Framed curves in the Euclidean space

Abstract: A framed curve in the Euclidean space is a curve with a moving frame. It is a generalization not only of regular curves with linear independent condition, but also of Legendre curves in the unit tangent bundle. We define smooth functions for a framed curve, called the curvature of the framed curve, similarly to the curvature of a regular curve and of a Legendre curve. Framed curves may have singularities. The curvature of the framed curve is quite useful to analyse the framed curves and their singularities. In fact, we give the existence and the uniqueness for the framed curves by using their curvature. As applications, we consider a contact between framed curves, and give a relationship between projections of framed space curves and Legendre curves.

Keywords: Framed curve, existence, uniqueness, curvature of a framed curve, congruent as framed curves.

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1 Introduction

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space equipped with the inner product $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a_i b_i$, where $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$. Let $\mathbf{a}_1, \ldots, \mathbf{a}_{n-1} \in \mathbb{R}^n$ be vectors $\mathbf{a}_i = (a_{i1}, \ldots, a_{in})$ for $i = 1, \ldots, n - 1$. We define the vector product

$$\mathbf{a}_1 \times \cdots \times \mathbf{a}_{n-1} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n-11} & \cdots & a_{n-1n} \\ e_1 & \cdots & e_n \end{vmatrix} = \sum_{i=1}^n \det(\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}, e_i) e_i,$$

where $e_1, \ldots, e_n$ are the canonical basis vectors of $\mathbb{R}^n$. Then $(\mathbf{a}_1 \times \cdots \times \mathbf{a}_{n-1}) \cdot \mathbf{a}_i = 0$ for $i = 1, \ldots, n - 1$. Note that for the case of $n = 3$,

$$\mathbf{a}_1 \times \mathbf{a}_2 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ e_1 & e_2 & e_3 \end{vmatrix} = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}.$$

The set

$$\Delta_{n-1} = \{ \mathbf{v} = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n | v_i \cdot v_j = \delta_{ij}, i, j = 1, \ldots, n - 1 \}$$

$$= \{ \mathbf{v} = (v_1, \ldots, v_{n-1}) \in S^{n-1} \times \cdots \times S^{n-1} | v_i \cdot v_j = 0, i \neq j, i, j = 1, \ldots, n - 1 \}$$

is an $n(n - 1)/2$-dimensional smooth manifold. If $\mathbf{v} = (v_1, \ldots, v_{n-1}) \in \Delta_{n-1}$, we define a unit vector $\mathbf{\mu} = (v_{11}, \ldots, v_{n-1})$ of $\mathbb{R}^n$. It follows that $(\mathbf{v}, \mathbf{\mu}) \in \Delta_n$ and $\det(\mathbf{v}, \mathbf{\mu}) = 1$.

A framed curve in the Euclidean space is a curve with a moving frame. It is a generalization not only of regular curves with linear independent condition, but also of Legendre curves in the unit tangent bundle.

Definition 1.1. We say that $(y, \mathbf{v}) : I \to \mathbb{R}^n \times \Delta_{n-1}$ is a framed curve if $y(t) \cdot v_i(t) = 0$ for all $t \in I$ and $i = 1, \ldots, n - 1$. We also say that $y : I \to \mathbb{R}^n$ is a framed curve (or a framed base curve) if there exists $\mathbf{v} : I \to \Delta_{n-1}$ such that $(y, \mathbf{v})$ is a framed curve.

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We define smooth functions for a framed curve similarly to the curvature of a regular curve and of a Legendre curve. Let \((y, \nu) : I \rightarrow \mathbb{R}^n \times \Delta_n\) be a framed curve. We define \(\mathbf{\mu} : I \rightarrow \mathbb{R}^{n-1}\) by \(\mathbf{\mu}(t) = \nu_1(t) \times \cdots \times \nu_{n-1}(t)\). By definition, \((\nu(t), \mathbf{\mu}(t)) \in \Delta_n\) for each \(t \in I\) and we call \((\nu(t), \mathbf{\mu}(t))\) a moving frame along the framed base curve \(y(t)\). Then we have the Frenet–Serret type formula
\[
\begin{pmatrix}
\dot{\nu}(t) \\
\dot{\mathbf{\mu}}(t)
\end{pmatrix} = A(t) \begin{pmatrix}
\nu(t) \\
\mathbf{\mu}(t)
\end{pmatrix},
\]
where \(A(t) = (a_{ij}(t)) \in \mathfrak{o}(n)\) for \(i, j = 1, \ldots, n\), and \(\mathfrak{o}(n)\) is the set of alternative matrices. Moreover, there exists a smooth mapping \(a : I \rightarrow \mathbb{R}\) such that
\[
\dot{y}(t) = a(t)\mathbf{\mu}(t).
\]
We call the functions \((a_{ij}(t), a(t))\) the curvature of the framed curve (with respect to the parameter \(t\)). Clearly, \(t_0\) is a singular point of \(y\) if and only if \(a(t_0) = 0\). The curvature of the framed curve is quite useful to analyse the framed curves and singularities, see Theorems 1.3 and 1.4.

**Definition 1.2.** Let \((y, \nu)\) and \((\bar{y}, \bar{\nu}) : I \rightarrow \mathbb{R}^n \times \Delta_{n-1}\) be framed curves. We say that \((y, \nu)\) and \((\bar{y}, \bar{\nu})\) are (positive) congruent as framed curves if there exists a matrix \(X \in \text{SO}(n)\) and a constant vector \(x \in \mathbb{R}^n\) such that
\[
\bar{y}(t) = X(y(t)) + x, \quad \bar{v}(t) = X(v(t))
\]
for all \(t \in I\), where \(\text{SO}(n)\) is the set of special orthogonal matrices.

The main results are the following (for \(n = 2\), see [6]).

**Theorem 1.3** (The Existence Theorem). Let \((a_{ij}, a) : I \rightarrow \mathfrak{o}(n) \times \mathbb{R}\) be a smooth mapping. There exists a framed curve \((y, \nu) : I \rightarrow \mathbb{R}^n \times \Delta_{n-1}\) whose associated curvature is \((a_{ij}, a)\).

**Theorem 1.4** (The Uniqueness Theorem). Let \((y, \nu)\) and \((\bar{y}, \bar{\nu}) : I \rightarrow \mathbb{R}^n \times \Delta_{n-1}\) be framed curves whose curvatures \((a_{ij}, a)\) and \((\bar{a}_{ij}, \bar{a})\) coincide. Then \((y, \nu)\) and \((\bar{y}, \bar{\nu})\) are congruent as framed curves.

We shall prove these theorems in \(\S 2\). We consider properties of the curvature of framed curves and concentrate in \(\S 3\) on the case \(n = 3\) of framed curves in \(\mathbb{R}^3\). We consider contact between framed curves, and give a relationship between projections of framed space curves and Legendre curves. Moreover, we give the arc-length parameter of framed immersions. In \(\S 4\), we give examples of framed curves in \(\mathbb{R}^3 \times \Delta_2\).

All maps and manifolds considered here are differentiable of class \(C^\infty\).

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## 2 Proofs for the existence and uniqueness theorems

First we prove the existence theorem by using the theorem of existence and uniqueness for a system of linear ordinary differential equations.

**Proof of Theorem 1.3.** Choose any fixed value \(t = t_0\) of the parameter. We consider the initial value problem
\[
\frac{d}{dt} F(t) = A(t) F(t), \quad F(t_0) = I_n,
\]
where \(F(t) \in M(n), A(t) = (a_{ij}(t)) \in \mathfrak{o}(n)\) for \(i, j = 1, \ldots, n\); here \(M(n)\) is the set of \(n \times n\) matrices and \(I_n\) is the identity matrix. By the existence and the uniqueness of the solution of a system of linear ordinary differential equations, there exists a solution \(F(t)\). Since \(A(t) \in \mathfrak{o}(n)\),
\[
\frac{d}{dt} \left( t F(t) F(t) \right) = \left( \frac{d}{dt} t F(t) \right) F(t) + t F(t) \left( \frac{d}{dt} F(t) \right) = t F(t) (A(t) + A(t)) F(t) = 0.
\]
It follows that \( \dot{F}(t)F(t) \) is constant. Thus \( \dot{F}(t)F(t) = iF(t_0)F(t_0) = I_n \), and \( F(t) \) is an orthogonal matrix. Let \( F(t) = \dot{\gamma}_1(t), \ldots, \dot{\gamma}_{n-1}(t), \mu(t) \). Since \((d/dt)(det F(t)) = 0\), we have \( det F(t) = det F(t_0) = det I_n = 1 \). Then \( \mu(t) = \dot{\gamma}_1(t) \times \cdots \times \dot{\gamma}_{n-1}(t) \). Next we consider the initial value problem
\[
\dot{\gamma}(t) = a(t)\mu(t), \quad \gamma(t_0) = x,
\]
where \( x \) is a point in \( \mathbb{R}^n \). By the existence and the uniqueness of the solution of a system of linear ordinary differential equations, there exists a solution \( \gamma(t) \). Therefore, there exists a framed curve \( (\gamma, v) : I \rightarrow \mathbb{R}^n \times \Delta_{n-1} \) whose associated curvature is \( (a_{ij}, \alpha) \).

In order to prove the Uniqueness Theorem (Theorem 1.4), we need two lemmas.

**Lemma 2.1.** Let \((\gamma, v)\) and \((\tilde{\gamma}, \tilde{v}) : I \rightarrow \mathbb{R}^n \times \Delta_{n-1} \) be congruent as framed curves. Then their curvatures coincide.

**Proof.** Since \((\gamma, v)\) and \((\tilde{\gamma}, \tilde{v})\) are congruent as framed curves, there exist a matrix \( X \in SO(n) \) and a constant vector \( x \in \mathbb{R}^n \) with the property that
\[
\tilde{\gamma}(t) = X(\gamma(t)) + x, \quad \tilde{v}(t) = X(v(t)).
\]
for all \( t \in I \). By definition of \( \mu \), we have \( \tilde{\mu}(t) = X(\mu(t)) \) for all \( t \in I \). By a direct calculation, we have
\[
\alpha_{ij}(t) = \tilde{\gamma}_i(t) \cdot \tilde{v}_j(t) = X(\gamma_i(t)) \cdot X(v_j(t)) = \tilde{v}_i(t) \cdot v_j(t) = a_{ij}(t),
\]
\[
\tilde{\gamma}(t) = X(\dot{\gamma}(t)) = X(a(t)\mu(t)) = a(t)xX(\mu(t)) = a(t)\tilde{\mu}(t).
\]
Hence we have \( a_{ij}(t) = \tilde{a}_{ij}(t) \) and \( a(t) = \tilde{a}(t) \).

**Lemma 2.2.** Let \((\gamma, v)\) and \((\tilde{\gamma}, \tilde{v}) : I \rightarrow \mathbb{R}^n \times \Delta_{n-1} \) be framed curves having equal curvature, that is, \( (a_{ij}(t), \alpha(t)) = (\tilde{a}_{ij}(t), \tilde{\alpha}(t)) \) for all \( t \in I \). If there exists a parameter \( t = t_0 \) for which \( (\gamma(t_0), v(t_0)) = (\tilde{\gamma}(t_0), \tilde{v}(t_0)) \), then \((\gamma, v)\) and \((\tilde{\gamma}, \tilde{v})\) coincide.

**Proof.** Here we put \( v_n(t) = \mu(t) \). Define a smooth function \( f : I \rightarrow \mathbb{R} \) by \( f(t) = \sum_{i=1}^{n} v_i(t) \dot{\gamma}_i(t) \). Since \( a_{ij}(t) = \tilde{a}_{ij}(t) \) and \( a(t) = \tilde{a}(t) \), we have
\[
\dot{f}(t) = \sum_{i=1}^{n} (v_i(t) \dot{\gamma}_i(t) + v_j(t) \dot{\tilde{\gamma}}_j(t))
= \sum_{i=1}^{n} \left\{ \left( \sum_{j=1}^{n} a_{ij}(t)v_j(t) \right) \dot{\gamma}_j(t) + v_j(t) \left( \sum_{i=1}^{n} \tilde{a}_{ij}(t) v_i(t) \right) \right\}
= 2 \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij}(t) + \tilde{a}_{ij}(t)) v_i(t) \dot{\gamma}_j(t) 
= 0.
\]
It follows that \( f \) is constant. Moreover, \( v(t_0) = \tilde{v}(t_0) \), so \( \mu(t_0) = \tilde{\mu}(t_0) \). Hence \( f(t_0) = n \) and the function \( f \) is constant with value \( n \). By the Cauchy–Schwarz inequality, we have
\[
v_i(t) \cdot \dot{\gamma}_i(t) \leq |v_i(t)||\dot{\gamma}_i(t)| = 1,
\]
for each \( i = 1, \ldots, n \). If one of these inequalities were strict, the value of \( f(t) \) would be less than \( n \). It follows that these inequalities are equalities, and we have \( v_i(t) \cdot \dot{\gamma}_i(t) = 1 \) for all \( t \in I \) and \( i = 1, \ldots, n \). Then we have
\[
|v_i(t) - \dot{\gamma}_i(t)|^2 = v_i(t) \cdot v_i(t) - 2v_i(t) \cdot \dot{\gamma}_i(t) + \dot{\gamma}_i(t) \cdot \dot{\gamma}_i(t) = 0.
\]
Hence \( v_i(t) = \dot{\gamma}_i(t) \) for all \( t \in I \) and \( i = 1, \ldots, n \). Since \( \gamma(t) = a(t)\mu(t), \dot{\gamma}(t) = \tilde{a}(t)\tilde{\mu}(t) \) and with the assumption \( a(t) = \tilde{a}(t) \) we obtain \( (d/dt)(\gamma(t) - \tilde{\gamma}(t)) = 0 \). It follows that \( \gamma(t) - \tilde{\gamma}(t) \) is constant. By the condition \( \gamma(t_0) = \tilde{\gamma}(t_0) \), we have \( \gamma(t) = \tilde{\gamma}(t) \) for all \( t \in I \).

**Proof of Theorem 1.4.** Choose any fixed value \( t = t_0 \) of the parameter. By using a matrix \( X \in SO(n) \) and a constant vector \( x \in \mathbb{R}^n \), we can assume that \( \tilde{\gamma}(t_0) = X(\gamma(t_0)) + x \) and \( \tilde{\gamma}(t_0) = X(v(t_0)) \). By Lemma 2.1, the curvatures of the framed curves \( (\gamma(t), v(t)) \) and \( (X(\gamma(t)) + x, X(v(t)) \) coincide. By Lemma 2.2, we have
\[
\tilde{\gamma}(t) = X(\gamma(t)) + x, \quad \tilde{v}(t) = X(v(t)),
\]
for all \( t \in I \). It follows that \( (\gamma, v) \) and \( (\tilde{\gamma}, \tilde{v}) \) are congruent as framed curves.

**Remark 2.3.** The Uniqueness Theorem 1.4 can be proved also by using the theorem of uniqueness of the solution of a system of ordinary differential equations.
3 Framed curves in $\mathbb{R}^3 \times \Delta_2$

In this section, we focus on space curves. One can extend the results to higher dimensional curves. However, it is rather tedious; we concentrate on the case of $n = 3$.

We fix the following notation throughout this section. Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve and $\mu(t) = \nu_1(t) \times \nu_2(t)$. The Frenet–Serret type formula is given by

$$
\begin{pmatrix}
\nu_1(t) \\
\nu_2(t) \\
\mu(t)
\end{pmatrix} = 
\begin{pmatrix}
0 & \ell(t) & m(t) \\
-\ell(t) & 0 & n(t) \\
-m(t) & -n(t) & 0
\end{pmatrix}
\begin{pmatrix}
\nu_1(t) \\
\nu_2(t) \\
\mu(t)
\end{pmatrix},
$$

where $\ell(t) = \nu_1(t) \cdot \nu_2(t)$, $m(t) = \nu_1(t) \cdot \mu(t)$ and $n(t) = \nu_2(t) \cdot \mu(t)$. Moreover, there exists a smooth mapping $\alpha : I \to \mathbb{R}$ such that

$$
\dot{\gamma}(t) = \alpha(t) \mu(t).
$$

**Example 3.1.** Typical examples of framed curves are regular curves with linear independent conditions. Let $\gamma : I \to \mathbb{R}^3$ be a regular curve with linear independent condition, namely, $\dot{\gamma}(t)$ and $\ddot{\gamma}(t)$ are linear independent for all $t \in I$. If we take $\nu_1(t) = \boldsymbol{n}(t)$ and $\nu_2(t) = \boldsymbol{b}(t)$, then $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta_2$ is a framed curve. Note that $\mu(t) = \nu_1(t) \times \nu_2(t) = \boldsymbol{t}(t)$. Here

$$
\boldsymbol{t}(t) = \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}, \quad \boldsymbol{n}(t) = \frac{(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \times \dddot{\gamma}(t)}{|(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \times \dddot{\gamma}(t)|}, \quad \boldsymbol{b}(t) = \frac{\dot{\gamma}(t) \times \dddot{\gamma}(t)}{|\dot{\gamma}(t) \times \dddot{\gamma}(t)|}.
$$

We give a relationship between regular curves and framed curves.

**Proposition 3.2.** With notation as in Example 3.1, the relationships between the curvature of the framed curve $(\ell(t), m(t), n(t), \alpha(t))$, and the curvature $\kappa(t)$ and torsion $\tau(t)$ of $\gamma$ are given by

$$
|\alpha(t)| \kappa(t) = \sqrt{\ell^2(t) + n^2(t)}, \quad \alpha(t)(m^2(t) + n^2(t)) \tau(t) = m(t)n(t) - \dot{m}(t)n(t) + (m^2(t) + n^2(t))\ell(t).
$$

**Proof.** By a direct calculation, we have

$$
\begin{align*}
\dot{\gamma}(t) &= \alpha(t) \mu(t), \\
\ddot{\gamma}(t) &= \alpha(t) \mu(t) - \alpha(t)m(t)\nu_1(t) - \alpha(t)n(t)\nu_2(t), \\
\dddot{\gamma}(t) &= (\dot{\alpha}(t) - \alpha(t)m^2(t) - \alpha(t)n^2(t))\mu(t) - (2\dot{\alpha}(t)m(t) + \alpha(t)\dot{m}(t) - \alpha(t)n(t)\ell(t))\nu_1(t) \\
&\quad - (2\dot{\alpha}(t)n(t) + \alpha(t)\dot{n}(t) + \alpha(t)m(t)\ell(t))\nu_2(t).
\end{align*}
$$

It follows that

$$
|\dddot{\gamma}(t)| = |\alpha(t)| \kappa(t) = \sqrt{\ell^2(t) + n^2(t)}
$$

and

$$
\det(\dot{\gamma}(t), \ddot{\gamma}(t), \dddot{\gamma}(t)) = \alpha^3(t)(m(t)n(t) - \dot{m}(t)n(t) + (m^2(t) + n^2(t))\ell(t)).
$$

Therefore, the curvature $\kappa(t)$ and the torsion $\tau(t)$ are given by

$$
\kappa(t) = \frac{|\dot{\gamma}(t) \times \ddot{\gamma}(t)|}{|\dot{\gamma}(t)|} = \frac{\sqrt{\ell^2(t) + n^2(t)}}{|\alpha(t)|}
$$

and

$$
\tau(t) = \frac{\det(\dot{\gamma}(t), \ddot{\gamma}(t), \dddot{\gamma}(t))}{|\dot{\gamma}(t) \times \ddot{\gamma}(t)|^2} = \frac{m(t)n(t) - \dot{m}(t)n(t) + (m^2(t) + n^2(t))\ell(t)}{\alpha(t)(m^2(t) + n^2(t))},
$$

$\square$
Let \( y : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0) \) be a space curve germ and write \( y(t) = (x(t), y(t), z(t)) \). It can be shown that, if \( y \) is not infinitely flat, namely, if either \( x(t), y(t) \) or \( z(t) \) does not belong to \( m^{\infty} \) (the ideal of infinitely flat function germs), then \( y \) is a framed base curve. Suppose without loss of generality that \( x(t) \) does not belong to \( m^{\infty} \) and that order \( x(t) \leq \min \{ \text{order } y(t), \text{order } z(t) \} \). Then there exist smooth function germs \( a(t) \) and \( b(t) \) such that \( y(t) = a(t)x(t) \) and \( z(t) = b(t)x(t) \). Thus if we take

\[
\begin{align*}
   v_1(t) &= \frac{1}{\sqrt{1 + a^2(t)}}(-a(t), 1, 0), \\
   v_2(t) &= \frac{1}{\sqrt{(1 + a^2(t))(1 + a^2(t) + b^2(t))}}(-b(t), -a(t)b(t), 1 + a^2(t)),
\end{align*}
\]

then \((y, v_1, v_2)\) is a framed curve. Note that

\[
\mu(t) = v_1(t) \times v_2(t) = \frac{1}{\sqrt{1 + a^2(t) + b^2(t)}}(1, a(t), b(t)).
\]

On the other hand, constant maps are also framed base curves, which do not satisfy the above sufficient condition. In particular an analytic curve germ is always a framed base curve, because if it is infinitely flat, then it is constant.

Let \((y, v_1, v_2) : I \to \mathbb{R}^3 \times \Delta_2\) be a framed curve with the curvature of the framed curve \((\ell, m, n, \alpha)\). By (3), (4) and (5) in the proof of Proposition 3.2, we have the following Taylor expansion of \( y \):

\[
y(t) = y(t_0) + (t - t_0)a(t_0)\mu(t_0) + \frac{(t - t_0)^2}{2}(\dot{a}(t_0)\mu(t_0) - a(t_0)m(t_0)v_1(t_0) - a(t_0)n(t_0)v_2(t_0)) \\
   + \frac{(t - t_0)^3}{3!}(\ddot{a}(t_0) - a(t_0)m^2(t_0) - a(t_0)n^2(t_0))\mu(t_0) - 2\dot{a}(t_0)m(t_0) + a(t_0)n(t_0) \\
   - a(t_0)n(t_0)\ell(t_0)v_1(t_0) - 2\dot{a}(t_0)n(t_0) + a(t_0)n(t_0)\ell(t_0)v_2(t_0)) + o(4).
\]

If \( t_0 \) is a singular point of \( y \), then we have

\[
y(t) = y(t_0) + \frac{(t - t_0)^2}{2}\dot{a}(t_0)\mu(t_0) + \frac{(t - t_0)^3}{3!}(\ddot{a}(t_0)\mu(t_0) - 2\dot{a}(t_0)m(t_0)v_1(t_0) - 2\dot{a}(t_0)n(t_0)v_2(t_0)) + o(4).
\]

Let \((y, v_1, v_2) : I \to \mathbb{R}^3 \times \Delta_2\) be a framed curve with the curvature of the framed curve \((\ell, m, n, \alpha)\). For the normal plane of \( y(t) \), spanned by \( v_1(t) \) and \( v_2(t) \), there is some ambient of framed curves similarly to the case of the Bishop frame of a regular space curve (cf. [3]). We define \((\bar{v}_1(t), \bar{v}_2(t)) \in \Delta_2\) by

\[
\begin{pmatrix}
   \bar{v}_1(t) \\
   \bar{v}_2(t)
\end{pmatrix} = \begin{pmatrix}
   \cos \theta(t) & -\sin \theta(t) \\
   \sin \theta(t) & \cos \theta(t)
\end{pmatrix} \begin{pmatrix}
   v_1(t) \\
   v_2(t)
\end{pmatrix},
\]

where \( \theta(t) \) is a smooth function. Then \((y, \bar{v}_1, \bar{v}_2) : I \to \mathbb{R}^3 \times \Delta_2\) is also a framed curve and

\[
\begin{align*}
   \bar{\mu}(t) &= \bar{v}_1(t) \times \bar{v}_2(t) = (\cos \theta(t)v_1(t) - \sin \theta(t)v_2(t)) \times (\sin \theta(t)v_1(t) + \cos \theta(t)v_2(t)) \\
   &= v_1(t) \times v_2(t) = \mu(t).
\end{align*}
\]

By a direct calculation, we have

\[
\begin{align*}
   \bar{v}_1(t) &= (\ell(t) - \dot{\theta}(t)) \sin \theta(t)v_1(t) + (\ell(t) - \dot{\theta}(t)) \cos \theta(t)v_2(t) + (m(t) \cos \theta(t) - n(t) \sin \theta(t))\mu(t), \\
   \bar{v}_2(t) &= -(\ell(t) - \dot{\theta}(t)) \cos \theta(t)v_1(t) + (\ell(t) - \dot{\theta}(t)) \sin \theta(t)v_2(t) + (m(t) \sin \theta(t) + n(t) \cos \theta(t))\mu(t).
\end{align*}
\]

If we take a smooth function \( \theta : I \to \mathbb{R} \) which satisfies \( \dot{\theta} = \ell(t) \), then we call the frame \([\bar{v}_1(t), \bar{v}_2(t), \mu(t)]\) an adapted frame along the framed base curve \( y(t) \). It follows that the Frenet–Serret type formula is given by

\[
\begin{pmatrix}
   \bar{v}_1(t) \\
   \bar{v}_2(t) \\
   \bar{\mu}(t)
\end{pmatrix} = \begin{pmatrix}
   0 & 0 & \bar{m}(t) \\
   0 & 0 & \bar{n}(t) \\
   -\bar{m}(t) & -\bar{n}(t) & 0
\end{pmatrix} \begin{pmatrix}
   v_1(t) \\
   v_2(t) \\
   \mu(t)
\end{pmatrix},
\]

where \( \bar{m}(t) \) and \( \bar{n}(t) \) are the components of \( \mu(t) \) along the directions \( \bar{v}_1(t) \) and \( \bar{v}_2(t) \), respectively.
where \( \overline{m}(t) \) and \( \overline{n}(t) \) are given by
\[
\begin{bmatrix}
\overline{m}(t) \\
\overline{n}(t)
\end{bmatrix} = 
\begin{bmatrix}
\cos \theta(t) & -\sin \theta(t) \\
\sin \theta(t) & \cos \theta(t)
\end{bmatrix}
\begin{bmatrix}
m(t) \\
n(t)
\end{bmatrix}.
\] (6)

We now consider framed curves in a plane. Let \((y, v_1, v_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2\) be a framed curve with the curvature of the framed curve \((\ell, m, n, \alpha)\). We denote a plane by
\[
P(v, c) = \{ x \in \mathbb{R}^3 \mid x \cdot v = c \},
\]
where \(v \in S^2\) and \(c \in \mathbb{R}\). If \(y(t) \in P(v, c)\), then we have \(\det(y(t), \dot{y}(t), \ddot{y}(t)) = 0\). It follows that
\[
a(t)(m(t)n(t) - \dot{m}(t)n(t) + (m^2(t) + n^2(t))\ell(t)) = 0
\]
for all \(t \in I\). Conversely, we have the following result.

**Proposition 3.3.** Let \((y, v_1, v_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2\) be a framed curve with curvature \((\ell, m, n, \alpha)\).

1. If \(a(t) = 0\) for all \(t \in I\), then \(y(t)\) is a point.
2. If \(m(t) = n(t) = 0\) for all \(t \in I\), then \(y(t)\) is a part of a straight line.
3. If \(m(t)n(t) - \dot{m}(t)n(t) + (m^2(t) + n^2(t))\ell(t) = 0\) and \(m^2(t) + n^2(t) \neq 0\) for all \(t \in I\), then there exist a vector \(v \in S^2\) and a constant \(c \in \mathbb{R}\) such that \(y(t) \in P(v, c)\).

**Proof.** (1) By \(y(t) = a(t)\ell(t)\) is a point.

(2) By the Frenet–Serret type formula, \(\mu(t) = 0\) for all \(t \in I\) and hence \(\ddot{y}(t) = a(t)\mu(t) = a(t)v\), where \(v \in S^2\) is a constant vector. Then there exists a constant vector \(x\) such that \(y(t) = \int a(t) dt v + x\). It follows that \(y(t)\) is a part of a straight line.

(3) We take an adapted frame \([\overline{v}_1(t), \overline{v}_2(t), \mu(t)]\) along the framed base curve \(y(t)\). By (6) and a direct calculation, we have
\[
\overline{m}(t)\overline{m}(t) - \overline{m}(t)\overline{n}(t) = m(t)n(t) - \dot{m}(t)n(t) + (m^2(t) + n^2(t))\ell(t) = 0
\]
and
\[
\overline{m}(t)\overline{n}(t) + \overline{n}(t)\overline{n}(t) = m^2(t) + n^2(t) \neq 0
\]
for all \(t \in I\). It follows that \(\overline{m}(t)\) and \(\overline{n}(t)\) are linear dependent on \(I\) (cf. [4; 9; 10]). Thus, there exists a non-zero constant vector \((c_1, c_2)\) such that \(c_1\overline{m}(t) + c_2\overline{n}(t) = 0\) for all \(t \in I\). Then \(\overline{v} = c_1\overline{v}_1(t) + c_2\overline{v}_2(t)\) is a non-zero constant vector. Let \(v = \overline{v}/(c_1^2 + c_2^2)\). Since \(y(t) \cdot v = a(t)\mu(t) \cdot v = 0\) for all \(t \in I\), there exists a constant \(c \in \mathbb{R}\) such that \(y(t) \in P(v, c)\).

**Remark 3.4.** If \((y, v_1, v_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2\) is an analytic framed curve, then \(\overline{m}(t)\) and \(\overline{n}(t)\) are also analytic functions. Hence if \(m(t)n(t) - \dot{m}(t)n(t) + (m^2(t) + n^2(t))\ell(t) = 0\) for all \(t \in I\), then \(\overline{m}(t)\) and \(\overline{n}(t)\) are linear dependent on \(I\) (cf. [4; 10]). It follows that there exist a vector \(v \in S^2\) and a constant \(c \in \mathbb{R}\) such that \(y(t) \in P(v, c)\).

We also define a Legendre curve on a plane.

**Definition 3.5.** We say that \((y, v) : I \rightarrow \mathbb{R}^3 \times S^2\) is a Legendre curve on the plane \(P(v, c)\) if \(y(t) \cdot v = c, \dot{y}(t) \cdot v(t) = 0\) and \(v(t) \cdot v = 0\) for all \(t \in I\).

**Proposition 3.6.** (1) If \((y, v) : I \rightarrow \mathbb{R}^3 \times S^2\) is a Legendre curve on the plane \(P(v, c)\), then \((y, v, v) : I \rightarrow \mathbb{R}^3 \times \Delta_2\) is a framed curve with \(\ell(t) = m(t) = 0\) for all \(t \in I\). Conversely, if \((y, v_1, v_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2\) is a framed curve with \(\ell(t) = m(t) = 0\) for all \(t \in I\), then there exist a constant vector \(v \in S^2\) and a constant \(c \in \mathbb{R}\) such that \((y, v_1) : I \rightarrow \mathbb{R}^3 \times S^2\) is a Legendre curve on the plane \(P(v, c)\).

(2) If \((y, v) : I \rightarrow \mathbb{R}^3 \times S^2\) is a Legendre curve on the plane \(P(v, c)\), then \((y, v, v) : I \rightarrow \mathbb{R}^3 \times \Delta_2\) is a framed curve with \(\ell(t) = n(t) = 0\) for all \(t \in I\). Conversely, if \((y, v_1, v_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2\) is a framed curve with \(\ell(t) = n(t) = 0\) for all \(t \in I\), then there exist a constant vector \(v \in S^2\) and a constant \(c \in \mathbb{R}\) such that \((y, v_1) : I \rightarrow \mathbb{R}^3 \times S^2\) is a Legendre curve on the plane \(P(v, c)\).
Proof. (1) By definition, we have \( \dot{y}(t) \cdot v = 0 \) and \((v, \dot{v}(t)) \in \Delta_2 \). Since \( v \) is a constant, we have \( \ell(t) = m(t) = 0 \) for all \( t \in I \). Conversely, by the Frenet–Serret type formula, \( v = v_1(t) \in S^2 \) is a constant vector. Moreover, since \( \dot{y}(t) \cdot v = a(t) \mu(t) \cdot \dot{v}_1(t) = 0 \) for all \( t \in I \), there exists a constant \( c \in \mathbb{R} \) such that \( \dot{y}(t) \cdot v = c \). It follows that \((y, v_2) : I \to \mathbb{R}^3 \times S^2 \) is a Legendre curve on the plane \( P(v, c) \).

Assertion (2) can be proved similarly.

### 3.1 Contact between framed curves

In this subsection, we discuss contact between framed curves. Let

\[
(y, v_1, v_2) : I \to \mathbb{R}^3 \times \Delta_2; t \mapsto (y(t), v_1(t), v_2(t)) \quad \text{and} \quad \bar{(y), \bar{v}_1, \bar{v}_2) : I \to \mathbb{R}^3 \times \Delta_2; u \mapsto (\bar{y}(u), \bar{v}_1(u), \bar{v}_2(u))
\]

be framed curves and let \( k \) be a natural number. We say that \((y, v_1, v_2) \) and \((\bar{y}, \bar{v}_1, \bar{v}_2) \) have \( k \)-th order contact at \( t = t_0, u = u_0 \) if

\[
(y, v_1, v_2)(t_0) = (\bar{y}, \bar{v}_1, \bar{v}_2)(u_0), \quad \frac{d}{dt}(y, v_1, v_2)(t_0) = \frac{d}{du}(\bar{y}, \bar{v}_1, \bar{v}_2)(u_0), \quad \ldots, \\
\frac{d^{k-1}}{dt^{k-1}}(y, v_1, v_2)(t_0) = \frac{d^{k-1}}{du^{k-1}}(\bar{y}, \bar{v}_1, \bar{v}_2)(u_0)
\]

(cf. [6; 8]). Moreover, we say that \((y, v_1, v_2) \) and \((\bar{y}, \bar{v}_1, \bar{v}_2) \) have at least \( k \)-th order contact at \( t = t_0, u = u_0 \) if

\[
(y, v_1, v_2)(t_0) = (\bar{y}, \bar{v}_1, \bar{v}_2)(u_0), \quad \frac{d}{dt}(y, v_1, v_2)(t_0) = \frac{d}{du}(\bar{y}, \bar{v}_1, \bar{v}_2)(u_0), \quad \ldots, \\
\frac{d^k}{dt^k}(y, v_1, v_2)(t_0) = \frac{d^k}{du^k}(\bar{y}, \bar{v}_1, \bar{v}_2)(u_0).
\]

In general, we may assume that \((y, v_1, v_2) \) and \((\bar{y}, \bar{v}_1, \bar{v}_2) \) have at least first order contact at any point \( t = t_0, u = u_0 \), up to congruence as framed curves. We denote the curvatures of the framed curves \((y(t), v_1(t), v_2(t)) \) by \( \mathcal{F}(t) = (\ell(t), m(t), n(t), a(t)) \) and that of \((\bar{y}(u), \bar{v}_1(u), \bar{v}_2(u)) \) by \( \bar{\mathcal{F}}(u) = (\bar{\ell}(u), \bar{m}(u), \bar{n}(u), \bar{a}(u)) \).

**Theorem 3.7.** Let \((y, v_1, v_2) \) and \((\bar{y}, \bar{v}_1, \bar{v}_2) \) as above. If \((y, v_1, v_2) \) and \((\bar{y}, \bar{v}_1, \bar{v}_2) \) have at least \((k+1)\)-th order contact at \( t = t_0, u = u_0 \) then

\[
\mathcal{F}(t_0) = \bar{\mathcal{F}}(u_0), \quad \frac{d}{dt}\mathcal{F}(t_0) = \frac{d}{du}\bar{\mathcal{F}}(u_0), \quad \ldots, \quad \frac{d^{k-1}}{dt^{k-1}}\mathcal{F}(t_0) = \frac{d^{k-1}}{du^{k-1}}\bar{\mathcal{F}}(u_0). \tag{7}
\]

Conversely, if conditions (7) hold, then \((y, v_1, v_2) \) and \((\bar{y}, \bar{v}_1, \bar{v}_2) \) have at least \((k+1)\)-th order contact at \( t = t_0, u = u_0 \), up to congruence as framed curves.

**Proof.** Suppose that \((y, v_1, v_2) \) and \((\bar{y}, \bar{v}_1, \bar{v}_2) \) have at least second order contact at \( t = t_0, u = u_0 \). Since \( v_1(t_0) = \bar{v}_1(u_0) \) and \( v_2(t_0) = \bar{v}_2(u_0) \), we have \( \mu(t_0) = \bar{\mu}(u_0) \). By the Frenet–Serret type formula,

\[
\frac{d}{dt}(y, v_1, v_2) = (a(t)\mu(t), \ell(t)v_2(t) + m(t)\mu(t), \ell(t)v_1(t) + n(t)\mu(t)), \\
\frac{d}{du}(\bar{y}, \bar{v}_1, \bar{v}_2) = (\bar{a}(u)\bar{\mu}(u), \bar{\ell}(u)\bar{v}_2(u) + \bar{m}(u)\bar{\mu}(u), \bar{\ell}(u)\bar{v}_1(u) + \bar{n}(u)\bar{\mu}(u)).
\]

It follows that \( \mathcal{F}(t_0) = \bar{\mathcal{F}}(u_0) \). Hence, the first assertion of Theorem 3.7 holds in the case of \( k = 1 \).

Suppose that the assumption is true up to the \( k \)-th order of contact. Let \((y, v_1, v_2) \) and \((\bar{y}, \bar{v}_1, \bar{v}_2) \) have at least \((k+1)\)-th order contact at \( t = t_0, u = u_0 \). Then they have at least \( k \)-th order of contact, so

\[
\mathcal{F}(t_0) = \bar{\mathcal{F}}(u_0), \quad \frac{d}{dt}\mathcal{F}(t_0) = \frac{d}{du}\bar{\mathcal{F}}(u_0), \quad \ldots, \quad \frac{d^{k-2}}{dt^{k-2}}\mathcal{F}(t_0) = \frac{d^{k-2}}{du^{k-2}}\bar{\mathcal{F}}(u_0).
\]

By the Frenet–Serret type formula, we have
We call the pair 
\[
dk{d}{k}\gamma(t) = \left( \frac{d^{k-1}}{dt^{k-1}} \alpha(t) \right) \mu(t) + f_1 \left( \mathcal{T}(t), \ldots, \frac{d^{k-2}}{dt^{k-2}} \mathcal{T}(t) \right) \nu_1(t) \\
+ f_2 \left( \mathcal{T}(t), \ldots, \frac{d^{k-2}}{dt^{k-2}} \mathcal{T}(t) \right) \nu_2(t) + f_3 \left( \mathcal{T}(t), \ldots, \frac{d^{k-2}}{dt^{k-2}} \mathcal{T}(t) \right) \mu(t), \\
dk{d}{k} \nu_1(t) = \left( \frac{d^{k-1}}{dt^{k-1}} \ell(t) \right) \nu_2(t) + \left( \frac{d^{k-1}}{dt^{k-1}} m(t) \right) \mu(t) + g_1 \left( \mathcal{T}(t), \ldots, \frac{d^{k-2}}{dt^{k-2}} \mathcal{T}(t) \right) \nu_2(t) \\
+ g_2 \left( \mathcal{T}(t), \ldots, \frac{d^{k-2}}{dt^{k-2}} \mathcal{T}(t) \right) \nu_2(t) + g_3 \left( \mathcal{T}(t), \ldots, \frac{d^{k-2}}{dt^{k-2}} \mathcal{T}(t) \right) \mu(t), \\
dk{d}{k} \nu_2(t) = - \left( \frac{d^{k-1}}{dt^{k-1}} \ell(t) \right) \nu_1(t) + \left( \frac{d^{k-1}}{dt^{k-1}} n(t) \right) \mu(t) + h_1 \left( \mathcal{T}(t), \ldots, \frac{d^{k-2}}{dt^{k-2}} \mathcal{T}(t) \right) \nu_1(t) \\
+ h_2 \left( \mathcal{T}(t), \ldots, \frac{d^{k-2}}{dt^{k-2}} \mathcal{T}(t) \right) \nu_2(t) + h_3 \left( \mathcal{T}(t), \ldots, \frac{d^{k-2}}{dt^{k-2}} \mathcal{T}(t) \right) \mu(t)
\]
for some smooth functions \( f_i, g_i, h_i \) \((i = 1, 2, 3)\). By the same calculations,
\[
\frac{d}{du} \gamma(u) = \left( \frac{d^{k-1}}{du^{k-1}} \alpha(u) \right) \mu(u) + f_1 \left( \mathcal{T}(u), \ldots, \frac{d^{k-2}}{du^{k-2}} \mathcal{T}(u) \right) \nu_1(u) \\
+ f_2 \left( \mathcal{T}(u), \ldots, \frac{d^{k-2}}{du^{k-2}} \mathcal{T}(u) \right) \nu_2(u) + f_3 \left( \mathcal{T}(u), \ldots, \frac{d^{k-2}}{du^{k-2}} \mathcal{T}(u) \right) \mu(u), \\
\frac{d}{du} \nu_1(u) = \left( \frac{d^{k-1}}{du^{k-1}} \ell(u) \right) \nu_2(u) + \left( \frac{d^{k-1}}{du^{k-1}} m(u) \right) \mu(u) + g_1 \left( \mathcal{T}(u), \ldots, \frac{d^{k-2}}{du^{k-2}} \mathcal{T}(u) \right) \nu_2(u) \\
+ g_2 \left( \mathcal{T}(u), \ldots, \frac{d^{k-2}}{du^{k-2}} \mathcal{T}(u) \right) \nu_2(u) + g_3 \left( \mathcal{T}(u), \ldots, \frac{d^{k-2}}{du^{k-2}} \mathcal{T}(u) \right) \mu(u), \\
\frac{d}{du} \nu_2(u) = - \left( \frac{d^{k-1}}{du^{k-1}} \ell(u) \right) \nu_1(u) + \left( \frac{d^{k-1}}{du^{k-1}} n(u) \right) \mu(u) + h_1 \left( \mathcal{T}(u), \ldots, \frac{d^{k-2}}{du^{k-2}} \mathcal{T}(u) \right) \nu_1(u) \\
+ h_2 \left( \mathcal{T}(u), \ldots, \frac{d^{k-2}}{du^{k-2}} \mathcal{T}(u) \right) \nu_2(u) + h_3 \left( \mathcal{T}(u), \ldots, \frac{d^{k-2}}{du^{k-2}} \mathcal{T}(u) \right) \mu(u).
\]
It follows that \((d^{k-1}/dt^{k-1})\mathcal{T}(t_0) = (d^{k-1}/du^{k-1})\mathcal{T}(u_0)\). By induction, we have the first assertion.

Conversely, suppose that Condition (7) holds. By the above calculations, we have \((d^{k-1}/dt^{k-1})(\gamma, \nu_1, \nu_2)(t_0) = (d^i/du^j)(\gamma, \nu_1, \nu_2)(u_0)\) for \(i = 1, \ldots, k\). Therefore, \((\gamma, \nu_1, \nu_2)\) and \((\gamma, \nu_1, \nu_2)\) have at least \((k + 1)\)-th order contact at \(t = t_0, u = u_0\), up to congruence as framed curves.

\[\square\]

### 3.2 Projections to planes and Legendre curves

We quickly review Legendre curves; for more detail see [6]. The Legendre curves correspond to the case of \(n = 2\) for framed curves. We say that \((\gamma, \nu) : I \to \mathbb{R}^2 \times S^1\) is a Legendre curve if \((\gamma, \nu)^* \theta = 0\) for all \(t \in I\), where \(\theta\) is a canonical contact 1-form on the unit tangent bundle \(T_1 \mathbb{R}^2 = \mathbb{R}^2 \times S^1\) (cf. [1; 2]). This condition is equivalent to \(\mathcal{J}(t) \cdot \nu(t) = 0\) for all \(t \in I\). We say that \(\gamma : I \to \mathbb{R}^2\) is a frontal if there exists a smooth mapping \(\nu : I \to S^1\) such that \((\gamma, \nu)\) is a Legendre curve.

Let \((\gamma, \nu) : I \to \mathbb{R}^2 \times S^1\) be a Legendre curve. Then we have the Frenet formula of the frontal \(\gamma\) as follows. We put \(\mu(t) = J(\nu(t))\), where \(J\) is the anti-clockwise rotation by \(\pi/2\) on \(\mathbb{R}^2\). We call the pair \((\nu(t), \mu(t))\) a moving frame along the frontal \(\gamma(t)\) in \(\mathbb{R}^2\). The Frenet formula of the frontal (or, the Legendre curve) is given by

\[
\begin{pmatrix}
\dot{\gamma}(t) \\
\dot{\mu}(t)
\end{pmatrix} = 
\begin{pmatrix}
0 & \ell(t) \\
-\ell(t) & 0
\end{pmatrix}
\begin{pmatrix}
\gamma(t) \\
\mu(t)
\end{pmatrix},
\]
where \(\ell(t) = \dot{\gamma}(t) \cdot \mu(t)\). Moreover, there exists a smooth function \(\beta(t)\) such that

\[
\mathcal{J}(t) = \beta(t) \mu(t).
\]
We call the pair \((\ell(t), \beta(t))\) the curvature of the Legendre curve (with respect to the parameter \(t\)).
Let \((γ, v_1, v_2) : I → R^3 × Δ_2\) be a framed curve with curvature \((\ell, m, n, α)\). For a fix point \(t_0 ∈ I\), we consider three orthogonal projections from \(R^3\) along the direction \(v_1(t_0), v_2(t_0)\) and \(μ(t_0)\).

First, we consider the projection of \(γ\) along the \(v_1(t_0)\) direction given by \(γ_{v_1} : I → R^2\) with \(γ_{v_1}(t) = (γ(t) · v_1(t_0), γ(t) · μ(t_0))\). Then \(γ_{v_1}(t) = α(t)(μ(t) · v_1(t_0), μ(t) · μ(t_0))\). There is a subinterval \(I_1\) of \(I\) around \(t_0\) such that \((μ(t) · v_1(t_0))^2 + (μ(t) · μ(t_0))^2 ≠ 0\) for all \(t ∈ I_1\). We define a smooth map \(v_{v_1} : I_1 → S^1\) by

\[
v_{v_1}(t) = \frac{1}{\sqrt{(μ(t) · v_1(t_0))^2 + (μ(t) · μ(t_0))^2}}(μ(t) · μ(t_0), -μ(t) · v_1(t_0)).
\]

Then \((γ_{v_1}, v_{v_1}) : I_1 → R^2 × S^1\) is a Legendre curve. Since \(μ_{v_1} : I_1 → S^1\) is given by

\[
μ_{v_1}(t) = f(v_{v_1}(t)) = \frac{1}{\sqrt{(μ(t) · v_1(t_0))^2 + (μ(t) · μ(t_0))^2}}(μ(t) · μ(t_0), μ(t) · v_1(t_0)),
\]

the curvature of the Legendre curve \((γ_{v_1}, v_{v_1})\) is given by

\[
ℓ_{v_1}(t) = \frac{1}{(μ(t) · v_1(t_0))^2 + (μ(t) · μ(t_0))^2} \left( \frac{m(t)((v_1(t) · v_1(t_0))(μ(t) · μ(t_0)) - (v_1(t) · μ(t_0))(μ(t) · v_1(t_0)))}{(μ(t) · v_1(t_0))^2 + (μ(t) · μ(t_0))^2} \right)
\]

and

\[
β_{v_1}(t) = α(t) \sqrt{(μ(t) · v_1(t_0))^2 + (μ(t) · μ(t_0))^2}.
\]

Note that \(ℓ_{v_1}(t_0) = n(t_0)\) and \(β_{v_1}(t_0) = α(t_0)\). The projection of \(γ\) along the \(v_2(t_0)\) direction is similar to the case of the \(v_1(t_0)\) direction.

Next, we consider the projection of \(γ\) along \(μ(t_0)\), given by \(γ_μ : I → R^2\) with \(γ_μ(t) = (γ(t) · v_1(t_0), γ(t) · v_2(t_0))\). Then \(γ_μ(t) = α(t)(μ(t) · v_1(t_0), μ(t) · v_2(t_0))\). In this case, \(γ_μ\) is not always a frontal, that is, there does not exist a smooth mapping \(ν_μ : I → S^1\) such that \((γ_μ, ν_μ) : I → R^2 × S^1\) is a Legendre curve, see Example 4.2.

However, if \(γ_μ\) is not infinitely flat around \(t_0\), namely, if either \(γ(t) · v_1(t_0)\) or \(γ(t) · v_2(t_0)\) does not belong to \(m^{2R}\), then \(γ_μ\) is a frontal (cf. [6]).

In general, let \(t_0 ∈ I\) and fix a positive orthonormal basis \(\{v_1, v_2, v_3\}\) on \(R^3\), with \(v_1, v_2 ∈ Δ_2\) and \(v_3 = v_1 × v_2\), such that \(v_3 ≠ ±μ(t_0)\). Then we consider the orthogonal projection along \(v_3\) to the \((v_1, v_2)\)-plane. We denote \(γ_ν : I → R^2\) given by \(γ_ν(t) = (γ(t) · v_1, γ(t) · v_2)\). Then \(γ_ν(t) = α(t)(μ(t) · v_1, μ(t) · v_2)\). By the assumption, there is a subinterval \(I_0\) of \(I\) around \(t_0\) such that \((μ(t) · v_1, μ(t) · v_2) ≠ (0, 0)\) for all \(t ∈ I_0\). We define a smooth map \(ν_ν : I_0 → S^1\) by

\[
ν_ν(t) = \frac{1}{\sqrt{(μ(t) · v_1)^2 + (μ(t) · v_2)^2}}(μ(t) · v_2, -μ(t) · v_1).
\]

Then \((γ_ν, ν_ν) : I_0 → R^2 × S^1\) is a Legendre curve. Since \(μ_ν : I → S^1\) is given by

\[
μ_ν(t) = f(ν_ν(t)) = \frac{1}{\sqrt{(μ(t) · v_1)^2 + (μ(t) · v_2)^2}}(μ(t) · v_1, μ(t) · v_2),
\]

the curvature of the Legendre curve \((γ_ν, ν_ν)\) is given by

\[
ℓ_ν(t) = \frac{1}{(μ(t) · v_1)^2 + (μ(t) · v_2)^2} \left( \frac{m(t)((v_1(t) · v_1(t_0))(μ(t) · v_2) - (v_1(t) · v_2)(μ(t) · v_1))}{(μ(t) · v_1)^2 + (μ(t) · v_2)^2} \right)
\]

and

\[
β_ν(t) = α(t) \sqrt{(μ(t) · v_1)^2 + (μ(t) · v_2)^2}.
\]

Remark 3.8. If we take a positive orthonormal basis \(\{v_1, v_2, v_3\}\) on \(R^3\) such that \(v_3 ∈ S^2 \setminus ±μ(I)\), then we may consider \(I = I_0\). In this case, the Legendre curve \((γ_ν, ν_ν)\) can be defined globally.
3.3 Framed immersions

Let I and \( \bar{I} \) be intervals. A smooth function \( s : \bar{I} \rightarrow I \) is a (positive) change of parameter when \( s \) is surjective and has a positive derivative at every point. It follows that \( s \) is a diffeomorphism.

Let \( (\gamma, v_1, v_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2 \) and \((\bar{\gamma}, \bar{v}_1, \bar{v}_2) : \bar{I} \rightarrow \mathbb{R}^3 \times \Delta_2 \) be framed curves whose curvatures are \((\ell, m, n, a)\) and \((\bar{\ell}, \bar{m}, \bar{n}, \bar{a})\) respectively. Suppose \((\gamma, v_1, v_2)\) and \((\bar{\gamma}, \bar{v}_1, \bar{v}_2)\) are parametrically equivalent via the change of parameter \( s : \bar{I} \rightarrow I \). Thus \((\bar{\gamma}(t), v_1(s(t)), v_2(s(t)))\) for all \( t \in \bar{I} \). By differentiation, we have

\[
\bar{\ell}(t) = \ell(s(t))\dot{s}(t), \quad \bar{m}(t) = m(s(t))\dot{s}(t), \quad \bar{n}(t) = n(s(t))\dot{s}(t), \quad \bar{a}(t) = a(s(t))\dot{s}(t).\]

Hence the curvature is dependent on the parametrization. Note that \((\gamma, v_1, v_2)\) is a framed immersion if and only if \((\ell(t), m(t), n(t), a(t)) \neq (0, 0, 0, 0)\) for all \( t \in I \). Moreover, for an adapted frame, we may assume that \( \ell(t) = 0 \) for all \( t \in I \).

In general, we cannot consider the arc-length parameter of the framed base curve \( \gamma \), since \( \gamma \) may have singularities. However, if \((\gamma, v_1, v_2)\) is an immersion, we introduce the arc-length parameter for the framed immersion \((\gamma, v_1, v_2)\). The speed \( s(t) \) of the framed immersion at the parameter \( t \) is defined to be the length of the tangent vector at \( t \), namely,

\[
s(t) = |(\dot{\gamma}(t), v_1(t), v_2(t))| = \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t) + v_1(t) \cdot v_1(t) + v_2(t) \cdot v_2(t)}.
\]

Given scalars \( a, b \in I \), we define the arc-length from \( t = a \) to \( t = b \) to be the integral of the speed,

\[
L(\gamma, v) = \int_a^b s(t) \, dt.
\]

By the same method for the arc-length parameter of a regular curve, one can prove the following (cf. [5; 7; 8]).

**Proposition 3.9.** Let \((\gamma, v_1, v_2)\) be a framed immersion, and let \( t_0 \in I \). Then \((\gamma, v_1, v_2)\) is parametrically equivalent to a unit speed curve \((\bar{\gamma}, \bar{v}_1, \bar{v}_2) : \bar{I} \rightarrow \mathbb{R}^3 \times \Delta_2 \) under a change of parameter \( t : \bar{I} \rightarrow I \) with \( t(0) = t_0 \) and with \( t'(s) > 0 \).

We call the parameter \( s \) in Proposition 3.9 the arc-length parameter for the framed immersion. Let \( s \) be the arc-length parameter for \((\gamma, v_1, v_2)\). By definition, we have \( y'(s) \cdot y'(s) + v_1'(s) \cdot v_1'(s) + v_2'(s) \cdot v_2'(s) = 1 \), where \( \cdot \) is the derivation with respect to \( s \). It follows that \( 2L(s)^2 + m(s)^2 + n(s)^2 + a(s)^2 = 1 \).

If we consider the framed immersion with an adapted frame, then \( \ell(s) = 0 \) for all \( s \in I \). It follows that we have \( m(s)^2 + n(s)^2 + a(s)^2 = 1 \).

4 Examples

**Example 4.1.** Let \( n_1, n_2, n_3, k_1 \) and \( k_2 \) be natural numbers with \( n_2 = n_1 + k_1 \) and \( n_3 = n_2 + k_2 \). Let \((\gamma, v_1, v_2) : \mathbb{R} \rightarrow \mathbb{R}^3 \times \Delta_2 \) be defined by

\[
\gamma(t) = \begin{pmatrix} t^{n_1} \\ t^{n_2} \\ t^{n_3} \end{pmatrix},
\]

\[
v_1(t) = \frac{1}{\sqrt{1 + t^{2k_1}}}(-t^{k_1}, 1, 0),
\]

\[
v_2(t) = \frac{1}{\sqrt{(1 + t^{2k_1})(1 + t^{2k_1} + t^{2k_1} + 2k_2})}(-t^{k_1} + k_2, -t^{2k_1} + k_2, 1 + t^{2k_1}).
\]

We see that \((\gamma, v_1, v_2)\) is a framed curve, and a framed immersion when \( n_1 = 1 \) or \( k_1 = 1 \). We say that \( \gamma \) is of type \((n_1, n_2, n_3)\). By definition, \( \mu : \mathbb{R} \rightarrow S^2 \) is given by

\[
\mu(t) = \frac{1}{\sqrt{1 + t^{2k_1} + t^{2k_1} + 2k_2}}(1, t^{k_1}, t^{k_1} + k_2).
\]
and the components of the curvature are
\[
\ell(t) = \frac{k_1 t^{2k_1+k_2-1}}{(1 + t^{2k_1}) \sqrt{1 + t^{2k_1} + t^{2k_1+2k_2}}},
\]
\[
m(t) = \frac{-k_1 t^{k_1-1}}{\sqrt{(1 + t^{2k_1})(1 + t^{2k_1} + t^{2k_1+2k_2})}},
\]
\[
n(t) = \frac{-k_1^{k_1+k_2-1}(k_1 + k_2 + k_2 t^{2k_1})}{(1 + t^{2k_1} + t^{2k_1+2k_2}) \sqrt{1 + t^{2k_1}}},
\]
\[
a(t) = t^{n-1} \sqrt{1 + t^{2k_1} + t^{2k_1+2k_2}}.
\]

**Example 4.2.** Let \( \gamma : \mathbb{R} \to \mathbb{R}^3 \) be the smooth mapping \( \gamma(t) = \begin{cases} (t, 0, e^{-1/t^2}) & \text{if } t > 0, \\ (0, 0, 0) & \text{if } t = 0, \\ (t, e^{-1/t^2}, 0) & \text{if } t < 0. \end{cases} \)

The curve \( \gamma \) is regular but does not satisfy the linear independent condition at \( t = 0 \). However, \( \gamma \) is a framed base curve. We have the smooth mapping \( (v_1, v_2) : \mathbb{R} \to \Delta_2 \) with

\[
v_1(t) = \begin{cases} (1/\sqrt{2 + \dot{f}(t)^2}) \dot{f}(t), -1, -1 & \text{if } t \neq 0, \\ (1/\sqrt{2}) (0, -1, -1) & \text{if } t = 0, \end{cases}
\]
\[
v_2(t) = \begin{cases} (1/\sqrt{(1 + \dot{f}(t)^2)(2 + \ddot{f}(t)^2)}) \dot{f}(t), 1 + \ddot{f}(t)^2, -1 & \text{if } t > 0, \\ (1/\sqrt{2}) (0, 1, -1) & \text{if } t = 0, \\ (1/\sqrt{(1 + \dot{f}(t)^2)(2 + \ddot{f}(t)^2)}) (-\dddot{f}(t), 1, 1 - \ddot{f}(t)^2) & \text{if } t < 0, \end{cases}
\]

where \( \dot{f}(t) = e^{-1/t^2} \) for \( t \neq 0 \). It is easy to see that \( (\gamma, v_1, v_2) \) is a framed curve. Since \( \mu : \mathbb{R} \to S^2 \) is given by

\[
\mu(t) = v_1(t) \times v_2(t) = \begin{cases} \left( 1/\sqrt{1 + \ddot{f}(t)^2} \right) (1, 0, \dddot{f}(t)) & \text{if } t > 0, \\ (1, 0, 0) & \text{if } t = 0, \\ \left( 1/\sqrt{1 + \ddot{f}(t)^2} \right) (1, \dddot{f}(t), 0) & \text{if } t < 0, \end{cases}
\]

the curvature of the framed curve is given by

\[
\ell(t) = \begin{cases} \dddot{f}(t) \dot{f}(t)/(2 + \ddot{f}(t)^2) \sqrt{1 + \dot{f}(t)^2} & \text{if } t > 0, \\ \dddot{f}(t) \dot{f}(t)/(2 + \ddot{f}(t)^2) \sqrt{1 + \dot{f}(t)^2} & \text{if } t = 0, \\ \dddot{f}(t) \dot{f}(t)/(2 + \ddot{f}(t)^2) \sqrt{1 + \dot{f}(t)^2} & \text{if } t < 0, \end{cases}
\]
\[
m(t) = \begin{cases} \dddot{f}(t)/(2 + \ddot{f}(t)^2) \sqrt{1 + \dot{f}(t)^2} & \text{if } t \neq 0, \\ \dddot{f}(t)/(2 + \ddot{f}(t)^2) \sqrt{1 + \dot{f}(t)^2} & \text{if } t = 0, \\ \dddot{f}(t)/(2 + \ddot{f}(t)^2) \sqrt{1 + \dot{f}(t)^2} & \text{if } t < 0, \end{cases}
\]
\[
n(t) = \begin{cases} \dddot{f}(t)/(2 + \ddot{f}(t)^2) \sqrt{1 + \dot{f}(t)^2} & \text{if } t > 0, \\ \dddot{f}(t)/(2 + \ddot{f}(t)^2) \sqrt{1 + \dot{f}(t)^2} & \text{if } t = 0, \\ \dddot{f}(t)/(2 + \ddot{f}(t)^2) \sqrt{1 + \dot{f}(t)^2} & \text{if } t < 0, \end{cases}
\]
\[
a(t) = \begin{cases} \dddot{f}(t)/(2 + \ddot{f}(t)^2) \sqrt{1 + \dot{f}(t)^2} & \text{if } t \neq 0, \\ 1 & \text{if } t = 0. \end{cases}
\]

Note that the projection to \( \mu(0) = (1, 0, 0) \) direction. Then \( \gamma_\mu : \mathbb{R} \to \mathbb{R}^2 \) is given by

\[
\gamma_\mu(t) = \begin{cases} (-1/\sqrt{2})(e^{-1/t^2}, e^{-1/t^2}) & \text{if } t > 0, \\ (0, 0) & \text{if } t = 0, \\ (1/\sqrt{2})(-e^{-1/t^2}, e^{-1/t^2}) & \text{if } t < 0. \end{cases}
\]

It follows that \( \gamma_\mu : \mathbb{R} \to \mathbb{R}^2 \) is not a frontal (cf. [6]).
References


