the Loess Plateau, China. The step in Figure 1d is immediately downstream of the step in Figure 1c. There were over 11 steps in the gully at the time of the photograph (2012). The spacing between steps was of the order of tens of meters. The bed consists of weakly cohesive silty loess with no obvious internal structure, and no sand or gravel, and with a grass cover that is too sparse to protect against the entrainment of dust by even a mild wind.

 studied in a laboratory flume [Yokokawa et al., 2016]. Although in principle the terminology could apply to any streamwise train of steps in a channel, here we mean steps that are self-formed by a morphodynamic interaction between the bed and the flow above it. In addition, the flow over these steps under formative conditions is Froude-transcritical; each step contains an upstream region of supercritical flow followed by a downstream region of subcritical flow and is bounded upstream and downstream by hydraulic jumps [Parker and Izumi, 2000; Kostic et al., 2010] (see also Figure 2). Finally, the train of steps tends to migrate upstream. Here we study the case of cyclic steps, as defined above, in mixed bedrock-alluvial rivers. It should be noted that there are indeed periodic trains of steps in bedrock rivers, driven allogenically by faults or variation in erodibility [Goode and Wohl, 2010a, 2010b], but these are not the subject of our analysis.

Examples of cyclic steps have been observed extensively in mixed bedrock-alluvial rivers [Duckson and Duckson, 1995]. One such example, the five steps of Nanatsugama in Nishizawa Canyon, Yamanashi, Japan, is illustrated in Figure 1b. Both exposed bedrock and alluvium can be seen just upstream of the lip of the step
farthest downstream. Although real-time documentation of the evolution of these steps is not available to us, direct inspection by one of us (Yokokawa) shows that (a) gravel deposited on the downstream lip of the plunge pool armorsthe bedrock and is covered by moss, suggesting that the clasts have not moved recently, whereas gravel deposited at the upstream end of the plunge pool shows evidence of having moved vigorously and collided with the bed during plunging flow, suggesting the potential for upstream migration of these steps, and (b) the apparent absence of externally imposed constraints (such as faults) suggests that these steps are indeed autogenic. Direct observation for formation of autogenic cyclic steps associated with hydraulic jumps can be seen in the experimental bedrock-alluvial stepsof Yokokawa et al. [2013], Scheingross et al. [2015], and Scheingross [2016, chap. 6 therein]; upstream migration is alsodescribed in the latter two references. Here we pursue a physically based analysis to explain these steps. More specifically, we assume that the mechanism for differential incision of the steps into bedrock is abrasion caused by alluvial particles colliding with the bedrock surface and that the spatial pattern of incision is governed by an interaction between alluvial transport over a partially covered bedrock surface and the fluid mechanics of Froude-transcritical flow.

The first analytical study on cyclic steps was performed by Parker and Izumi [2000]. They considered the case of incisional steps in a cohesive bed which could be eroded by the force of the flow alone, without the aid of tools such as stones to cause wear. The cyclic steps that form in such a setting have also been termed “discontinuous gullies” [Reid, 1989]. Figures 1c and 1d illustrate the form of these steps in a gully on the Loess Plateau of China. The bed consists of poorly consolidated silt with an absence of internal structure. This material is easily eroded by the direct action of flow alone. The channel bed consists of sparse grass growing directly from the loess, with no layer of loose sediment. Parker and Izumi [2000] showed that the interaction of Froude-transcritical flow over a bed eroded by the direct action of water allows the formation of upstream-migrating trains of steps. The analytical framework admits a solution of permanent form, in which the train of steps incises and migrates upstream both at constant rates but otherwise does not change in morphology. The analysis we consider here falls within the same framework as Parker and Izumi [2000], but the way in which incision is treated is fundamentally different. The importance of tools for incision in the case of steps in mixed bedrock-alluvial streams, and their lack of relevance in the case of gullies in cohesive material, can be appreciated by comparing Figures 1b–1d.

Since the work of Parker and Izumi [2000], a number of studies have been devoted to the analytical and numerical modeling of cyclic steps in a variety of environments. For example, Sun and Parker [2005] found solutions of permanent form for cyclic steps in alluvium. Kostic and Parker [2006] and Fildani et al. [2006] have numerically modeled cyclic steps created by turbidity currents. Yokokawa et al. [2016] provide an analytical framework which describes incipient (but not fully formed) cyclic steps at ice-water interfaces.

The key to the analysis of cyclic steps in the case of bedrock-alluvial channels is in the treatment of alluvial-bedrock interaction. Here we use a modified version of the saltation-abrasion model of bedrock wear due to colliding bed load particles first proposed by Sklar and Dietrich [2004]. We embed this model into the Macro-Roughness Saltation Abrasion Alluviation (MRSAA) model of bedrock-alluvial morphodynamics of Zhang et al. [2015].
2. Formulation

2.1. Erosion of Bedrock

In bedrock rivers, one way that bedrock can be eroded is by the process of abrasion due to gravel bed load particles striking exposed bedrock. Formulations of this abrasion process have been proposed by several researchers. Among them, Sklar and Dietrich [2004] described the erosion rate $E_d$ (length/time) in the form

$$E_d = \beta q_{ad} \left( \frac{1 - q_{ad}}{q_{acd}} \right)$$

where $\beta$ is an abrasion coefficient with the dimension of (length scale)$^{-1}$ and $q_{ad}$ is the volumetric gravel transport rate per unit width. In addition, $q_{acd}$ is the volumetric gravel transport capacity per unit width that would prevail were the bed completely alluviated with no exposed bedrock, a parameter which we assume to be determined solely by the local bed shear stress. In the above equation, then, the term $1 - q_{ad}/q_{acd}$ reflects the fact that some fraction of bed surface area may be composed of exposed bedrock rather than alluvium. The bedrock erosion function defined above, including both effects of sediment availability and partial alluvial cover is a landmark contribution to the morphodynamics of bedrock rivers. However, there is a limitation on the application of the model of Sklar and Dietrich [2004], in the sense that $q_{ad}$ represents the sediment transport rate limited by the sediment supply which is not directly connected to local morphodynamics [Zhang et al., 2015].

Any global tendency toward riverbed incision is likely determined by conditions of sediment supply from upstream. However, when we focus on a local river reach, the net sediment discharge should be strongly affected by local hydraulic parameters and bed topography of the surrounding area. Based on this idea, Izumi and Yokokawa [2011], Izumi et al. [2012], and Tanaka and Izumi [2013] have made a modification of Sklar and Dietrich’s model [Sklar and Dietrich, 2004]. They assume that the cover fraction $p$, defined as the local mean areal fraction of bedrock surface covered with alluvium, is determined by the balance between the rates at which sediment enters and leaves the site in question. They employ a form of the Exner equation of sediment continuity adapted to include partial cover to describe the spatial variation of $p$. Recently, Zhang et al. [2015] have introduced the concept of a macroroughness intrinsic to bedrock into this modified formulation. Their model, which is termed the Macro-Roughness Saltation Abrasion Alluviation (MRSAA) model, is explained in detail below.

In the MRSAA model, bedrock is assumed to have a geometric roughness with the characteristic length scale $L_{mr}$, here called a macroroughness (so as to distinguish it from the roughness height that enters into the logarithmic velocity law for turbulent flow over a hydraulically rough bed). Where the local sediment supply is small relative to the transport capacity, any deposition that occurs will be discontinuous, causing infilling of local interstices, but not completely covering the bedrock, which remains widely exposed. As the local sediment transport rate increases, the exposed area of bedrock decreases, and when the local sediment supply increases above some value corresponding to a capacity state, the bedrock is fully covered with sediment.

The elevation of the bedrock at the base of the macroroughness elements is denoted by $\eta_{bd}$. The time variation of $\eta_{bd}$ corresponds to the incision rate $-E_d$, where $E_d$ can be described by (1) rewritten by the use of the cover fraction $p$ representing the areal fraction of the bed that is covered with alluvium [Zhang et al., 2015]. That is,

$$\frac{\partial \eta_{bd}}{\partial t_d} = -\beta p q_{acd} (1 - p)$$

where $t_d$ is time. In these variables, the subscript $d$ indicates a dimensional variable that will later be represented in dimensionless form with the subscript removed.

Denoting the local mean thickness of alluvium deposited on bedrock by $\eta_{ad}$, the time variation of $\eta_{ad}$ can be assumed to be expressed by the sediment continuity equation of the form

$$p \frac{\partial \eta_{ad}}{\partial x_d} = -\frac{1}{1 - \lambda} \frac{\partial p q_{ad}}{\partial x_d}$$

where $\lambda$ is the porosity of the alluvium and $x_d$ is the streamwise coordinate. The cover fraction $p$ on the left in the above equation expresses the fact that the time variation of $\eta_{ad}$ takes place only in part of the bed...
covered with alluvium. As described above, the cover fraction \( p \) can be expected to increase with the alluvium thickness \( \eta_{ad} \). The simplest quantification that captures this tendency is as follows:

\[
p = \begin{cases} 
\frac{\eta_{ad}}{L_{mrd}} & \text{when } 0 \leq \eta_{ad} \leq L_{mrd} \\
1 & \text{when } \eta_{ad} > L_{mrd}
\end{cases}
\]  

(4)

Here \( L_{mrd} \) is the macroroughness height. The implication of the above equation is that when the alluvial thickness is smaller than the macroroughness height, the local areal fraction of the bed that is covered with sediment rather than exposed bedrock, i.e., the areal cover fraction \( p \), is equal to the ratio of the alluvial thickness to the macroroughness height, and \( p \) is unity when the alluvial thickness is larger than the macroroughness height (thus completely burying the roughness elements). The total bed elevation \( \eta_d \) is then expressed by

\[
\eta_d = \eta_{ad} + \eta_{bd}
\]  

(5)

### 2.2. Flow Equations

As noted above, cyclic steps are a series of upstream-migrating steps, each of which is composed of a gently (or adversely) sloping Froude-subcritical upstream portion, and a steeply sloping Froude-supercritical downstream portion as shown in Figure 2. Though the downstream transition from the mild to steep bed slope is continuous, the transition from the steep to mild bed slope is discontinuous. Correspondingly, subcritical flow on the mild slope gradually accelerates to supercritical flow as slope increases downstream, whereas the transition from supercritical flow on the steep slope to subcritical flow on the mild slope is abrupt and is accompanied by a hydraulic jump. In this paper, the upstream end of a step is defined as the location just downstream of a hydraulic jump, and the downstream end is defined as the location just upstream of the next hydraulic jump.

When the characteristic length scale of change in the streamwise direction is much larger than that in the upward normal (approximately vertical) direction, the flow can be described by the shallow water equations of momentum and mass balance:

\[
\frac{\partial u_d}{\partial x_d} = -g \frac{\partial h_d}{\partial x_d} - g \frac{\partial \eta_d}{\partial x_d} - \frac{\tau_{bd}}{\rho h_d}
\]  

(6)

\[
u_d h_d = q_{wd}
\]  

(7)

where \( u_d, h_d, \eta_d \) and \( x_d \) are the depth-averaged velocity, flow depth, total bed elevation, and horizontal streamwise coordinate, respectively, \( g \) is the gravitational acceleration (\( = 9.8 \text{ m/s}^2 \)), \( \rho \) is the water density (\( = 1000 \text{ kg/m}^3 \)), and \( q_{wd} \) is the flow discharge per unit width, here assumed to be constant. Note that here we have applied the standard quasi-steady assumption of alluvial morphodynamics, according to which the characteristic time for morphological change of the bed (such as the formation of cyclic steps) is very slow compared to the characteristic time for the flow to respond to changed bed, thus allowing us to drop the time derivative terms in the above equations. In (6), \( \tau_{bd} \) is the bed shear stress, which we relate to the bed friction coefficient \( C_f \) in the form

\[
\tau_{bd} = \rho C_f u_d^2
\]  

(8)

The bed friction coefficient \( C_f \) is in general a weak function of relative flow depth (flow depth/roughness height), but here we assume it to be a constant for simplicity. This issue is discussed in more detail below.

The volumetric sediment transport rate per unit width over a completely alluviated bed is here denoted as \( q_{acd} \), where the subscript \( c \) denotes capacity. At this point it is not necessary to specify a relation for this sediment transport rate. It is enough to assume that the transport rate increases monotonically with bed shear stress \( \tau_{bd} \), and thus in accordance with (8), flow velocity \( u_d \).
3. Theoretical Development

3.1. Threshold State for Incision

As illustrated in Figure 3a, a steady, uniform (normal) threshold state for incision prevails when the bedrock is just barely drowned in alluvium, so that \( p = 1 \) and according to (2), \( \partial \eta_{ad} / \partial t_j = 0 \). Once water discharge \( q_{wd} \), alluvial sediment supply rate \( q_{asd} \), and a sediment discharge relation are specified, the threshold slope \( S_t \) at which this supply rate corresponds to the capacity rate can be computed. In this steady, uniform threshold state for incision, (6) reduces with (7) and (8) to

\[
C_f F_r^2 = S_t, \tag{9a}
\]

\[
F_r^2 = \frac{u_{td}^3}{g q_{wd}} \tag{9b}
\]

where \( u_{td} \) denotes the flow velocity at this threshold state. The corresponding depth \( h_{td} \) at the threshold state is then given as \( q_{wd} / u_{td} \).

As assumed implicitly at the beginning of section 1, and in the above paragraph, we assume that the sediment supply rate \( q_{asd} \) is constant at least during floods when significant bedrock incision takes place. However, it is recognized that sediment supply in steep mountain rivers can be episodic, exhibiting both short-term variation and long-term variation [Ashida et al., 1981; Benda and T. Dunne, 1997a, 1997b; Montgomery and Buffington, 1997; Istanbulluoglu et al., 2004]. As such, our assumption of constant sediment supply is not strictly valid. Nevertheless, for our purposes the issue is the relative difference in time scale between the variation of sediment supply and the bed evolution due to bedrock incision. If the time scale of bed evolution is much longer than that of the variation of sediment supply, the assumption of constant sediment supply is expected to be a good approximation. This notwithstanding, even if the time scale of the bed evolution is equivalent to or shorter than that of the variation of sediment supply, the assumption of constant sediment supply corresponds to the first and simplest approximation to employ in studying a problem with more complex elements, i.e., the process of formation of incisional cyclic steps on bedrock.

3.2. Nondimensionalization and Quasi-Steady Assumption of Alluvial Processes

We use the above-defined threshold state to render the governing equations dimensionless. As noted above, no incision can occur at the threshold state, because \( p = 1 \). This notwithstanding, the parameter \( \beta q_{asd} \), the alluvial sediment supply rate multiplied by the abrasion coefficient, can serve as a scale for the incision rate, in so far as \( p \) is a nondimensional variable between 0 and 1 in (2). The time for the bed to erode one depth \( h_{td} \)
Thus scales as $h_{td}/(\beta q_{asd})$. We use the backwater length $h_{td}/S_t$ as a horizontal length scale. We then employ the following equations to nondimensionalize the problem:

\begin{align}
  t &= \frac{t_d}{h_{td}/(\beta q_{asd})}, \\
  x &= \frac{x_d}{h_{td}/S_t}, \\
  q_{ac} &= \frac{q_{acd}}{q_{asd}}, \\
  (\eta, \eta_a, \eta_b, L_{mr}) &= \left( \frac{\eta_d, \eta_{ad}, \eta_{bd}, L_{mr}}{h_{td}} \right)
\end{align}

Here the parameters $t, x, q_{ac}, \eta, \eta_a, \eta_b, \text{ and } L_{mr}$ are the dimensionless versions of their dimensioned corresponding counterparts $t_d, x_d, q_{acd}, \eta_d, \eta_{ad}, \eta_{bd}$, and $L_{mr}$. With the use of the above nondimensionalization, (2)–(5) can be rewritten in the forms

\begin{align}
  \frac{\partial \eta_b}{\partial t} &= -pq_{ac}(1-p) \\
  \gamma p \frac{\partial \eta_a}{\partial t} &= -\frac{\partial q_{ac}}{\partial x} \\
  p &= \begin{cases} 
  \frac{\eta_a}{L_{mr}} & \text{when } 0 \leq \eta_a \leq L_{mr} \\
  1 & \text{when } \eta_a > L_{mr}
\end{cases}
\end{align}

where the nondimensional parameter $\gamma$ is

$$\gamma = \frac{\beta q_{asd}}{q_{acd}/((1-\lambda)h_{td}/S_t)} = \frac{(1-\lambda)\beta h_{td}}{S_t}$$

The numerator and the denominator of $\gamma$ scale the characteristic erosion rate of bedrock, and the typical speed of alluvial sediment deposition and erosion, respectively; therefore, $\gamma$ represents the ratio between the two (referred to as the incisional-alluvial speed ratio, hereafter). Here we consider the common case of bedrock that is so strong that the bedrock erosion is much slower than alluvial processes, so that $\gamma$ is expected to be small. (This condition is not universal: in some cases the bedrock is so weak and erosion prone that it is alluvial cover that prevents rapid degradation [Tanise et al., 2008].) Thus, assuming $\gamma = 0$ and dropping the terms with $\gamma$ in (12) is equivalent to ignoring the unsteadiness of the alluvial process. This corresponds to a second quasi-steady condition in addition to the standard quasi-steady assumption for the flow introduced above, according to which flow can be assumed to be steady in morphodynamic problems. According to this second condition, alluvial thickness $\eta_{ad}$ is assumed to respond quickly to changes in bedrock elevation $\eta_{bd}$. Under this constraint, at the time scale of bedrock erosion, the alluvial thickness $\eta_{ad}$ responds “immediately,” so that alluvial processes can be assumed to be steady. In this paper, the assumption that alluvial processes can be assumed to be steady is referred to as the “quasi-steady assumption of alluvial processes.”

Dropping terms multiplied by $\gamma$ with the use of the quasi-steady assumption of alluvial processes in (12), we obtain the result that the nondimensional net volumetric gravel transport rate $q_a$, equivalent to $pq_{ac}$, must be constant in space. Because this constant volumetric gravel transport rate is nothing other than the dimensionless alluvial sediment supply rate $q_{as}$, which is equivalent to unity as all the quantities of sediment transport rate are nondimensionalized by the dimensionless alluvial sediment supply rate $q_{as}$. We obtain

$$pq_{ac} = 1$$

The above equation implies that the areal cover fraction $p$ is inversely proportional to the dimensionless volumetric gravel transport capacity $q_{ac}$. In so far as it is assumed that the sediment supply is constant, and the volumetric gravel transport rate increases with increasing bed shear stress, it follows that the cover fraction $p$ decreases with increasing bed shear stress and increases with decreasing bed shear stress. In other words,
under the condition that the sediment transport rate is constant but below capacity ($q_{acd} > q_{asd}$, or $q_{ac} > 1$ in the dimensionless form), a relatively large alluvial cover fraction ($p$ close to the maximum value of unity) prevails in reaches where the dimensionless sediment transport capacity is small ($q_{acd}$ is close to the minimum value of unity in the dimensionless form), whereas a relatively small cover fraction ($p \ll 1$) prevails in reaches where the dimensionless sediment transport capacity is large ($q_{acd} \gg q_{asd}$, or $q_{ac} \gg 1$ in the dimensionless form).

From (13) and (16), the alluvial thickness $\eta_a$ is expressed in the form

$$\eta_a = \frac{L_{mr}}{q_{ac}}$$

(17)

With the use of (16), (11) reduces to

$$\frac{\partial \eta}{\partial t} = -(1 - q_{ac}^{-1})$$

(18)

From (14), (17), and (18), we find that the time variation of the total bed elevation $\eta$ is expressed by the relation

$$\frac{\partial \eta}{\partial t} = L_{mr} \frac{\partial q_{ac}^{-1}}{\partial t} - (1 - q_{ac}^{-1})$$

(19)

### 3.3. Conditions at the Upstream End of a Step

*Parker and Izumi* [2000] employed the assumption that a threshold condition is achieved at the upstream end of a step in the analysis of cyclic steps by flow over a cohesive bed. This is because, even though solutions for cyclic steps are possible without this assumption, such solutions may represent unstable finite amplitude equilibria. That is, any small perturbation would result in the initiation of erosion at the upstream end of each step, which would then only be completely stabilized when the threshold condition for bed erosion is achieved just beyond the hydraulic jump. The same assumption is made in this treatment of bedrock incision as well. (This is only a local condition; it is shown below that the train of steps as a whole can continue to incise downstream, because the point of zero incision corresponds to an upstream moving boundary.) According to (2), bedrock incision vanishes either when the net volumetric gravel transport rate vanishes ($pq_{acd} = 0$), or when the bedrock is fully covered with sediment ($p = 1$). Because $pq_{acd}$ is constant in space, if $pq_{acd}$ were to vanish at the upstream end, there would be no sediment transport anywhere on the step, thus resulting in no bed erosion anywhere either. In so far as we are not interested in this case, we assume that at the upstream end of the reach (just downstream of a hydraulic jump), the bedrock is fully but barely covered with sediment, and thus $p = 1$ there. In addition, the condition $p = 1$ must be achieved only precisely at the upstream end because, if there were a finite reach where no erosion takes place, cyclic steps of permanent form migrating upstream with constant wave velocity (see Figure 5 subsequently shown) could not exist. (Indeed, were a train of steps subject to erosion without downcutting, not only could a permanent form not be maintained, but the steps themselves would be completely obliterated.) As a result, the following incision threshold condition (not to be confused with the threshold of motion of the gravel) is satisfied right at the upstream end of a step:

$$q_{ac} = 1$$

(20)

Since the upstream end of a step is immediately downstream of a hydraulic jump, it follows that the Froude number at this point of incision threshold should be less than unity, i.e., $Fr = Fr_1 < 1$.

### 3.4. Equilibrium Incision in the Absence of Steps

In addition to the threshold state for incision, it is of value to define one more steady, uniform (normal) state, i.e., that corresponding to incision in the absence of steps. Nondimensional parameters defined in these two steady uniform states are helpful for facilitating the subsequent analysis. Variables in this normal state are denoted by the subscript $n$, which denotes “normal.” To define this uniform state, we consider a flow for which all parameters (including water discharge per unit width $q_{wd}$ and sediment supply rate $q_{asd}$) are the same as the threshold state except the bed slope $S$, which is now taken to be higher than $S_\ast$. As shown in Figure 3b, this perfors corresponds to a cover fraction at this normal state satisfying the condition $p_n < 1$, so that the bed
Figure 4. Conceptual diagram for bed erosion in the absence of steps.

Incises everywhere. The normal flow velocity at this state \( u_n \) accordingly exceeds \( u_t \). Denoting Froude number corresponding to this equilibrium state by \( F_{rn} \), we obtain the relations

\[
C_f F_{rn}^2 = S, \quad (21a)
\]

\[
F_{rn}^2 = \frac{u_{nd}^3}{g q_{nad}}, \quad (21b)
\]

Note that since \( S > S_t \), it follows that \( F_{rn} > F_{rt} \). Since the bed is only partially covered with sediment, it undergoes incision at the dimensionless vertical rate \( w_n \) and corresponding dimensioned rate \( w_{nd} \), where

\[
w_n = \frac{w_{nd}}{\beta q_{nad}} = p_r q_{acn} (1 - p_n) = 1 - \frac{1}{q_{acn}}, \quad (22)
\]

In the above equation, \( q_{acn} \) denotes the nondimensional capacity sediment transport rate at the equilibrium (normal-flow) incision condition in the absence of steps. The condition that the sediment transport rate be equal to the supply yields the dimensioned relation \( q_{sad} = p_r q_{acn} \) where \( q_{acn} \) is the dimensioned version of \( q_{acn} \), or rewriting in dimensionless form with (10c), \( 1 = p_r q_{acn} \).

As shown in Figure 4, there is a simple geometrical relation \( w_{nd} = c_{nd} S \) between the constant (dimensioned) vertical bed degradation rate \( w_{nd} \) and the constant (dimensioned) rate of upstream migration of the bed profile in the absence of steps \( c_{nd} \). Nondimensionalizing \( c_{nd} \) by \( \beta q_{sad} / S_t \), we obtain the following relation between \( w_n \) and \( c_n \):

\[
w_n = c_n S_t, \quad (23a)
\]

\[
S_t = \frac{S}{S_t} \quad (23b)
\]

In addition, from (9a) and (9b), and (21a) and (21b), \( S_t \) can be expressed in terms of \( F_{r} \) and \( F_{rn} \) by the relation

\[
S_t = \frac{F_{r}^2}{F_{rn}^2} \quad (24)
\]

3.5. Cyclic Steps of Permanent Form

Certain systems of equations admit solutions of permanent form. That is, after translation to account for constant migration rates in the streamwise and vertical direction, the governing equations reduce to a form in which only spatial variation remains. A classical case of a solution of permanent form is that for roll waves: there is a state at which the waves are periodic in space and migrate upstream at a constant speed [e.g., Balmforth and Vakil, 2012]. Parker and Izumi [2000] have shown that the equations governing cyclic steps in cohesive material admit a solution of permanent form. Here we demonstrate that a similar solution exists for the case of cyclic steps in mixed bedrock-alluvial streams. That is, the wave train of steps migrates upstream and downward at constant rates, but otherwise does not change in time. In order to describe a stepped bed migrating
upstream with nondimensional wave speed $c$, and a nondimensional additive degradation rate additional to that caused by a parallel shift in the horizontal direction $w_a$, we introduce the expression for $\eta$ in the form

$$\eta(x, t) = \tilde{\eta}(\tilde{x}) - w_a t.$$  \hspace{1cm} (25a)

$$\tilde{x} = x + ct.$$ \hspace{1cm} (25b)

where the tilde indicates a coordinate moving with steps and $c$ and $w_a$ are parameters that have been made dimensionless from their corresponding dimensional parameters $c_d$ and $w_a$, using the scales $\beta q_a S$ and $\beta q_a$, respectively. In general, the upstream migration rate in the presence of steps $c$ is different from the value $c_d$ in the absence of steps. In addition, $w_a$ is not a total vertical degradation rate, but instead a vertical degradation rate additional to that associated with degradation caused by purely horizontal migration (see Figure 5). Denoting the wavelength nondimensionalized by $h_{vd}/S$ by $L$, and the wave height nondimensionalized by $h_{vd}$ by $\Delta \eta$, we obtain the following relation:

$$\Delta \eta = \tilde{\eta}(\tilde{x}) - \tilde{\eta}(\tilde{x} + L).$$ \hspace{1cm} (26)

In this paper, we idealize the problem to an infinitely long river channel with a constant mean bed slope, so that upstream and downstream influences never extend into the target domain. When this assumption is applied to the case of bedrock erosion considered in this paper, the mean bed slope is a prescribed value equal to the bed slope $S$ in the absence of steps, as shown in Figure 4. Keeping in mind the vertical and horizontal length scales used in the nondimensionalization, the slope $S$ can be written, with the use of $\Delta \eta$ and $L$, in the form

$$\frac{S}{S_i} = S_r = \frac{\Delta \eta}{L}.$$ \hspace{1cm} (27)

The total nondimensional vertical degradation rate $w_r$ is then

$$w_r = -\frac{\partial \bar{\eta}}{\partial t}$$ \hspace{1cm} (28)

where the overbar denotes averaging over one wavelength, defined as $\bar{F} = \int_{\tilde{x}}^\tilde{x+L} F(\xi) \, d\xi / L$. Substituting (25a) and (25b) into (28), we obtain

$$w_r = w_a - c \frac{\partial \bar{\eta}}{\partial \tilde{x}} = w_a + c S_r.$$ \hspace{1cm} (29)

It is found from the above equation that the total vertical degradation rate $w_r$ is the sum of the vertical degradation caused by horizontal migration $c S_r$ and the additional vertical degradation rate $w_a$ as shown in Figure 5.
Nondimensionalizing the velocity and flow depth in (6) and (7) by \(u_{ad}\) and \(h_{ad}\) respectively, substituting with (25a) and (25b), and eliminating \(h\), we obtain

\[
(F r_t^2 u - u^{-2}) \frac{d u}{d x} = \frac{d \eta}{d x} - u^3
\]

(30)

where the \(\sim\) has been (and is henceforth) dropped for simplicity. Substituting (25a) and (25b) into (19), and dropping \(\sim\) again, we obtain

\[
\frac{c}{L_m} \frac{d \eta}{d x} = w_a + c L_m \frac{d q_{ac}^{-1}}{d x} - (1 - q_{ac}^{-1})
\]

(31)

Substitution of the above equation into (30) yields

\[
\frac{d u}{d x} = c^{-1} \left(1 - q_{ac}^{-1} - w_a\right) - u^3
\]

(32)

where \(q_{ac,u} = \partial q_{ac} / \partial u\). The spatial distribution of velocity \(u\) can be obtained by integrating the above equation with respect to \(x\).

Taking the origin of \(x\) at the upstream end of a step, where the threshold condition for bed incision (barely complete alluvial cover) is achieved, we have the following boundary condition:

\[
u(0) = 1
\]

(33)

At the downstream end of a step, the flow condition just before the next hydraulic jump downstream is realized. The flow velocity must thus be equal to the conjugate value of the velocity at \(x = 0\). That is,

\[
u(L) = \left[\frac{1 + 8 F r_t^2}{2}\right]^{-1}
\]

(34)

Once \(u\) is obtained as a function of \(x\), (31) can be integrated with respect to \(x\), and the distribution of \(\eta\) can be obtained in the form

\[
\eta(x) = L_m \left[q_{ac}^{-1}(u(x)) - q_{ac}^{-1}(u_L)\right] + \frac{1 - w_a}{c} \int_x^L q_{ac}^{-1}(u(\xi)) d\xi
\]

(35)

where \(u_L = u(L)\). Substituting \(x = 0\) in the above equation, we obtain the following relation for step height \(\Delta \eta\):

\[
\Delta \eta = L_m \left[1 - q_{ac}^{-1}(u_L)\right] + \frac{1 - w_a}{c} L - \frac{c}{L_m} q_{ac}^{-1}
\]

(36)

Modifying the above equation with the use of (24), (26), and (27), we obtain

\[
w_a = 1 - q_{ac}^{-1} + c \left[\frac{L_m}{L} \left(1 - q_{ac}^{-1}\right) - \frac{F r_n^2}{F r_t^2}\right]
\]

(37)

This equation specifies the relation between \(w_a\), \(c\), \(L\), \(F r_n\), and \(F r_t\).

### 3.6. Bed Load Function

In order to integrate (32), the capacity transport rate \(q_{ac}\) has to be specified as a function of velocity \(u\). Here we assume that the mode of sediment transport is bed load. In addition, we assume for simplicity that the sediment can be characterized by a single grain size and use the Meyer-Peter and Müller [1948] formula to predict the bed load transport rate. This relation takes the form

\[
q_{ac} = 8 \left(\theta - \theta_c\right)^{1/2} \sqrt{R_s g d^3}
\]

(38)

where \(\theta\) denotes the dimensionless Shields number, defined here as \(\theta = C_j u_d^2 / (R_s g d)\), and \(\theta_c\) denotes the critical value of \(\theta\) for incipient motion of sediment. In the above equation \(R_s\) is the submerged specific gravity.
of sediment (= 1.65 for quartz), and \( d_{ac} \) is the characteristic sediment diameter. The parameters \( q_{ac} \) and \( q_{ac,wc} \), i.e., the derivative of \( q_{ac} \) with respect to \( u \), can now be written in the dimensionless forms

\[
q_{ac} = \left( \frac{u^2 - u_c^2}{1 - u_c^2} \right)^{3/2}, \tag{39a}
\]

\[
q_{ac,wc} = \frac{3u(u^2 - u_c^2)^{1/2}}{(1 - u_c^2)^{3/2}}, \tag{39b}
\]

where \( u_c \) is the critical velocity for incipient sediment motion nondimensionalized by \( u_{ac} \), and it is defined in terms of a nondimensional critical bed shear stress \( \theta_c \), and the nondimensional bed shear stress \( \theta \), corresponding to \( u_{ac} \). The parameters \( u_c \) and \( \theta_c \) can be rewritten in the forms

\[
u_c = \sqrt{\frac{\theta_c}{\theta}}, \tag{40a}\]

\[
\theta_t = \frac{Ct_{id}u_c^2}{R_s g d_{id}} \tag{40b}
\]

In the succeeding section, we obtain \( u \) and \( \eta \) by solving (32) numerically.

### 3.7. Regularity Condition

As described above, the flow is in the Froude-subcritical regime in the upstream portion of a step, and in the Froude-supercritical regime in the downstream portion. Therefore, there must exist a point of transition from subcritical to supercritical regime somewhere on the step. The mathematical implication of this transition is that the denominator of the differential equation (32) changes sign from negative to positive and vanishes subcritical to supercritical regimes somewhere on the step. The mathematical implication of this transition is

Froude-supercritical regime in the downstream portion. Therefore, there must exist a point of transition from subcritical to supercritical regime somewhere on the step. The mathematical implication of this transition is that the denominator of the differential equation (32) changes sign from negative to positive and vanishes subcritical to supercritical regimes somewhere on the step. The mathematical implication of this transition is

\[
u_c = \sqrt{\frac{\theta_c}{\theta}}, \tag{40a}\]

\[
\theta_t = \frac{Ct_{id}u_c^2}{R_s g d_{id}} \tag{40b}
\]

In the succeeding section, we obtain \( u \) and \( \eta \) by solving (32) numerically.

In Figure 6, the upper limit of \( Fr_c \) for the existence of steps is shown on the \( u_c-Fr_c \) plane. The solid, dashed, and dotted curves correspond to the cases \( L_{mr} = 0.05, 0.1, \) and \( 0.2 \), respectively. In the region above each curve, cyclic steps cannot be uniquely specified. It is found that the upper limit of \( Fr_c \) approaches unity as \( u_c \) approaches unity as well and takes a nearly constant value when \( u_c \) is sufficiently below unity. In addition,

\[
Fr_c^2 u_c - u_c^2 - L_{mr}q_{ac,wc}(u_{ac})q_{ac}^2(u_c) > 0 \tag{42}
\]
the upper limit of $F_{tn}$ decreases with increasing $L_{mr}$. The sediment transport rate becomes large, and thus, the effect of alluvial processes in (42) (i.e., the effect of terms containing the parameter $q_{ac}$) becomes significant when $F_{tn}$ and $L_{mr}$ are large, and $u_c$ is sufficiently below unity. Therefore, the physical implication of Figure 6 is that cyclic steps cannot form when the effect of alluvial process becomes so significant that (42) cannot be satisfied. It should be recalled in this regard that no bedrock-alluvial cyclic steps can form over a completely alluviated bed, as illustrated in Figure 3a.

3.8. Upper and Lower Limits of $w_a$

We now consider the character of the boundary value problem associated with the first-order ordinary differential equation (32). For each set of specified values of the sediment supply rate per unit width $q_{sd}$, the water discharge per unit width $q_{wd}$, the average slope $S$, the bed friction coefficient $C_f$, and the two Froude numbers $F_{tn}$ and $Fr_n$ are specified as well. Once the two Froude numbers are specified, the first-order differential equation (32) for $u(x)$ with the three unknown parameters $c$, $w_a$, and $L$ can now be solved under the two boundary conditions (33) and (34), and the two additional constraints (37) and (41). The numerical solution of differential equation (32) for a set of specified values of two Froude numbers requires an iterative process. It is convenient to specify $F_{tn}$ and $w_a$ and then compute $Fr_n$ and the remaining parameters. In the numerical solution, $w_a$ cannot be selected arbitrarily. There is an upper bound $w_{au}$ and also a lower bound $w_{al}$, outside of which a solution cannot be realized. These bounds are specified below.

At the transition point from the subcritical to supercritical regime, (32) in conjunction with (41) takes an indeterminate form $du/dx = 0/0$. Applying L'Hôpital's rule, we obtain

$$\left.\frac{du}{dx}\right|_{u=u_1} = \frac{u_1^2 q_{ac,u}(u_1) q_{ac}^{-1}(u_1)}{(1 - q_{ac}^{-1}(u_1) - w_a) Q} - \frac{3u_1^2}{Q}$$

(43)

$$Q = F_{tn}^2 + 2u_1^2 + L_{mr}q_{ac}^{-2}(u_1) \left(2q_{ac,u}(u_1)q_{ac}^{-1}(u_1) - q_{ac,uu}(u_1)\right)$$

(44)

In (44), the term containing $L_{mr}$ is generally small compared with $F_{tn}^2 + 2u_1^2$, and, therefore, $Q$ is found to take a positive value. Therefore, $du/dx|_{u_1}$ becomes infinite as $w_a$ approaches $1 - q_{ac}^{-1}(u_1)$ from below. When $w_a$ is slightly larger than $1 - q_{ac}^{-1}(u_1)$, $du/dx|_{u_1}$ takes a large negative value, which is not physically realistic because this means that the flow decelerates at the transition from the subcritical to supercritical regime. Thus, the following upper bound holds:

$$w_{au} = 1 - q_{ac}^{-1}(u_1)$$

(45)

The above equation in conjunction with (41) also implies that $c$ must be positive, so that in accordance with the coordinate system defined above, cyclic steps always migrate in the upstream direction according to the present model.
Figure 7. Lower and upper limits \( w_{al} \) and \( w_{au} \) on \( w_a \) as functions of \( Fr_t \) for the cases (a) \( L_{mr} = 0.1, u_c = 0.2 \), and (b) \( L_{mr} = 0.1, u_c = 0.8 \). The solid and dashed lines correspond to \( w_{al} \) and \( w_{au} \), respectively.

The lower bound \( w_{al} \) is derived from the condition that the flow must accelerate over the whole domain of a step. As \( w_a \) decreases, the condition \( du/dx < 0 \) is first satisfied at the downstream end of the domain. Therefore, we require that \( du/dx\big|_{u_L} > 0 \). This condition may be rewritten in the form

\[
\frac{1-q_{ac}^{-1}(u_1) - q_{ac}^{-1}(u_L)}{1-q_{ac}^{-1}(u_1) - q_{ac}^{-1}(u_L)} > 0
\]

Thus, the lower bound satisfying the above equation \( w_{al} \) is

\[
w_{al} = 1 + \frac{u_1^2/q_{ac}(u_1) - u_L^2/q_{ac}(u_L)}{u_1^2 - u_L^2} > 0
\]

Figures 7a and 7b show \( w_{al} \) and \( w_{au} \) as functions of \( Fr_t \) for the cases \((L_{mr}, u_c) = (0.1, 0.2) \) and \((0.1, 0.8) \), respectively. In both figures, the solid and dashed curves indicate \( w_{al} \) and \( w_{au} \) respectively. Cyclic steps of permanent form are possible only between the solid and dashed curves. In both cases, cyclic steps of permanent form can exist only within an ever-narrower range of values \( w_a \) close to unity when \( Fr_t \) becomes smaller than 0.5.

3.9. Solution Domains on the \( Fr_t-Fr_n \) Plane

The solution domain on the \( Fr_t-Fr_n \) plane for cyclic steps for the case \( L_{mr} = 0.1 \) and \( u_c = 0.5 \) is illustrated in Figure 8. It can be seen from Figure 8 that a necessary but insufficient condition for step formation is \( Fr_t < 1 \) and \( Fr_n > 1 \). The figure is obtained by calculating values of \( Fr_n \) as \( w_a \) is changed from \( w_{al} \) to \( w_{au} \) for a given value of \( Fr_t \). The region in which cyclic steps are not solvable due to the lack of a regularity condition shown in Figure 6 is also shown on the right-hand side of Figure 8, i.e., in the range where \( Fr_t \) becomes greater than a value that is somewhat smaller than unity. While this upper limit on \( Fr_t \) changes somewhat with changing values of \( L_{mr} \) and \( u_c \), the boundary between steps and no steps on the left-hand side of the figure depends only slightly on the values of \( L_{mr} \) and \( u_c \).
According to Figure 8, near the upper limit $Fr_t = 0.9$, cyclic steps can exist over a wide range of $Fr_n$ larger than unity. The solution domain is confined to the range of larger values of $Fr_n$ as $Fr_t$ decreases, however. It is found from (9a) and (9b) that $u_{ed}$ decreases, and thus, the sediment transport rate $q_{sd}$ decreases with decreasing $Fr_t$. In addition, large values of $Fr_n$ correspond to large values of the bed slope $S$ according to (21a) and (21b). It follows that as the sediment transport rate becomes ever smaller, cyclic steps can be formed only in ever steeper channels.

As seen in Figure 10 and as is subsequently shown in this paper, when $Fr_t$ is small, the cover fraction $p$ is almost zero everywhere except in the vicinity of the upstream end of each step. Although a small value of $p$ implies a small amount of sediment transport that may be insufficient to erode bedrock, steep bed slopes can enhance erosional efficiency. Therefore, a large bed slope may be required. Parker and Izumi [2000] have expressed the nondimensional erosion rate of a bed composed of cohesive soil in the following form:

$$E = (u^2 - 1)^a$$

(48)

They found that if $a$ is larger than 1.5, cyclic steps can be formed over the whole domain of values of $Fr_t \in (0, 1)$ and $Fr_t < Fr_n$. On the other hand, if $a$ is smaller than 1.5, cyclic steps can be formed only in a more restricted range of $Fr_n$. In the case of bedrock incision studied in this analysis, if the alluvial layer is ignored, the nondimensional incision function reduces to

$$E = 1 - q_{ac}^{-1}(u) = 1 - \left( \frac{1 - u_c^2}{u^2 - u_c^2} \right)^{3/2}$$

(49)

In the above equation, $E$ asymptotically approaches unity as $u$ increases. The behavior of the incision function (49) in the range of large $u$ is similar to that of (48) with under the condition of relatively small $a$, so that cyclic steps can exist only within a restricted range of $Fr_n$ in the case of bedrock incision.

4. Results and Discussion

4.1. Parameters and Step Profiles as Functions of $Fr_m$, $Fr_r$, and $w_a$

In Figures 9a–9c, the wavelength $L$, horizontal migration speed $c$, step height $\Delta \eta$, vertical additional degradation rate $w_a$, and cover fraction averaged over one wavelength $\bar{p}(= \bar{q}_{ac})$ are plotted against $Fr_m$ for the respective values of $Fr_t$ of 0.2, 0.4, and 0.6, and assuming $L_m = 0.1$ and $u_c = 0.5$. It is found that in all these cases, $L$, $c$, and $\Delta \eta$ increase rapidly to approach $\infty$ as $w_a$ approaches its lower bound $w_{al}$. Meanwhile, $w_a$ and $\bar{p}$ increase to approach almost constant values in the range of sufficiently large $Fr_m$.

In accordance with Figure 8, when $Fr_t$ is small, cyclic steps can exist only in the range of large values of $Fr_n$. In the case $Fr_t = 0.2$ in particular, cyclic steps cannot form if $Fr_m$ is smaller than about 10. It is also found that the possible range of values of $L$, $c$, and $\bar{p}$ increase with increasing $Fr_t$. 

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Figure 8. Solution domains on the $Fr_t$-$Fr_n$ plane for cyclic steps for the case for which $L_{mr} = 0.1$ and $u_c = 0.5$. 

Cyclic steps are not solvable in this region.
Figure 9. Parameters for cyclic steps as functions of $Fr_n$ for the cases (a) $Fr_t = 0.2$, (b) $Fr_t = 0.4$, and (c) $Fr_t = 0.6$; $L_{mr} = 0.1$, $u_c = 0.5$. Note that while values of the vertical axis are constant, the variables are multiplied by different coefficients in the three plots in order to show all the variables in one figure.

As seen in Figure 7, the possible range of values of $w_a$ becomes ever narrower when $Fr_t$ becomes smaller than 0.5. Accordingly, in Figures 9a and 9b corresponding to the cases $Fr_t = 0.2$ and 0.5, respectively, $w_a$ barely changes as $Fr_n$ varies. In Figure 9c corresponding to the case $Fr_t = 0.6$, $w_a$ is nearly constant in $Fr_n$, varying only slightly in the vicinity of the lower limit of $Fr_n$. In addition, in the range of $Fr_n$ close to its lower limit, $w_a$ can take two different values for a specified value of $Fr_n$. Consequently, the other corresponding parameters also become double valued near the lower limit of $Fr_n$. This implies that within a narrow range of values of $Fr_n$, two different states of cyclic steps are possible for a pair of specified values of $Fr_t$ and $Fr_n$.

Although the issue of double-valued behavior is a minor one due the restricted conditions under which it occurs, it merits some explanation. The step height $\Delta \eta$ and wavelength $L$ are both single-valued functions that increase monotonically with decreasing $w_a$. In the range of relatively large values of $w_a$, $L$ increases faster than $\Delta \eta$, so that the slope $S$ decreases monotonically with decreasing $w_a$. However, in the vicinity of the lower limit of $w_a$, there appears a region where the increase in $\Delta \eta$ overcomes the increase in $L$. In this region,
Figure 10. Profiles of the bed elevation \( \eta \), the water surface elevation \( \eta + h \), and the cover fraction \( p \) over a single cyclic step for \( L_{mr} = 0.1, u_c = 0.5 \), and (a) \( Fr_t = 0.2, Fr_n = 9.54, w_a = 0.9732 \); (b) \( Fr_t = 0.2, Fr_n = 12.5, w_a = 0.9734 \); (c) \( Fr_t = 0.4, Fr_n = 2.83, w_a = 0.8427 \); (d) \( Fr_t = 0.4, Fr_n = 2.83, w_a = 0.8472 \); (e) \( Fr_t = 0.4, Fr_n = 4.4, w_a = 0.8553 \); (f) \( Fr_t = 0.6, Fr_n = 1.6, w_a = 0.656 \); (g) \( Fr_t = 0.6, Fr_n = 1.6, w_a = 0.715 \); (h) \( Fr_t = 0.6, Fr_n = 2.27, w_a = 0.740 \). The solid, dashed and thick dotted lines indicate \( \eta \), \( \eta + h \), and \( p \), respectively, and the dot denotes the point where the Froude-critical condition is achieved. Figures 10a and 10b show cases of relatively small sediment supply, and relatively mild and steep cases, respectively. Figures 10c–10e show cases of medium sediment supply; Figures 10c and 10d are cases of the same mean bed slope but different values of \( w_a \), and Figure 10e is a case of a steeper slope. Figures 10f–10h are cases of large sediment supply; Figures 10f and 10g are cases of the same slope but different values of \( w_a \), and Figure 10h is a case of a relatively steep slope. Note that the vertical scale varies between panels.

S consequently increases with decreasing \( w_a \). In order to know which state of the two is actually realized, the stability of those two solutions should be studied. Further detailed analysis of this problem is left as a future problem.

In order to obtain a view of the variation of step shape as the two Froude numbers \( Fr_t \) and \( Fr_n \) are varied, a total of eight cases of profiles of bed elevation \( \eta \), water surface elevation \( \eta + h \), and cover fraction \( p \) over a single cyclic step are shown in Figures 10a–10h. The values \( L_{mr} = 0.1 \) and \( u_c = 0.5 \) have been used in the
computations. In the figures, the solid, dashed, and thick dotted curves illustrate $\eta$, $\eta + h$, and $p$, respectively. The dot on the bed denotes the point at which the Froude-critical condition is achieved. As described above, the vertical length scale for the nondimensionalization is smaller than the horizontal length scale by a factor of $S_t$. Correspondingly, the plots of $\eta$, $\eta + h$, and $p$ against $x$ have an inherent distortion. In order to remove the effect of this distortion from the plots, we renormalize the horizontal coordinate as follows:

$$\hat{x} = \frac{x}{S_t} = \frac{x_{sd}}{h_{sd}}$$

(50)

Assuming $C_r = 0.04$ in (9a) and (9b), the values of $S_t$ are calculated to be 0.0016, 0.0064, and 0.0144 for the cases $F_{rc} = 0.2$, 0.4, and 0.6, respectively. These values have been used in the computation of the parameter $\hat{x}$ in the plots.

Figures 10a and 10b correspond to the cases $(F_{rc}, F_{rn}, w_o) = (0.2, 9.54, 0.976)$, and $(0.2, 12.5, 0.973)$, respectively. It is seen from the figures that the bed slope is rather large when $F_{rc}$ is small and $F_{rn}$ is large, as discussed in regard to Figure 8. The present analysis uses forms of governing equations that apply only when the bed slope is not too high, i.e., when $\cos \phi \approx 1$ and $\sin \phi \approx \tan \phi = S$, with $\phi$ denoting the slope angle. At sufficiently steep slopes, even the longwave approximation breaks down. Results at very high slopes are nevertheless shown in order to illustrate the general characteristics of the solution. It is expected, however, that these results have some qualitative significance even outside the range of quantitative application.

Figures 10c–10e correspond to the cases $(F_{rc}, F_{rc}, w_o) = (0.4, 2.83, 0.843)$, $(0.4, 2.83, 0.8472)$, and $(0.4, 4.40, 0.855)$, respectively. In Figures 10c and 10d, two different forms of steps are obtained for the same value of $F_{rc}$ and the different values of $w_o$, for $F_{rc} = 0.4$. It can be seen that though the step in Figure 10c with a larger value of $w_o$ has a longer wavelength than that in 10d with a smaller $w_o$, both have an identical average bed slope. An example with a larger value of $F_{rc}$ is shown in Figure 10e. It is found that as $F_{rc}$ increases, the average bed slope becomes too large for the present analysis to apply even if $F_{rc}$ is larger than 0.2.

In Figures 10f–10h, the profiles of $\eta$, $\eta + h$, and $p$ are shown for the cases $(F_{rc}, F_{rc}, w_o) = (0.6, 1.60, 0.656)$, $(0.6, 1.60, 0.716)$, and $(0.6, 2.27, 0.740)$, respectively. The set of Figures 10f and 10g indicates that there exist two different forms of steps for the same value of $F_{rc} = 1.6$ for $F_{rc} = 0.6$. It is found that the wavelength and wave height of the step in Figure 10f are both approximately 10 times larger than those of the step in Figure 10g, so resulting in the same mean slope for both.

According to this analysis, then, the profile of a step is thus characterized by a relatively short, upstream subcritical portion which has an adverse bed slope in all the cases in Figure 10. According to Parker and Izumi [2000], in the case of purely erosional cohesive soil a local adverse slope can be preserved as long as $w_o$ is positive at the upstream end of the step. In the present case of bedrock erosion, however, (31) does not immediately reduce to the same result as the purely erosional case; $w_o$ must be positive at least for an adverse slope to be preserved at the upstream end. This is consistent with Figure 7, which shows that $w_o$ takes a positive value over most of the range of $F_{rc}$.

The shallow water formulation used in this analysis is insufficient to describe the formation of a plunge pool just downstream of the hydraulic jump. In reality, however, erosion may be activated by gravel deposited in the adversely sloping upstream portion, ultimately resulting in a fully excavated plunge pool. As seen in Figure 1b, cyclic steps observed in the field are often accompanied by plunge pools. The condition that the upstream portion of all the steps shown in Figure 10 have an adverse bed slope might be thought to abort the formation of a plunge pool. The morphodynamics of plunge pool excavation itself is a fascinating problem that is one step beyond the present analysis. Recent developments in this line of research can be found in Scheingross et al. [2015], Scheingross and Lamb [2016], and Scheingross [2016, chap. 6 therein].

Another important feature of step shape deserves emphasis. The downstream supercritical region is a relatively long and has a nearly constant bed gradient. As shown in (49), the nondimensional erosion function approaches a constant value of unity as $u$ becomes sufficiently large. It is expected, therefore, that the erosion rate sufficiently far downstream, where the velocity becomes large, should approach a constant value. The slope profile of permanent form subject to uniform erosion can be nothing but a flat slope with a constant gradient. It follows that as long as (49) is used to describe the erosion process, a flat slope with a constant gradient can be expected to appear in the downstream portion of a step, as shown in Figure 10.
Figure 11. The boundary between the ranges where \( w_r > 1 \) and \( w_r < 1 \), and the boundary between the ranges where \( c_r > 1 \) and \( c_r < 1 \). \( L_m = 0.1 \) and \( u_c = 0.5 \). The range \( w_r > 1 \) \((w_r < 1)\) corresponds to an incision rate that is enhanced (suppressed) by the presence of steps, and the range \( c_r > 1 \) \((c_r < 1)\) corresponds to an upstream migration rate that is enhanced (suppressed) by the presence of steps.

4.2. Effects of Steps on Bedrock Erosion

The question arises as to whether or not steps abet or inhibit incision, as compared to the case with no steps described in Figure 4. In order to study this issue, we define the following two ratios:

\[
c_r = \frac{c_r}{c_n}, \quad w_r = \frac{w_r}{w_n}
\]

That is, \( c_r \) and \( w_r \) denote the ratio of wave speed and total degradation rate, respectively, with and without steps. Once the two Froude numbers \( F_{rt} \) and \( F_{rn} \) are specified, \( c_r, w_r, c_n, \) and \( w_n \) are uniquely determined. Therefore, we can calculate the values of \( c_r \) and \( w_r \) in the \( F_{rt}-F_{rn} \) plane. Figure 11 is an extended version of Figure 8, in which zones where \( c_r < 1, c_r > 1, w_r < 1, \) and \( w_r > 1 \) are shown. It can be seen therein that for a given value of \( F_{rn} \), smaller values of \( F_{rt} \) favor stepped beds that both incise and migrate upstream more slowly than the case of no steps, and larger values of \( F_{rt} \) favor the opposite behavior. Of interest is the fact that the lines where \( c_r = 1 \) and \( w_r = 1 \) nearly coincide. This means that the upstream migration and the vertical incision are both amplified by the formation of steps in more or less the same region of the \( F_{rt}-F_{rn} \) plane, where \( F_{rt} \) is large and therefore the sediment supply is large. In this region, bedrock erosion is generally activated by the formation of steps. On the other hand, if the sediment supply is relatively small, total bedrock erosion is reduced by the formation of cyclic steps in bedrock.

4.3. Cover Fraction \( p \), Friction Coefficient \( C_f \), and Direction of Incision

Several issues merit further discussion. In this paper, the cover fraction \( p \) and the ratio \( n_s/L_m \) of alluvial thickness to macroroughness are related by (13). According to this relation, \( p = 0 \) when \( n_s = 0 \). Zhang et al. [2015] use a somewhat different relation, according to which \( p \) takes a small residual value when \( n_s = 0 \). The small residual value prevents the speed of an alluvial wave from taking an infinite value when \( n_s = 0 \). The physics corresponding to the need for this residual value are discussed in Zhang et al. [2015]; sediment particles can roll at high speed over a very smooth bed. It is not necessary to include this residual value in the present formulation, because the solution of permanent form is predicated on the condition that alluvial cover nowhere drops to zero.

Here it is assumed that the resistance coefficient \( C_f \) takes a prescribed value. Many authors, including Nelson and Seminara [2011], Tanaka and Izumi [2013], Inoue et al. [2015], and Johnson [2014] have suggested that \( C_f \) should be computed as a weighted average based on the resistance offered by alluvial particles in covered areas, and the resistance offered by bedrock macroroughness in exposed areas. The present analysis is easily extended to this case, which might yield interesting new results. But a first-order theory that captures the basic mechanisms of incisional steps of permanent form does not require this detail.
Similarly, the alluvial cover in typical bedrock rivers corresponds to a mixture of sizes. The present analysis could be extended to sediment mixtures using an appropriate sediment transport formulation, such as the Ashida and Michiue [1972] relation and an active layer formulation for the bed. We again point out, however, that a first-order theory of incisional cyclic steps does not require a consideration of mixtures. Having said this, a consideration of mixtures might result in noticeably different results for step morphology. If the volumetric transport capacity depends on the sediment size, the cover fraction \( p \) is a function of the sediment size as well. Therefore, the fraction of each sediment size range in the bed may depend on the location within a step. Because the ability to erode bedrock is determined by the sediment size, the erosion rate for a sediment mixture may differ from that for uniform sediment. This problem is worth consideration for the future.

We note that the present analysis is formally restricted to slopes that are not so steep that the \( \cos \phi \) cannot be approximated as unity. In order to extend the analysis to such high slopes, not only do the shallow water equations need to be modified but also the direction of incision needs to be changed from (approximately) vertical downward to downward normal from the bed. Such changes, and also the changes necessary to capture plunge pool dynamics [e.g., Scheingross and Lamb, 2016], might best be implemented with a full numerical model.

### 4.4. Features of Cyclic Steps Predicted by the Model and Future Work for Validation

The results of our analysis warrant further investigation and validation through field observations and laboratory experiments. In particular, the following findings should be examined:

1. Our analysis predicts that incisional cyclic steps can form only within a specific range in the \( F_{t}-F_{n} \) plane. Therefore, if the data pertaining to \( F_{t} \) and \( F_{n} \) can be obtained for sites where incisional cyclic steps are observed, as well as sites where they are not observed, the solution domain predicted by the analysis can be validated.

2. According to our analysis, the wavelength \( L \) is determined by the values of \( F_{t} \) and \( F_{n} \). Therefore, if data for \( L, F_{t}, \) and \( F_{n} \) can be obtained in the field or in experiments, the results of the analysis can be validated by comparing the observed wavelength \( L \) with the predicted one.

3. The bed profile over a single cyclic step predicted by the analysis is characterized by a relatively short upstream portion with an adverse gradient, and a long flat downstream portion with a constant slope. Although pertinent data are available for testing our predictions [e.g., Yokokawa et al., 2015; Scheingross et al., 2015; Scheingross, 2016, chap. 6 therein], the data are fairly limited, precluding confident evaluation at this time.

### 5. Conclusion

We propose a mathematical model to explain the formation of incisional cyclic steps in mixed bedrock-alluvial channels. The model is composed of the Macro-Roughness Saltation Abrasion Alluviation model for incisional and alluvial processes and the shallow water momentum and continuity equations. To perform the analysis, we employ several constraints, such as the cover factor being less than unity in the downstream steep portion of steps. These constraints are originally based on our conceptualization of the problem but are found to be consistent with field observations. The model is reduced to a first-order ordinary differential equation for the streamwise velocity profile and three unknown parameters. This equation is solved under two boundary conditions and two additional constraints. The analysis is greatly simplified with the use of the quasi-steady assumption for alluvial processes which we introduce here for the first time.

The salient results of our analysis are listed below.

1. Solutions for incisional cyclic steps of permanent form in mixed bedrock-alluvial rivers can be found within the range of \( F_{t} \) from 0 to 1 and \( F_{n} \) larger than 1. In the range of small \( F_{t} \), such solutions exist only in the range of sufficiently large \( F_{n} \), while in the range of large \( F_{t} \), they exist over a wide range of values of \( F_{n} \) larger than unity. The parameters \( F_{t} \) and \( F_{n} \) are related to the sediment supply and the average slope, respectively. Therefore, if the sediment supply is sufficiently small cyclic steps of permanent form can be expressed only on steep slopes. If the sediment transport rate is sufficiently large on the other hand, cyclic steps can be expressed on lower slopes. This is probably because, when the sediment supply is not sufficiently large, small amounts of sediment cannot effectively drive incision, so that large slopes are required to enhance the work done by the available sediment in forming cyclic steps.
2. Step shape is characterized by a relatively short upstream portion with an adverse gradient, and a long, rather flat downstream portion with a constant gradient. The former characteristic of the step shape is caused by the fact that the dimensionless vertical degradation rate additional to that caused by horizontal migration is positive over most of the range of values of $Fr$. We suggest that an adverse gradient in a relatively short upstream portion might easily evolve into the plunge pool commonly observed at the downstream end of each cyclic step as illustrated in Figure 1b, and as observed experimentally by Yokokawa et al. [2013], Scheingross et al. [2015], and Scheingross [2016, chap. 6 therein]. The latter characteristic of the step shape is associated with the nondimensional erosion function (49). The erosion rate approaches a constant value of unity as the flow velocity becomes sufficiently large and is expected to be a constant value sufficiently far downstream as well. The slope profile of permanent form under a uniform erosion rate must thus be linear with a constant gradient.

3. The upstream migration and the vertical incision are both reduced by the formation of cyclic steps in the range of small $Fr$ and increased in the range of large $Fr$. Larger values of $Fr$ correspond to larger amounts of sediment supply. Therefore, the bedrock incision is intensified by the formation of cyclic steps when the amount of sediment supply is large, while it is reduced when the amount of sediment supply is small.

### Notation

- $C_f$: bed friction coefficient.
- $c$: dimensionless horizontal migration rate of the bed profile.
- $c_d$: dimensional version of $c$.
- $c_n$: $c$ at the equilibrium incision condition in the absence of steps.
- $c_{nd}$: dimensional version of $c_n$.
- $c_r$: ratio of horizontal migration rate with and without steps ($= c/c_n$).
- $d_{ch}$: characteristic sediment diameter.
- $E$: dimensionless erosion or incision rate.
- $E_d$: dimensional erosion or incision rate.
- $Fr$: Froude number.
- $Fr_n$: $Fr$ at the equilibrium incision condition in the absence of steps.
- $Fr_t$: $Fr$ at the incisional threshold state.
- $g$: gravitational acceleration ($= 9.8 \text{ m/s}^2$).
- $h$: dimensionless flow depth.
- $h_d$: dimensional version of $h$.
- $h_t$: dimensionless flow depth at the incisional threshold state.
- $h_{td}$: dimensional version of $h_t$.
- $L$: dimensionless wavelength.
- $L_{mr}$: dimensionless macroroughness height.
- $L_{md}$: dimensional version of $L_{mr}$.
- $p$: areal fraction of the bed covered with alluvium.
- $p_n$: $p$ at the equilibrium incision condition in the absence of steps.
- $Q$: derivative of the denominator of the right-hand side of (32) with respect to $u$ evaluated at the Froude critical point ($u = u_t$).
- $q_a$: dimensionless net volumetric gravel transport per unit width.
- $q_{ad}$: dimensional version of $q_a$.
- $q_{ac}$: dimensionless volumetric gravel transport capacity per unit width.
- $q_{acd}$: dimensional version of $q_{ac}$.
- $q_{acn}$: $q_{ac}$ at the equilibrium incision condition in the absence of steps.
- $q_{acnd}$: dimensional version of $q_{acn}$.
- $q_{ad}$: dimensional net volumetric gravel transport rate per unit width.
- $q_{as}$: dimensionless sediment supply rate per unit width ($= 1$).
- $q_{asd}$: dimensional version of $q_{as}$.
- $q_{ad}$: dimensionless flow discharge per unit width.
- $R_s$: submerged specific gravity of sediment ($= 1.65$).
- $S$: bed slope.
- $S_t$: bed slope relative to the threshold bed slope ($= S/S_t$).
\emph{S} \_ threshold bed slope.
\emph{t} \_ dimensionless time.
\emph{t_d} \_ dimensional version of \emph{t}.
\emph{u} \_ dimensionless depth-averaged flow velocity.
\emph{u_1} \_ dimensionless Froude critical velocity.
\emph{u_c} \_ dimensionless critical flow velocity for incipient sediment motion.
\emph{u_d} \_ dimensional version of \emph{u}.
\emph{u_i} \_ \emph{u} just upstream of a hydraulic jump.
\emph{u_e} \_ \emph{u} at the equilibrium incision condition in the absence of steps.
\emph{u_{ad}} \_ dimensional version of \emph{u_e}.
\emph{u_r} \_ dimensionless flow velocity at the incisional threshold state.
\emph{u_{adr}} \_ dimensional version of \emph{u_r}.
\emph{u_1} \_ dimensionless Froude critical velocity.
\emph{u_c} \_ dimensionless critical flow velocity for incipient sediment motion.
\emph{u_d} \_ dimensional version of \emph{u}.
\emph{u_n} \_ \emph{u} at the equilibrium incision condition in the absence of steps.
\emph{u_{ad}} \_ dimensional version of \emph{u_n}.
\emph{u_1} \_ dimensionless Froude critical velocity.
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\emph{u_c} \_ dimensionless critical flow velocity for incipient sediment motion.
\emph{u_d} \_ dimensional version of \emph{u}.
\emph{u_n} \_ \emph{u} at the equilibrium incision condition in the absence of steps.
\emph{u_{ad}} \_ dimensional version of \emph{u_n}.
\emph{u_r} \_ ratio of vertical degradation rate with and without steps \((= \frac{w_s}{w_n})\).
\emph{w_s} \_ dimensionless total vertical degradation rate in the presence of steps.
\emph{\lambda} \_ porosity of alluvium.
\emph{\rho} \_ density of water \((= 1000 \text{ kg/m}^3)\).
\emph{\phi} \_ bed slope angle.
\emph{\tau_{bd}} \_ dimensional bed shear stress.
\emph{\{\}}_{d} \_ dimensional variables.
\emph{\{\}}_{u} \_ partial derivative with respect to \emph{u}.

\textbf{References}


Izumi, N., and M. Yokokawa (2011), Cyclic Steps Formed in Bedrock Rivers, IAHR, Beijing, China, 6–8 Sept.


