



Title	Stability of flat zero-energy states at the dirty surface of a nodal superconductor
Author(s)	Ikegaya, Satoshi; Asano, Yasuhiro
Citation	Physical Review B, 95(21), 214503-1-214503-11 <a href="https://doi.org/10.1103/PhysRevB.95.214503">https://doi.org/10.1103/PhysRevB.95.214503</a>
Issue Date	2017-06-05
Doc URL	<a href="http://hdl.handle.net/2115/66588">http://hdl.handle.net/2115/66588</a>
Rights	©2017 American Physical Society
Type	article
File Information	PhysRevB.95.214503.pdf



[Instructions for use](#)

# Stability of flat zero-energy states at the dirty surface of a nodal superconductor

Satoshi Ikegaya<sup>1,\*</sup> and Yasuhiro Asano<sup>1,2,3</sup><sup>1</sup>*Department of Applied Physics, Hokkaido University, Sapporo 060-8628, Japan*<sup>2</sup>*Center of Topological Science and Technology, Hokkaido University, Sapporo 060-8628, Japan*<sup>3</sup>*Moscow Institute of Physics and Technology, 141700 Dolgoprudny, Russia*

(Received 19 January 2017; published 5 June 2017)

We discuss the stability of highly degenerate zero-energy states that appear at the surface of a nodal superconductor preserving time-reversal symmetry. The existence of such surface states is a direct consequence of the nontrivial topological numbers defined in the restricted Brillouin zones in the clean limit. In experiments, however, potential disorder is inevitable near the surface of a real superconductor, which may lift the high degeneracy at zero energy. We show that an index defined in terms of the chiral eigenvalues of the zero-energy states can be used to measure the degree of degeneracy at zero energy in the presence of potential disorder. We also discuss the relationship between the index and the topological numbers.

DOI: [10.1103/PhysRevB.95.214503](https://doi.org/10.1103/PhysRevB.95.214503)

## I. INTRODUCTION

The discovery of a topological insulator [1,2] had an impact on researchers studying the physics of superconductivity. The gapped band structures in a superconductor can also be topologically nontrivial. The bulk-boundary correspondence, which is the interrelationship between the nontrivial topological invariant in superconducting states in the bulk and the number of gapless states at its surface, immediately ensures the existence of surface bound states. Recently, such topologically nontrivial superconductors have attracted enormous attention due to the existence of exotic surface bound states, some of which are composed of Majorana particles [3]. Using the 10-fold classification as a basis [4], the early studies in this context were devoted to *fully gapped* topological superconductors.

The 10-fold topological classification, however, covers only some real superconductors. A number of real unconventional superconductors display nodes in the superconducting gap. Nevertheless, such a *nodal* superconductor can often host highly degenerate surface bound states at zero energy (Fermi level). The sign change in the pair potential on the Fermi surface, which is possible only in the presence of nodes in the gap functions, is the source of a topologically nontrivial superconducting phase preserving time-reversal symmetry [5,6]. When a three- (two-) dimensional nodal superconductor has the superconducting line (point) nodes, the surface bound states have flat dispersion in terms of two- (one-) dimensional wave number parallel to the surface [7,8]. A prescription called dimensional reduction enables us to topologically characterize such flat zero-energy states of a nodal superconductor. In a  $d$ -dimensional superconductor, it is possible to choose a one-dimensional Brillouin zone by fixing a  $(d - 1)$ -dimensional wave number at a certain point (say  $k$ ). When the energy spectra in the one-dimensional Brillouin zone at  $k$  have gaps, we can define the winding number  $W(k)$  [7,8]. According to the bulk-boundary correspondence for each Brillouin zone, a nodal superconductor often hosts degenerate zero-energy states (ZESs) at its clean surface. Namely,  $\sum_k |W(k)|$ -fold degenerate ZESs are expected at a

surface parallel to  $k$ . Actually, the existence of degenerate surface ZESs has been suggested in time-reversal unconventional superconductors [5,6,9,10], noncentrosymmetric superconductors [11–15], semiconductor/superconductor heterostructures [16–19], superconductor/topological insulator heterostructures [20], and superconducting Weyl semimetals [21]. It is widely accepted that flat ZESs cause various anomalies in low-energy transport such as the zero-bias anomaly in the conductance of a normal-metal/superconductor junction [5,6,19,22,23] and the fractional Josephson effect in a superconductor/insulator/superconductor junction [24–28]. These phenomena are unique to topologically nontrivial superconductors.

In experiments, however, nonmagnetic potential disorder is inevitable in the vicinity of the surface or junction interface of a superconductor. The one-dimensional Brillouin zone is ill defined with the disordered potential breaking the translational symmetry. Therefore, the winding number  $W(k)$  can no longer use to predict the number of ZESs at a dirty surface [29–32]. In other words, the potential disorder may lift the high degeneracy in the surface ZESs and may wash out the characteristic transport properties. Such a situation requires a theoretical tool that measures the stability of degenerate ZESs in the presence of potential disorder. This paper addresses this issue and will provide experimentalists with helpful information.

By paying attention to the chiral symmetry of a Bogoliubov–de Gennes (BdG) Hamiltonian [19,33–36], we show that a mathematical index,  $N_{\text{ZES}}$ , well characterizes the number of ZESs at a dirty surface. The index  $N_{\text{ZES}}$  is an invariant defined in terms of the chirality of the surface ZESs and is closely related to the one-dimensional winding number  $W(k)$  [7]. We conclude that the index  $N_{\text{ZES}}$  calculated in a clean superconductor exactly predicts the degree of degeneracy in ZESs at the dirty surface of a nodal superconductor. Numerical simulations for several nodal superconductors ensure the validity of the conclusion.

The organization of this paper is as follows. In Sec. II, we discuss the one-dimensional winding number for a nodal superconductor preserving chiral symmetry. The index  $N_{\text{ZES}}$  is defined in terms of the chiral eigenvalues of ZESs and is connected to the one-dimensional winding number through the index theorem in mathematics. In Sec. III, we confirm the

\*satoshi-ikegaya@eng.hokudai.ac.jp

validity of our conclusion for several superconductors such as  $p$ -,  $d$ - and  $f$ -wave unconventional superconductors, and two noncentrosymmetric superconductors. Section IV provides our conclusion.

## II. CHIRAL SYMMETRY AND INDEX THEOREM

### A. Winding number in a clean superconductor

We begin our discussion with a brief summary of the topological property of a nodal superconductor in the clean limit. The BdG Hamiltonian in momentum space is represented by

$$H_0(\mathbf{k}) = \begin{bmatrix} \xi_0(\mathbf{k}) & \Delta(\mathbf{k}) \\ -\Delta^*(-\mathbf{k}) & -\xi_0^*(-\mathbf{k}) \end{bmatrix}, \quad (1)$$

where  $\xi_0(\mathbf{k})$  denotes the  $M \times M$  Hamiltonian for an electron,  $\Delta(\mathbf{k})$  is the  $M \times M$  pair potential, and where  $M$  represents the number of degrees of freedom for an electron such as spin and band. The BdG Hamiltonian intrinsically preserves particle-hole symmetry

$$\Xi H_0(\mathbf{k}) \Xi^{-1} = -H_0(-\mathbf{k}), \quad (2)$$

$$\Xi = C\mathcal{K}, \quad C = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad (3)$$

where  $I$  is the  $M \times M$  unit matrix and  $\mathcal{K}$  denotes the complex-conjugation operator. We assume that the BdG Hamiltonian preserves time-reversal or time-reversal-like symmetry as

$$\mathcal{T}_\pm H_0(\mathbf{k}) \mathcal{T}_\pm^{-1} = H_0(-\mathbf{k}), \quad (4)$$

$$\mathcal{T}_\pm = U_\pm \mathcal{K}, \quad U_\pm = \begin{bmatrix} u_\pm & 0 \\ 0 & u_\pm^* \end{bmatrix}, \quad (5)$$

where  $u_\pm$  is an  $M \times M$  unitary matrix satisfying  $u_\pm u_\pm^* = \pm I$ . Time-reversal symmetry is denoted with  $u_- u_-^* = -I$ , while time-reversal-like symmetry is denoted with  $u_+ u_+^* = I$ . By combining the particle-hole symmetry operator  $\Xi$  and the time-reversal symmetry operator  $\mathcal{T}_-$ , we can define an operator

$$\Gamma = -iC\mathcal{U}_- = \begin{bmatrix} 0 & -iu_-^* \\ -iu_- & 0 \end{bmatrix}. \quad (6)$$

The BdG Hamiltonian satisfies

$$\Gamma H_0(\mathbf{k}) \Gamma^{-1} = -H_0(\mathbf{k}), \quad (7)$$

which represents chiral symmetry of the Hamiltonian. The chiral symmetry for a case of time-reversal-like symmetry is also defined in a similar way:

$$\Gamma H_0(\mathbf{k}) \Gamma^{-1} = -H_0(\mathbf{k}), \quad (8)$$

$$\Gamma = -C\mathcal{U}_+ = \begin{bmatrix} 0 & -u_+^* \\ -u_+ & 0 \end{bmatrix}. \quad (9)$$

The pair potential under consideration has nodes. Namely,  $\Delta(\mathbf{k}_{\text{node}}) = 0$  is satisfied at nodal points  $\mathbf{k}_{\text{node}}$  on the Fermi surface. Therefore, it is impossible to characterize such superconducting states topologically in terms of the wave function of the whole Brillouin zone. Alternatively, we define a winding number in a one-dimensional Brillouin zone by fixing  $\mathbf{k}_\parallel$  at a certain point. The momentum  $k_\perp$  indicates a

superconducting state in a one-dimensional Brillouin zone. The winding number is defined by [7]

$$W(\mathbf{k}_\parallel) = \frac{i}{4\pi} \int dk_\perp \text{Tr}[\Gamma H_0^{-1}(\mathbf{k}) \partial_{k_\perp} H_0(\mathbf{k})]. \quad (10)$$

Since  $\mathbf{k}_{\text{node}}$  represents nodal points on the Fermi surface, the relation  $\xi(\mathbf{k}_{\text{node}}) = 0$  holds simultaneously. The winding number  $W(\mathbf{k}_\parallel)$  is ill defined when the integration path along  $k_\perp$  in Eq. (10) intersects  $\mathbf{k}_{\text{node}}$ . Therefore, we have to choose  $\mathbf{k}_\parallel$  so that  $k_\perp$  can be kept away from the nodal points. When  $W(\mathbf{k}_\parallel)$  is nonzero in a finite region of  $\mathbf{k}_\parallel$ , dispersionless ZESs with respect to  $\mathbf{k}_\parallel$  appear at a clean surface parallel to  $\mathbf{k}_\parallel$  [5,6,9–21]. The number of ZESs at a clean surface is represented by

$$N_{\text{clean}} = \sum'_{\mathbf{k}_\parallel} |W(\mathbf{k}_\parallel)|, \quad (11)$$

where  $\sum'_{\mathbf{k}_\parallel}$  denotes a summation over  $\mathbf{k}_\parallel$  excluding the nodal points. In what follows, we describe the degree of degeneracy in the ZESs at a *dirty* surface in the presence of potential disorder. The random impurity potential in the bulk region strongly suppresses the unconventional superconducting pair potential. Thus, we consider the effects of the potential disorder only near a surface.

### B. Zero-energy states at a dirty surface

We consider a semi-infinite superconductor that occupies  $x_\perp \leq 0$  as shown in Fig. 1(a). We apply the periodic boundary condition in a direction parallel to the surface  $x_\parallel$ . The BdG Hamiltonian in real space  $H_0(\mathbf{r})$  is obtained by replacing the momentum  $\mathbf{k}$  by  $-i\nabla_r$ . The nonmagnetic random impurity

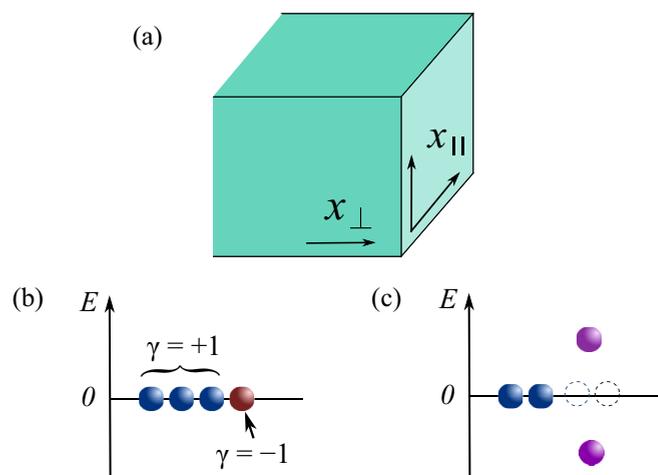


FIG. 1. (a) Schematic image of a semi-infinite nodal superconductor. (b) Fourfold degenerate zero-energy states (ZESs) in the absence of random potential. Three of them belong to the positive chiral eigenvalue (i.e.,  $N_+ = 3$ ). One remaining ZES belongs to the negative chiral eigenvalue (i.e.,  $N_- = 1$ ). The fourfold degeneracy is protected by translational symmetry. (c) In the presence of random potential, a positive and a negative chiral ZES form a pair and departs from zero energy. However, two positive chiral ZESs remain at zero energy. The index  $N_{\text{ZES}} = N_+ - N_- = 2$  represents the number of ZESs remaining at zero energy in the presence of random potential.

potential in the vicinity of the surface is represented by

$$V_{\text{imp}}(\mathbf{r}) = \begin{bmatrix} v(\mathbf{r})I & 0 \\ 0 & -v(\mathbf{r})I \end{bmatrix}, \quad (12)$$

where the random potential  $v(\mathbf{r})$  disappears rapidly with increases in  $x_{\perp}$  from the surface. The total Hamiltonian is given by

$$H(\mathbf{r}) = H_0(\mathbf{r}) + V_{\text{imp}}(\mathbf{r}). \quad (13)$$

The momentum  $\mathbf{k}_{\parallel}$  is no longer a good quantum number because the impurity potential breaks the translational symmetry. As a result, it is impossible to define the one-dimensional winding number  $W(\mathbf{k}_{\parallel})$  in the presence of the potential disorder. However, the Hamiltonian  $H(\mathbf{r})$  preserves the chiral symmetry of the Hamiltonian in Eqs. (7) or (8), which is the most important factor in the argument below.

The central ingredient of our theory consists of the following two important properties of the eigenstates in the presence of chiral symmetry [7] (i.e.,  $\{H, \Gamma\} = 0$ ).

(i) First, the zero-energy states of  $H$  are simultaneously the eigenstates of the chiral operator  $\Gamma$ . Since  $\Gamma^2 = 1$ , the eigenvalue of  $\Gamma$  is either  $\gamma = +1$  or  $-1$ . Namely, a ZES satisfying  $H_0(\mathbf{r})\varphi_{\gamma}(\mathbf{r}) = 0$  also satisfies  $\Gamma\varphi_{\gamma}(\mathbf{r}) = \gamma\varphi_{\gamma}(\mathbf{r})$ . We refer to  $\gamma$  as the chirality in the following.

(ii) Second, the nonzero energy states of  $H$  are described by the linear combination of the two states: one has  $\gamma = +1$  and the other has  $\gamma = -1$ . Namely, a nonzero energy state is described as  $\varphi_{E \neq 0}(\mathbf{r}) = c_+\chi_+(\mathbf{r}) + c_-\chi_-(\mathbf{r})$ , where  $\Gamma\chi_{\pm}(\mathbf{r}) = \pm\chi_{\pm}(\mathbf{r})$ . Moreover, the relation  $|c_+| = |c_-|$  always holds [19].

Here, we define two integer numbers  $N_+$  and  $N_-$  in the clean limit. According to property (i), we can immediately conclude that each ZES at a clean surface belongs to either the positive or the negative chiral state. The integer  $N_+$  ( $N_-$ ) is the number of ZESs that have the positive (negative) chiral eigenvalue. [See also Fig. 1(b).] The total number of ZESs at a clean surface is represented by  $N_+ + N_-$ , which must be identical to  $N_{\text{clean}}$  in Eq. (11).

The stability of the flat ZESs in the presence of impurities can be discussed by using property (ii). A ZES departs from zero energy only when it can form a pair with its chiral partner. When  $N_+ > N_-$ , for example,  $N_-$  negative chiral ZESs can couple to  $N_-$  positive chiral ZESs under potential disorder. As a result, they form nonzero energy states whose number is  $2N_-$ . However,  $N_+ - N_-$  positive chiral states remain at zero energy even in the presence of impurities because their chiral partner is absent. The integer number defined by

$$N_{\text{ZES}} = N_+ - N_- \quad (14)$$

represents the number of ZESs that remain at a dirty surface. When  $N_+ < N_-$ , the number of ZESs at a dirty surface is given by  $N_- - N_+$ . Therefore, in general,  $|N_{\text{ZES}}|$  is the degree of degeneracy at zero energy in the presence of potential disorder. The essence of this argument is illustrated in Figs. 1(b) and 1(c) with  $N_+ = 3$  and  $N_- = 1$ . In Fig. 1(b), we consider four ZESs at a clean surface. In Fig. 1(c), we introduce the impurity potential at the surface. Although the index  $N_{\text{ZES}}$  is defined in the presence of translational symmetry, it represents the degree of the degeneracy at zero energy in the absence of translational symmetry. This is the main conclusion of our paper.

### C. Relation with topological number

At the end of this section, we discuss the topological aspect of  $N_{\text{ZES}}$ . As examined in Ref. [7], the index theorem relates to the winding number  $W(\mathbf{k}_{\parallel})$  and the number of ZESs on a clean surface as follows:

$$W(\mathbf{k}_{\parallel}) = \pm[n_+(\mathbf{k}_{\parallel}) - n_-(\mathbf{k}_{\parallel})], \quad (15)$$

where  $n_+(\mathbf{k}_{\parallel})$  [ $n_-(\mathbf{k}_{\parallel})$ ] denotes the number of positive (negative) chiral zero-energy states at  $\mathbf{k}_{\parallel}$ . There are two possible choices for the sign on the right-hand side of Eq. (15). When we consider the surface of a semi-infinite superconductor occupying  $x_{\perp} \leq 0$  as shown in Fig. 1(a), we should choose the positive sign. On the other hand, we should choose the negative sign at the surface of a semi-infinite superconductor occupying  $x_{\perp} \geq 0$  [7]. However, this sign has no physical meaning because the number of ZESs is always given by  $|n_+(\mathbf{k}_{\parallel}) - n_-(\mathbf{k}_{\parallel})|$ . As discussed in the previous subsection, the index  $N_{\text{ZES}}$  is represented by the difference between the total numbers of positive and negative chiral ZESs. Therefore, by taking Eq. (15) into account, we find an important relation

$$N_{\text{ZES}} = \sum'_{\mathbf{k}_{\parallel}} W(\mathbf{k}_{\parallel}) = \pm(N_+ - N_-) \quad (16)$$

as a result of the index theorem. More specifically, the index  $N_{\text{ZES}}$  is a topological invariant defined in terms of the wave function in the superconducting states. Simultaneously, it is an invariant defined in terms of the zero-energy solutions in a differential equation. The index theorem mathematically bridges the two different invariants. In physics, the index  $N_{\text{ZES}}$  is an invariant that measures the degree of degeneracy of the ZESs staying at the dirty surface of a nodal superconductor. In the next section, we check the validity of our conclusion by performing numerical simulations on tight-binding model.

## III. NUMERICAL RESULTS

### A. Unconventional superconductors

We apply the general argument in Sec. II to the several time-reversal superconductors in two dimensions. The first example is the three types of unconventional superconductors characterized by  $p_x$ -,  $d_{xy}$ -, and  $f$ -wave pairing symmetry. We describe the present superconductors by the  $2 \times 2$  BdG Hamiltonian

$$\hat{H}_0(\mathbf{k}) = \begin{bmatrix} \xi(\mathbf{k}) & \Delta_{\mu}(\mathbf{k}) \\ \Delta_{\mu}(\mathbf{k}) & -\xi(\mathbf{k}) \end{bmatrix}, \quad (17)$$

$$\xi(\mathbf{k}) = \frac{\hbar^2 \mathbf{k}^2}{2m} - \mu_{\text{F}}, \quad (18)$$

$$\Delta_{p_x}(\mathbf{k}) = \frac{\Delta_0}{k_{\text{F}}} k_x, \quad (19)$$

$$\Delta_{d_{xy}}(\mathbf{k}) = \frac{\Delta_0}{k_{\text{F}}^2} k_x k_y, \quad (20)$$

$$\Delta_f(\mathbf{k}) = \frac{\Delta_0}{k_{\text{F}}^3} k_x (k_x^2 - 2k_y^2), \quad (21)$$

where the subscript  $\mu = p_x, d_{xy}, f$  labels the pairing symmetry,  $m$  denotes the effective mass of an electron,  $\mu_{\text{F}}$  is the chemical potential,  $\Delta_0$  is the amplitude of the pair potential

at zero temperature, and  $k_F = \sqrt{2m\mu_F}/\hbar$  represents the Fermi wave number. The Hamiltonian satisfies

$$\hat{\Gamma} \hat{H}_0(\mathbf{k}) \hat{\Gamma}^{-1} = -\hat{H}_0(\mathbf{k}), \quad \hat{\Gamma} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad (22)$$

which represents chiral symmetry of the Hamiltonian.

The one-dimensional winding number in Eq. (10) can be further simplified to [7,8]

$$W_\mu(k_y) = \frac{1}{2} \sum_{\xi(\mathbf{k})=0} \text{sgn}[\partial_{k_x} \xi(\mathbf{k})] \text{sgn}[\Delta_\mu(\mathbf{k})], \quad (23)$$

where the summation is carried out for  $k_x$  satisfying  $\xi(\mathbf{k}) = 0$  with fixed  $k_y$ . From Eq. (23), the winding number for each pairing symmetry is calculated as

$$W_{p_x}(k_y) = \begin{cases} 1 & \text{for } |k_y| < k_F, \\ 0 & \text{for } |k_y| > k_F, \end{cases} \quad (24)$$

$$W_{d_{xy}}(k_y) = \begin{cases} 1 & \text{for } 0 < k_y < k_F, \\ -1 & \text{for } 0 > k_y > -k_F, \\ 0 & \text{for } |k_y| > k_F, \end{cases} \quad (25)$$

$$W_f(k_y) = \begin{cases} 1 & \text{for } |k_y| < k_f, \\ -1 & \text{for } k_f < |k_y| < k_F, \\ 0 & \text{for } |k_y| > k_F, \end{cases} \quad (26)$$

where  $k_f = k_F/\sqrt{2}$ . The total number of topologically protected ZESs at the clean surface is calculated as

$$N_{\text{clean}} = \sum_{k_y} |W_\mu(k_y)|, \quad (27)$$

according to the bulk-boundary correspondence. The number of ZESs at the dirty surface, on the other hand, is evaluated by the index  $N_{\text{ZES}}$ . By substituting Eqs. (24)–(26) into Eq. (16), we obtain the index  $N_{\text{ZES}}$  for each pairing symmetry as

$$|N_{\text{ZES}}| = \begin{cases} \sum_{|k_y| < k_F} = N_{\text{clean}} & \text{for } p_x \text{ wave,} \\ 0 & \text{for } d_{xy} \text{ wave,} \\ \sum_{|k_y| < k_f} - \sum_{k_f < |k_y| < k_F} \neq 0 & \text{for } f \text{ wave.} \end{cases} \quad (28)$$

To check the validity of Eq. (28), we numerically calculate the eigenenergy of an isolating unconventional superconductor on the two-dimensional tight-binding model as shown in Fig. 2. A lattice site is indicated by a vector  $\mathbf{r} = j a_0 \mathbf{x} + m a_0 \mathbf{y}$ , where  $a_0$  denotes the lattice constant and  $\mathbf{x}$  ( $\mathbf{y}$ ) is a unit vector

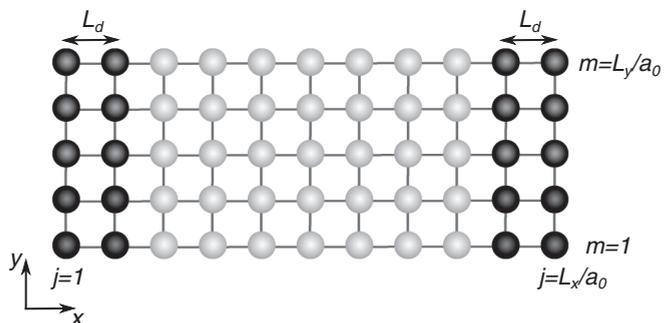


FIG. 2. Schematic picture of a superconductor on the tight-binding lattice.

in the  $x$  ( $y$ ) direction. The number of the lattice site in the  $x$  and  $y$  directions is denoted by  $L_x/a_0$  and  $L_y/a_0$ , respectively. In the  $y$  direction, the periodic boundary condition is applied. In the  $x$  direction, we apply the hard-wall boundary condition. The wave functions of zero-energy surface states decay exponentially into the bulk. Therefore, when the system length  $L_x$  is much longer than the decay length of the zero-energy states  $\xi_S$ , we can ignore the couplings between the zero-energy states at the left surface (i.e.,  $j = 1$ ) and the right surface (i.e.,  $j = L_x$ ). As a consequence, when  $L_x \gg \xi_S$  is satisfied for all the zero-energy states, we can expect  $2 \times N_{\text{clean}}$  ( $2 \times N_{\text{ZES}}$ ) zero-energy states in the finite-size system with the clean (dirty) surfaces, where the factor 2 is derived from the contribution from two different surfaces in the  $x$  direction. In the following numerical simulations, we only consider the enough long system which enables us to ignore the couplings between the zero-energy states at the different surfaces. We introduce the nonmagnetic impurity potential by adding the random onsite potentials  $v(\mathbf{r})$  in the outermost  $L_d/a_0$  layers in the  $x$  direction as shown in Fig. 2. The amplitude of  $v(\mathbf{r})$  is given randomly in the range of  $-V_1/2 \leq v(\mathbf{r}) \leq V_1/2$ . We numerically diagonalize the BdG Hamiltonian on the tight-binding model which is shown in Appendix A 1. We fix several parameters as  $\mu_F = 1.5t$ ,  $\Delta_0 = 1.0t$ ,  $L_x = 60a_0$ , and  $L_y = 18a_0$  where  $t$  denotes the nearest-neighbor hopping integral. The number of the zero-energy states at a clean surface is evaluated from  $N_{\text{clean}}$  in Eq. (27). This parameter choice leads to  $N_{\text{clean}} = 7$  for a  $p_x$ - and a  $f$ -wave superconductor, and  $N_{\text{clean}} = 6$  for a  $d_{xy}$ -wave superconductor. The number of the zero-energy states at a dirty surface is evaluated from the index  $N_{\text{ZES}}$  in Eq. (28). In the present parameters choice, we obtain  $N_{\text{ZES}} = 7, 0$ , and 3 for  $p_x$ -,  $d_{xy}$ -, and  $f$ -wave pairing symmetry, respectively. In Figs. 3(a)–3(c), we show the numerical results of energy eigenvalues, where the eigenenergy is labeled by an integer  $n$ . The open symbols and the filled symbols respectively denote the energy eigenvalues in a superconductor with clean surfaces and those with dirty surfaces. We chose  $V_1 = 3.0t$  and  $L_d = 5a_0$  to realize the dirty surfaces. As shown in the open symbols of Figs. 3(a)–3(c), we find  $2 \times N_{\text{clean}} = 14$  zero-energy states at the clean surfaces of  $p_x$ -wave and  $f$ -wave superconductor and  $2 \times N_{\text{clean}} = 12$  zero-energy states at the clean surfaces of  $d_{xy}$ -wave superconductor. Therefore, our parameter choice satisfies the condition  $L_x \gg \xi_S$  for all the zero-energy states. As shown in filled symbols of Fig. 3(c), for instance, eight ZESs of the  $f$ -wave superconductor move away from the zero energy by introducing the impurity potentials, even though we find that six ZESs still keep staying at zero energy. Since  $2 \times |N_{\text{ZES}}| = 6$  under the present parameter choice, the argument in Sec. II predicts the number of ZESs at a dirty surface exactly. Figures 3(a) and 3(b) show the perfect agreement between our theory and numerical results. For a  $p_x$ -wave superconductor,  $2 \times |N_{\text{ZES}}| = 14$  states remain at zero energy. In a  $d_{xy}$ -wave case, ZESs are absent at the dirty surfaces.

## B. Noncentrosymmetric superconductor I

Second, we apply the argument in Sec. II to the noncentrosymmetric superconductors (NCSs) in two dimensions. The

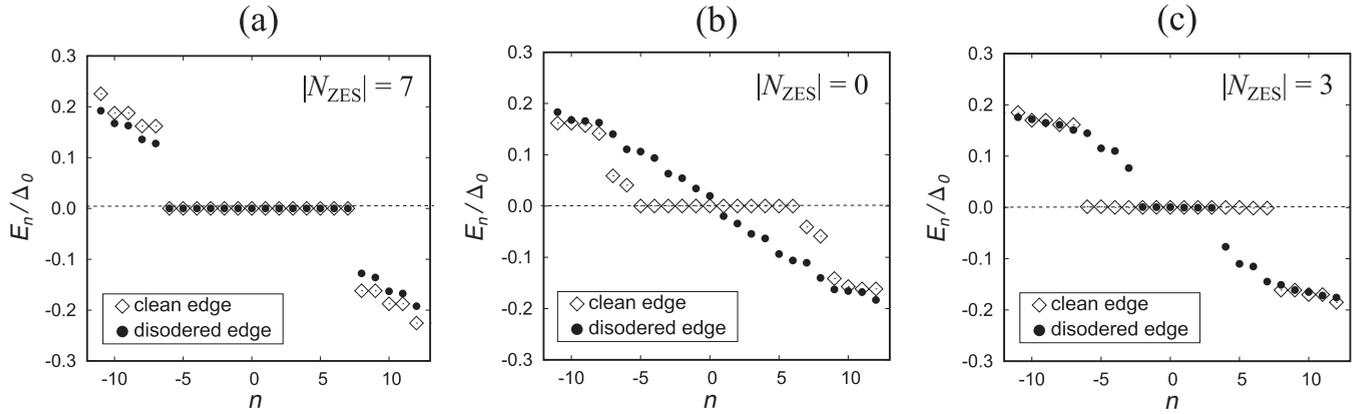


FIG. 3. Energy eigenvalues of (a)  $p_x$ -, (b)  $d_{xy}$ -, and (c)  $f$ -wave superconductor are plotted. In numerical simulation, eigenvalues are calculated in decreasing order labeled by  $n$  in the horizontal axis. In the clean limit, as shown with the open symbols, we find  $2 \times N_{\text{clean}} = 14$  zero-energy states (ZESs) for  $p_x$ - and  $f$ -wave cases. For a  $d_{xy}$  symmetry, the two states at a gap nodal point  $k_y = 0$  move from zero energy due to the finite-size effect. Thus, the number of ZESs becomes  $2 \times N_{\text{clean}} = 12$ . In the presence of potential disorder at two surfaces, as shown with the filled symbols, the number of the ZESs is identical to  $2 \times |N_{\text{ZES}}|$  which is 14 in (a), 0 in (b), and 6 in (c).

BdG Hamiltonian for a NCS is given by

$$\check{H}(\mathbf{k}) = \begin{bmatrix} \hat{h}(\mathbf{k}) & \hat{\Delta}(\mathbf{k}) \\ -\hat{\Delta}^*(-\mathbf{k}) & -\hat{h}^*(-\mathbf{k}) \end{bmatrix}, \quad (29)$$

$$\hat{h}(\mathbf{k}) = \xi(\mathbf{k})\hat{\sigma}_0 + \mathbf{g}(\mathbf{k}) \cdot \hat{\boldsymbol{\sigma}}, \quad (30)$$

$$\hat{\Delta}(\mathbf{k}) = i[\psi(\mathbf{k}) + \mathbf{d}(\mathbf{k}) \cdot \hat{\boldsymbol{\sigma}}]\hat{\sigma}_y, \quad (31)$$

where  $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$  and  $\hat{\sigma}_0$  denote Pauli matrices in spin space and the  $2 \times 2$  unit matrix, respectively. The absence of inversion symmetry leads the spin-orbit coupling (SOC) potential denoted by  $\mathbf{g}(\mathbf{k}) = -\mathbf{g}(-\mathbf{k})$ . Furthermore, the pair potential becomes the admixture of the even-parity spin-singlet component  $\psi(\mathbf{k}) = \psi(-\mathbf{k})$  and the odd-parity spin-triplet pair component  $\mathbf{d}(\mathbf{k}) = -\mathbf{d}(-\mathbf{k})$  because parity is no longer a good quantum index [37,38]. The spin-triplet pairing vector  $\mathbf{d}(\mathbf{k})$  is set to be parallel to the polarization vector of the SOC [37] [i.e.,  $\mathbf{d}(\mathbf{k}) \parallel \mathbf{g}(\mathbf{k})$ ]. The BdG Hamiltonian satisfies

$$\check{\Gamma} \check{H}_0(\mathbf{k}) \check{\Gamma}^{-1} = -\check{H}_0(\mathbf{k}), \quad \check{\Gamma} = \begin{bmatrix} 0 & \hat{\sigma}_y \\ \hat{\sigma}_y & 0 \end{bmatrix}, \quad (32)$$

which represents the chiral symmetry of the Hamiltonian.

A superconductor with  $(d_{xy} + p)$ -wave pairing symmetry is an example of NCS which host the flat ZES at its clean surface [11,12,29]. Under the Rashba-type SOC  $\mathbf{g}_r(\mathbf{k}) = \alpha(k_x \hat{x} - k_y \hat{y})$  with  $\alpha$  being the coupling amplitude, the normal-state Fermi surface splits into the two circles as illustrated in Fig. 4, where the two wave numbers

$$k_{\pm} = \mp \frac{m\alpha}{\hbar^2} + \sqrt{k_F^2 + \left(\frac{m\alpha}{\hbar^2}\right)^2} \quad (33)$$

characterize the two Fermi surfaces. The pair potential of the  $(d_{xy} + p)$ -wave superconductor is given as [11,12,39]

$$\psi(\mathbf{k}) = \Delta_s f(\mathbf{k}), \quad \mathbf{d}(\mathbf{k}) = \Delta_t f(\mathbf{k}) \frac{\mathbf{g}_r(\mathbf{k})}{\alpha k}, \quad (34)$$

with  $f(\mathbf{k}) = (k_x k_y / k^2)$  and  $k = \sqrt{k_x^2 + k_y^2}$ . In this pair potential, there are eight nodal points which are located at  $(\pm k_{\pm}, 0)$  and  $(0, \pm k_{\pm})$  as illustrated in Fig. 4.

By applying a unitary transformation shown in Appendix B, it is possible to deform the BdG Hamiltonian of the  $(d_{xy} + p)$ -wave superconductor as

$$\check{H}'_0(\mathbf{k}) = \begin{bmatrix} \hat{H}_+(\mathbf{k}) & 0 \\ 0 & \hat{H}_-(\mathbf{k}) \end{bmatrix}, \quad (35)$$

$$\hat{H}_{\pm}(\mathbf{k}) = \begin{bmatrix} \xi_{\pm}(\mathbf{k}) & -\Delta_{\pm}(\mathbf{k}) \\ -\Delta_{\pm}(\mathbf{k}) & -\xi_{\pm}(\mathbf{k}) \end{bmatrix}, \quad (36)$$

$$\xi_{\pm}(\mathbf{k}) = \xi(\mathbf{k}) \pm |\mathbf{g}_r(\mathbf{k})|, \quad (37)$$

$$\Delta_{\pm}(\mathbf{k}) = f(\mathbf{k})[\Delta_t \pm \Delta_s]. \quad (38)$$

The chiral symmetry in this new basis is represented as

$$\hat{\Gamma}_{\pm} \hat{H}_{\pm}(\mathbf{k}) \hat{\Gamma}_{\pm}^{-1} = -\hat{H}_{\pm}(\mathbf{k}), \quad \hat{\Gamma}_{\pm} = \mp \hat{\sigma}_y. \quad (39)$$

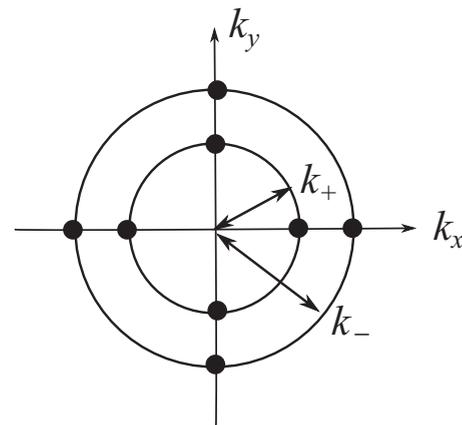


FIG. 4. Two Fermi surfaces under the Rashba SOC are illustrated. The eight nodal points are indicated by the black dots.

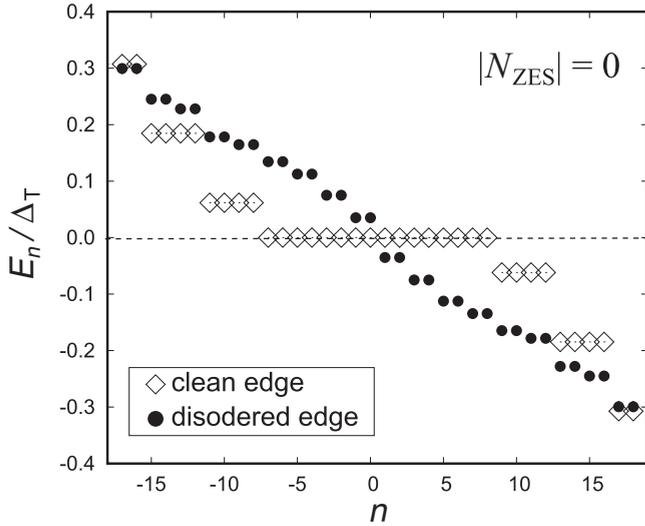


FIG. 5. Energy eigenvalues of a  $(d_{xy} + p)$ -wave superconductor are plotted in the same manner as in Fig. 3. An energy is normalized by  $\Delta_T = \Delta_s + \Delta_t$ . In the clean limit, there are 16 zero-energy states as shown with the open symbols. In the presence of random potential at the two surfaces, on the other hand, the ZESs are absent as shown with the filled symbols in agreement with  $N_{ZES} = 0$ .

By using Eqs. (36) and (39), the relevant winding number can be calculated as [7,8]

$$W(k_y) = W_+(k_y) - W_-(k_y), \quad (40)$$

$$W_{\pm}(k_y) = \frac{1}{2} \sum_{\xi_{\pm}(\mathbf{k})=0} \text{sgn}[\partial_{k_x} \xi_{\pm}(\mathbf{k})] \text{sgn}[\Delta_{\pm}(\mathbf{k})], \quad (41)$$

where the summation is carried out for  $k_x$  that satisfies  $\xi_{\pm}(\mathbf{k}) = 0$  with fixed  $k_y$ . From Eq. (40), we obtain

$$W(k_y) = \begin{cases} 2 \text{sgn}(k_y) & \text{for } |k_y| < k_+, \\ \text{sgn}(k_y) & \text{for } k_+ < |k_y| < k_-, \\ 0 & \text{for } |k_y| > k_- \end{cases} \quad (42)$$

for  $\Delta_s > \Delta_t$ , and

$$W(k_y) = \begin{cases} 0 & \text{for } |k_y| < k_+, \\ -\text{sgn}(k_y) & \text{for } k_+ < |k_y| < k_-, \\ 0 & \text{for } |k_y| > k_- \end{cases} \quad (43)$$

for  $\Delta_t > \Delta_s$ . The index  $N_{ZES}$  is calculated from Eq. (16). Since the winding number satisfies  $W(k_y) = -W(-k_y)$ , we immediately find  $N_{ZES} = 0$  for both  $\Delta_s > \Delta_t$  and  $\Delta_t > \Delta_s$ .

Figure 5 shows the eigenvalues of the BdG Hamiltonian for a  $(d_{xy} + p)$ -wave superconductor. The expression of the Hamiltonian on the tight-binding model is given in Appendix A 2. We chose parameters as  $\mu_F = 2.0t$ ,  $\alpha = 0.1t$ ,  $\Delta_s = 0.8t$ ,  $\Delta_t = 0.2t$ ,  $L_x = 50a_0$ ,  $L_y = 10a_0$ , and  $L_d = 5a_0$ . The open and the filled symbols denote the energy eigenvalues of a superconductor with the clean surface ( $V_l = 0$ ) and the dirty surfaces ( $V_l = 3.0t$ ), respectively. The present parameter choice leads  $N_{\text{clean}} = \sum_{k_y} |W_{\mu}(k_y)| = 8$ . As shown with the open symbol, we indeed find the  $2 \times N_{\text{clean}} = 16$  zero-energy states at the clean surface. The random potential at the surfaces completely lifts the degeneracy at zero energy as shown

with the filled symbols. The numerical results agree with the argument in Sec. II. Since  $N_{ZES} = 0$ , the flat ZES in a  $(d_{xy} + p)$ -wave superconductor is fragile under the potential disorder. At finite energies, the eigenvalues for dirty surfaces are always doubly degenerate, which corresponds to the Kramers doublets protected by time-reversal symmetry.

### C. Noncentrosymmetric superconductor II

In a zinc-blende semiconductor quantum well confined in the [110] crystal direction, the Dresselhaus [110] type spin-orbit coupling described by  $\mathbf{g}_d(\mathbf{k}) = \beta k_x \mathbf{z}$  becomes dominant. The Hamiltonian for the normal states is given by

$$\hat{h}_p(\mathbf{k}) = \xi(\mathbf{k}) \hat{\sigma}_0 + \beta k_x \hat{\sigma}_z. \quad (44)$$

The electronic states described by Eq. (44) have been well studied in spintronics because they show an unusual spin property called persistent spin helix [40–43]. As shown in Appendix C, the persistent spin-helix states can be also obtained in the thin film growing along the [001] crystal direction [40,41]. In what follows, we discuss the flat ZESs appearing at a surface of a proximity-induced superconducting Dresselhaus[110] thin film described by

$$\check{H}_0(\mathbf{k}) = \begin{bmatrix} \hat{h}_p(\mathbf{k}) & \hat{\Delta}_p(\mathbf{k}) \\ -\hat{\Delta}_p^*(-\mathbf{k}) & -\hat{h}_p^*(-\mathbf{k}) \end{bmatrix}, \quad (45)$$

$$\hat{\Delta}_p = i \left[ \Delta_s + \Delta_t \frac{k_x}{k_F} \hat{\sigma}_z \right] \hat{\sigma}_y, \quad (46)$$

where we assume the  $s$ -wave pairing symmetry for the spin-singlet component. The Dresselhaus [110] SOC potential shifts the Fermi surfaces in the  $k_x$  direction as illustrated in Fig. 6. The center of the Fermi surfaces is located at  $(\pm Q, 0)$  with  $Q = m\beta/\hbar^2$ . The superconducting gap has four nodes on the Fermi surface when the condition

$$\frac{\beta k_F}{\mu_F} > \frac{\Delta_s^2 - \Delta_t^2}{\Delta_s \Delta_t} \quad (47)$$

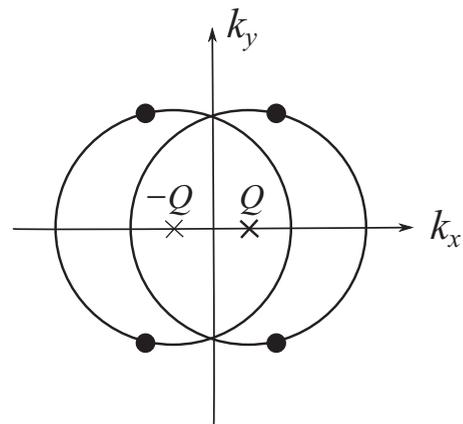


FIG. 6. Fermi surfaces under the Dresselhaus [110] SOC are illustrated. The eight nodal points are indicated by the black dots. Strictly speaking, the positions of the nodal points depend on the parameters as shown in Eq. (48).

is satisfied. The four nodal points are located at

$$(\pm r_{st} k_F, \pm k_Q), \quad (48)$$

$$k_Q = k_F \sqrt{1 - r_{st}^2 + r_{st}(\beta k_F / \mu_F)}, \quad (49)$$

as indicated by filled circle in Fig. 6, where  $r_{st} = (\Delta_s / \Delta_t)$ .

The BdG Hamiltonian preserves both time-reversal and time-reversal-like symmetry as

$$\check{T}_{\pm} \check{H}_0(\mathbf{k}) \check{T}_{\pm}^{-1} = \check{H}_0(-\mathbf{k}), \quad (50)$$

$$\check{T}_+ = \begin{bmatrix} i\hat{\sigma}_x & 0 \\ 0 & -i\hat{\sigma}_x \end{bmatrix} \mathcal{K}, \quad \check{T}_- = \begin{bmatrix} i\hat{\sigma}_y & 0 \\ 0 & i\hat{\sigma}_y \end{bmatrix} \mathcal{K}, \quad (51)$$

where  $\check{T}_{\pm}^2 = \pm 1$ . Therefore, we obtain two different chiral symmetry operators as

$$\check{\Gamma}_{\pm} \check{H}_0(\mathbf{k}) \check{\Gamma}_{\pm}^{-1} = -\check{H}_0(\mathbf{k}), \quad (52)$$

$$\check{\Gamma}_+ = \begin{bmatrix} 0 & -i\hat{\sigma}_x \\ i\hat{\sigma}_x & 0 \end{bmatrix}, \quad \check{\Gamma}_- = \begin{bmatrix} 0 & \hat{\sigma}_y \\ \hat{\sigma}_y & 0 \end{bmatrix}. \quad (53)$$

By applying the unitary transformation as

$$\check{H}'(\mathbf{k}) = \check{U}_0^\dagger \check{H}_0(\mathbf{k}) \check{U}_0, \quad \check{U}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (54)$$

the BdG Hamiltonian is block diagonalized into the two  $2 \times 2$  sectors as  $\check{H}' = \text{diag}(\hat{H}_1, \hat{H}_2)$ ,

$$\hat{H}_j(\mathbf{k}) = \begin{bmatrix} \xi_j(\mathbf{k}) & -\Delta_j(\mathbf{k}) \\ -\Delta_j(\mathbf{k}) & -\xi_j(\mathbf{k}) \end{bmatrix} \quad (55)$$

for  $j = 1 - 2$  with

$$\xi_{1(2)}(\mathbf{k}) = \xi(\mathbf{k}) + (-)\beta k_x, \quad (56)$$

$$\Delta_{1(2)}(\mathbf{k}) = \left[ \Delta_t \frac{k_x}{k_F} + (-)\Delta_s \right]. \quad (57)$$

The chiral symmetry of each block component is represented as

$$\hat{\Gamma}_{\pm, j} \hat{H}_j(\mathbf{k}) \hat{\Gamma}_{\pm, j}^{-1} = -\hat{H}_j(\mathbf{k}) \quad (58)$$

for  $j = 1 - 2$ , where

$$\hat{\Gamma}_{+, 1} = \hat{\sigma}_y, \quad \hat{\Gamma}_{+, 2} = \hat{\sigma}_y \quad (59)$$

is originated from time-reversal-like symmetry, and

$$\hat{\Gamma}_{-, 1} = \hat{\sigma}_y, \quad \hat{\Gamma}_{-, 2} = -\hat{\sigma}_y \quad (60)$$

is originated from time-reversal symmetry. The definition of the winding number depends on the form of the chiral symmetry operator. From the chiral symmetry operator originated from the time-reversal symmetry in Eq. (60), the winding number is given by

$$W(k_y) = W_1(k_y) - W_2(k_y), \quad (61)$$

with

$$W_j(k_y) = \frac{1}{2} \sum_{\xi_j(\mathbf{k})=0} \text{sgn}[\partial_{k_x} \xi_j(\mathbf{k})] \text{sgn}[\Delta_j(\mathbf{k})] \quad (62)$$

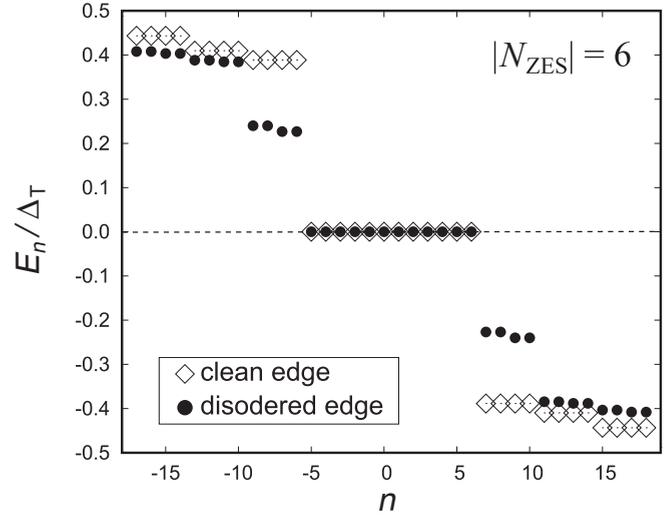


FIG. 7. Energy eigenvalues of a NCS with Dresselhaus [110] SOC are plotted in the same manner as Fig. 3. The results are normalized by  $\Delta_T = \Delta_s + \Delta_t$ . The number of ZESs is 12 in the clean case as shown with the open symbols. All of the ZESs keep staying at zero energy even in the presence of random potential at the two surfaces as predicted by the index  $2 \times |N_{\text{ZES}}| = 12$ .

for  $j = 1 - 2$ , where the summation is carried out for  $k_x$  satisfying  $\xi_j(\mathbf{k}) = 0$  at a fixed  $k_y$ . The winding number in each sector is calculated to be

$$W_1(k_y) = W_2(k_y) = \begin{cases} 1 & \text{for } |k_y| < k_Q, \\ 0 & \text{for } |k_y| > k_Q. \end{cases} \quad (63)$$

Although the winding number in each sector  $W_{1(2)}(k_y)$  is nontrivial, the relation  $W(k_y) = 0$  always holds. As a consequence, we find  $N_{\text{ZES}} = 0$ . The results suggest the degeneracy at zero energy would be fragile under the potential disorder. However, the winding number originated from time-reversal-like symmetry can be nontrivial because the winding number defined with Eq. (59) is given as

$$W(k_y) = W_1(k_y) + W_2(k_y). \quad (64)$$

As a consequence, we find

$$W(k_y) = \begin{cases} 2 & \text{for } |k_y| < k_Q, \\ 0 & \text{for } |k_y| > k_Q, \end{cases} \quad (65)$$

$$N_{\text{clean}} = N_{\text{ZES}} = 2 \sum_{|k_y| < k_Q} 1. \quad (66)$$

In Fig. 7, we show the energy eigenvalues of the NCS with the Dresselhaus [110] SOC on the tight-binding model. The BdG Hamiltonian used in the numerical simulation is shown in Appendix A 3. We chose parameters as  $\mu_F = 1.0t$ ,  $\beta = 0.1t$ ,  $\Delta_s = 0.1t$ ,  $\Delta_t = 0.9t$ ,  $L_x = 50a_0$ , and  $L_y = 10a_0$ . This parameter choice leads  $|N_{\text{ZES}}| = N_{\text{clean}} = 6$ . The results for a superconductor with clean surface show  $2 \times N_{\text{clean}} = 12$  ZES as shown with the open symbols. Although we introduce random impurity potential at its surfaces, the flat ZESs remain unchanged as shown with the filled symbols in agreement with the relation  $|N_{\text{ZES}}| = N_{\text{clean}}$ . This suggests the validity of our conclusion in Sec. II. The degeneracy at zero energy

is protected by chiral symmetry originated from the time-reversal-like symmetry.

#### IV. CONCLUSION

We have discussed the effects of the random impurity potential on the degenerate zero-energy states appearing at the surface of a nodal superconductor preserving chiral symmetry. A method called dimensional reduction enables us to topologically characterize nodal superconductors in the presence of translational symmetry. The number of zero-energy bound states at a clean surface,  $N_{\text{clean}}$ , is calculated by using a winding number defined in a one-dimensional Brillouin zone and is usually much larger than unity proportional to the surface width. By focusing on the chiral symmetry of the Bogoliubov–de Gennes Hamiltonian, we show that an index  $N_{\text{ZES}}$  characterizes the number of zero-energy states at a dirty surface. We confirmed our conclusion with numerical simulations on the tight-binding model. The index  $N_{\text{ZES}}$  is defined by the chiral eigenvalues of zero-energy states. Simultaneously,  $N_{\text{ZES}}$  is calculated from the winding number in a one-dimensional Brillouin zone. The index theorem

explains the coincidence of two  $N_{\text{ZES}}$  calculated in the two different ways. We conclude that  $N_{\text{ZES}}$  measures degree of the degeneracy of zero-energy states at a dirty superconductor surface. In experiments, potential disorder is inevitable in the vicinity of the surface. Therefore, our conclusion sends a useful message to experimentalists in this field as regards choosing a target material.

#### ACKNOWLEDGMENTS

We are grateful to S. Kobayashi and S.-I. Suzuki for useful discussions. This work was supported by Topological Materials Science (Nos. JP15H05852 and JP15K21717) and KAKENHI (No. JP15H03525) from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan, JSPS Core-to-Core Program (A. Advanced Research Networks), Japan-RFBR JSPS Bilateral Joint Research Projects/Seminars (2717G8334b), and by the Ministry of Education and Science of the Russian Federation (Grant No. 14Y.26.31.0007). S.I. is supported in part by a Grant-in-Aid for JSPS Fellows (Grant No. JP16J00956) provided by the Japan Society for the Promotion of Science (JSPS).

#### APPENDIX A: BOGOLIUBOV–DE GENNES HAMILTONIAN ON THE TIGHT-BINDING MODEL

We present the BdG Hamiltonian on a two-dimensional tight-binding lattice. The eigenvalues in Sec. III are by diagonalizing the tight-binding Hamiltonian. The kinetic energy part and the impurity potential are common for all the superconductors and are given by

$$H_{\text{kin}} = \sum_{m=1}^{l_y} \sum_{\alpha=\uparrow,\downarrow} \left[ -t \sum_{j=1}^{l_x-1} (\psi_{r+\hat{x},\alpha}^\dagger \psi_{r,\alpha} + \psi_{r,\alpha}^\dagger \psi_{r+\hat{x},\alpha}) - t \sum_{j=1}^{l_x} (\psi_{r+\hat{y},\alpha}^\dagger \psi_{r,\alpha} + \psi_{r,\alpha}^\dagger \psi_{r+\hat{y},\alpha}) + \sum_{j=1}^{l_x} (4t - \mu) \psi_{r,\alpha}^\dagger \psi_{r,\alpha} \right], \quad (\text{A1})$$

$$H_{\text{imp}} = \left( \sum_{j=1}^{l_d} + \sum_{j=l_x-l_d+1}^{l_x} \right) \sum_{m=1}^{l_y} \sum_{\alpha=\uparrow,\downarrow} v(\mathbf{r}) \psi_{r,\alpha}^\dagger \psi_{r,\alpha}, \quad (\text{A2})$$

where  $\psi_{r,\alpha}$  ( $\psi_{r,\alpha}^\dagger$ ) is the annihilation (creation) operator of an electron at  $\mathbf{r} = ja_0\mathbf{x} + ma_0\mathbf{y}$  with spin  $\alpha$ ,  $t$  is the hopping integral, and  $\mu$  is the chemical potential. The unit lattice vectors in the  $x$  and  $y$  directions are defined by  $\hat{x}$  and  $\hat{y}$ , respectively. The number of the lattice sites in the  $x$  ( $y$ ) direction is represented by  $l_x = L_x/a_0$  ( $l_y = L_y/a_0$ ). We introduce the impurity potential  $v(\mathbf{r})$  in the outermost  $l_d = L_d/a_0$  layers in the  $x$  direction. The nonmagnetic impurity potential is described by  $v(\mathbf{r})$  which is given randomly in the range of  $-V_1/2 \leq v(\mathbf{r}) \leq V_1/2$ .

##### 1. Unconventional superconductors

The total Hamiltonian of the unconventional superconductors is represented as  $H = H_{\text{kin}} + H_\mu + H_{\text{imp}}$ , where  $H_\mu$  for  $\mu = p, d$ , and  $f$  depends on the pairing symmetry as

$$H_p = \sum_{j=1}^{l_x-1} \sum_{m=1}^{l_y} \sum_{\alpha} \frac{i\Delta_0}{2} (\psi_{r+\hat{x},\alpha}^\dagger \psi_{r,\bar{\alpha}}^\dagger - \psi_{r,\alpha}^\dagger \psi_{r+\hat{x},\bar{\alpha}}^\dagger) + \text{H.c.}, \quad (\text{A3})$$

$$H_d = \sum_{j=1}^{l_x-1} \sum_{m=1}^{l_y} \sum_{\alpha} \frac{s_\alpha \Delta_0}{4} (\psi_{r+\hat{x}+\hat{y},\alpha}^\dagger \psi_{r,\bar{\alpha}}^\dagger + \psi_{r,\alpha}^\dagger \psi_{r+\hat{x}+\hat{y},\bar{\alpha}}^\dagger - \psi_{r+\hat{x}-\hat{y},\alpha}^\dagger \psi_{r,\bar{\alpha}}^\dagger - \psi_{r,\alpha}^\dagger \psi_{r+\hat{x}-\hat{y},\bar{\alpha}}^\dagger) + \text{H.c.}, \quad (\text{A4})$$

$$H_f = \sum_{j=1}^{l_x-1} \sum_{m=1}^{l_y} \sum_{\alpha} \frac{i\Delta_0}{4} (\psi_{r+\hat{x}+2\hat{y},\alpha}^\dagger \psi_{r,\bar{\alpha}}^\dagger + \psi_{r+\hat{x}-2\hat{y},\alpha}^\dagger \psi_{r,\bar{\alpha}}^\dagger - \psi_{r,\alpha}^\dagger \psi_{r+\hat{x}+2\hat{y},\bar{\alpha}}^\dagger - \psi_{r,\alpha}^\dagger \psi_{r+\hat{x}-2\hat{y},\bar{\alpha}}^\dagger) + \text{H.c.}, \quad (\text{A5})$$

where  $\bar{\alpha}$  is the opposite spin of  $\alpha$  and  $\Delta_0$  is the pair potential at zero temperature. The factor  $s_\alpha$  is  $+1$  for  $\alpha = \uparrow$  and is  $-1$  for  $\alpha = \downarrow$ . For spin-triplet case, we assume a Cooper pair consists of two electrons with the opposite spin directions. In Sec. III, we diagonalize the reduced BdG Hamiltonian into  $2 \times 2$  Nambu space.

## 2. ( $d_{xy} + p$ )-wave superconductor

The Hamiltonian of the ( $d_{xy} + p$ )-wave superconductor discussed in Sec. III B is described by adding the Rashba spin-orbit interaction  $H_R$  and the pair potential  $H_{d_{xy}+p}$  to  $H_{\text{kin}} + H_{\text{imp}}$ . The spin-orbit coupling term is represented by

$$H_R = -i \frac{\lambda_R}{2} \sum_{\alpha, \beta} \sum_{m=1}^{l_y} \left[ \sum_{j=1}^{l_x-1} (\sigma_y)_{\alpha, \beta} (\psi_{r+\hat{x}, \alpha}^\dagger \psi_{r, \beta} - \psi_{r, \alpha}^\dagger \psi_{r+\hat{x}, \beta}) - \sum_{j=1}^{l_x} (\sigma_x)_{\alpha, \beta} (\psi_{r+\hat{y}, \alpha}^\dagger \psi_{r, \beta} - \psi_{r, \alpha}^\dagger \psi_{r+\hat{y}, \beta}) \right]. \quad (\text{A6})$$

The pair potential consists of five parts:  $H_{d_{xy}+p} = H_{\Delta 1} + H_{\Delta 2} + H_{\Delta 3} + H_{\Delta 4} + H_{\Delta 5}$ . Each part is represented by

$$H_{\Delta 1} = i \frac{\Delta_s}{4} \sum_{\alpha, \beta} \sum_{j=1}^{l_x-1} \sum_{m=1}^{l_y} (\sigma_y)_{\alpha, \beta} [\psi_{r+\hat{x}+\hat{y}, \alpha}^\dagger \psi_{r, \beta} + \psi_{r, \alpha}^\dagger \psi_{r+\hat{x}+\hat{y}, \beta} - \psi_{r+\hat{x}-\hat{y}, \alpha}^\dagger \psi_{r, \beta} - \psi_{r, \alpha}^\dagger \psi_{r+\hat{x}-\hat{y}, \beta}] + \text{H.c.}, \quad (\text{A7})$$

$$H_{\Delta 2} = -i \frac{\Delta_t}{4} \sum_{\alpha, \beta} \sum_{j=1}^{l_x-1} \sum_{m=1}^{l_y} (\sigma_z)_{\alpha, \beta} [\psi_{r+\hat{x}, \alpha}^\dagger \psi_{r, \beta} - \psi_{r, \alpha}^\dagger \psi_{r+\hat{x}, \beta}] + \text{H.c.}, \quad (\text{A8})$$

$$H_{\Delta 3} = \frac{\Delta_t}{4} \sum_{\alpha} \sum_{j=1}^{l_x} \sum_{m=1}^{l_y} [\psi_{r+\hat{y}, \alpha}^\dagger \psi_{r, \alpha} - \psi_{r, \alpha}^\dagger \psi_{r+\hat{y}, \alpha}] + \text{H.c.}, \quad (\text{A9})$$

$$H_{\Delta 4} = i \frac{\Delta_t}{8} \sum_{\alpha, \beta} \sum_{j=1}^{l_x-1} \sum_{m=1}^{l_y} (\sigma_z)_{\alpha, \beta} [\psi_{r+\hat{x}+2\hat{y}, \alpha}^\dagger \psi_{r, \beta} + \psi_{r+\hat{x}-2\hat{y}, \alpha}^\dagger \psi_{r, \beta} - \psi_{r, \alpha}^\dagger \psi_{r+\hat{x}+2\hat{y}, \beta} - \psi_{r, \alpha}^\dagger \psi_{r+\hat{x}-2\hat{y}, \beta}] + \text{H.c.}, \quad (\text{A10})$$

$$H_{\Delta 5} = -\frac{\Delta_t}{8} \sum_{\alpha} \sum_{j=1}^{l_x-2} \sum_{m=1}^{l_y} [\psi_{r+2\hat{x}+\hat{y}, \alpha}^\dagger \psi_{r, \alpha} + \psi_{r-2\hat{x}+\hat{y}, \alpha}^\dagger \psi_{r, \alpha} - \psi_{r, \alpha}^\dagger \psi_{r+2\hat{x}+\hat{y}, \alpha} - \psi_{r, \alpha}^\dagger \psi_{r-2\hat{x}+\hat{y}, \alpha}] + \text{H.c.} \quad (\text{A11})$$

The amplitude of the pair potential for the spin-singlet (-triplet) component is represented by  $\Delta_s$  ( $\Delta_t$ ). The Pauli matrices in spin space are represented by  $\sigma_\nu$  ( $\nu = x, y, z$ ).

## 3. Noncentrosymmetric superconductor with the persistent helix states

The BdG Hamiltonian for a NCS with the persistent spin-helix states discussed in Sec. III C is described by  $H = H_{\text{kin}} + H_D^{110} + H_{\Delta_p} + H_{\text{imp}}$ . The spin-orbit coupling and the pair potential are given by

$$H_D^{110} = i \frac{\lambda_D}{2} \sum_{\alpha, \beta} \sum_{j=1}^{l_x-1} \sum_{m=1}^{l_y} (\sigma_z)_{\alpha, \beta} (\psi_{r+\hat{x}, \alpha}^\dagger \psi_{r, \beta} - \psi_{r, \alpha}^\dagger \psi_{r+\hat{x}, \beta}), \quad (\text{A12})$$

$$H_{\Delta_p} = \sum_{\alpha, \beta} \sum_{m=1}^{l_y} \left[ i \Delta_s \sum_{j=1}^{l_x} (\sigma_y)_{\alpha, \beta} \psi_{r, \alpha}^\dagger \psi_{r, \beta} + i \frac{\Delta_t}{2} \sum_{j=1}^{l_x-1} (\sigma_x)_{\alpha, \beta} (\psi_{r+\hat{x}, \alpha}^\dagger \psi_{r, \beta} - \psi_{r, \alpha}^\dagger \psi_{r+\hat{x}, \beta}) \right] + \text{H.c.} \quad (\text{A13})$$

## APPENDIX B: UNITARY TRANSFORMATION FOR THE ( $d_{xy} + p$ )-WAVE SUPERCONDUCTOR

The BdG Hamiltonian of a ( $d_{xy} + p$ ) superconductor in Sec. III B is represented by

$$H_{\mathbf{k}} = \begin{bmatrix} \hat{h}_{\mathbf{k}} & \hat{\Delta}_{\mathbf{k}} \\ -\hat{\Delta}_{-\mathbf{k}}^* & -\hat{h}_{-\mathbf{k}} \end{bmatrix}, \quad (\text{B1})$$

$$\hat{h}_{\mathbf{k}} = \xi(\mathbf{k}) \hat{\sigma}_0 + \mathbf{g}_r(\mathbf{k}) \cdot \hat{\boldsymbol{\sigma}}, \quad (\text{B2})$$

$$\hat{\Delta}_{\mathbf{k}} = i f(\mathbf{k}) \left[ \Delta_s + \Delta_t \frac{\Delta_t}{\alpha k} \mathbf{g}_r(\mathbf{k}) \cdot \hat{\boldsymbol{\sigma}} \right] \hat{\sigma}_2, \quad (\text{B3})$$

where  $\mathbf{g}_r(\mathbf{k}) = \alpha(k_y \mathbf{x} - k_x \mathbf{y})$ ,  $f(\mathbf{k}) = (k_x k_y / k^2)$ , and  $k = \sqrt{k_x^2 + k_y^2}$ . We first apply a unitary transformation to

$$U_{\mathbf{k}}^\dagger H_{\mathbf{k}} U_{\mathbf{k}} = H'_{\mathbf{k}} \text{ with}$$

$$U_{\mathbf{k}} = \begin{bmatrix} \hat{u}_{\mathbf{k}} & 0 \\ 0 & \hat{u}_{-\mathbf{k}}^* \end{bmatrix}, \quad (\text{B4})$$

$$\hat{u}_{\mathbf{k}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i e^{-i\theta_{\mathbf{k}}} \\ -i e^{i\theta_{\mathbf{k}}} & -1 \end{bmatrix}, \quad \theta_{\mathbf{k}} = \arctan \left[ \frac{k_y}{k_x} \right]. \quad (\text{B5})$$

The second unitary transformation

$$H_{\nu}(\mathbf{k}) = U_0^\dagger H'_{\mathbf{k}} U_0, \quad U_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (\text{B6})$$

results in

$$H_V(\mathbf{k}) = \begin{bmatrix} \hat{H}_+(\mathbf{k}) & 0 \\ 0 & \hat{H}_-(\mathbf{k}) \end{bmatrix}, \quad (\text{B7})$$

$$\hat{H}_\pm(\mathbf{k}) = \begin{bmatrix} E_\pm(\mathbf{k}) & -\Delta_\pm(\mathbf{k}) \\ -\Delta_\pm(\mathbf{k}) & -E_\pm(\mathbf{k}) \end{bmatrix}, \quad (\text{B8})$$

$$E_\pm(\mathbf{k}) = \xi(\mathbf{k}) \pm |\mathbf{g}_R(\mathbf{k})|, \quad (\text{B9})$$

$$\Delta_\pm(\mathbf{k}) = f(\mathbf{k})[\Delta_t \pm \Delta_s]. \quad (\text{B10})$$

### APPENDIX C: PERSISTENT SPIN-HELIX STATES WITH COEXISTENCE OF RASHBA AND DRESSSELHAUS [100] SPIN-ORBIT COUPLING

We shown an alternative way to realize the persistent spin-helix states [40,41]. Let us consider the thin film growing along the [001] crystal direction. In such two-dimensional electron system, the Rashba type SOC  $\mathbf{g}_r(\mathbf{k}) = \alpha(k_y\mathbf{x} - k_x\mathbf{y})$  and the Dresselhaus [001] type SOC  $\mathbf{g}'_d(\mathbf{k}) = \beta'(k_x\mathbf{x} - k_y\mathbf{y})$  coexist. The Hamiltonian is described as

$$\hat{h}_{\text{RD}}(\mathbf{k}) = \xi(\mathbf{k})\hat{\sigma}_0 + \hat{h}_R(\mathbf{k}) + \hat{h}_D^{100}(\mathbf{k}), \quad (\text{C1})$$

$$\hat{h}_R = \alpha(k_y\hat{\sigma}_x - k_x\hat{\sigma}_y), \quad (\text{C2})$$

$$\hat{h}_D^{100} = \beta'(k_x\hat{\sigma}_x - k_y\hat{\sigma}_y). \quad (\text{C3})$$

When we define

$$k_\pm = \frac{1}{\sqrt{2}}(k_x \pm k_y), \quad (\text{C4})$$

the Hamiltonian is rewritten as

$$\hat{h}_{\text{RD}}(\mathbf{k}) = \xi'(\mathbf{k})\hat{\sigma}_0 + \hat{h}_+ + \hat{h}_-, \quad (\text{C5})$$

where

$$\xi'(\mathbf{k}) = \frac{\hbar^2}{2m}(k_+^2 + k_-^2) - \mu_F, \quad (\text{C6})$$

$$\hat{h}_\pm = \lambda_\pm k_\pm \hat{\sigma}_\pm, \quad (\text{C7})$$

$$\lambda_\pm = \frac{1}{\hbar}(\beta \pm \alpha), \quad \hat{\sigma}_\pm = \frac{1}{\sqrt{2}}(\hat{\sigma}_x \mp \hat{\sigma}_y). \quad (\text{C8})$$

The strength of the Rashba SOC is tunable by an externally applied electric field. When we consider a special case of  $\alpha = \beta'$ , which can be experimentally accessible [42,43], the Hamiltonian is deformed as

$$\hat{h}(\mathbf{k}) = \left[ \frac{\hbar^2}{2m}(k_+^2 + k_-^2) - \mu_F \right] \hat{\sigma}_0 + \beta' p_+ \hat{\sigma}_+. \quad (\text{C9})$$

The Hamiltonian in Eq. (C9) is unitary equivalent to that in Eq. (44) [40]. Therefore, the persistent spin-helix states can be also obtained in the thin film growing along the [001] crystal direction. Moreover, as discussed in Sec. III C, we can expect the flat ZESs at a dirty surface of a superconducting thin film with coexistence of Rashba and Dresselhaus [001] spin-orbit coupling.

- 
- [1] C. L. Kane and E. J. Mele, *Phys. Rev. Lett.* **95**, 146802 (2005).  
[2] M. Z. Hasan and C. L. Kane, *Rev. Mod. Phys.* **82**, 3045 (2010).  
[3] F. Wilczek, *Nat. Phys.* **5**, 614 (2009).  
[4] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, *Phys. Rev. B* **78**, 195125 (2008).  
[5] Y. Tanaka and S. Kashiwaya, *Phys. Rev. Lett.* **74**, 3451 (1995).  
[6] Y. Asano, Y. Tanaka, and S. Kashiwaya, *Phys. Rev. B* **69**, 134501 (2004).  
[7] M. Sato, Y. Tanaka, K. Yada, and T. Yokoyama, *Phys. Rev. B* **83**, 224511 (2011).  
[8] A. P. Schnyder and P. M. R. Brydon, *J. Phys.: Condens. Matter* **27**, 243201 (2015).  
[9] L. J. Buchholtz and G. Zwicknagl, *Phys. Rev. B* **23**, 5788 (1981).  
[10] C. R. Hu, *Phys. Rev. Lett.* **72**, 1526 (1994).  
[11] Y. Tanaka, Y. Mizuno, T. Yokoyama, K. Yada, and M. Sato, *Phys. Rev. Lett.* **105**, 097002 (2010).  
[12] K. Yada, M. Sato, Y. Tanaka, and T. Yokoyama, *Phys. Rev. B* **83**, 064505 (2011).  
[13] P. M. R. Brydon, A. P. Schnyder, and C. Timm, *Phys. Rev. B* **84**, 020501(R) (2011).  
[14] A. P. Schnyder and S. Ryu, *Phys. Rev. B* **84**, 060504(R) (2011).  
[15] A. P. Schnyder, P. M. R. Brydon, and C. Timm, *Phys. Rev. B* **85**, 024522 (2012).  
[16] J. Alicea, *Phys. Rev. B* **81**, 125318 (2010).  
[17] J. You, C. H. Oh, and V. Vedral, *Phys. Rev. B* **87**, 054501 (2013).  
[18] C. L. M. Wong, J. Liu, K. T. Law, and P. A. Lee, *Phys. Rev. B* **88**, 060504(R) (2013).  
[19] S. Ikegaya, Y. Asano, and Y. Tanaka, *Phys. Rev. B* **91**, 174511 (2015).  
[20] S. Deng, G. Ortiz, A. Poudel, and L. Viola, *Phys. Rev. B* **89**, 140507(R) (2014).  
[21] A. Chen and M. Franz, *Phys. Rev. B* **93**, 201105(R) (2016).  
[22] Y. Tanaka and S. Kashiwaya, *Phys. Rev. B* **70**, 012507 (2004).  
[23] Y. Asano, Y. Tanaka, A. A. Golubov, and S. Kashiwaya, *Phys. Rev. Lett.* **99**, 067005 (2007).  
[24] Y. Tanaka and S. Kashiwaya, *Phys. Rev. B* **53**, R11957(R) (1996).  
[25] Y. S. Barash, H. Burkhardt, and D. Rainer, *Phys. Rev. Lett.* **77**, 4070 (1996).  
[26] H.-J. Kwon, K. Sengupta, and V. M. Yakovenko, *Eur. Phys. J. B* **37**, 349 (2004).  
[27] Y. Asano, Y. Tanaka, and S. Kashiwaya, *Phys. Rev. Lett.* **96**, 097007 (2006).  
[28] S. Ikegaya and Y. Asano, *J. Phys.: Condens. Matter* **28**, 375702 (2016).  
[29] R. Queiroz and A. P. Schnyder, *Phys. Rev. B* **89**, 054501 (2014).  
[30] S. V. Bakurskiy, A. A. Golubov, M. Yu. Kupriyanov, K. Yada, and Y. Tanaka, *Phys. Rev. B* **90**, 064513 (2014).  
[31] S.-I. Suzuki and Y. Asano, *Phys. Rev. B* **91**, 214510 (2015).

- [32] B. Lu, P. Buset, Y. Tanuma, A. A. Golubov, Y. Asano, and Y. Tanaka, *Phys. Rev. B* **94**, 014504 (2016).
- [33] S. Tewari and J. D. Sau, *Phys. Rev. Lett.* **109**, 150408 (2012).
- [34] Y. Niu, S.-B. Chung, C.-H. Hsu, I. Mandal, S. Raghu, and S. Chakravarty, *Phys. Rev. B* **85**, 035110 (2012).
- [35] M. Diez, J. P. Dahlhaus, M. Wimmer, and C. W. J. Beenakker, *Phys. Rev. B* **86**, 094501 (2012).
- [36] S. Ikegaya, S.-I. Suzuki, Y. Tanaka, and Y. Asano, *Phys. Rev. B* **94**, 054512 (2016).
- [37] P. A. Frigeri, D. F. Agterberg, A. Koga, and M. Sigrist, *Phys. Rev. Lett.* **92**, 097001 (2004).
- [38] S. Fujimoto, *J. Phys. Soc. Jpn.* **76**, 051008 (2007).
- [39] K. Yada, S. Onari, Y. Tanaka, and J.-I. Inoue, *Phys. Rev. B* **80**, 140509(R) (2009).
- [40] B. A. Bernevig, J. Orenstein, and S. C. Zhang, *Phys. Rev. Lett.* **97**, 236601 (2006).
- [41] M.-H. Liu, K.-W. Chen, S.-H. Chen, and C.-R. Chang, *Phys. Rev. B* **74**, 235322 (2006).
- [42] M. Kohda, V. Lechner, Y. Kunihashi, T. Dollinger, P. Olbrich, C. Schonhuber, I. Caspers, V. V. Bel'kov, L. E. Golub, D. Weiss, K. Richter, J. Nitta, and S. D. Ganichev, *Phys. Rev. B* **86**, 081306(R) (2012).
- [43] M. P. Walser, C. Reichl, W. Wegscheider, and G. Salis, *Nat. Phys.* **8**, 757 (2012).