Normal developable surfaces of surfaces along curves

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October 6, 2015

Abstract

We consider a developable surface normal to a surface along a curve on the surface. We call it a normal developable surface along the curve on the surface. We investigate the uniqueness and the singularities of such developable surfaces. We discover two new invariants of curves on a surface which characterize these singularities.

1 Introduction

In this paper we consider a curve on a surface in the Euclidean 3-space and a developable surface normal to the surface along the curve. Such a developable surface is called a normal developable surface along the curve if it exists. The notion of Darboux frames along curves on surfaces is well known. We have a special direction in the Darboux frame at each point of the curve which is directed by a vector in the rectifying plane of the surface along the curve. We call this vector field a rectifying Darboux vector field along the curve. On the other hand, there are three invariants associated with the Darboux frame of a curve on a surface. With a certain condition of those invariants, we can show that there exists a normal developable surface along the curve which is given as the envelope of rectifying spaces of the surface along the curve. It follows that a normal developable surface is a ruled surface whose rulings are directed by the rectifying Darboux vector field along the curve. By using the above invariants, we introduce two new invariants which are related to the singularities of normal developable surfaces. Actually, one of these invariants is constantly equal to zero if and only if the rectifying Darboux vector field has a constant direction which means that the normal developable surface is a cylindrical surface. We give a classification of the singularities of the normal developable surface along a curve on a surface by using those two invariants (Theorem 3.3). In §6 we consider the uniqueness of the normal developable surface. If the uniqueness does not hold, then the curve is a straight line (Corollary 6.3). In §7 we consider special curves on surfaces. If we consider a geodesic on a surface, the normal developable surface is a tangent surface of the curve. We also consider lines of curvatures of a surface. In this case the director curve of the normal developable surface is given by the normal vector field of the surface along the curve. It is known that the ruled surface along a curve on a surface whose director curve is the normal vector field of the surface is a developable surface if and only if the curve is a line.
of curvature. If we consider a parametrization of a surface such that each coordinate curve is a line of curvature, then the locus of the singular values of normal developable surfaces for all coordinate curves is the focal surface of the surface. We give an example of an ellipsoid in §7.

In [4] the second author and Saki Otani introduced an osculating developable surface of a surface along a curve. Such a developable surface is tangent to the surface along the curve and it gives a flat approximation of the surface along the curve. The method we use in this paper is analogous to the method in the above paper. However, we consider the normal developable surface of the surface along the curve. Therefore, the situations are different.

Throughout this paper all curves, surfaces and maps will be of class $C^\infty$.

## 2 Basic concepts

We consider a surface $M = X(U)$ given locally by an embedding $X : U \to \mathbb{R}^3$, where $\mathbb{R}^3$ is the Euclidean space and $U \subset \mathbb{R}^2$ is an open set. Let $\gamma : I \to U$ be an embedding, where $\gamma(t) = (u(t), v(t))$ and $I$ is an open interval. Then we have a regular curve $\gamma = X \circ \gamma : I \to M \subset \mathbb{R}^3$ on the surface $M$. On the surface, we have the unit normal vector field $n$ defined by

$$n(p) = \frac{X_u \times X_v}{||X_u \times X_v||}(u, v),$$

where $p = X(u, v)$. Here, $a \times b$ is the exterior product of $a, b$ in $\mathbb{R}^3$. Since $\gamma$ is a space curve in $\mathbb{R}^3$, we adopt the arc-length parameter as usual and write $\gamma(s) = X(u(s), v(s))$. Then we have the unit tangent vector field $t(s) = \gamma'(s)$ of $\gamma(s)$, where $\gamma'(s) = d\gamma/ds(s)$. We have $n_\gamma(s) = n \circ \gamma(s)$ which is the unit normal vector field of $M$ along $\gamma$. Moreover, we define $b(s) = n_\gamma(s) \times t(s)$. Then we have an orthonormal frame $\{t(s), n_\gamma(s), b(s)\}$ along $\gamma$, which is called the Darboux frame along $\gamma$. Then we have the following Frenet-Serret type formulae:

$$\begin{cases} t'(s) = \kappa_g(s)b(s) + \kappa_n(s)n_\gamma(s), \\ b'(s) = -\kappa_g(s)t(s) + \tau_g(s)n_\gamma(s), \\ n'_\gamma(s) = -\kappa_n(s)t(s) - \tau_g(s)b(s). \end{cases}$$

By using the matrix representation, we have

$$\begin{pmatrix} t' \\ b' \\ n'_\gamma \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} t \\ b \\ n_\gamma \end{pmatrix}.$$  

Here,

$$\kappa_g(s) = \langle t'(s), b(s) \rangle = \det(\gamma'(s), \gamma''(s), n_\gamma(s)),$$

$$\kappa_n(s) = \langle t'(s), n_\gamma(s) \rangle = \langle \gamma'(s), n_\gamma(s) \rangle,$$

$$\tau_g(s) = \langle b'(s), n_\gamma(s) \rangle = \det(\gamma'(s), n_\gamma(s), n_\gamma'(s))$$

and $\langle a, b \rangle$ is the canonical inner product of $\mathbb{R}^3$. We call $\kappa_g(s)$ a geodesic curvature, $\kappa_n(s)$ a normal curvature and $\tau_g(s)$ a geodesic torsion of $\gamma$ respectively. It is known that

1) $\gamma$ is an asymptotic curve of $M$ if and only if $\kappa_n = 0$,

2) $\gamma$ is a geodesic of $M$ if and only if $\kappa_g = 0$,

3) $\gamma$ is a principal curve of $M$ if and only if $\tau_g = 0$.  

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We define a vector field $D_r(s)$ along $\gamma$ by

$$D_r(s) = \tau_g(s) t(s) + \kappa_g(s) n_\gamma(s),$$

which is called a rectifying Darboux vector along $\gamma$. If $\kappa_g^2 + \tau_g^2 \neq 0$, we can define the spherical rectifying Darboux image by

$$D_r(s) = \frac{\tau_g(s) t(s) + \kappa_g(s) n_\gamma(s)}{\sqrt{\kappa_g^2(s) + \tau_g^2(s)}}.$$

On the other hand, we briefly review the notions and basic properties of ruled surfaces and developable surfaces. Let $\gamma : I \to \mathbb{R}^3$ and $\xi : I \to \mathbb{R}^3 \setminus \{0\}$ be $C^\infty$-mappings. Then we define a mapping $F_{(\gamma, \xi)} : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$F_{(\gamma, \xi)}(u, v) = \gamma(u) + v\xi(u).$$

We call the image of $F_{(\gamma, \xi)}$ a ruled surface, the mapping $\gamma$ a base curve and the mapping $\xi$ a director curve. The line defined by $\gamma(u) + v\xi(u)$ for a fixed $u \in I$ is called a ruling. We call the ruled surface with vanishing Gaussian curvature on the regular part a developable surface. It is known that a ruled surface $F_{(\gamma, \xi)}$ is a developable surface if and only if

$$\det \left( \gamma(u), \xi(u), \dot{\gamma}(u) \right) = 0,$$

where $\dot{\gamma}(u) = (d\gamma/du)(u)$ (cf., [3]). If the direction of the director curve $\xi$ is constant, we call $F_{(\gamma, \xi)}$ a (generalized) cylinder. If we write $\tilde{\xi}(u) = \xi(u)/||\xi(u)||$, then we have $F_{(\gamma, \xi)}(I \times \mathbb{R}) = F_{(\gamma, \tilde{\xi})}(I \times \mathbb{R})$. In this case $F_{(\gamma, \xi)}$ is a cylinder if and only if $\tilde{\xi}(u) \equiv 0$. We say that $F_{(\gamma, \xi)}$ is non-cylindrical if $\tilde{\xi}(u) \neq 0$. Suppose that $F_{(\gamma, \xi)}$ is non-cylindrical. Then a striction curve is defined to be

$$\sigma(u) = \gamma(u) - \frac{\langle \gamma(u), \tilde{\xi}(u) \rangle \tilde{\xi}(u)}{\langle \tilde{\xi}(u), \tilde{\xi}(u) \rangle}.$$  

It is known that a singular point of the non-cylindrical ruled surface is located on the striction curve [3]. A non-cylindrical ruled surface $F_{(\gamma, \xi)}$ is a cone if the striction curve $\sigma$ is constant. In general, a wave front in $\mathbb{R}^3$ is a (singular) surface which is a projection image of a Legendrian submanifold in the projective cotangent bundle $\pi : PT^*(\mathbb{R}^3) \to \mathbb{R}^3$. It is known (cf., [3]) that a non-cylindrical developable surface $F_{(\gamma, \xi)}$ is a wave front if and only if

$$\det \left( \xi(u), \dot{\xi}(u), \ddot{\xi}(u) \right) \neq 0.$$  

In this case we call $F_{(\gamma, \xi)}$ a (non-cylindrical) developable front.

Let $M \subset \mathbb{R}^3$ be a surface. We say that a developable surface $N$ is a normal developable surface of $M$ if $N \cap M \neq \emptyset$ and $T_pN$ and $T_pM$ are orthogonal at any point $p \in N \cap M$. If $N$ is a cylinder, we say that $N$ is a normal cylinder of $M$. We also say that $N$ is a normal cone of $M$ if $N$ is a cone. For a normal developable surface $N$ of $M$, the intersection $N \cap M$ is a regular curve. In particular, we say that the intersection $N \cap M$ is a normal cylindrical slice if $N$ is a normal cylinder of $M$. Moreover, $N \cap M$ is said to be a normal conical slice if $N$ is a normal cone of $M$. 

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3 Normal developable surfaces

In this section we investigate normal developable surfaces of a given surface. For a regular curve \( \gamma = X \circ \eta : I \rightarrow M \subset \mathbb{R}^3 \) on a surface \( M \) with \( \kappa_g^2(s) + \tau_g^2(s) \neq 0 \), we define a map \( ND_\gamma : I \times \mathbb{R} \rightarrow \mathbb{R}^3 \) by

\[
ND_\gamma(s, u) = \gamma(s) + u \overrightarrow{D_r}(s) = \gamma(s) + u \frac{\tau_g(s) \mathbf{t}(s) + \kappa_g(s) \mathbf{n}_\gamma(s)}{\sqrt{\kappa_g(s)^2 + \tau_g(s)^2}}.
\]

This is a ruled surface and

\[
\overrightarrow{D_r}' = \left( \kappa_n - \frac{\kappa_g \tau'_g - \kappa'_g \tau_g}{\kappa_g^2 + \tau_g^2} \right) \frac{-\kappa_g \mathbf{t} + \tau_g \mathbf{n}_\gamma}{\sqrt{\kappa_g^2 + \tau_g^2}},
\]

so that we have

\[
\det(\gamma', \overrightarrow{D_r}, \overrightarrow{D_r}') = \det \left( \mathbf{t}, \frac{\tau_g \mathbf{t} + \kappa_g \mathbf{n}_\gamma}{\sqrt{\kappa_g^2 + \tau_g^2}}, \left( \kappa_n - \frac{\kappa_g \tau'_g - \kappa'_g \tau_g}{\kappa_g^2 + \tau_g^2} \right) \frac{-\kappa_g \mathbf{t} + \tau_g \mathbf{n}_\gamma}{\sqrt{\kappa_g^2 + \tau_g^2}} \right)
= 0.
\]

This means that \( ND_\gamma(I \times \mathbb{R}) \) is a developable surface. We call \( ND_\gamma \) a normal developable surface of \( M \) along \( \gamma \). Moreover, we introduce two invariants \( \delta_r(s), \sigma_r(s) \) of \( (M, \gamma) \) as follows:

\[
\delta_r(s) = \kappa_n(s) - \frac{\kappa_g(s) \tau'_g(s) - \kappa'_g(s) \tau_g(s)}{\kappa_g^2(s) + \tau_g^2(s)},
\]

\[
\sigma_r(s) = \frac{\tau_g(s)}{\sqrt{\kappa_g^2(s) + \tau_g^2(s)}} + \left( \frac{\kappa_g(s)}{\delta_r(s) \sqrt{\kappa_g^2(s) + \tau_g^2(s)}} \right)' , \quad (\text{when } \delta_r(s) \neq 0).
\]

By the above calculation, \( \delta_r(s) = 0 \) if and only if \( \overrightarrow{D_r}'(s) = 0 \). We can also calculate that

\[
\frac{\partial ND_\gamma}{\partial s} \times \frac{\partial ND_\gamma}{\partial u} = \left( u \delta_r - \frac{\kappa_g}{\sqrt{\kappa_g^2 + \tau_g^2}} \right) \mathbf{b}.
\]

Therefore, \((s_0, u_0) \in I \times \mathbb{R}\) is a singular point of \( ND_\gamma \) if and only if \( \delta_r(s_0) \neq 0 \) and

\[
u_0 = \frac{\kappa_g(s_0)}{\delta_r(s_0) \sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}}.
\]

If \((s_0, 0)\) is a regular point (i.e., \( \kappa_g(s_0) \neq 0 \)), the normal vector of \( ND_\gamma \) at \( ND_\gamma(s_0, 0) = \gamma(s_0) \) is orthogonal to the normal vector of \( M \) at \( \gamma(s_0) \). This is the reason why we call \( ND_\gamma \) the normal developable surface of \( M \) along \( \gamma \). On the other hand, these two invariants characterize normal cylindrical slices and normal conical slices of \( M \), respectively.

**Theorem 3.1** Let \( \gamma : I \rightarrow M \subset \mathbb{R}^3 \) be a unit speed curves on \( M \) with \( \kappa_g^2(s) + \tau_g^2(s) \neq 0 \). Then we have the following:

(A) The following are equivalent:

1. \( ND_\gamma \) is a cylinder.
(2) $\delta_r(s) \equiv 0$,

(3) $\gamma$ is a normal cylindrical slice of $M$.

(B) If $\delta_r(s) \neq 0$, then the following are equivalent:

1. $\mathbf{ND}_\gamma$ a cone,
2. $\mathbf{r}_r(s) \equiv 0$,
3. $\gamma$ is a normal conical slice of $M$.

Proof. (A) By definition, $\mathbf{ND}_\gamma$ is a cylinder if and only if $\overline{D}_r(s)$ is constant. Since

$$\overline{D}_r'(s) = \delta_r(s) \frac{-\kappa_n(s) t(s) + \tau_g(s) n_\gamma(s)}{\sqrt{\kappa_g^2(s) + \tau_g^2(s)}},$$

$\overline{D}_r(s)$ is constant if and only if $\delta_r(s) \equiv 0$. Therefore, (1) is equivalent to (2). Suppose that (3) holds. Then there exists a vector $\mathbf{k} \in S^2$ such that $\langle \mathbf{b}(s), \mathbf{k} \rangle \equiv 0$, where $\mathbf{k}$ is the director of the normal cylinder. Then there exist $\lambda, \mu \in \mathbb{R}$ such that $\mathbf{k} = \lambda \mathbf{t}(s) + \mu n_\gamma(s)$. Since $\langle \mathbf{b'}(s), \mathbf{k} \rangle \equiv 0$, we have $-\kappa_g(s) \lambda + \tau_g(s) \mu = 0$, so that we have $\mathbf{k} = \overline{D}_r(s)$. Condition (1) holds. It is clear that (1) implies (3).

(B) Condition (1) means that the singular value set of $\mathbf{ND}_\gamma$ is a constant vector. We consider a vector valued function $\mathbf{f}(s)$ defined by

$$\mathbf{f}(s) = \gamma(s) - \frac{\kappa_g(s)}{\delta_r(s) \sqrt{\kappa_g^2(s) + \tau_g^2(s)}} \overline{D}_r(s).$$

Then the condition (1) is equivalent to the condition that $\mathbf{f}'(s) \equiv 0$. We can calculate that

$$\mathbf{f}' = \mathbf{t} + \left( \frac{\kappa_g}{\delta \sqrt{\kappa_g^2 + \tau_g^2}} \right) ' \overline{D}_r + \frac{\kappa_g}{\delta \sqrt{\kappa_g^2 + \tau_g^2}} \overline{D}_r'$$

$$= \mathbf{t} + \left( \frac{\kappa_g}{\delta \sqrt{\kappa_g^2 + \tau_g^2}} \right) ' \overline{D}_r + \frac{\kappa_g}{\sqrt{\kappa_g^2 + \tau_g^2}} \frac{\kappa_g \mathbf{t} + \tau_g \mathbf{b}}{\sqrt{\kappa_g^2 + \tau_g^2}}$$

$$= \left( \frac{\tau_g}{\sqrt{\kappa_g^2 + \tau_g^2}} + \left( \frac{\kappa_g}{\delta \sqrt{\kappa_g^2 + \tau_g^2}} \right) ' \right) \overline{D}_r$$

$$= \mathbf{r}_r \overline{D}_o.$$

It follows that (1) and (2) are equivalent. By the definition of the conical slice, (3) means that there exists $\mathbf{c} \in \mathbb{R}^3$ such that $\langle \gamma(s) - \mathbf{c}, \mathbf{b}(s) \rangle \equiv 0$. If (1) holds, then $\mathbf{f}(s)$ is constant. For the constant point $\mathbf{c} = \mathbf{f}(s) \in \mathbb{R}^3$, we have

$$\langle \gamma(s) - \mathbf{c}, \mathbf{b}(s) \rangle = \langle \gamma(s) - \mathbf{f}(s), \mathbf{b}(s) \rangle = \left\langle \frac{\kappa_g(s)}{\delta_r(s) \sqrt{\kappa_g^2(s) + \tau_g^2(s)}} \overline{D}_r(s), \mathbf{b}(s) \right\rangle = 0.$$

This means that (3) holds. For the converse, by (3), there exists a point $\mathbf{c} \in \mathbb{R}^3$ such that $\langle \gamma(s) - \mathbf{c}, \mathbf{b}(s) \rangle = 0$. Taking the derivative of both sides, we have $0 = \langle \gamma(s) - \mathbf{c}, \mathbf{b}(s) \rangle' = \langle \gamma(s) - \mathbf{c}, -\kappa_g \mathbf{t}(s) + \tau_g n_\gamma(s) \rangle$. Then there exists $\lambda \in \mathbb{R}$ such that $\gamma(s) - \mathbf{c} = \lambda \overline{D}_r(s)$. Taking the derivative again, we have

$$0 = \langle \gamma - \mathbf{c}, \mathbf{b} \rangle'' = \langle \mathbf{t}, -\kappa_g \mathbf{t} + \tau_g n_\gamma \rangle + \langle \gamma - \mathbf{c}, (-\kappa_g \mathbf{t} + \tau_g n_\gamma) \rangle' = \kappa_g + \lambda \delta_r \sqrt{\kappa_g^2 + \tau_g^2}.$$
It follows that
\[ c = \gamma(s) - \lambda \overrightarrow{D_o}(s) = \gamma(s) + \frac{\kappa_g(s)}{\delta_r(s)\sqrt{\kappa_g^2(s) + \tau_g^2(s)}} \overrightarrow{D_r}(s) = f(s). \]

Therefore, \( f(s) \) is constant, so that (1) holds. This completes the proof. \( \square \)

**Corollary 3.2** The normal developable surface \( ND_\gamma \) is non-cylindrical if and only if \( \delta_r(s) \neq 0 \).

We remark that developable surfaces are classified into cylinders, cones or tangent surfaces of space curves (cf., \([6]\)). Hartman and Nirenberg \([2]\) showed that a cylinder is only one non-singular (complete) developable surface. Hence, (complete) tangent surfaces have always singularities. By the results of Theorem 3.1, two invariants \( \delta_r(s) \) and \( \sigma_r(s) \) might be related to the singularities of normal developable surfaces. Actually, we can classify the singularities of normal developable surfaces of \( M \) along curves by using these two invariants \( \delta_r(s) \) and \( \sigma_r(s) \).

**Theorem 3.3** Let \( \gamma : I \rightarrow M \subset \mathbb{R}^3 \) be a unit speed curve with \( \kappa_g^2(s) + \tau_g^2(s) \neq 0 \). Then we have the following:

1. The image of the normal developable surface \( ND_\gamma \) of \( M \) along \( \gamma \) is non-singular at \((s_0, u_0)\) if and only if
   \[ \frac{\kappa_g(s_0)}{\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}} - u_0 \delta_r(s_0) \neq 0. \]

2. The image of the normal developable surface \( ND_\gamma \) of \( M \) along \( \gamma \) is locally diffeomorphic to the cuspidal edge \( C \times \mathbb{R} \) at \((s_0, u_0)\) if
   (i) \( \delta_r(s_0) \neq 0, \sigma_r(s_0) \neq 0 \) and
   \[ u_0 = \frac{\kappa_g(s_0)}{\delta_r(s_0)\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}} , \]
   or
   (ii) \( \delta_r(s_0) = \kappa_g(s_0) = 0, \delta'_r(s_0) \neq 0 \) and
   \[ u_0 = -\frac{\kappa'_g(s_0)}{2\kappa_n(s_0)\tau'_g(s_0) + \kappa'_n(s_0)\tau_g(s_0) + \kappa''_g(s_0)}, \]
   or
   (iii) \( \delta_r(s_0) = \kappa_g(s_0) = 0 \) and \( \kappa'_g(s_0) \neq 0 \).

We remark that if \( \delta'_r(s_0) \neq 0 \), then
\[ 2\kappa_n(s_0)\tau'_g(s_0) + \kappa'_n(s_0)\tau_g(s_0) + \kappa''_g(s_0) \neq 0. \]

3. The image of the normal developable surface \( ND_\gamma \) of \( M \) along \( \gamma \) is locally diffeomorphic to the swallowtail \( SW \) at \((s_0, u_0)\) if \( \delta_r(s_0) \neq 0, \sigma_r(s_0) = 0, \sigma'_r(s_0) \neq 0 \) and
   \[ u_0 = -\frac{\kappa_g(s_0)}{\delta_r(s_0)\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}} . \]

Here, \( C \times \mathbb{R} = \{(x_1, x_2, x_3)|x_1^2 = x_2^3\} \) is the cuspidal edge (c.f., Fig.1) and \( SW = \{(x_1, x_2, x_3)|x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\} \) is the swallowtail (c.f., Fig.2).
4 Support functions

In this section we introduce a family of functions on a curve which is useful for the study of invariants of curves on surfaces. For a unit speed curve \( \gamma : I \rightarrow M \subset \mathbb{R}^3 \), we define a function \( G : I \times \mathbb{R}^3 \rightarrow \mathbb{R} \) by \( G(s, x) = \langle x - \gamma(s), b(s) \rangle \). We call \( G \) a support function on \( \gamma \) with respect to \( b \). We write \( g_{x_0}(s) = G(s, x_0) \) for any \( x_0 \in \mathbb{R}^3 \). Then we have the following proposition.

**Proposition 4.1** Let \( \gamma : I \rightarrow M \subset \mathbb{R}^3 \) be a unit speed curve with \( \kappa_\gamma^2 + \tau_\gamma^2 \neq 0 \). Then we have the following:

1. \( g_{x_0}(s_0) = 0 \) if and only if there exist \( u, v \in \mathbb{R} \) such that \( x_0 - \gamma(s_0) = u t(s_0) + v n_\gamma(s_0) \)
2. \( g_{x_0}(s_0) = g'_{x_0}(s_0) = 0 \) if and only if there exists \( u \in \mathbb{R} \) such that
   \[
   x_0 - \gamma(s_0) = \frac{u \tau_\gamma(s_0) t(s_0) + \kappa_\gamma(s_0) n_\gamma(s_0)}{\sqrt{\kappa_\gamma^2(s_0) + \tau_\gamma^2(s_0)}}.
   \]

(A) Suppose that \( \delta_r(s_0) \neq 0 \). Then we have the following:

3. \( g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = 0 \) if and only if
   \[
   x_0 - \gamma(s_0) = \frac{\kappa_\gamma(s_0)}{\delta_r(s_0) \sqrt{\kappa_\gamma^2(s_0) + \tau_\gamma^2(s_0)}} t(s_0) + \frac{\tau_\gamma(s_0) t(s_0) + \kappa_\gamma(s_0) n_\gamma(s_0)}{\sqrt{\kappa_\gamma^2(s_0) + \tau_\gamma^2(s_0)}}.
   \]

4. \( g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = g'^{(3)}(s_0) = \frac{\tau_\gamma}{\delta_r(s_0) \sqrt{\kappa_\gamma^2(s_0) + \tau_\gamma^2(s_0)}} t(s_0) + \kappa_\gamma(s_0) n_\gamma(s_0) \) if and only if \( \sigma_r(s_0) = 0 \) and 

(B) Suppose that \( \delta_r(s_0) = 0 \). Then we have the following:

5. \( g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = g'^{(3)}(s_0) = \frac{\tau_\gamma}{\delta_r(s_0) \sqrt{\kappa_\gamma^2(s_0) + \tau_\gamma^2(s_0)}} t(s_0) + \kappa_\gamma(s_0) n_\gamma(s_0) \) if and only if \( \sigma_r(s_0) = 0 \) and 

6. \( g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = 0 \) if and only if
   \[
   \kappa_\gamma(s_0) = 0 \quad (i.e., \kappa_\gamma(s_0) = 0, \kappa_\gamma'(s_0) = -\kappa_\gamma(s_0) \tau_\gamma(s_0)) \quad \text{and there exists } u \in \mathbb{R} \text{ such that}
   \]
   \[
   x_0 - \gamma(s_0) = u t(s_0).
   \]
(7) $g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = g^{(3)}_{x_0}(s_0) = 0$ if and only if one of the following conditions holds:

(a) $\delta'_r(s_0) \neq 0, \kappa_g(s_0) = 0$

(i.e., $\kappa_g(s_0) = 0$, $\kappa'_g(s_0) = -\kappa_n(s_0)\tau_g(s_0), 2\kappa_n(s_0)\tau'_g(s_0) + \kappa'_n(s_0)\tau_g(s_0) + \kappa''_g(s_0) \neq 0$) and

$$x_0 - \gamma(s_0) = -\frac{\kappa'_n(s_0)}{2\kappa_n(s_0)\tau'_g(s_0) + \kappa'_n(s_0)\tau_g(s_0) + \kappa''_g(s_0)} t(s_0).$$

(b) $\delta'_r(s_0) = 0, \kappa_g(s_0) = \kappa'_g(s_0) = 0$

(i.e., $\kappa_n(s_0) = \kappa_g(s_0) = \kappa'_g(s_0) = 0, \kappa''_g(s_0) = -\kappa'_n(s_0)\tau_g(s_0)$) and there exists $u \in \mathbb{R}$ such that

$$x_0 - \gamma(s_0) = ut(s_0).$$

Proof. Since $g_{x_0}(s) = \langle x_0 - \gamma(s), b(s) \rangle$, we have the following calculations:

(\alpha) $g_{x_0} = \langle x_0 - \gamma, b \rangle$,

(\beta) $g'_{x_0} = \langle x_0 - \gamma, -\kappa_g t + \tau_g n_\gamma \rangle$,

(\gamma) $g''_{x_0} = \kappa_g + \langle x_0 - \gamma, -(\kappa'_g + \tau_g \kappa_n) t - (\kappa'^2_g + \tau'^2_g) b - (\kappa_g \kappa_n + \tau'_g) n_\gamma \rangle$,

(\delta) $g^{(3)}_{x_0} = 2\kappa'_g + \kappa_n \tau_g$

$$+ \langle x_0 - \gamma, (\kappa_g(\kappa'^2_g + \kappa'^2_n + \tau'^2_g) + (\kappa'_n \tau_g + 2\kappa_n \tau'_g) - \kappa''_g) t$$

$$- 3(\kappa_g \kappa'_g + \tau_g \tau'_g) b + (\tau_g(\kappa'^2_g + \kappa'^2_n + \tau'^2_g) - (\kappa'_n \kappa_g + 2\kappa_n \kappa_n g - \tau''_g) n_\gamma \rangle,$$

(\epsilon) $g^{(4)}_{x_0} = 3\kappa'_g + 2\kappa'_n \tau_g + 3\kappa_n \tau'_g - \kappa_g(\kappa_g^2 + \kappa_n^2 + \tau_g^2)$

$$+ \langle x_0 - \gamma, (\kappa'_g(3\kappa'_g^2 + \kappa'_n^2 + \tau'_g) + \kappa_g(3\kappa_n \kappa'_n + 5\kappa_n \kappa'_g + 5\tau_g \tau'_g)$$

$$+ \kappa_n \tau_g(\kappa'_n^2 + \kappa_n^2 + \tau_g^2) + (\kappa''_n \tau_g + 3\kappa'_n \tau'_g + 3\kappa_n \tau''_g) - \kappa''_g) t$$

$$+ ((\kappa^2_g + \tau^2_g)(\kappa'_g + \kappa'_n + \tau'_g) + 2\kappa_n(\kappa'_g \tau_g - \kappa_g \tau'_g) - 3(\kappa'^2_g + \tau'^2_g) - 4(\kappa_g \kappa''_g + \tau_g \tau''_g)) b$$

$$+ (-\tau'_g(3\kappa'_n^2 + \kappa'_n^2 + \tau'_g) - \tau_g(3\kappa_n \kappa'_n + 5\kappa_g \kappa'_g + 5\tau_g \tau'_g)$$

$$+ \tau_g \kappa_n \kappa'^2_n + \kappa^2_n + \tau^2_g) - (\kappa''_n \kappa_g + 3\kappa'_n \kappa'_g + 3\kappa_n \kappa''_g + \tau'_g) n_\gamma \rangle.$$ 

By definition and (\alpha), assertion (1) follows.

By (\beta), $g_{x_0}(s_0) = g'_{x_0}(s_0) = 0$ if and only if $x_0 - \gamma(s_0) = ut(s_0) + vn_\gamma(s_0)$ and $-\kappa_g(s_0)u + \tau_g(s_0)v = 0$. If $\kappa_n(s_0) \neq 0, \tau_g(s_0) \neq 0$, then we have

$$u = v \frac{\tau_g(s_0)}{\kappa_g(s_0)}, v = u \frac{\kappa_g(s_0)}{\tau_g(s_0)}.$$

so that there exists $\lambda \in \mathbb{R}$ such that

$$x_0 - \gamma(s_0) = \lambda \frac{\tau_g(s_0) t(s_0) + \kappa_g(s_0) n_\gamma(s_0)}{\sqrt{\kappa^2_g(s_0) + \tau^2_g(s_0)}}.$$

Suppose that $\kappa_g(s_0) = 0$. Then we have $\tau_g(s_0) \neq 0$, so that $\tau_g(s_0)v = 0$. Therefore we have

$$x_0 - \gamma(s_0) = ut(s_0) = \pm u \frac{\tau_g(s_0) t(s_0) + \kappa_g(s_0) n_\gamma(s_0)}{\sqrt{\kappa^2_g(s_0) + \tau^2_g(s_0)}}.$$
If \( \tau_g(s_0) = 0 \), then we have \( x_0 - \gamma(s_0) = \nu \eta \gamma(s_0) \). Therefore the assertion (2) holds.

By (\( \gamma \)), \( g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = 0 \) if and only if

\[
x_0 - \gamma(s_0) = \lambda \tau_g(s_0) \frac{t(s_0) + \kappa_g(s_0) \eta \gamma(s_0)}{\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}}.
\]

and

\[
\kappa_g(s_0) + \frac{-\tau_g(\kappa_n + \kappa_g')}{\sqrt{\kappa_g^2 + \tau_g^2}} - \frac{\kappa_g(\kappa_n + \kappa_g')}{\sqrt{\kappa_g^2 + \tau_g^2}}(s_0) = 0.
\]

It follows that

\[
\frac{\kappa_g}{\sqrt{\kappa_g^2 + \tau_g^2}}(s_0) + \lambda \left( \kappa_n - \frac{\kappa_g \tau_g' - \kappa_g' \tau_g}{\kappa_g^2 + \tau_g^2} \right)(s_0) = 0.
\]

Thus,

\[
\delta_r(s_0) = \kappa_n(s_0) - \frac{\kappa_g \tau_g' - \kappa_g' \tau_g}{\kappa_g^2 + \tau_g^2}(s_0) \neq 0 \quad \text{and} \quad \lambda = -\frac{\kappa_g}{\delta_r \sqrt{\kappa_g^2 + \tau_g^2}(s_0)}
\]

or \( \delta_r(s_0) = 0, \kappa_g(s_0) = 0 \). This completes the proof of the assertion (A), (3) and (B), (6).

Suppose that \( \delta_r(s_0) \neq 0 \). By (\( \delta \)), \( g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = g^{(3)}_{x_0}(s_0) = 0 \) if and only if

\[
2\kappa' + \kappa_n \tau_g - \kappa_g \tau_g\begin{pmatrix}
\frac{\tau_g}{\sqrt{\kappa_n^2 + \tau_g^2}} \left( \kappa_g^2 + \kappa_n^2 + \tau_g^2 \right) + (\kappa_n \tau_g + 2\kappa_n \tau_g') - \kappa_g \\
\frac{kappa}{\sqrt{\kappa_n^2 + \tau_g^2}} \left( \tau_g^2 + \kappa_n^2 + \tau_g^2 \right) - (\kappa_n \kappa_n + 2\kappa_n \kappa_n') - \kappa_g^2 \right)
\end{pmatrix} = 0
\]

at \( s = s_0 \). It follows that

\[
2\kappa'(s_0) + \kappa_g(s_0) \tau_g(s_0) - \frac{\kappa_n(s_0)}{\delta_r(s_0)} \left( \kappa_n + 2\kappa_n \frac{\kappa_g \kappa_n' + \tau_g \tau_g'}{\kappa_g^2 + \tau_g^2} - \frac{\kappa_g \tau_g'' - \kappa_g' \tau_g}{\kappa_g^2 + \tau_g^2} \right) (s_0) = 0.
\]

Since

\[
2\kappa'(s_0) + \kappa_g(s_0) \tau_g(s_0) - \kappa_g(s_0) \frac{\delta_r(s_0)}{\delta_r(s_0)} - 2\kappa_g(s_0) \frac{\kappa_g(s_0) \kappa_g(s_0) \tau_g(s_0) \tau_g(s_0)}{\kappa_g^2(s_0) + \tau_g^2(s_0)} = 0.
\]

Moreover, we apply the relation

\[
\left( \frac{\kappa_g}{\sqrt{\kappa_g^2 + \tau_g^2}} \right)' = \frac{\tau_g}{\sqrt{\kappa_g^2 + \tau_g^2}} \frac{kappa_kappa' + \tau_g \tau_g'}{\kappa_g^2 + \tau_g^2} = \frac{\tau_g}{\sqrt{\kappa_g^2 + \tau_g^2}} (\delta_r - \kappa_n)
\]

to the above. Then we have

\[
\delta(s_0) \sqrt{\kappa_g^2 + \tau_g^2(s_0)} \left( \frac{\tau_g}{\sqrt{\kappa_g^2 + \tau_g^2}} + \left( \frac{\kappa_g}{\delta \sqrt{\kappa_g^2 + \tau_g^2}} \right)' \right)(s_0) = \delta_r(s_0) \sigma_r(s_0) \sqrt{\kappa_g^2 + \tau_g^2(s_0)} = 0,
\]

so that \( \sigma_r(s_0) = 0 \). The converse assertion also holds.
Suppose that $\delta_r(s_0) = 0$. Then by $(\delta)$, $g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = g^{(3)}_{x_0}(s_0) = 0$ if and only if $\kappa_g(s_0) = 0$ (i.e., $\kappa_g(s_0) = 0, \kappa'_g(s_0) = -\kappa_n(s_0)\tau_g(s_0)$), there exists $u \in \mathbb{R}$ such that

$$x_0 - \gamma(s_0) = ut(s_0)$$

and

$$2\kappa'_g(s_0) + \kappa_n(s_0)\tau_g(s_0) - u(2\kappa_n(s_0)\tau'_g(s_0) + \kappa'_n(s_0)\tau_g(s_0) + \kappa''_g(s_0)) = 0.$$

Since $\delta_r(s_0) = 0$ and $\kappa_g(s_0) = 0$, we have $\kappa_n(s_0)\tau_g(s_0) = -\kappa'_g(s_0)$, so that

$$\kappa'_g(s_0) - u(2\kappa_n(s_0)\tau'_g(s_0) + \kappa'_n(s_0)\tau_g(s_0) + \kappa''_g(s_0)) = 0.$$

It follows that

$$2\kappa_n(s_0)\tau'_g(s_0) + \kappa'_n(s_0)\tau_g(s_0) + \kappa''_g(s_0) \neq 0$$

and

$$u = \frac{\kappa'_g(s_0)}{2\kappa_n(s_0)\tau'_g(s_0) + \kappa'_n(s_0)\tau_g(s_0) + \kappa''_g(s_0)}$$

or

$$2\kappa_n(s_0)\tau'_g(s_0) + \kappa'_n(s_0)\tau_g(s_0) + \kappa''_g(s_0) = 0 \text{ and } \kappa'_g(s_0) = 0.$$

Therefore we have (B), (7), (a) or (b).

By the similar arguments to the above, we have the assertion (A), (5). This completes the proof. \qed

In order to prove Theorem 3.3, we use some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book[1]. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be a function germ. We call $F$ an $r$-parameter unfolding of $f$, where $f(s) = F_{x_0}(s, x_0)$. We say that $f$ has an $A_k$-singularity at $s_0$ if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(s_0) \neq 0$. Let $F$ be an unfolding of $f$ and $f(s)$ has an $A_k$-singularity ($k \geq 1$) at $s_0$. We write the $(k - 1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at $s_0$ by $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s, x_0))(s_0) = \sum_{j=0}^{k-1} \alpha_{ji}(s - s_0)^j$ for $i = 1, \ldots, r$. Then $F$ is called an $\mathcal{R}$-versal unfolding if the $k \times r$ matrix of coefficients $(\alpha_{ji})_{j=0,\ldots,k-1;i=1,\ldots,r}$ has rank $k$ ($k \leq r$). We introduce an important set concerning the unfoldings relative to the above notions. The discriminant set of $F$ is the set

$$\mathcal{D}_F = \{x \in \mathbb{R}^r|\text{there exists } s \text{ with } F = \frac{\partial F}{\partial s} = 0 \text{ at } (s, x)\}.$$

Then we have the following classification (cf., [1]).

**Theorem 4.2** Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an $r$-parameter unfolding of $f(s)$ which has an $A_k$ singularity at $s_0$. Suppose that $F$ is an $\mathcal{R}$-versal unfolding.

1. If $k = 2$, then $\mathcal{D}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-1}$.
2. If $k = 3$, then $\mathcal{D}_F$ is locally diffeomorphic to $SW \times \mathbb{R}^{r-2}$.

For the proof of Theorem 3.3, we have the following propositions.

**Proposition 4.3** Let $\gamma : I \to M \subset \mathbb{R}^3$ be a unit speed curve with $\kappa_n^2 + \kappa_g^2 \neq 0$ and $G : I \times \mathbb{R}^3 \to \mathbb{R}$ the support function on $\gamma$ with respect to $b$. If $g_{x_0}$ has an $A_k$-singularity ($k = 2, 3$) at $s_0$, then $G$ is an $\mathcal{R}$-versal unfolding of $g_{x_0}$. Here, we assume that $\delta_r(s_0) \neq 0$ for $k = 3$. 

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Therefore the 2-jet is

\[ j^2 \frac{\partial G}{\partial x_i}(s, x_0) = b_i(s_0) + b'_i(s_0)(s - s_0) + \frac{1}{2} b''_i(s_0)(s - s_0)^2. \]

We consider the following matrix:

\[
A = \begin{pmatrix}
  b_1(s_0) & b_2(s_0) & b_3(s_0) \\
  b'_1(s_0) & b'_2(s_0) & b'_3(s_0) \\
  b''_1(s_0) & b''_2(s_0) & b''_3(s_0)
\end{pmatrix} = \begin{pmatrix}
  b(s_0) \\
  b'(s_0) \\
  b''(s_0)
\end{pmatrix}.
\]

By the Frenet-Serret type formulae, we have

\[
b' = -\kappa_g t + \tau_g n_\gamma \quad \text{and} \quad b'' = (\kappa_n \tau_g - \kappa_g') t - (\kappa_n^2 + \tau_g^2) b + (\tau_g' - \kappa_\gamma \kappa_g) n_\gamma.
\]

Since \{t, b, n_\gamma\} is an orthonormal basis of \(\mathbb{R}^3\), the rank of

\[
A = \begin{pmatrix}
  b(s_0) \\
  -\kappa_g(s_0) t(s_0) + \tau_g(s_0) n_\gamma(s_0) \\
  (-\kappa_n(s_0) \tau_g(s_0) - \kappa_g'(s_0)) t(s_0) - (\kappa_n^2(s_0) + \tau_g^2(s_0)) b(s_0) + (\tau_g'(s_0) - \kappa_\gamma(s_0) \kappa_g(s_0)) n_\gamma(s_0)
\end{pmatrix}
\]

is equal to the rank of

\[
\begin{pmatrix}
  0 & 0 & 1 \\
  -\kappa_g(s_0) & +\tau_g(s_0) & 0 \\
  -\kappa_n(s_0) \tau_g(s_0) - \kappa_g'(s_0) & \tau_g'(s_0) - \kappa_\gamma(s_0) \kappa_g(s_0) & -(\kappa_n^2(s_0) + \tau_g^2(s_0))
\end{pmatrix}.
\]

Therefore rank \(A = 3\) if and only if

\[
0 \neq -\kappa_g(\tau_g' - \kappa_\gamma \kappa_g) + \tau_g(\kappa_n \tau_g + \kappa_g') = \kappa_n(\kappa_n^2 + \tau_g^2) - (\kappa_\gamma \tau_g' - \kappa_\gamma' \tau_g)
\]

st \(s = s_0\). The last condition is equivalent to the condition \(\delta_r(s_0) \neq 0\). Moreover, the rank of

\[
\begin{pmatrix}
  b(s_0) \\
  b'(s_0)
\end{pmatrix} = \begin{pmatrix}
  b(s_0) \\
  -\kappa_g(s_0) t(s_0) + \tau_g(s_0) n_\gamma(s_0)
\end{pmatrix}
\]

is always two.

If \(g_{x_0}\) has an \(A_k\)-singularity (\(k = 2, 3\)) at \(s_0\), then \(G\) is \(\mathcal{R}\)-versal unfolding of \(g_{x_0}\). This completes the proof. \(\square\)

**Proof of Theorem 3.3.** By a straightforward calculation, we have

\[
\frac{\partial N D_3}{\partial s} \times \frac{\partial N D_2}{\partial u} = -\left( \frac{\kappa_g}{\sqrt{\kappa_n^2 + \tau_g^2}} + w \delta_r \right) n_\gamma.
\]

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Therefore, $(s_0, u_0)$ is non-singular if and only if
\[
\frac{\partial ND_\gamma}{\partial s} \times \frac{\partial ND_\gamma}{\partial u} \neq 0.
\]
This condition is equivalent to
\[
\frac{\kappa_g(s_0)}{\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}} + u_0 \delta_r(s_0) \neq 0.
\]
This completes the proof of assertion (1).

By Proposition 4.1, (2), the discriminant set $D_G$ of the support function $G$ of $\gamma$ with respect to $b$ is the image of the normal developable surface of $M$ along $\gamma$.

Suppose that $\delta_r(s_0) \neq 0$. It follows from Proposition 4.1, A, (3), (4) and (5) that $g_{x_0}$ has the $A_2$-type singularity (respectively, the $A_3$-type singularity) at $s = s_0$ if and only if
\[
u_0 = \frac{\kappa_g(s_0)}{\delta_r(s_0)\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}}
\]
and $\sigma_r(s_0) \neq 0$ (respectively, $\sigma_r(s_0) = 0$ and $\sigma'_r(s_0) \neq 0$). By Theorem 4.2 and Proposition 3.3, we have the assertions (2), (α) and (3).

Suppose that $\delta_r(s_0) = 0$. It follows from Proposition 4.1, B, (6) and (7) that $g_{x_0}$ has the $A_3$-type singularity if and only if $\delta_r(s_0) = 0, \kappa_g(s_0) = 0$ and
\[
\kappa'_g(s_0) \neq 0 \text{ or } \kappa'_g(s_0) + u_0 \left(2\kappa_n(s_0)\tau'_g(s_0) + \kappa'_n(s_0)\tau_g(s_0) + \kappa''_g(s_0)\right) \neq 0.
\]
By Theorem 4.3 and Proposition 4.3, we have the assertion (2), (β). This completes the proof. □

5 Invariants of curves on surfaces

In this section we consider geometric meanings of the invariant $\sigma_r$. Let $\Gamma : I \rightarrow \mathbb{R}^3 \times S^2$ be a regular curve and $F : \mathbb{R}^3 \times S^2 \rightarrow \mathbb{R}$ a submersion. We say that $\Gamma$ and $F^{-1}(0)$ have contact of at least order $k$ for $t = t_0$ if the function $g(t) = F \circ \Gamma(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k)}(t_0) = 0$. If $\gamma$ and $F^{-1}(0)$ have contact of at least order $k$ for $t = t_0$ and satisfies the condition that $g^{(k+1)}(t_0) \neq 0$, then we say that $\Gamma$ and $F^{-1}(0)$ have contact of order $k$ for $t = t_0$. For any $x \in \mathbb{R}^3$, we define a function $g_x : \mathbb{R}^3 \times S^2 \rightarrow \mathbb{R}$ by $g_x(u, v) = \langle x - u, v \rangle$. Then we have
\[
g_x^{-1}(0) = \{(u, v) \in \mathbb{R}^3 \times S^2 \mid \langle u, v \rangle = \langle x, v \rangle\}.
\]
If we fix $v \in S^2$, then $g_x^{-1}(0) |_{\mathbb{R}^3 \times \{v\}}$ is an affine plane defined by $\langle u, v \rangle = c$, where $c = \langle x, v \rangle$. Since this plane is orthogonal to $v$, it is parallel to the tangent plane $T_v S^2$ at $v$. Here we have a representation of the tangent bundle of $S^2$ as follows:
\[
T S^2 = \{(u, v) \in \mathbb{R}^3 \times S^2 \mid \langle u, v \rangle = 1\}.
\]
We consider the canonical projection $\pi_2 | g_x^{-1}(0) : g_x^{-1}(0) \rightarrow S^2$, where $\pi_2 : \mathbb{R}^3 \times S^2 \rightarrow S^2$. Then $\pi_2 | g_x^{-1}(0) : g_x^{-1}(0) \rightarrow S^2$ is a plane bundle over $S^2$. Moreover, we define a map $\Psi :$
If there exists \( x_0 \in \mathbb{R}^3 \) such that \( g_{x_0}(\gamma(s), b(s)) = 0 \), then we have
\[
\gamma(s) - x_0 = \frac{\kappa_g(s)}{\delta_r(s) \sqrt{\kappa_g^2(s) + \tau_g^2(s)}} D_r(s).
\]
Therefore
\[
g_{x_0}(\gamma(s), b(s)) = g_{x_0}(s) = \langle \gamma(s) - x_0, b(s) \rangle = 0.
\]
If \( \sigma_r(s) \equiv 0 \), we consider a regular curve \((\gamma, b) : I \rightarrow \mathbb{R}^3 \times S^2\). Then we have the following proposition.

**Proposition 5.1** Let \( \gamma : I \rightarrow M \subset \mathbb{R}^3 \) be a unit speed curve on \( M \) with \( \kappa_g^2(s) + \tau_g^2(s) \neq 0 \) and \( \delta_r(s) \neq 0 \). Then there exists \( x_0 \in \mathbb{R}^3 \) such that \( \langle \gamma(s), b(s) \rangle = 0 \) if and only if \( \sigma_r(s) \equiv 0 \).

The result of the above proposition explains that the geometric meaning of the singularities of \( ND_\gamma \) is related not only to the curve as a space curve but also to the behavior of the curve in the surface. Let \( \gamma : I \rightarrow M \subset \mathbb{R}^3 \) be a unit speed curve with \( \kappa_g^2(s) + \tau_g^2(s) \neq 0 \). Then we consider the support function \( g_{x_0}(s) = g_{x_0}(\gamma(s), b(s)) \). By assertion (2) of Proposition 4.1, \((\gamma, b)\) is tangent to \( TS^2(x_0) \) at \( s = s_0 \) if and only if \( x_0 = ND_\gamma(s_0, u_0) \) for some \( u_0 \in \mathbb{R} \). Then we have the following proposition.

**Proposition 5.2** Let \( \gamma : I \rightarrow M \subset \mathbb{R}^3 \) be a unit speed curve with \( \kappa_g^2(s) + \tau_g^2(s) \neq 0 \) and \( \delta_r(s) \neq 0 \). For \( x_0 = ND_\gamma(s_0, u_0) \), we have the following:

1. The order of contact of \((\gamma, b)\) with \( TS^2(x_0) \) at \( s = s_0 \) is two if and only if
   \[
   u_0 = -\frac{\kappa_g(s_0)}{\delta_r(s_0) \sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}}.
   \]
   and \( \sigma_r(s_0) \neq 0 \).

2. The order of contact of \((\gamma, b)\) with \( TS^2(x_0) \) at \( s = s_0 \) is three if and only if \((***) \) and \( \sigma_r(s_0) = 0 \) and \( \sigma'_r(s_0) \neq 0 \).

**Proof.** By assertions (3), (4) of Proposition 4.1, the conditions \( g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = 0 \) and \( g^{(3)}_{x_0}(s_0) \neq 0 \) if and only if \((***) \) and \( \sigma_r(s_0) \neq 0 \). Since \( g_{x_0} = g_{x_0} \circ (\gamma, b) \), the above condition means that \((\gamma, b)\) and \( TS^2(x_0) \) have contact of order two at \( s = s_0 \). For the proof assertion (2), we use assertions (4), (5) of Proposition 4.1 exactly the same way as the above case. \( \square \)

Therefore the geometric meaning of the classification results of Theorem 3.3 are given as follows.
Theorem 5.3 Let \( \gamma : I \rightarrow M \subset \mathbb{R}^3 \) be a unit speed curve with \( \kappa_2^2(s) + \tau_2^2(s) \neq 0 \) and \( \delta_r(s) \neq 0 \).

(1) The image of the normal developable surface \( ND_\gamma \) of \( M \) along \( \gamma \) is locally diffeomorphic to the cuspidal edge \( C \times \mathbb{R} \) at \( (s_0, u_0) \) if

\[
u_0 = -\frac{\kappa_2(s_0)}{\delta(s_0) \sqrt{\kappa_2^2(s_0) + \tau_2^2(s_0)}}
\]

and the order of contact of \( (\gamma, b) \) with \( TS^2(x_0) \) at \( s = s_0 \) is two.

(2) The image of the normal developable surface \( ND_\gamma \) of \( M \) along \( \gamma \) is locally diffeomorphic to the swallowtail \( SW \) at \( (s_0, u_0) \) if

\[
u_0 = -\frac{\kappa_2(s_0)}{\delta_r(s_0) \sqrt{\kappa_2^2(s_0) + \tau_2^2(s_0)}}
\]

and the order of contact of \( (\gamma, b) \) with \( TS^2(x_0) \) at \( s = s_0 \) is three.

6 Uniqueness of normal developable surfaces

In this section we consider existence and uniqueness of normal developable surfaces along curves on a surface.

**Theorem 6.1** Let \( M \subset \mathbb{R}^3 \) be a regular surface and \( \gamma : I \rightarrow M \subset \mathbb{R}^3 \) a unit speed curve with \( \kappa_2^2(s) + \tau_2^2(s) \neq 0 \). Then there exists a unique developable surface which is normal to \( M \) along \( \gamma \).

**Proof.** For existence, we have the normal developable surface \( ND_\gamma \) along \( \gamma \). On the other hand, let \( \mathcal{N}_\gamma \) be a developable surface which is normal to \( M \) along \( \gamma \). Since \( \mathcal{N}_\gamma \) is a ruled surface, we assume that

\[\mathcal{N}_\gamma(s, u) = \gamma(s) + u\xi(s),\]

Here we write

\[\xi(s) = \lambda(s)t(s) + \mu(s)b(s) + \nu(s)n_\gamma(s),\]

so that

\[\xi' = (\lambda' - \mu_\kappa_g - \nu_\kappa_n) t + (\mu' + \lambda_\kappa_g - \nu_\tau_g) b + (\nu' + \lambda_\kappa_n + \mu_\tau_g) n_\gamma.\]

Since \( \mathcal{N}_\gamma \) is a developable surface, we have \( \det (\gamma', \xi, \xi') = 0 \). The last condition is equivalent to

\[
\det \begin{pmatrix}
\xi & \xi' \\
\lambda t + \mu b + \nu n_\gamma & (\lambda' - \mu_\kappa_g - \nu_\kappa_n) t + (\mu' + \lambda_\kappa_g - \nu_\tau_g) b + (\nu' + \lambda_\kappa_n + \mu_\tau_g) n_\gamma
\end{pmatrix} = 0,
\]

which is also equivalent to

\[
\mu (\nu' + \lambda_\kappa_n + \mu_\tau_g) - \nu (\mu' + \lambda_\kappa_g - \nu_\tau_g) = 0.
\] (1)

On the other hand, since \( \mathcal{N}_\gamma \) is a developable surface which is normal to \( M \) along \( \gamma \), we have

\[
\frac{\partial \mathcal{N}_\gamma}{\partial s}(s, u) \times \frac{\partial \mathcal{N}_\gamma}{\partial u}(s, u) = \theta(s, u)b(s).
\] (2)
If \( N_\gamma \) is nonsingular at \((s,0)\), then \( \theta(s,0) \neq 0 \). By a straightforward calculation, we have
\[
\frac{\partial N_\gamma}{\partial s} = \gamma' + u\xi'
\]
\[
= (1 + u(\lambda' - \mu\kappa_g - \nu\kappa_n)) t + u (\mu' + \lambda\kappa_g - \nu\tau_g) b + u (\nu' + \lambda\kappa_n + \mu\tau_g) n_\gamma
\]
\[
\frac{\partial N_\gamma}{\partial u} = \xi = \lambda t + \mu b + \nu n_\gamma,
\]
so that
\[
\frac{\partial N_\gamma}{\partial s} \times \frac{\partial N_\gamma}{\partial u} = u (\mu' + \lambda\kappa_n + \mu\tau_g) - \nu (\mu' + \lambda\kappa_g - \nu\tau_g)) t
\]
\[
+ (u\lambda (\nu' + \lambda\kappa_g + \mu\tau_g) - \nu \{1 + u (\mu' + \lambda\kappa_g - \nu\tau_g))\} b
\]
\[
+ (\mu \{1 + u (\lambda' - \mu\kappa_g - \nu\kappa_n)) - u\lambda (\mu' + \lambda\kappa_g - \nu\tau_g)) n_\gamma.
\]
If we substitute \( u = 0 \), then
\[
\frac{\partial N_\gamma}{\partial s} \times \frac{\partial N_\gamma}{\partial u} = -\nu b + \mu n_\gamma.
\]
By (2), we have \(-\nu(s) = \theta(s,0), \mu(s) = 0\). It also follows from (1) that
\[
\nu(s) (\lambda(s)\kappa_g(s) - \nu(s)\tau_g(s)) = 0.
\]
Suppose that \( N_\gamma \) is non-singular along \( \gamma \). Then \( \theta(s,0) \neq 0 \), so that \( \nu(s) \neq 0 \). Thus, we have \( \lambda(s)\kappa_g(s) - \nu(s)\tau_g(s) = 0 \). If \( \kappa_g(s) \neq 0 \), then \( \lambda(s) = \frac{\tau_g(s)}{\kappa_g(s)} \nu(s) \). Therefore, we have
\[
\xi(s) = \frac{\tau_g(s)}{\kappa_g(s)} \nu(s) t(s) + \nu(s) n_\gamma(s)
\]
\[
= \nu(s) \sqrt{\kappa_g^2(s) + \tau_g^2(s)} \left( \frac{\tau_g(s) t(s) + \kappa_g(s) n_\gamma(s)}{\sqrt{\kappa_g^2(s) + \tau_g^2(s)}} \right)
\]
\[
= \nu(s) \sqrt{\kappa_g^2(s) + \tau_g^2(s)} \frac{\kappa_g(s)}{\kappa_g(s)} D_r(s).
\]
This means that the direction of \( \xi(s) \) is equal to the direction of \( D_r(s) \). If \( \tau_g(s) \neq 0 \), we have the same result as the above case. On the other hand, suppose that \( N_\gamma \) has a singular point at \((s_0,0)\). Then \( \theta(s_0,0) = 0 \). It follows that \( \nu(s_0) = 0, \mu(s_0) = 0 \). Thus we have \( \xi(s_0) = \lambda(s_0) t(s_0) \). If the singular point \( \gamma(s_0) \) is in the closure of the set of points where the normal developable is regular on \( \gamma \), then there exists a point \( s \) in any neighborhood of \( s_0 \) such that the uniqueness of the normal developable surface holds at \( s \). Passing to the limit \( s \to s_0 \), uniqueness of the normal developable surface holds at \( s_0 \). Suppose that there exists an open interval \( J \subset I \) such that \( N_\gamma \) is singular at \( \gamma(s) \) for any \( s \in J \). Then \( N_\gamma(s) = \gamma(s) + u\lambda(s) t(s) \) for any \( s \in J \). This means that \( \mu(s) = \nu(s) = 0 \) for \( s \in J \). It follows that
\[
\frac{\partial N_\gamma}{\partial s} \times \frac{\partial N_\gamma}{\partial u}(s,u) = u\lambda^2(s)(\kappa_n(s)b(s) - \kappa_g(s)n_\gamma(s)).
\]
The above vector is directed to \( b(s) \), so that \( \kappa_g(s) = 0 \) for any \( s \in J \). In this case, \( D_r(s) = \pm t(s) \).
This means that uniqueness holds.
Proposition 6.2 Let $\gamma : I \to M$ be a regular curve with $\kappa_g(s) \equiv \tau_g(s) \equiv 0$. Then $N_\gamma$ is a normal developable along $\gamma$ if and only if $\gamma$ is a normal slice of $M$.

Proof. We remark that the torsion of the curve $\gamma$ as a space curve is

$$\tau = \tau_g + \frac{\kappa_g \kappa_n' - \kappa'_g \kappa_n}{\kappa_g^2 + \tau_g^2}.$$  

If $\kappa_g(s) \equiv \tau_g(s) \equiv 0$, then $\tau \equiv 0$, so that $\gamma$ is a plane curve. Moreover, we have $b' = -\kappa_g t + \tau_g n$, so that $N_\gamma$ is a plane normal to $M$. Since $\gamma$ is the intersection of $M$ and $N_\gamma$, $\gamma$ is a normal slice of $M$. For the converse, if $\gamma$ is a normal slice of $M$, then there exists a plane $\Pi$ such that $\gamma(I) = M \cap \Pi$ and $n_\gamma(s), t(s) \in \Pi$ for any $s \in I$. Therefore, $\Pi$ is orthogonal to $b(s)$ for any $s \in I$. Since $\Pi$ is a plane (i.e. a developable surface), $\Pi$ is a normal developable surface of $M$ along $\gamma$. \hfill $\square$

Corollary 6.3 Let $\gamma : I \to M \subset \mathbb{R}^3$ be a unit speed curve on $M$. If there are two normal developable surfaces along $\gamma$, then $\gamma$ is a straight line.

Proof. Under the assumption of $\kappa_g^2 + \tau_g^2 \neq 0$, the normal developable surface along $\gamma$ is unique by Theorem 6.1. If $\kappa_g \equiv 0$ and $\tau_g \equiv 0$, $\gamma$ is a normal slice. In this case, a normal plane $\Pi$ of $M$ at $\gamma(s_0)$ is a normal developable surface along $\gamma$. If there is another normal developable surface $N_\gamma$ along $\gamma$, then $N_\gamma$ is tangent to $\Pi$ along $\gamma$. This means that $\Pi$ is a tangent plane of the developable surface $N_\gamma$. By definition, $\Pi$ is tangent to $N_\gamma$ along a ruling of $N_\gamma$, which is $\gamma$. Thus $\gamma$ is a line. If $\kappa_g = \tau_g = 0$ at a point $s_0$ in the closure of the set of points where $\kappa_g^2 + \tau_g^2 \neq 0$, then there exists a point $s$ in any neighborhood of $s_0$ such that the uniqueness of the normal developable surface holds at $s$. Passing to the limit $s \to s_0$, uniqueness of the normal developable surface holds at $s_0$. This completes the proof. \hfill $\square$

7 Special curves on a surface

In this section we consider special curves on a surface.

7.1 Geodesics

We consider geodesics on surfaces. Let $\gamma : I \to M$ be a unit speed curve on a surface $M \subset \mathbb{R}^3$. It is a geodesic of $M$ if and only if $\kappa_g \equiv 0$. Therefore, if $\tau_g \neq 0$, then

$$ND_\gamma(s, u) = \gamma(s) + ut(s),$$

which is known as the tangent surface of $\gamma$. In this case

$$\delta_r(s) = \kappa_n(s), \quad \sigma_r(s) = \pm 1, \quad \sigma'_r(s) \equiv 0.$$  

By Theorem 3.3, we have the following proposition.

Proposition 7.1 Let $\gamma : I \to M$ be a geodesic on a surface $M$ with $\tau_g(s) \neq 0$. Then we have the following:
1. $ND_\gamma$ is a regular surface at $(s_0, u_0)$ if $u_0 \neq 0$,

2. The image of $(ND_\gamma, (s_0, u_0))$ is locally diffeomorphic to the cuspidal edge if 
   (a) $\kappa_n(s_0) \neq 0$ and $u_0 = 0$, or
   (b) $\kappa_n(s_0) = 0$ and $u_0 \left(2\tau'_n\kappa_n + \tau_g\kappa'_n\right)(s_0) \neq 0$.

Since $\sigma'_g(s) \equiv 0$, swallowtails never appear. We now consider the following example.

**Example 7.2** We consider a surface parametrized by

\[ X(t, u) = (t \cos(t), t \sin(t), 5t) + u \left( \frac{\cos(t) - t \sin(t)}{\sqrt{t^2 + 26}}, \frac{\sin(t) + t \cos(t)}{\sqrt{t^2 + 26}}, \frac{5}{\sqrt{t^2 + 26}} \right). \]

This surface is a ruled surface such that the base curve is $\gamma(t) = (t \cos(t), t \sin(t), 5t)$. Thus, $\gamma$ is a regular curve on the surface $M = \text{Im} \ X$. It follows that

\[ \dot{\gamma}(t) = \left( -2 \sin(t) - t \cos(t), 2 \cos(t) - t \sin(t), 0 \right) \]

\[ \frac{\partial X}{\partial t} \times \frac{\partial X}{\partial u} = \frac{\alpha(t) + \beta(t, u)}{(t^4 + 29t^2 + 104)^{7/2}} \left( (t^2 + 52) \sin(t) + t(t^2 + 27) \cos(t), \right. \]

\[ \left. t(t^2 + 27) \sin(t) - (t^2 + 52) \cos(t), 5t \right), \]

where

\[ \alpha(t) = t^{12} + 87t^{10} + 2835t^8 + 42485t^6 + 294840t^4 + 940992t^2 + 1124864, \]

\[ \beta(t, u) = u \left( 31460t \sqrt{t^4 + 29t^2 + 104} - 5t^7 \sqrt{t^4 + 29t^2 + 104} + 260t^5 \sqrt{t^4 + 29t^2 + 104} - 5850t^3 \sqrt{t^4 + 29t^2 + 104} \right). \]

In this case the Darboux frame of $\gamma$ is

\[ n_\gamma(t) = \frac{X_t \times X_u}{\|X_t \times X_u\|} \]

\[ = \frac{1}{t^6 + 55t^4 + 858t^2 + 2704} \left( (t^2 + 52) \sin(t) + t(t^2 + 27) \cos(t), \right. \]

\[ \left. t(t^2 + 27) \sin(t) + (t^2 - 52) \cos(t), 5t \right), \]

\[ t(t) = \frac{\gamma(t)}{\|\gamma(t)\|} = \left( \frac{\cos(t) - t \sin(t)}{\sqrt{t^2 + 26}}, \frac{\sin(t) + t \cos(t)}{\sqrt{t^2 + 26}}, \frac{5}{\sqrt{t^2 + 26}} \right), \]

\[ b(t) = n_\gamma(t) \times t(t). \]

Since $\dot{\gamma}(t) = (-2 \sin(t) - t \cos(t), 2 \cos(t) - t \sin(t), 0)$, we have

\[ \kappa_g(t) = \frac{\det (\dot{\gamma}(t), \ddot{\gamma}(t), n_\gamma(t))}{\|\dot{\gamma}(t)\|^3} = 0, \]

so that $\gamma$ is a geodesic of $M$. Moreover, we have

\[ \kappa_n(t) = \frac{\langle \gamma(t), n_\gamma(t) \rangle}{\|\gamma(t)\|^2} = - \frac{1}{(t^2 + 26)^2}, \quad \kappa'_n(t) = \frac{4t}{(t^2 + 26)^3}, \]

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\[ \tau_g(t) = \frac{\det(\gamma(t), n_\gamma(t), \dot{n}_\gamma(t))}{\|\gamma(t)\|^2} = \frac{5(t^2 + 6)}{(t^2 + 26)(t^4 + 29t^2 + 104)^2}, \]

\[ \tau'_g(t) = \frac{-20t(2t^6 + 83t^4 + 950t^2 + 3484)}{(t^2 + 26)^2(t^4 + 29t^2 + 104)^3}. \]

We can draw the pictures of \( \gamma \) and \( M \) in Fig. 3, 4, 5.

![Figure 3: M and γ](image1)

![Figure 4: ND_γ](image2)

![Figure 5: M and ND_γ](image3)

### 7.2 Lines of curvature

We consider lines of curvature. Let \( \gamma : I \rightarrow M \) be a unit speed line of curvature of \( M \) with \( \kappa_g(s) \neq 0 \). Since \( \tau_g(s) \equiv 0 \), we have

\[ ND_\gamma(s, u) = \gamma(s) + u n_\gamma(s) \]

and \( \delta_r(s) = \kappa_n(s) \). Moreover, if \( \kappa_n(s) \neq 0 \), then

\[ \sigma_r(s) = -\frac{\kappa'_n(s)}{\kappa_n^2(s)}, \quad \sigma'_r(s) = \frac{2\kappa^n - \kappa''_n \kappa_n(s)}{\kappa^3_n(s)} \]

Then we have the following proposition as a corollary of Theorem 3.3.

**Proposition 7.3** Let \( \gamma : I \rightarrow M \) be a line of curvature on a surface \( M \) with \( \kappa_g(s) \neq 0 \). Then we have the following:

1) \( ND_\gamma \) is a regular surface at \( (s_0, u_0) \) if \( u_0\kappa_n(s_0) \neq \pm 1 \).

2) \( ND_\gamma \) is locally diffeomorphic to the cuspidal edge at \( (s_0, u_0) \) if \( \kappa_n(s_0) \neq 0 \) and \( u_0 = \pm \frac{1}{\kappa_n(s_0)} \).

3) \( ND_\gamma \) is locally diffeomorphic to the swallowtail at \( (s_0, u_0) \) if \( \kappa_n(s_0) \neq 0, \kappa'_n(s_0) = 0, \kappa''_n(s_0) \neq 0 \) and \( u_0 = \pm \frac{1}{\kappa_n(s_0)} \).

**Example 7.4** We now consider an ellipsoid as an example. The following parametrization of an ellipsoid is known. Let \( X(u, v) = (x(u, v), y(u, v), z(u, v)) \) be a surface in \( \mathbb{R}^3 \) defined by

\[
\begin{align*}
x(u, v) &= \pm \sqrt{\frac{a(u-a)(u-v)}{(a-b)(a-c)}}, \\
y(u, v) &= \pm \sqrt{\frac{b(b-u)(b-v)}{(b-c)(b-a)}}, \\
z(u, v) &= \pm \sqrt{\frac{c(c-u)(c-v)}{(c-a)(c-b)}},
\end{align*}
\]

where \( 0 < c \leq v \leq b \leq u \leq a \). We can easily show that the image of \( X \) is an ellipsoid defined by \( \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1 \). Moreover, it is classically known that each \( u \)-curve and \( v \)-curve are
We can observe that there appears the swallowtail on the normal developable surface. We also show as follows: For a $u$-curve defined by $\gamma(u) = X(u, v_0) = (x(u, v_0), y(u, v_0), z(u, v_0))$, the unit normal vector along $\gamma$ is

$$n_\gamma(u) = \frac{X_u \times X_v}{\|X_u \times X_v\|(u, v_0)} = \begin{pmatrix} -y_u(u, v_0)z_u(u, v_0)(b - c)(u - v_0) \\ \omega_u(u, v_0)(u - b)(u - c)(v_0 - b)(v_0 - c) \\ z_u(u, v_0)x_u(u, v_0)(c - a)(u - v_0) \\ \omega_u(u, v_0)(u - a)(u - c)(v_0 - a)(v_0 - c) \\ x_u(u, v_0)y_u(u, v_0)(a - b)(u - v_0) \\ \omega_u(u, v_0)(u - a)(u - b)(v_0 - a)(v_0 - b) \end{pmatrix},$$

where

$$\omega(u, v) = \sqrt{u v(u - v)^2 - (u + v)(a - v)(u - b)(v - c)(u - c)}.$$ Since

$$\dot{\gamma}(u) = \begin{pmatrix} a(a - v_0) \\ 2x_u(u, v_0)(a - b)(a - c) \\ 2y_u(u, v_0)(b - v_0) \\ 2z_u(u, v_0)(c - v_0) \\ c(c - v_0) \end{pmatrix},$$

$$\ddot{\gamma}(u) = \begin{pmatrix} a^2(a - v_0)^2 \\ 4x_u(u, v_0)^{3/2}(a - b)^2(a - c)^2 \\ 4y_u(u, v_0)^{3/2}(b - a)^2(b - c)^2 \\ 4z_u(u, v_0)^{3/2}(c - a)^2(c - b)^2 \end{pmatrix},$$

we have $\tau_g(u) = \frac{\det(\ddot{\gamma}(u), n_\gamma(u), \dot{n}_\gamma(u))}{\|\ddot{\gamma}(u)\|^2} = 0$. This means that each $u$-curve is a line of curvature. By the same calculation as the above, we can show that each $v$-curve is a line of curvature.

Suppose that $a = 20, b = 10, c = 5$. Then we can draw the pictures of the ellipsoid, the $u$-curve for $v = 7$ and the normal developable surface along the $u$-curve with $v = 7$.

We can observe that there appears the swallowtail on the normal developable surface. We also have

$$\kappa_n(u) = \frac{\langle \dot{\gamma}(u), n_\gamma(u) \rangle}{\|\dot{\gamma}(u)\|^2} = -\frac{bcx_u(u, v_0)(u - v_0)}{4uy_u(u, v_0)z_u(u, v_0)(a - u)(a - v_0)(b - c)}$$

and show that the singular point is the swallowtail. We can also show that the normal developable surface along a $v$-curve has the swallowtail. Since the director of the normal developable
surface along a line of curvature $\gamma$ is $n_\gamma$, if we consider all $u$-curves and $v$-curves, then we have the evolute as the trajectory of the singular value sets of corresponding normal developable surfaces. We can draw the picture of the trajectory of the singular value sets of normal developable surfaces along $u$-curves in Fig. 9, which gives one of the branch of the evolute of the ellipsoid. The other branch of the evolute is depicted in Fig. 10, which is given by the normal developable suffices along $v$-curves.

Figure 9: $u$-curves and the trajectory of the singular values

Figure 10: $v$-curves and trajectory of the singular values

The picture of the union of two branches of the evolute is drawn in Fig. 11. It is known that the purse $PS$ appears as a singular point of the evolute of the ellipsoid [5]. Here

$$PS = \{(3u^2 + uv, 3w^2 + wu, w) \in \mathbb{R}^3 \mid w^2 = 36uv \}.$$ 

Moreover, the purse corresponds to an umbilical point of the ellipsoid. We can observe the purse in Fig.12.

Figure 11: The union of the Fig.9 and Fig. 10

Figure 12: The purse of the evolute

8 Curves on a surface of revolution

In this section we consider curves on surfaces of revolution. A surface of revolution is defined to be

$$X(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$$

for $(u, v) \in U \subset \mathbb{R}^2$. We assume that $f(u) \neq 0$. It is easy to show that the unit normal vector field alone $M = X(U)$ is

$$n(u, v) = \left( -\frac{f(u)g'(u) \cos v}{\sqrt{f(u)^2 (f'(u)^2 + g'(u)^2)}}, -\frac{f(u)g'(u) \sin v}{\sqrt{f(u)^2 (f'(u)^2 + g'(u)^2)}}, \frac{f(u)f'(u)}{\sqrt{f(u)^2 (f'(u)^2 + g'(u)^2)}} \right).$$
For a curve
\[ \gamma(t) = (f(u(t)) \cos v(t), f(u(t)) \sin v(t), g(u(t))) \]
on \(M\), the Darboux frame is given by
\[ n(t) = \frac{1}{\sqrt{f^2 h}} (-fg' \cos t_v, -fg' \sin t_v, ff') \]
\[ t(t) = \frac{1}{\sqrt{f^2 v^2 + h u^2}} (f' \dot{v} \cos t_v - f \dot{v} \sin t_v, f' \dot{u} \sin t_v + f \dot{v} \cos t_v, g' \ddot{u}) \]
\[ b(t) = \frac{1}{\sqrt{h (f^2 v^2 + h u^2)}} (-h u \dot{v} \cos t_v - f f' \dot{v} \cos t_v, h \dot{u} \cos t_v - f f' \dot{v} \sin t_v, -f g' \dot{v}) \]
where
\[ f' = \frac{d}{du} f, \quad g' = \frac{d}{du} g, \quad h(t) = f'(u(t))^2 + g'(u(t))^2, \quad \ddot{u}(t) = \frac{d}{dt} \dot{u}(t), \quad \dot{v}(t) = \frac{d}{dt} v(t), \quad t_v = v(t). \]
Moreover, we can calculate that
\[ \kappa_g(t) = \frac{\det(\gamma(t), \dot{\gamma}(t), n(t))}{||\dot{\gamma}(t)||^3} = \frac{1}{\sqrt{h (f^2 v^2 + h u^2)^3}} \left( 2 \dot{u}^2 \dot{v} f^3 + f f'^2 (\dddot{v} - \dddot{u}) + \dot{v} f' \left( f \dddot{v}^2 - \dddot{u} \dddot{f} + 2 \dddot{u} g'^2 \right) + f g' \left( -\dddot{u} \dddot{v} g' - \dddot{u} \dddot{v} g' + \dddot{u} \dddot{v} g' \right) \right) \]
\[ \kappa_n(t) = \frac{\langle \gamma(t), n(t) \rangle}{||\gamma(t)||^2} = \frac{\dddot{u} \left( f f'^2 - f f' g' + f \dot{v} \dddot{g}' \right)}{\sqrt{h \left( \dddot{u}^2 f^2 + f^2 \dddot{v}^2 + \dddot{u}^2 g'^2 \right)}}, \]
\[ \tau_g(t) = \frac{\det(\gamma(t), n(t), \dot{n}(t))}{||\dot{\gamma}(t)||^2} = \frac{\dddot{u} \left( f f'^2 g' - f f' g'' + f \dot{v} \dddot{g}' + g'^3 \right)}{h \left( \dddot{u}^2 f^2 + \dddot{u}^2 g'^2 \right)}. \]
For a meridian curve \( \gamma(u) = X(u, v_0) = (f(u) \cos v_0, f(u) \sin v_0, g(u)) \), we have
\[ \kappa_g(u) \equiv 0, \quad \tau_g(u) \equiv 0, \quad \kappa_n(u) = \frac{f' g'' - f f'' g'}{(f^2 + g'^2)^{3/2}}. \]
In this case the normal developable surface along \( \gamma \) is a normal slice of \( M \) (cf. Fig.13 and 14).
For a parallel circle $\gamma(v) = X(u_0, v) = (f(u_0) \cos v, f(u_0) \sin v, g(u_0))$, we have

$$\kappa_g(v) = \frac{10v}{(v^2 + 2)^2} , \tau_g(v) = 0 , \kappa_n(v) = 1.$$  

Since $n(\gamma(v)) = -\gamma(v)$, the normal developable surface along $\gamma$ is $ND_\gamma(v, w) = w\gamma(v)$. It is the circular cone through the origin (cf. Fig. 15, 16 and 17).

Figure 15: A parallel circle  Figure 16: The circular cone  Figure 17: The union of Fig.15 and 16

References


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