Normal developable surfaces of surfaces along curves

Satoshi HANANOI and Shyuichi IZUMIYA

October 6, 2015

Abstract

We consider a developable surface normal to a surface along a curve on the surface. We call it a *normal developable surface* along the curve on the surface. We investigate the uniqueness and the singularities of such developable surfaces. We discover two new invariants of curves on a surface which characterize these singularities.

1 Introduction

In this paper we consider a curve on a surface in the Euclidean 3-space and a developable surface normal to the surface along the curve. Such a developable surface is called a *normal developable surface along* the curve if it exists. The notion of Darboux frames along curves on surfaces is well known. We have a special direction in the Darboux frame at each point of the curve which is directed by a vector in the rectifying plane of the surface along the curve. We call this vector field a *rectifying Darboux vector* field along the curve. On the other hand, there are three invariants associated with the Darboux frame of a curve on a surface. With a certain condition of those invariants, we can show that there exists a normal developable surface along the curve which is given as the envelope of rectifying spaces of the surface along the curve. It follows that a normal developable surface is a ruled surface whose rulings are directed by the rectifying Darboux vector field along the curve. By using the above invariants, we introduce two new invariants which are related to the singularities of normal developable surfaces. Actually, one of these invariants is constantly equal to zero if and only if the rectifying Darboux vector field has a constant direction which means that the normal developable surface is a cylindrical surface. We give a classification of the singularities of the normal developable surface along a curve on a surface by using those two invariants (Theorem 3.3). In §6 we consider the uniqueness of the normal developable surface. If the uniqueness does not hold, then the curve is a straight line (Corollary 6.3). In §7 we consider special curves on surfaces. If we consider a geodesic on a surface, the normal developable surface is a tangent surface of the curve. We also consider lines of curvatures of a surface. In this case the director curve of the normal developable surface is given by the normal vector field of the surface along the curve. It is known that the ruled surface along a curve on a surface whose director curve is the normal vector field of the surface is a developable surface if and only if the curve is a line

2010 Mathematics Subject classification. Primary 57R45; Secondary 58Kxx
Key Words and Phrases. Normal developable surfaces, Curves on surfaces

of curvature. If we consider a parametrization of a surface such that each coordinate curve is a line of curvature, then the locus of the singular values of normal developable surfaces for all coordinate curves is the focal surface of the surface. We give an example of an ellipsoid in §7.

In [4] the second author and Saki Otani introduced an osculating developable surface of a surface along a curve. Such a developable surface is tangent to the surface along the curve and it gives a flat approximation of the surface along the curve. The method we use in this paper is analogous to the method in the above paper. However, we consider the normal developable surface of the surface along the curve. Therefore, the situations are different.

Throughout this paper all curves, surfaces and maps will be of class C^∞.

2 Basic concepts

We consider a surface $M = \boldsymbol{X}(U)$ given locally by an embedding $\boldsymbol{X} : U \longrightarrow \mathbb{R}^3$, where $\mathbb{R}^3$ is the Euclidean space and $U \subset \mathbb{R}^2$ is an open set. Let $\overline{\boldsymbol{\gamma}} : I \to U$ be an embedding, where $\overline{\boldsymbol{\gamma}}(t) = (u(t), v(t))$ and I is an open interval. Then we have a regular curve $\boldsymbol{\gamma} = \boldsymbol{X} \circ \overline{\boldsymbol{\gamma}} : I \longrightarrow M \subset \mathbb{R}^3$ on the surface M. On the surface, we have the unit normal vector field $\boldsymbol{n}$ defined by

$$\boldsymbol{n}(p) = \frac{\boldsymbol{X}_u \times \boldsymbol{X}_v}{\|\boldsymbol{X}_u \times \boldsymbol{X}_v\|}(u, v),$$

where $p = \boldsymbol{X}(u, v)$. Here, $\boldsymbol{a} \times \boldsymbol{b}$ is the exterior product of $\boldsymbol{a}, \boldsymbol{b}$ in $\mathbb{R}^3$. Since $\boldsymbol{\gamma}$ is a space curve in $\mathbb{R}^3$, we adopt the arc-length parameter as usual and write $\boldsymbol{\gamma}(s) = \boldsymbol{X}(u(s), v(s))$. Then we have the unit tangent vector field $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ of $\boldsymbol{\gamma}(s)$, where $\boldsymbol{\gamma}'(s) = d\boldsymbol{\gamma}/ds(s)$. We have $\boldsymbol{n}_{\boldsymbol{\gamma}}(s) = \boldsymbol{n} \circ \boldsymbol{\gamma}(s)$ which is the unit normal vector field of M along $\boldsymbol{\gamma}$. Moreover, we define $\boldsymbol{b}(s) = \boldsymbol{n}_{\boldsymbol{\gamma}}(s) \times \boldsymbol{t}(s)$. Then we have an orthonormal frame $\{\boldsymbol{t}(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s), \boldsymbol{b}(s)\}$ along $\boldsymbol{\gamma}$, which is called the *Darboux frame* along $\boldsymbol{\gamma}$. Then we have the following Frenet-Serret type formulae:

$$\begin{cases} \boldsymbol{t}'(s) &= \kappa_g(s)\boldsymbol{b}(s) + \kappa_n(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s), \\ \boldsymbol{b}'(s) &= -\kappa_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s), \\ \boldsymbol{n}'_{\boldsymbol{\gamma}}(s) &= -\kappa_n(s)\boldsymbol{t}(s) - \tau_g(s)\boldsymbol{b}(s). \end{cases}$$

By using the matrix representation, we have

$$\begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{b}' \\ \boldsymbol{n}'_{\boldsymbol{\gamma}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{b} \\ \boldsymbol{n}_{\boldsymbol{\gamma}} \end{pmatrix}.$$

Here,

$$\begin{aligned} \kappa_g(s) &= \langle \boldsymbol{t}'(s), \boldsymbol{b}(s) \rangle = \det\left(\boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s)\right), \\ \kappa_n(s) &= \langle \boldsymbol{t}'(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s) \rangle = \langle \boldsymbol{\gamma}''(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s) \rangle, \\ \tau_g(s) &= \langle \boldsymbol{b}'(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s) \rangle = \det(\boldsymbol{\gamma}'(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s), \boldsymbol{n}'_{\boldsymbol{\gamma}}(s)) \end{aligned}$$

and $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ is the canonical inner product of $\mathbb{R}^3$. We call $\kappa_g(s)$ a *geodesic curvature*, $\kappa_n(s)$ a *normal curvature* and $\tau_g(s)$ a *geodesic torsion* of $\boldsymbol{\gamma}$ respectively. It is known that

1) $\boldsymbol{\gamma}$ is an asymptotic curve of M if and only if $\kappa_n = 0$,
2) $\boldsymbol{\gamma}$ is a geodesic of M if and only if $\kappa_g = 0$,
3) $\boldsymbol{\gamma}$ is a principal curve of M if and only if $\tau_g = 0$.

We define a vector field $D_r(s)$ along $\boldsymbol{\gamma}$ by

$$D_r(s) = \tau_g(s)\boldsymbol{t}(s) + \kappa_g(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s),$$

which is called a *rectifying Darboux vector* along $\boldsymbol{\gamma}$. If $\kappa_g^2 + \tau_g^2 \neq 0$, we can define the *spherical rectifying Darboux image* by

$$\overline{D_r}(s) = \frac{\tau_g(s)\boldsymbol{t}(s) + \kappa_g(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s)}{\sqrt{\kappa_g^2(s) + \tau_g^2(s)}}.$$

On the other hand, we briefly review the notions and basic properties of ruled surfaces and developable surfaces. Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ and $\boldsymbol{\xi} : I \longrightarrow \mathbb{R}^3 \setminus \{\boldsymbol{0}\}$ be C^∞-mappings. Then we define a mapping $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})} : I \times \mathbb{R} \longrightarrow \mathbb{R}^3$ by

$$F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}(u, v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\xi}(u).$$

We call the image of $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ a *ruled surface*, the mapping $\boldsymbol{\gamma}$ a *base curve* and the mapping $\boldsymbol{\xi}$ a *director curve*. The line defined by $\boldsymbol{\gamma}(u) + v\boldsymbol{\xi}(u)$ for a fixed $u \in I$ is called a *ruling*. We call the ruled surface with vanishing Gaussian curvature on the regular part a *developable surface*. It is known that a ruled surface $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a developable surface if and only if

$$\det\Big(\dot{\boldsymbol{\gamma}}(u), \boldsymbol{\xi}(u), \dot{\boldsymbol{\xi}}(u)\Big) = 0,$$

where $\dot{\boldsymbol{\gamma}}(u) = (d\boldsymbol{\gamma}/du)(u)$(cf., [3]). If the direction of the director curve $\boldsymbol{\xi}$ is constant, we call $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ a (*generalized*) *cylinder*. If we write $\widetilde{\boldsymbol{\xi}}(u) = \boldsymbol{\xi}(u)/\|\boldsymbol{\xi}(u)\|$, then we have $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}(I \times \mathbb{R}) = F_{(\boldsymbol{\gamma},\widetilde{\boldsymbol{\xi}})}(I \times \mathbb{R})$. In this case $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a cylinder if and only if $\dot{\widetilde{\boldsymbol{\xi}}}(u) \equiv 0$. We say that $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is *non-cylindrical* if $\dot{\widetilde{\boldsymbol{\xi}}}(u) \neq 0$. Suppose that $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is non-cylindrical. Then a *striction curve* is defined to be

$$\boldsymbol{\sigma}(u) = \boldsymbol{\gamma}(u) - \frac{\langle \dot{\boldsymbol{\gamma}}(u), \dot{\widetilde{\boldsymbol{\xi}}}(u)\rangle}{\langle \dot{\widetilde{\boldsymbol{\xi}}}(u), \dot{\widetilde{\boldsymbol{\xi}}}(u)\rangle}\widetilde{\boldsymbol{\xi}}(u).$$

It is known that a singular point of the non-cylindrical ruled surface is located on the striction curve [3]. A non-cylindrical ruled surface $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a *cone* if the striction curve $\boldsymbol{\sigma}$ is constant. In general, a *wave front* in $\mathbb{R}^3$ is a (singular) surface which is a projection image of a Legendrian submanifold in the projective cotangent bundle $\pi : PT^*(\mathbb{R}^3) \longrightarrow \mathbb{R}^3$. It is known (cf., [3]) that a non-cylindrical developable surface $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a wave front if and only if

$$\det\Big(\boldsymbol{\xi}(u), \dot{\boldsymbol{\xi}}(u), \ddot{\boldsymbol{\xi}}(u)\Big) \neq 0.$$

In this case we call $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ a (*non-cylindrical*) *developable front*.

Let $M \subset \mathbb{R}^3$ be a surface. We say that a developable surface N is a *normal developable surface* of M if $N \cap M \neq \emptyset$ and T_pN and T_pM are orthogonal at any point $p \in N \cap M$. If N is a cylinder, we say that N is a *normal cylinder* of M. We also say that N is a *normal cone* of M if N is a cone. For a normal developable surface N of M, the intersection $N \cap M$ is a regular curve. In particular, we say that the intersection $N \cap M$ is a *normal cylindrical slice* if N is a normal cylinder of M. Moreover, $N \cap M$ is said to be a *normal conical slice* if N is a normal cone of M.

3 Normal developable surfaces

In this section we investigate normal developable surfaces of a given surface. For a regular curve $\boldsymbol{\gamma} = \boldsymbol{X} \circ \overline{\boldsymbol{\gamma}} : I \longrightarrow M \subset \mathbb{R}^3$ on a surface M with $\kappa_g^2(s) + \tau_g^2(s) \neq 0$, we define a map $ND_{\boldsymbol{\gamma}} : I \times \mathbb{R} \longrightarrow \mathbb{R}^3$ by

$$ND_{\boldsymbol{\gamma}}(s,u) = \boldsymbol{\gamma}(s) + u\overline{D_r}(s) = \boldsymbol{\gamma}(s) + u\frac{\tau_g(s)\boldsymbol{t}(s) + \kappa_g(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s)}{\sqrt{\kappa_g(s)^2 + \tau_g(s)^2}}.$$

This is a ruled surface and

$$\overline{D_r}' = \left(\kappa_n - \frac{\kappa_g\tau_g' - \kappa_g'\tau_g}{\kappa_g^2 + \tau_g^2}\right)\frac{-\kappa_g\boldsymbol{t} + \tau_g\boldsymbol{n}_{\boldsymbol{\gamma}}}{\sqrt{\kappa_g^2 + \tau_g^2}},$$

so that we have

$$\begin{aligned}\det(\boldsymbol{\gamma}', \overline{D_r}, \overline{D_r}') &= \det\left(\boldsymbol{t}, \frac{\tau_g\boldsymbol{t} + \kappa_g\boldsymbol{n}_{\boldsymbol{\gamma}}}{\sqrt{\kappa_g^2 + \tau_g^2}}, \left(\kappa_n - \frac{\kappa_g\tau_g' - \kappa_g'\tau_g}{\kappa_g^2 + \tau_g^2}\right)\frac{-\kappa_g\boldsymbol{t} + \tau_g\boldsymbol{n}_{\boldsymbol{\gamma}}}{\sqrt{\kappa_g^2 + \tau_g^2}}\right) \\ &= 0.\end{aligned}$$

This means that $ND_{\boldsymbol{\gamma}}(I \times \mathbb{R})$ is a developable surface. We call $ND_{\boldsymbol{\gamma}}$ a *normal developable surface* of M along $\boldsymbol{\gamma}$. Moreover, we introduce two invariants $\delta_r(s), \sigma_r(s)$ of $(M, \boldsymbol{\gamma})$ as follows:

$$\begin{aligned}\delta_r(s) &= \kappa_n(s) - \frac{\kappa_g(s)\tau_g'(s) - \kappa_g'(s)\tau_g(s)}{\kappa_g^2(s) + \tau_g^2(s)}, \\ \sigma_r(s) &= \frac{\tau_g(s)}{\sqrt{\kappa_g^2(s) + \tau_g^2(s)}} + \left(\frac{\kappa_g(s)}{\delta_r(s)\sqrt{\kappa_g^2(s) + \tau_g^2(s)}}\right)', \ (\text{when } \delta_r(s) \neq 0).\end{aligned}$$

By the above calculation, $\delta_r(s) = 0$ if and only if $\overline{D_r}'(s) = \boldsymbol{0}$. We can also calculate that

$$\frac{\partial ND_{\boldsymbol{\gamma}}}{\partial s} \times \frac{\partial ND_{\boldsymbol{\gamma}}}{\partial u} = \left(u\delta_r - \frac{\kappa_g}{\sqrt{\kappa_g^2 + \tau_g^2}}\right)\boldsymbol{b}.$$

Therefore, $(s_0, u_0) \in I \times \mathbb{R}$ is a singular point of $ND_{\boldsymbol{\gamma}}$ if and only if $\delta_r(s_0) \neq 0$ and

$$u_0 = \frac{\kappa_g(s_0)}{\delta_r(s_0)\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}}.$$

If $(s_0, 0)$ is a regular point (i.e., $\kappa_g(s_0) \neq 0$), the normal vector of $ND_{\boldsymbol{\gamma}}$ at $ND_{\boldsymbol{\gamma}}(s_0, 0) = \boldsymbol{\gamma}(s_0)$ is orthogonal to the normal vector of M at $\boldsymbol{\gamma}(s_0)$. This is the reason why we call $ND_{\boldsymbol{\gamma}}$ the normal developable surface of M along $\boldsymbol{\gamma}$. On the other hand, these two invariants characterize normal cylindrical slices and normal conical slices of M, respectively.

Theorem 3.1 *Let $\gamma : I \longrightarrow M \subset \mathbb{R}^3$ be a unit speed curves on M with $\kappa_g^2(s) + \tau_g^2(s) \neq 0$. Then we have the following*:

(A) *The following are equivalent*:

(1) *ND_γ is a cylinder,*

(2) $\delta_r(s) \equiv 0$,

(3) $\boldsymbol{\gamma}$ *is a normal cylindrical slice of* M.

(B) *If* $\delta_r(s) \neq 0$, *then the following are equivalent*:

(1) ND_γ *a cone,*

(2) $\sigma_r(s) \equiv 0$,

(3) $\boldsymbol{\gamma}$ *is a normal conical slice of* M.

Proof. (A) By definition, $ND_{\boldsymbol{\gamma}}$ is a cylinder if and only if $\overline{D_r}(s)$ is constant. Since

$$\overline{D_r}'(s) = \delta_r(s)\frac{-\kappa_n(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s)}{\sqrt{\kappa_g^2(s) + \tau_g^2(s)}},$$

$\overline{D_r}(s)$ is constant if and only if $\delta_r(s) \equiv 0$. Therefore, (1) is equivalent to (2). Suppose that (3) holds. Then there exists a vector $\boldsymbol{k} \in S^2$ such that $\langle \boldsymbol{b}(s), \boldsymbol{k}\rangle \equiv 0$, where $\boldsymbol{k}$ is the director of the normal cylinder. Then there exist $\lambda, \mu \in \mathbb{R}$ such that $\boldsymbol{k} = \lambda\boldsymbol{t}(s) + \mu\boldsymbol{n}_{\boldsymbol{\gamma}}(s)$. Since $\langle \boldsymbol{b}'(s), \boldsymbol{k}\rangle \equiv 0$, we have $-\kappa_g(s)\lambda + \tau_g(s)\mu = 0$, so that we have $\boldsymbol{k} = \overline{D_r}(s)$. Condition (1) holds. It is clear that (1) implies (3).

(B) Condition (1) means that the singular value set of $ND_{\boldsymbol{\gamma}}$ is a constant vector. We consider a vector valued function $\boldsymbol{f}(s)$ defined by

$$\boldsymbol{f}(s) = \boldsymbol{\gamma}(s) - \frac{\kappa_g(s)}{\delta_r(s)\sqrt{\kappa_g^2(s) + \tau_g^2(s)}}\overline{D_r}(s).$$

Then the condition (1) is equivalent to the condition that $\boldsymbol{f}'(s) \equiv 0$. We can calculate that

$$\begin{aligned}
\boldsymbol{f}' &= \boldsymbol{t} + \left(\frac{\kappa_g}{\delta\sqrt{\kappa_g^2 + \tau_g^2}}\right)'\overline{D_r} + \frac{\kappa_g}{\delta_r\sqrt{\kappa_g^2 + \tau_g^2}}\overline{D_r}' \\
&= \boldsymbol{t} + \left(\frac{\kappa_g}{\delta\sqrt{\kappa_g^2 + \tau_g^2}}\right)'\overline{D_r} + \frac{\kappa_g}{\sqrt{\kappa_g^2 + \tau_g^2}}\frac{\kappa_g\boldsymbol{t} + \tau_g\boldsymbol{b}}{\sqrt{\kappa_g^2 + \tau_g^2}} \\
&= \left(\frac{\tau_g}{\sqrt{\kappa_g^2 + \tau_g^2}} + \left(\frac{\kappa_g}{\delta_r\sqrt{\kappa_g^2 + \tau_g^2}}\right)'\right)\overline{D_r} \\
&= \sigma_r\overline{D_o}.
\end{aligned}$$

It follows that (1) and (2) are equivalent. By the definition of the conical slice, (3) means that there exists $\boldsymbol{c} \in \mathbb{R}^3$ such that $\langle \boldsymbol{\gamma}(s) - \boldsymbol{c}, \boldsymbol{b}(s)\rangle \equiv 0$. If (1) holds, then $\boldsymbol{f}(s)$ is constant. For the constant point $\boldsymbol{c} = \boldsymbol{f}(s) \in \mathbb{R}^3$, we have

$$\langle \boldsymbol{\gamma}(s) - \boldsymbol{c}, \boldsymbol{b}(s)\rangle = \langle \boldsymbol{\gamma}(s) - \boldsymbol{f}(s), \boldsymbol{b}(s)\rangle = \left\langle \frac{\kappa_g(s)}{\delta_r(s)\sqrt{\kappa_g^2(s) + \tau_g^2(s)}}\overline{D_r}(s), \boldsymbol{b}(s)\right\rangle = 0.$$

This means that (3) holds. For the converse, by (3), there exists a point $\boldsymbol{c} \in \mathbb{R}^3$ such that $\langle \boldsymbol{\gamma}(s) - \boldsymbol{c}, \boldsymbol{b}(s)\rangle = 0$. Taking the derivative of the both side, we have $0 = \langle \boldsymbol{\gamma}(s) - \boldsymbol{c}, \boldsymbol{b}(s)\rangle' = \langle \boldsymbol{\gamma}(s) - \boldsymbol{c}, -\kappa_g\boldsymbol{t}(s) + \tau_g\boldsymbol{n}_{\boldsymbol{\gamma}}(s)\rangle$. Then there exists $\lambda \in \mathbb{R}$ such that $\boldsymbol{\gamma}(s) - \boldsymbol{c} = \lambda\overline{D_r}(s)$. Taking the derivative again, we have

$$0 = \langle \boldsymbol{\gamma} - \boldsymbol{c}, \boldsymbol{b}\rangle'' = \langle \boldsymbol{t}, -\kappa_g\boldsymbol{t} + \tau_g\boldsymbol{n}_{\boldsymbol{\gamma}}\rangle + \langle \boldsymbol{\gamma} - \boldsymbol{c}, (-\kappa_g\boldsymbol{t} + \tau_g\boldsymbol{n}_{\boldsymbol{\gamma}})'\rangle = \kappa_g + \lambda\delta_r\sqrt{\kappa_g^2 + \tau_g^2}.$$

It follows that

$$\boldsymbol{c} = \boldsymbol{\gamma}(s) - \lambda \overline{D_o}(s) = \boldsymbol{\gamma}(s) + \frac{\kappa_g(s)}{\delta_r(s)\sqrt{\kappa_g^2(s) + \tau_g^2(s)}} \overline{D_r}(s) = \boldsymbol{f}(s).$$

Therefore, $\boldsymbol{f}(s)$ is constant, so that (1) holds. This completes the proof. □

Corollary 3.2 *The normal developable surface $ND_{\boldsymbol{\gamma}}$ is non-cylindrical if and only if $\delta_r(s) \neq 0$.*

We remark that developable surfaces are classified into cylinders, cones or tangent surfaces of space curves (cf., [6]). Hartman and Nirenberg [2] showed that a cylinder is only one non-singular (complete) developable surface. Hence, (complete) tangent surfaces have always singularities. By the results of Theorem 3.1, two invariants $\delta_r(s)$ and $\sigma_r(s)$ might be related to the singularities of normal developable surfaces. Actually, we can classify the singularities of normal developable surfaces of M along curves by using theses two invariants $\delta_r(s)$ and $\sigma_r(s)$.

Theorem 3.3 *Let $\boldsymbol{\gamma} : I \longrightarrow M \subset \mathbb{R}^3$ be a unit speed curve with $\kappa_g^2(s) + \tau_g^2(s) \neq 0$. Then we have the following*:
(1) *The image of the normal developable surface $ND_{\boldsymbol{\gamma}}$ of M along $\boldsymbol{\gamma}$ is non-singular at (s_0, u_0) if and only if*

$$\frac{\kappa_g(s_0)}{\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}} - u_0 \delta_r(s_0) \neq 0.$$

(2) *The image of the normal developable surface $ND_{\boldsymbol{\gamma}}$ of M along $\boldsymbol{\gamma}$ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0, u_0) if*

(i) $\delta_r(s_0) \neq 0, \sigma_r(s_0) \neq 0$ and

$$u_0 = \frac{\kappa_g(s_0)}{\delta_r(s_0)\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}},$$

or

(ii) $\delta_r(s_0) = \kappa_g(s_0) = 0$, $\delta_r'(s_0) \neq 0$ and

$$u_0 \neq -\frac{\kappa_g'(s_0)}{2\kappa_n(s_0)\tau_g'(s_0) + \kappa_n'(s_0)\tau_g(s_0) + \kappa_g''(s_0)},$$

or

(iii) $\delta_r(s_0) = \kappa_g(s_0) = 0$ and $\kappa_g'(s_0) \neq 0$.

We remark that if $\delta_r'(s_0) \neq 0$, then

$$2\kappa_n(s_0)\tau_g'(s_0) + \kappa_n'(s_0)\tau_g(s_0) + \kappa_g''(s_0) \neq 0.$$

(3) *The image of the normal developable surface $ND_{\boldsymbol{\gamma}}$ of M along $\boldsymbol{\gamma}$ is locally diffeomorphic to the swallowtail SW at (s_0, u_0) if $\delta_r(s_0) \neq 0, \sigma_r(s_0) = 0, \sigma_r'(s_0) \neq 0$ and*

$$u_0 = -\frac{\kappa_g(s_0)}{\delta_r(s_0)\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}}.$$

Here, $\boldsymbol{C} \times \mathbb{R} = \{(x_1, x_2, x_3) | {x_1}^2 = {x_2}^3\}$ is the *cuspidal edge* (c.f., Fig.1) and $\boldsymbol{SW} = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the *swallowtail* (c.f., Fig.2).

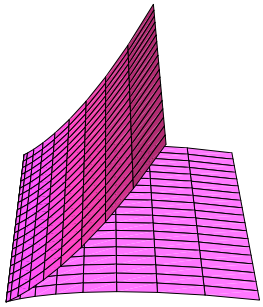

Figure 1: The cuspidal edge

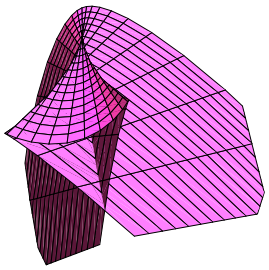

Figure 2: The swallowtail

4 Support functions

In this section we introduce a family of functions on a curve which is useful for the study of invariants of curves on surfaces. For a unit speed curve $\boldsymbol{\gamma} : I \longrightarrow M \subset \mathbb{R}^3$, we define a function $G : I \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ by $G(s, \boldsymbol{x}) = \langle \boldsymbol{x} - \boldsymbol{\gamma}(s), \boldsymbol{b}(s) \rangle$. We call G a *support function* on $\boldsymbol{\gamma}$ with respect to $\boldsymbol{b}$. We write $g_{\boldsymbol{x}_0}(s) = G(s, \boldsymbol{x}_0)$ for any $\boldsymbol{x}_0 \in \mathbb{R}^3$. Then we have the following proposition.

Proposition 4.1 *Let $\boldsymbol{\gamma} : I \longrightarrow M \subset \mathbb{R}^3$ be a unit speed curve with $\kappa_g^2 + \tau_g^2 \neq 0$. Then we have the following*:
(1) $g_{\boldsymbol{x}_0}(s_0) = 0$ *if and only if there exist $u, v \in \mathbb{R}$ such that $\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = u\boldsymbol{t}(s_0) + v\boldsymbol{n}_{\boldsymbol{\gamma}}(s_0)$*
(2) $g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = 0$ *if and only if there exists $u \in \mathbb{R}$ such that*

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = u\frac{\tau_g(s_0)\boldsymbol{t}(s_0) + \kappa_g(s_0)\boldsymbol{n}_{\boldsymbol{\gamma}}(s_0)}{\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}}.$$

(A) *Suppose that $\delta_r(s_0) \neq 0$. Then we have the following*:
(3) $g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = g''_{\boldsymbol{x}_0}(s_0) = 0$ *if and only if*

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = -\frac{\kappa_g(s_0)}{\delta_r(s_0)\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}}\frac{\tau_g(s_0)\boldsymbol{t}(s_0) + \kappa_g(s_0)\boldsymbol{n}_{\boldsymbol{\gamma}}(s_0)}{\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}}. \qquad (*)$$

(4) $g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = g''_{\boldsymbol{x}_0}(s_0) = g^{(3)}_{\boldsymbol{x}}(s_0) = 0$ *if and only if $\sigma_r(s_0) = 0$ and* (*).
(5) $g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = g''_{\boldsymbol{x}_0}(s_0) = g^{(3)}_{\boldsymbol{x}}(s_0) = d^{(4)}_{\boldsymbol{x}_0}(s_0) = 0$ *if and only if*

$$\sigma_r(s_0) = \sigma'_r(s_0) = 0 \text{ and } (*).$$

(B) *Suppose that $\delta_r(s_0) = 0$. Then we have the following*:
(6) $g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = g''_{\boldsymbol{x}_0}(s_0) = 0$ *if and only if*
$\kappa_g(s_0) = 0$ (*i.e.*, $\kappa_g(s_0) = 0, \kappa'_g(s_0) = -\kappa_n(s_0)\tau_g(s_0)$) *and there exists $u \in \mathbb{R}$ such that*

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = u\boldsymbol{t}(s_0).$$

(7) $g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = g''_{\boldsymbol{x}_0}(s_0) = g^{(3)}_{\boldsymbol{x}}(s_0) = 0$ *if and only if one of the following conditions holds*:

(a) $\delta'_r(s_0) \neq 0, \kappa_g(s_0) = 0$

(*i.e.*, $\kappa_g(s_0) = 0, \kappa'_g(s_0) = -\kappa_n(s_0)\tau_g(s_0), 2\kappa_n(s_0)\tau'_g(s_0) + \kappa'_n(s_0)\tau_g(s_0) + \kappa''_g(s_0) \neq 0$) *and*

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = -\frac{\kappa'_n(s_0)}{2\kappa_n(s_0)\tau'_g(s_0) + \kappa'_n(s_0)\tau_g(s_0) + \kappa''_g(s_0)}\boldsymbol{t}(s_0).$$

(b) $\delta'_r(s_0) = 0, \kappa_g(s_0) = \kappa'_g(s_0) = 0$

(*i.e.*, $\kappa_n(s_0) = \kappa_g(s_0) = \kappa'_g(s_0) = 0, \kappa''_g(s_0) = -\kappa'_n(s_0)\tau_g(s_0)$) *and there exists* $u \in \mathbb{R}$ *such that*

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = u\boldsymbol{t}(s_0).$$

Proof. Since $g_{\boldsymbol{x}_0}(s) = \langle \boldsymbol{x}_0 - \boldsymbol{\gamma}(s), \boldsymbol{b}(s)\rangle$, we have the following calculations:

$$\begin{aligned}
&(\alpha) \quad g_{\boldsymbol{x}_0} = \langle \boldsymbol{x}_0 - \boldsymbol{\gamma}, \boldsymbol{b}\rangle,\\
&(\beta) \quad g'_{\boldsymbol{x}_0} = \langle \boldsymbol{x}_0 - \boldsymbol{\gamma}, -\kappa_g\boldsymbol{t} + \tau_g\boldsymbol{n}_{\boldsymbol{\gamma}}\rangle,\\
&(\gamma) \quad g''_{\boldsymbol{x}_0} = \kappa_g + \langle \boldsymbol{x}_0 - \boldsymbol{\gamma}, -(\kappa'_g + \tau_g\kappa_n)\boldsymbol{t} - (\kappa_g^2 + \tau_g^2)\boldsymbol{b} - (\kappa_g\kappa_n + \tau'_g)\boldsymbol{n}_{\boldsymbol{\gamma}}\rangle,\\
&(\delta) \quad g^{(3)}_{\boldsymbol{x}_0} = 2\kappa'_g + \kappa_n\tau_g\\
&\qquad\qquad +\langle \boldsymbol{x}_0 - \boldsymbol{\gamma}, \left(\kappa_g(\kappa_g^2 + \kappa_n^2 + \tau_g^2) + (\kappa'_n\tau_g + 2\kappa_n\tau'_g) - \kappa''_g\right)\boldsymbol{t}\\
&\qquad\qquad -3(\kappa_g\kappa'_g + \tau_g\tau'_g)\boldsymbol{b} + \left(\tau_g(\kappa_g^2 + \kappa_n^2 + \tau_g^2) - (\kappa'_n\kappa_g + 2\kappa_n\kappa_n g) - \tau''_g\right)\boldsymbol{n}_{\boldsymbol{\gamma}}\rangle,
\end{aligned}$$

$$\begin{aligned}
(\epsilon) \quad & g^{(4)}_{\boldsymbol{x}_0} = 3\kappa''_g + 2\kappa'_n\tau_g + 3\kappa_n\tau'_g - \kappa_g(\kappa_g^2 + \kappa_n^2 + \tau_g^2)\\
&+\langle \boldsymbol{x}_0 - \boldsymbol{\gamma}, \left(\kappa'_g(3\kappa_n^2 + \kappa_g^2 + \tau_g^2) + \kappa_g(3\kappa_n\kappa'_n + 5\kappa_g\kappa'_g + 5\tau_g\tau'_g)\right.\\
&\qquad\qquad \left. +\kappa_n\tau_g(\kappa_g^2 + \kappa_n^2 + \tau_g^2) + (\kappa''_n\tau_g + 3\kappa'_n\tau'_g + 3\kappa_n\tau''_g) - \kappa'''_g\right)\boldsymbol{t}\\
&+\left((\kappa_g^2 + \tau_g^2)(\kappa_g^2 + \kappa_n^2 + \tau_g^2) + 2\kappa_n(\kappa'_g\tau_g - \kappa_g\tau'_g) - 3(\kappa'^2_g + \tau'^2_g) - 4(\kappa_g\kappa''_g + \tau_g\tau''_g)\right)\boldsymbol{b}\\
&+\left(-\tau'_g(3\kappa_n^2 + \kappa_g^2 + \tau_g^2) - \tau_g(3\kappa_n\kappa'_n + 5\kappa_g\kappa'_g + 5\tau_g\tau'_g)\right.\\
&\qquad\qquad \left. +\tau_g\kappa_n(\kappa_g^2 + \kappa_n^2 + \tau_g^2) - (\kappa''_n\kappa_g + 3\kappa'_g\kappa'_n + 3\kappa_n\kappa''_g) + \tau'''_g\right)\boldsymbol{n}_{\boldsymbol{\gamma}}\rangle
\end{aligned}$$

By definition and (α), assertion (1) follows.

By (β), $g_{\boldsymbol{x}_0}(s_0) = g'_{\boldsymbol{x}_0}(s_0) = 0$ if and only if $\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = u\boldsymbol{t}(s_0) + v\boldsymbol{n}_{\boldsymbol{\gamma}}(s_0)$ and $-\kappa_g(s_0)u + \tau_g(s_0)v = 0$. If $\kappa_n(s_0) \neq 0$, $\tau_g(s_0) \neq 0$, then we have

$$u = v\frac{\tau_g(s_0)}{\kappa_g(s_0)}, \ v = u\frac{\kappa_g(s_0)}{\tau_g(s_0)},$$

so that there exists $\lambda \in \mathbb{R}$ such that

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = \lambda\frac{\tau_g(s_0)\boldsymbol{t}(s_0) + \kappa_g(s_0)\boldsymbol{n}_{\boldsymbol{\gamma}}(s_0)}{\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}}.$$

Suppose that $\kappa_g(s_0) = 0$. Then we have $\tau_g(s_0) \neq 0$, so that $\tau_g(s_0)v = 0$. Therefore we have

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = u\boldsymbol{t}(s_0) = \pm u\frac{\tau_g(s_0)\boldsymbol{t}(s_0) + \kappa_g(s_0)\boldsymbol{n}_{\boldsymbol{\gamma}}(s_0)}{\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}}.$$

If $\tau_g(s_0)=0$, then we have $\boldsymbol{x}_0-\gamma(s_0)=v\boldsymbol{n}_{\boldsymbol{\gamma}}(s_0)$. Therefore the assertion (2) holds.

By (γ), $g_{\boldsymbol{x}_0}(s_0)=g'_{\boldsymbol{x}_0}(s_0)=g''_{\boldsymbol{x}_0}(s_0)=0$ if and only if

$$\boldsymbol{x}_0-\boldsymbol{\gamma}(s_0)=\lambda\frac{\tau_g(s_0)\boldsymbol{t}(s_0)+\kappa_g(s_0)\boldsymbol{n}_{\boldsymbol{\gamma}}(s_0)}{\sqrt{\kappa_g^2(s_0)+\tau_g^2(s_0)}}.$$

and

$$\kappa_g(s_0)+\lambda\frac{-\tau_g(\kappa_n\tau_g+\kappa_g')-\kappa_g(\kappa_g\kappa_n+\tau_g')}{\sqrt{\kappa_g^2+\tau_g^2}}(s_0)=0.$$

It follows that

$$\frac{\kappa_g}{\sqrt{\kappa_g^2+\tau_g^2}}(s_0)+\lambda\left(\kappa_n-\frac{\kappa_g\tau_g'-\kappa_g'\tau_g}{\kappa_g^2+\tau_g^2}\right)(s_0)=0.$$

Thus,

$$\delta_r(s_0)=\kappa_n(s_0)-\frac{\kappa_g\tau_g'-\kappa_g'\tau_g}{\kappa_g^2+\tau_g^2}(s_0)\neq 0 \text{ and } \lambda=-\frac{\kappa_g}{\delta_r\sqrt{\kappa_g^2+\tau_g^2}}(s_0)$$

or $\delta_r(s_0)=0,\kappa_g(s_0)=0$. This completes the proof of the assertion (A), (3) and (B),(6).

Suppose that $\delta_r(s_0)\neq 0$. By (δ), $g_{\boldsymbol{x}_0}(s_0)=g'_{\boldsymbol{x}_0}(s_0)=g''_{\boldsymbol{x}_0}(s_0)=g^{(3)}_{\boldsymbol{x}_0}(s_0)=0$ if and only if

$$\begin{aligned}2\kappa_g'+\kappa_n\tau_g-\frac{\kappa_g}{\delta_r\sqrt{\kappa_g^2+\tau_g^2}}\Bigg(&\frac{\tau_g}{\sqrt{\kappa_n^2+\tau_g^2}}\left(\kappa_g(\kappa_g^2+\kappa_n^2+\tau_g^2)+(\kappa_n'\tau_g+2\kappa_n\tau_g')-\kappa_g''\right)\\&+\frac{\kappa_g}{\sqrt{\kappa_n^2+\tau_g^2}}\left(\tau_g(\kappa_g^2+\kappa_n^2+\tau_g^2)-(\kappa_n'\kappa_g+2\kappa_g'\kappa_n)-\tau_g''\right)\Bigg) &= 0\end{aligned}$$

at $s=s_0$. It follows that

$$2\kappa_g'(s_0)+\kappa_g(s_0)\tau_g(s_0)-\frac{\kappa_n(s_0)}{\delta_r(s_0)}\left(\kappa_n'+2\kappa_n\frac{\kappa_g\kappa_g'+\tau_g\tau_g'}{\kappa_g^2+\tau_g^2}-\frac{\kappa_g\tau_g''-\kappa_g''\tau_g}{\kappa_g^2+\tau_g^2}\right)(s_0)=0.$$

Since

$$\delta_r'=\kappa_n'-2\frac{(\kappa_g\kappa_g'+\tau_g\tau_g')(\kappa_g\tau_g'-\kappa_g'\tau_g)}{(\kappa_g^2+\tau_g^2)^2}-\frac{\kappa_g\tau_g''-\kappa_g''\tau_g}{\kappa_g^2+\tau_g^2},$$

$$2\kappa_g'(s_0)+\kappa_n(s_0)\tau_g(s_0)-\kappa_g(s_0)\frac{\delta_r'(s_0)}{\delta_r(s_0)}-2\kappa_g(s_0)\frac{\kappa_g(s_0)\kappa_g'(s_0)+\tau_g(s_0)\tau_g'(s_0)}{\kappa_g^2(s_0)+\tau_g^2(s_0)}=0.$$

Moreover, we apply the relation

$$\left(\frac{\kappa_g}{\sqrt{\kappa_g^2+\tau_g^2}}\right)'=\frac{\tau_g}{\sqrt{\kappa_g^2+\tau_g^2}}\frac{\kappa_g\tau_g'-\kappa_g'\tau_g}{\kappa_g^2+\tau_g^2}=\frac{\tau_g}{\sqrt{\kappa_g^2+\tau_g^2}}(\delta_r-\kappa_n)$$

to the above. Then we have

$$\delta(s_0)\sqrt{\kappa_g^2+\tau_g^2}(s_0)\left(\frac{\tau_g}{\sqrt{\kappa_g^2+\tau_g^2}}+\left(\frac{\kappa_g}{\delta\sqrt{\kappa_g^2+\tau_g^2}}\right)'\right)(s_0)=\delta_r(s_0)\sigma_r(s_0)\sqrt{\kappa_g^2+\tau_g^2}(s_0)=0,$$

so that $\sigma_r(s_0)=0$. The converse assertion also holds.

Suppose that $\delta_r(s_0)=0$. Then by (δ), $g_{\boldsymbol{x}_0}(s_0)=g'_{\boldsymbol{x}_0}(s_0)=g''_{\boldsymbol{x}_0}(s_0)=g^{(3)}_{\boldsymbol{x}_0}(s_0)=0$ if and only if $\kappa_g(s_0)=0$ (i.e., $\kappa_g(s_0)=0, \kappa'_g(s_0)=-\kappa_n(s_0)\tau_g(s_0)$), there exists $u\in\mathbb{R}$ such that

$$\boldsymbol{x}_0-\boldsymbol{\gamma}(s_0)=u\boldsymbol{t}(s_0)$$

and

$$2\kappa'_g(s_0)+\kappa_n(s_0)\tau_g(s_0)-u\left(2\kappa_n(s_0)\tau'_g(s_0)+\kappa'_n(s_0)\tau_g(s_0)+\kappa''_g(s_0)\right)=0.$$

Since $\delta_r(s_0)=0$ and $\kappa_g(s_0)=0$, we have $\kappa_n(s_0)\tau_g(s_0)=-\kappa'_g(s_0)$, so that

$$\kappa'_g(s_0)-u(2\kappa_n(s_0)\tau'_g(s_0)+\kappa'_n(s_0)\tau_g(s_0)+\kappa''_g(s_0))=0.$$

It follows that

$$2\kappa_n(s_0)\tau'_g(s_0)+\kappa'_n(s_0)\tau_g(s_0)+\kappa''_g(s_0)\neq 0 \text{ and } u=-\frac{\kappa'_g(s_0)}{2\kappa_n(s_0)\tau'_g(s_0)+\kappa'_n(s_0)\tau_g(s_0)+\kappa''_g(s_0)}$$

or

$$2\kappa_n(s_0)\tau'_g(s_0)+\kappa'_n(s_0)\tau_g(s_0)+\kappa''_g(s_0)=0 \text{ and } \kappa'_g(s_0)=0.$$

Therefore we have (B), (7), (a) or (b).

By the similar arguments to the above, we have the assertion (A), (5). This completes the proof. □

In order to prove Theorem 3.3, we use some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book[1]. Let $F:(\mathbb{R}\times\mathbb{R}^r,(s_0,x_0))\to\mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f, where $f(s)=F_{x_0}(s,x_0)$. We say that f has an *A_k-singularity* at s_0 if $f^{(p)}(s_0)=0$ for all $1\leq p\leq k$, and $f^{(k+1)}(s_0)\neq 0$. Let F be an unfolding of f and $f(s)$ has an A_k-singularity $(k\geq 1)$ at s_0. We write the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s,x_0))(s_0)=\sum_{j=0}^{k-1}\alpha_{ji}(s-s_0)^j$ for $i=1,\ldots,r$. Then F is called an *$\mathcal{R}$-versal unfolding* if the $k\times r$ matrix of coefficients $(\alpha_{ji})_{j=0,\ldots,k-1;i=1,\ldots,r}$ has rank k $(k\leq r)$. We introduce an important set concerning the unfoldings relative to the above notions. The *discriminant set* of F is the set

$$\mathcal{D}_F=\{x\in\mathbb{R}^r|\text{there exists } s \text{ with } F=\frac{\partial F}{\partial s}=0 \text{ at } (s,x)\}.$$

Then we have the following classification (cf., [1]).

Theorem 4.2 *Let $F:(\mathbb{R}\times\mathbb{R}^r,(s_0,x_0))\to\mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has an A_k singularity at s_0. Suppose that F is an $\mathcal{R}$-versal unfolding.*

(1) *If $k=2$, then $\mathcal{D}_F$ is locally diffeomorphic to $C\times\mathbb{R}^{r-1}$.*

(2) *If $k=3$, then $\mathcal{D}_F$ is locally diffeomorphic to $SW\times\mathbb{R}^{r-2}$.*

For the proof of Theorem 3.3, we have the following propositions.

Proposition 4.3 *Let $\boldsymbol{\gamma}:I\longrightarrow M\subset\mathbb{R}^3$ be a unit speed curve with $\kappa_n^2+\tau_g^2\neq 0$ and $G:I\times\mathbb{R}^3\longrightarrow\mathbb{R}$ the support function on γ with respect to $\boldsymbol{b}$. If $g_{\boldsymbol{x}_0}$ has an A_k-singularity $(k=2,3)$ at s_0, then G is an $\mathcal{R}$-versal unfolding of $g_{\boldsymbol{x}_0}$. Here, we assume that $\delta_r(s_0)\neq 0$ for $k=3$.*

Proof. We write that $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{\gamma}(s) = (x_1(s), x_2(s), x_3(s))$ and $\boldsymbol{b}(s) = (b_1(s), b_2(s), b_3(s))$. Then we have

$$G(s, \boldsymbol{x}) = b_1(s)(x_1 - x_1(s)) + b_2(s)(x_2 - x_2(s)) + b_3(s)(x_3 - x_3(s)),$$

so that

$$\frac{\partial G}{\partial x_i}(s, \boldsymbol{x}) = b_i(s), \quad (i = 1, 2, 3).$$

Therefore the 2-jet is

$$j^2 \frac{\partial G}{\partial x_i}(s_0, \boldsymbol{x}_0) = b_i(s_0) + b_i'(s_0)(s - s_0) + \frac{1}{2} b_i''(s_0)(s - s_0)^2.$$

We consider the following matrix:

$$A = \begin{pmatrix} b_1(s_0) & b_2(s_0) & b_3(s_0) \\ b_1'(s_0) & b_2'(s_0) & b_3'(s_0) \\ b_1''(s_0) & b_2''(s_0) & b_3''(s_0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}(s_0) \\ \boldsymbol{b}'(s_0) \\ \boldsymbol{b}''(s_0) \end{pmatrix}.$$

By the Frenet-Serret type formulae, we have

$$\boldsymbol{b}' = -\kappa_g \boldsymbol{t} + \tau_g \boldsymbol{n}_{\boldsymbol{\gamma}} \text{ and } \boldsymbol{b}'' = (\kappa_n \tau_g - \kappa_g')\boldsymbol{t} - (\kappa_n^2 + \tau_g^2)\boldsymbol{b} + (\tau_g' - \kappa_g \kappa_n)\boldsymbol{n}_{\boldsymbol{\gamma}}.$$

Since $\{\boldsymbol{t}, \boldsymbol{b}, \boldsymbol{n}_{\boldsymbol{\gamma}}\}$ is an orthonormal basis of $\mathbb{R}^3$, the rank of

$$A = \begin{pmatrix} \boldsymbol{b}(s_0) \\ -\kappa_g(s_0)\boldsymbol{t}(s_0) + \tau_g(s_0)\boldsymbol{n}_{\boldsymbol{\gamma}}(s_0) \\ (-\kappa_n(s_0)\tau_g(s_0) - \kappa_g'(s_0))\boldsymbol{t}(s_0) - (\kappa_n^2(s_0) + \tau_g^2(s_0))\boldsymbol{b}(s_0) + (\tau_g'(s_0) - \kappa_g(s_0)\kappa_n(s_0))\boldsymbol{n}_{\boldsymbol{\gamma}}(s_0) \end{pmatrix}$$

is equal to the rank of

$$\begin{pmatrix} 0 & 0 & 1 \\ -\kappa_g(s_0) & +\tau_g(s_0) & 0 \\ -\kappa_n(s_0)\tau_g(s_0) - \kappa_g'(s_0) & \tau_g'(s_0) - \kappa_n(s_0)\kappa_g(s_0) & -(\kappa_g^2(s_0) + \tau_g^2(s_0)) \end{pmatrix}.$$

Therefore $\operatorname{rank} A = 3$ if and only if

$$0 \neq -\kappa_g(\tau_g' - \kappa_n \kappa_g) + \tau_g(\kappa_n \tau_g + \kappa_g') = \kappa_n(\kappa_g^2 + \tau_g^2) - (\kappa_g \tau_g' - \kappa_g' \tau_g)$$

st $s = s_0$. The last condition is equivalent to the condition $\delta_r(s_0) \neq 0$. Moreover, the rank of

$$\begin{pmatrix} \boldsymbol{b}(s_0) \\ \boldsymbol{b}'(s_0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}(s_0) \\ -\kappa_g(s_0)\boldsymbol{t}(s_0) + \tau_g(s_0)\boldsymbol{n}_{\boldsymbol{\gamma}}(s_0) \end{pmatrix}$$

is always two.

If $g_{\boldsymbol{x}_0}$ has an A_k-singularity (k=2,3) at s_0, then G is $\mathcal{R}$-versal unfolding of $g_{\boldsymbol{x}_0}$. This completes the proof. □

Proof of Theorem 3.3. By a straightforward calculation, we have

$$\frac{\partial ND_{\boldsymbol{\gamma}}}{\partial s} \times \frac{\partial ND_{\boldsymbol{\gamma}}}{\partial u} = -\left(\frac{\kappa_g}{\sqrt{\kappa_g^2 + \tau_g^2}} + u\delta_r \right) \boldsymbol{n}_{\boldsymbol{\gamma}}.$$

Therefore, (s_0, u_0) is non-singular if and only if

$$\frac{\partial ND_{\boldsymbol{\gamma}}}{\partial s} \times \frac{\partial ND_{\boldsymbol{\gamma}}}{\partial u} \neq \mathbf{0}.$$

This condition is equivalent to

$$\frac{\kappa_g(s_0)}{\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}} + u_0 \delta_r(s_0) \neq 0.$$

This completes the proof of assertion (1).

By Proposition 4.1, (2), the discriminant set $\mathcal{D}_G$ of the support function G of $\boldsymbol{\gamma}$ with respect to $\boldsymbol{b}$ is the image of the normal developable surface of M along $\boldsymbol{\gamma}$.

Suppose that $\delta_r(s_0) \neq 0$. It follows from Proposition 4.1, A, (3), (4) and (5) that $g_{\boldsymbol{x}_0}$ has the A_2-type singularity (respectively, the A_3-type singularity) at $s = s_0$ if and only if

$$u_0 = -\frac{\kappa_g(s_0)}{\delta_r(s_0)\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}}$$

and $\sigma_r(s_0) \neq 0$ (respectively, $\sigma_r(s_0) = 0$ and $\sigma_r'(s_0) \neq 0$). By Theorem 4.2 and Proposition 3.3, we have the assertions (2), (α) and (3).

Suppose that $\delta_r(s_0) = 0$. It follows from Proposition 4.1, B, (6) and (7) that $g_{\boldsymbol{x}_0}$ has the A_3-type singularity if and only if $\delta_r(s_0) = 0, \kappa_g(s_0) = 0$ and

$$\kappa_g'(s_0) \neq 0 \text{ or } \kappa_g'(s_0) + u_0 \left(2\kappa_n(s_0)\tau_g'(s_0) + \kappa_n'(s_0)\tau_g(s_0) + \kappa_g''(s_0)\right) \neq 0.$$

By Theorem 4.3 and Proposition 4.3, we have the assertion (2), (β). This completes the proof. □

5 Invariants of curves on surfaces

In this section we consider geometric meanings of the invariant σ_r. Let $\boldsymbol{\Gamma} : I \longrightarrow \mathbb{R}^3 \times S^2$ be a regular curve and $F : \mathbb{R}^3 \times S^2 \longrightarrow \mathbb{R}$ a submersion. We say that $\boldsymbol{\Gamma}$ and $F^{-1}(0)$ have contact of *at least order* k for $t = t_0$ if the function $g(t) = F \circ \boldsymbol{\Gamma}(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k)}(t_0) = 0$. If $\boldsymbol{\gamma}$ and $F^{-1}(0)$ have contact of at least order k for $t = t_0$ and satisfies the condition that $g^{(k+1)}(t_0) \neq 0$, then we say that $\boldsymbol{\Gamma}$ and $F^{-1}(0)$ have *contact of order* k for $t = t_0$. For any $\boldsymbol{x} \in \mathbb{R}^3$, we define a function $\mathfrak{g}_{\boldsymbol{x}} : \mathbb{R}^3 \times S^2 \longrightarrow \mathbb{R}$ by $\mathfrak{g}_{\boldsymbol{x}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{x} - \boldsymbol{u}, \boldsymbol{v} \rangle$. Then we have

$$\mathfrak{g}_{\boldsymbol{x}}^{-1}(0) = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^3 \times S^2 \mid \langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{x}, \boldsymbol{v} \rangle\}.$$

If we fix $\boldsymbol{v} \in S^2$, then $\mathfrak{g}_{\boldsymbol{x}}^{-1}(0)|\mathbb{R}^3 \times \{\boldsymbol{v}\}$ is an affine plane defined by $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = c$, where $c = \langle \boldsymbol{x}, \boldsymbol{v} \rangle$. Since this plane is orthogonal to $\boldsymbol{v}$, it is parallel to the tangent plane $T_{\boldsymbol{v}}S^2$ at $\boldsymbol{v}$. Here we have a representation of the tangent bundle of S^2 as follows:

$$TS^2 = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^3 \times S^2 \mid \langle \boldsymbol{u}, \boldsymbol{v} \rangle = 1\}.$$

We consider the canonical projection $\pi_2|\mathfrak{g}_{\boldsymbol{x}}^{-1}(0) : \mathfrak{g}_{\boldsymbol{x}}^{-1}(0) \longrightarrow S^2$, where $\pi_2 : \mathbb{R}^3 \times S^2 \longrightarrow S^2$. Then $\pi_2|\mathfrak{g}_{\boldsymbol{x}}^{-1}(0) : \mathfrak{g}_{\boldsymbol{x}}^{-1}(0) \longrightarrow S^2$ is a plane bundle over S^2. Moreover, we define a map Ψ :

$\mathfrak{g}_{\boldsymbol{x}}^{-1}(0) \longrightarrow TS^2$ by $\Phi(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}/\langle \boldsymbol{x}, \boldsymbol{v}\rangle, \boldsymbol{v})$. Then Φ is a bundle isomorphism. Therefore, we write $TS^2(\boldsymbol{x}) = \mathfrak{g}_{\boldsymbol{x}}^{-1}(0)$ and call it an *affine tangent bundle over* S^2 *through* $\boldsymbol{x}$. Let $\gamma : I \longrightarrow M \subset \mathbb{R}^3$ be a unit speed curve on M with $\kappa_g^2(s) + \tau_g^2(s) \neq 0$. Suppose that $\delta_r(s) \neq 0$. By the proof of the assertion (B) of Theorem 3.1, the derivative of the vector valued function $\boldsymbol{f}$ of $ND_{\boldsymbol{\gamma}}$ is $\boldsymbol{f}'(s) = \sigma_r(s)\overline{D}_r(s)$. If we assume that $\sigma_r(s) \equiv 0$, then $\boldsymbol{f}$ is a constant vector $\boldsymbol{x}_0$. Then

$$\boldsymbol{\gamma}(s) - \boldsymbol{x}_0 = \frac{\kappa_g(s)}{\delta_r(s)\sqrt{\kappa_g^2(s) + \tau_g^2(s)}}\overline{D}_r(s).$$

Therefore

$$\mathfrak{g}_{\boldsymbol{x}_0}(\boldsymbol{\gamma}(s), \boldsymbol{b}(s)) = g_{\boldsymbol{x}_0}(s) = \langle \boldsymbol{\gamma}(s) - \boldsymbol{x}_0, \boldsymbol{b}(s)\rangle = 0.$$

If there exists $\boldsymbol{x}_0 \in \mathbb{R}^3$ such that $\mathfrak{g}_{\boldsymbol{x}_0}(\boldsymbol{\gamma}(s), \boldsymbol{b}(s)) = 0$, then we have

$$\boldsymbol{\gamma}(s) - \boldsymbol{x}_0 = \frac{\kappa_g(s)}{\delta_r(s)\sqrt{\kappa_g^2(s) + \tau_g^2(s)}}\overline{D}_r(s).$$

and $\sigma_r(s) \equiv 0$. We consider a regular curve $(\boldsymbol{\gamma}, \boldsymbol{b}) : I \longrightarrow \mathbb{R}^3 \times S^2$. Then we have the following proposition.

Proposition 5.1 *Let $\gamma : I \longrightarrow M \subset \mathbb{R}^3$ be a unit speed curve on M with $\kappa_g^2(s) + \tau_g^2(s) \neq 0$ and $\delta_r(s) \neq 0$. Then there exists $\boldsymbol{x}_0 \in \mathbb{R}^3$ such that $(\boldsymbol{\gamma}, \boldsymbol{b})(I) \subset TS^2(\boldsymbol{x}_0)$ if and only if $\sigma_r(s) \equiv 0$.*

The result of the above proposition explains that the geometric meaning of the singularities of $ND_{\boldsymbol{\gamma}}$ is related not only to the curve as a space curve but also to the behavior of the curve in the surface. Let $\boldsymbol{\gamma} : I \longrightarrow M \subset \mathbb{R}^3$ be a unit speed curve with $\kappa_g^2(s) + \tau_g^2(s) \neq 0$. Then we consider the support function $g_{\boldsymbol{x}_0}(s) = \mathfrak{g}_{\boldsymbol{x}_0}(\boldsymbol{\gamma}(s), \boldsymbol{b}(s))$. By assertion (2) of Proposition 4.1, $(\boldsymbol{\gamma}, \boldsymbol{b})$ is tangent to $TS^2(\boldsymbol{x}_0)$ at $s = s_0$ if and only if $\boldsymbol{x}_0 = ND_{\boldsymbol{\gamma}}(s_0, u_0)$ for some $u_0 \in \mathbb{R}$. Then we have the following proposition.

Proposition 5.2 *Let $\boldsymbol{\gamma} : I \longrightarrow M \subset \mathbb{R}^3$ be a unit speed curve with $\kappa_g^2(s) + \tau_g^2(s) \neq 0$ and $\delta_r(s) \neq 0$. For $\boldsymbol{x}_0 = ND_{\boldsymbol{\gamma}}(s_0, u_0)$, we have the following*:
(1) *The order of contact of $(\boldsymbol{\gamma}, \boldsymbol{b})$ with $TS^2(\boldsymbol{x}_0)$ at $s = s_0$ is two if and only if*

$$u_0 = -\frac{\kappa_g(s_0)}{\delta_r(s_0)\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}}. \tag{**}$$

and $\sigma_r(s_0) \neq 0$.
(2) *The order of contact of $(\boldsymbol{\gamma}, \boldsymbol{b})$ with $TS^2(\boldsymbol{x}_0)$ at $s = s_0$ is three if and only if* (**) *and $\sigma_r(s_0) = 0$ and $\sigma_r'(s_0) \neq 0$.*

Proof. By assertions (3), (4) of Proposition 4.1, the conditions $g_{\boldsymbol{x}_0}(s_0) = g_{\boldsymbol{x}_0}'(s_0) = g_{\boldsymbol{x}_0}''(s_0) = 0$ and $g_{\boldsymbol{x}_0}^{(3)}(s_0) \neq 0$ if and only if (**) and $\sigma_r(s_0) \neq 0$. Since $g_{\boldsymbol{x}_0} = \mathfrak{g}_{\boldsymbol{x}_0} \circ (\boldsymbol{\gamma}, \boldsymbol{b})$, the above condition means that $(\boldsymbol{\gamma}, \boldsymbol{b})$ and $TS^2(\boldsymbol{x}_0)$ have contact of order two at $s = s_0$. For the proof assertion (2), we use assertions (4), (5) of Proposition 4.1 exactly the same way as the above case. □

Therefore the geometric meaning of the classification results of Theorem 3.3 are given as follows.

Theorem 5.3 *Let $\boldsymbol{\gamma} : I \longrightarrow M \subset \mathbb{R}^3$ be a unit speed curve with $\kappa_g^2(s)+\tau_g^2(s) \neq 0$ and $\delta_r(s) \neq 0$.*
(1) The image of the normal developable surface $ND_{\boldsymbol{\gamma}}$ of M along $\boldsymbol{\gamma}$ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0, u_0) if

$$u_0 = -\frac{\kappa_g(s_0)}{\delta(s_0)\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}}$$

and the order of contact of $(\boldsymbol{\gamma}, \boldsymbol{b})$ with $TS^2(\boldsymbol{x}_0)$ at $s = s_0$ is two.
(2) The image of the normal developable surface $ND_{\boldsymbol{\gamma}}$ of M along $\boldsymbol{\gamma}$ is locally diffeomorphic to the swallowtail SW at (s_0, u_0) if

$$u_0 = -\frac{\kappa_g(s_0)}{\delta_r(s_0)\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}}$$

and the order of contact of $(\boldsymbol{\gamma}, \boldsymbol{b})$ with $TS^2(\boldsymbol{x}_0)$ at $s = s_0$ is three.

6 Uniqueness of normal developable surfaces

In this section we consider existence and uniqueness of normal developable surfaces along curves on a surface.

Theorem 6.1 *Let $M \subset \mathbb{R}^3$ be a regular surface and $\gamma : I \longrightarrow M \subset \mathbb{R}^3$ a unit speed curve with $\kappa_g^2(s) + \tau_g^2(s) \neq 0$. Then there exists a unique developable surface which is normal to M along γ.*

Proof. For existence, we have the normal developable surface $ND_{\boldsymbol{\gamma}}$ along $\boldsymbol{\gamma}$. On the other hand, let $\mathcal{N}_\gamma$ be a developable surface which is normal to M along $\boldsymbol{\gamma}$. Since $\mathcal{N}_\gamma$ is a ruled surface, we assume that

$$\mathcal{N}_\gamma(s, u) = \boldsymbol{\gamma}(s) + u\xi(s).$$

Here we write

$$\xi(s) = \lambda(s)\boldsymbol{t}(s) + \mu(s)\boldsymbol{b}(s) + \nu(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s),$$

so that

$$\xi' = (\lambda' - \mu\kappa_g - \nu\kappa_n)\,\boldsymbol{t} + (\mu' + \lambda\kappa_g - \nu\tau_g)\,\boldsymbol{b} + (\nu' + \lambda\kappa_n + \mu\tau_g)\,\boldsymbol{n}_{\boldsymbol{\gamma}}.$$

Since $\mathcal{N}_\gamma$ is a developable surface, we have $\det(\boldsymbol{\gamma}',\ \xi,\ \xi') = 0$. The last condition is equivalent to

$$det \begin{pmatrix} \boldsymbol{t} \\ \lambda\boldsymbol{t} + \mu\boldsymbol{b} + \nu\boldsymbol{n}_{\boldsymbol{\gamma}} \\ (\lambda' - \mu\kappa_g - \nu\kappa_n)\,\boldsymbol{t} + (\mu' + \lambda\kappa_g - \nu\tau_g)\,\boldsymbol{b} + (\nu' + \lambda\kappa_n + \mu\tau_g)\,\boldsymbol{n}_{\boldsymbol{\gamma}} \end{pmatrix} = 0,$$

which is also equivalent to

$$\mu\,(\nu' + \lambda\kappa_n + \mu\tau_g) - \nu\,(\mu' + \lambda\kappa_g - \nu\tau_g) = 0. \tag{1}$$

On the other hand, since $\mathcal{N}_\gamma$ is a developable surface which is normal to M along γ, we have

$$\frac{\partial \mathcal{N}_\gamma}{\partial s}(s, u) \times \frac{\partial \mathcal{N}_\gamma}{\partial u}(s, u) \ = \ \theta(s, u)\boldsymbol{b}(s). \tag{2}$$

If $\mathcal{N}_\gamma$ is nonsingular at $(s,0)$, then $\theta(s,0)\neq 0$. By a straightforward calculation, we have

$$\begin{aligned}\frac{\partial \mathcal{N}_\gamma}{\partial s} &= \boldsymbol{\gamma}' + u\xi' \\ &= (1+u(\lambda'-\mu\kappa_g-\nu\kappa_n))\,\boldsymbol{t} + u(\mu'+\lambda\kappa_g-\nu\tau_g)\,\boldsymbol{b} + u(\nu'+\lambda\kappa_n+\mu\tau_g)\,\boldsymbol{n}_{\boldsymbol{\gamma}}\end{aligned}$$

$$\frac{\partial \mathcal{N}_\gamma}{\partial u} = \xi = \lambda\boldsymbol{t}+\mu\boldsymbol{b}+\nu\boldsymbol{n}_{\boldsymbol{\gamma}},$$

so that

$$\begin{aligned}\frac{\partial \mathcal{N}_\gamma}{\partial s}\times\frac{\partial \mathcal{N}_\gamma}{\partial u} &= u(\mu(\nu'+\lambda\kappa_n+\mu\tau_g)-\nu(\mu'+\lambda\kappa_g-\nu\tau_g))\,\boldsymbol{t} \\ &\quad + (u\lambda(\nu'+\lambda\kappa_n+\mu\tau_g)-\nu\{1+u(\mu'+\lambda\kappa_g-\nu\tau_g)\})\,\boldsymbol{b} \\ &\quad + (\mu\{1+u(\lambda'-\mu\kappa_g-\nu\kappa_n)\}-u\lambda(\mu'+\lambda\kappa_g-\nu\tau_g))\,\boldsymbol{n}_{\boldsymbol{\gamma}}.\end{aligned}$$

If we substitute $u=0$, then

$$\frac{\partial \mathcal{N}_\gamma}{\partial s}\times\frac{\partial \mathcal{N}_\gamma}{\partial u} = -\nu\boldsymbol{b}+\mu\boldsymbol{n}_{\boldsymbol{\gamma}}.$$

By (2), we have $-\nu(s)=\theta(s,0)$, $\mu(s)=0$. It also follows from (1) that

$$\nu(s)\,(\lambda(s)\kappa_g(s)-\nu(s)\tau_g(s))=0.$$

Suppose that $\mathcal{N}_\gamma$ is non-singular along $\boldsymbol{\gamma}$. Then $\theta(s,0)\neq 0$, so that $\nu(s)\neq 0$. Thus, we have $\lambda(s)\kappa_g(s)-\nu(s)\tau_g(s)=0$. If $\kappa_g(s)\neq 0$, then $\lambda(s)=\frac{\tau_g(s)}{\kappa_g(s)}\nu(s)$. Therefore, we have

$$\begin{aligned}\xi(s) &= \frac{\tau_g(s)}{\kappa_g(s)}\nu(s)\boldsymbol{t}(s)+\nu(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s) \\ &= \nu(s)\frac{\sqrt{\kappa_g^2(s)+\tau_g^2(s)}}{\kappa_g(s)}\left(\frac{\tau_g(s)\boldsymbol{t}(s)+\kappa_g(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s)}{\sqrt{\kappa_g^2(s)+\tau_g^2(s)}}\right) \\ &= \nu(s)\frac{\sqrt{\kappa_g^2(s)+\tau_g^2(s)}}{\kappa_g(s)}\overline{D}_r(s).\end{aligned}$$

This means that the direction of $\xi(s)$ is equal to the direction of $\overline{D}_r(s)$. If $\tau_g(s)\neq 0$, we have the same result as the above case. On the other hand, suppose that $\mathcal{N}_\gamma$ has a singular point at $(s_0,0)$. Then $\theta(s_0,0)=0$. It follows that $\nu(s_0)=0,\mu(s_0)=0$. Thus we have $\xi(s_0)=\lambda(s_0)\boldsymbol{t}(s_0)$. If the singular point $\boldsymbol{\gamma}(s_0)$ is in the closure of the set of points where the normal deveoplable is regular on $\boldsymbol{\gamma}$, then there exists a point s in any neighborhood of s_0 such that the uniqueness of the normal developable surface holds at s. Passing to the limit $s\to s_0$, uniqueness of the normal developable surface holds at s_0. Suppose that there exists an open interval $J\subset I$ such that $\mathcal{N}_\gamma$ is singular at $\boldsymbol{\gamma}(s)$ for any $s\in J$. Then $\mathcal{N}_\gamma(s)=\gamma(s)+u\lambda(s)\boldsymbol{t}(s)$ for any $s\in J$. This means that $\mu(s)=\nu(s)=0$ for $s\in J$. It follows that

$$\frac{\partial \mathcal{N}_\gamma}{\partial s}\times\frac{\partial \mathcal{N}_\gamma}{\partial u}(s,u)=u\lambda^2(s)(\kappa_n(s)\boldsymbol{b}(s)-\kappa_g(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s)).$$

The above vector is directed to $\boldsymbol{b}(s)$, so that $\kappa_g(s)=0$ for any $s\in J$. In this case, $\overline{D}_r(s)=\pm\boldsymbol{t}(s)$. This means that uniqueness holds. □

Proposition 6.2 *Let $\boldsymbol{\gamma} : I \longrightarrow M$ be a regular curve with $\kappa_g(s) \equiv \tau_g(s) \equiv 0$. Then $\mathcal{N}_\gamma$ is a normal developable along $\boldsymbol{\gamma}$ if and only if $\boldsymbol{\gamma}$ is a normal slice of M.*

Proof. We remark that the torsion of the curve $\boldsymbol{\gamma}$ as a space curve is

$$\tau = \tau_g + \frac{\kappa_g \kappa_n' - \kappa_g' \kappa_n}{\kappa_g^2 + \tau_g^2}.$$

If $\kappa_g(s) \equiv \tau_g(s) \equiv 0$, then $\tau \equiv 0$, so that $\boldsymbol{\gamma}$ is a plane curve. Moreover, we have $\boldsymbol{b}' = -\kappa_g \boldsymbol{t} + \tau_g \boldsymbol{n}_\gamma \equiv 0$. Thus, $\mathcal{N}_\gamma$ is a plane normal to M. Since $\boldsymbol{\gamma}$ is the intersection of M and $\mathcal{N}_\gamma$, $\boldsymbol{\gamma}$ is a normal slice of M. For the converse, if $\boldsymbol{\gamma}$ is a normal slice of M, then there exists a plane Π such that $\boldsymbol{\gamma}(I) = M \cap \Pi$ and $\boldsymbol{n}_\gamma(s), \boldsymbol{t}(s) \in \Pi$ for any $s \in I$. Therefore, Π is orthogonal to $\boldsymbol{b}(s)$ for any $s \in I$. Since Π is a plane (i.e. a developable surface), Π is a normal developable surface of M along $\boldsymbol{\gamma}$. □

Corollary 6.3 *Let $\boldsymbol{\gamma} : I \longrightarrow M \subset \mathbb{R}^3$ be a unit speed curve on M. If there are two normal developable surfaces along $\boldsymbol{\gamma}$, then $\boldsymbol{\gamma}$ is a straight line.*

Proof. Under the assumption of $\kappa_g^2 + \tau_g^2 \neq 0$, the normal developable surface along $\boldsymbol{\gamma}$ is unique by Theorem 6.1. If $\kappa_g \equiv 0$ and $\tau_g \equiv 0$, $\boldsymbol{\gamma}$ is a normal slice. In this case, a normal plane Π of M at $\boldsymbol{\gamma}(s_0)$ is an normal developable surface along $\boldsymbol{\gamma}$. If there is another normal developable surface $\mathcal{N}_\gamma$ along $\boldsymbol{\gamma}$, then $\mathcal{N}_\gamma$ is tangent to Π along $\boldsymbol{\gamma}$. This means that Π is a tangent plane of the developable surface $\mathcal{N}_\gamma$. By definition, Π is tangent to $\mathcal{N}_\gamma$ along a ruling of $\mathcal{N}_\gamma$, which is $\boldsymbol{\gamma}$. Thus $\boldsymbol{\gamma}$ is a line. If $\kappa_g = \tau_g = 0$ at a point s_0 in the closure of the set of points where $\kappa_g^2 + \tau_g^2 \neq 0$, then there exists a point s in any neighborhood of s_0 such that the uniqueness of the normal developable surface holds at s. Passing to the limit $s \to s_0$, uniqueness of the normal developable surface holds at s_0. This completes the proof. □

7 Special curves on a surface

In this section we consider special curves on a surface.

7.1 Geodesics

We consider geodesics on surfaces. Let $\boldsymbol{\gamma} : I \longrightarrow M$ be a unit speed curve on a suface $M \subset \mathbb{R}^3$. It is a geodesic of M if and only if $\kappa_g \equiv 0$. Therefore, if $\tau_g \neq 0$, then

$$ND_{\boldsymbol{\gamma}}(s, u) = \boldsymbol{\gamma}(s) + u\boldsymbol{t}(s),$$

which is known as the *tangent surface* of $\boldsymbol{\gamma}$. In this case

$$\delta_r(s) = \kappa_n(s) \ , \ \sigma_r(s) = \pm 1 \ , \ {\sigma_r}'(s) \equiv 0.$$

By Theorem 3.3, we have the following proposition.

Proposition 7.1 *Let $\boldsymbol{\gamma} : I \longrightarrow M$ be a geodesic on a surface M with $\tau_g(s) \neq 0$. Then we have the following:*

1. *ND_γ is a regular surface at (s_0, u_0) if $u_0 \neq 0$,*

2. *The image of $(ND_\gamma, (s_0, u_0))$ is locally diffeomorphic to the cuspidal edge if*

   (a) *$\kappa_n(s_0) \neq 0$ and $u_0 = 0$, or*

   (b) *$\kappa_n(s_0) = 0$ and $u_0 \left(2\tau_g'\kappa_n + \tau_g\kappa_n'\right)(s_0) \neq 0$.*

Since $\sigma_r{}'(s) \equiv 0$, swallowtails never appear. We now consider the following example.

Example 7.2 We consider a surface parametrized by

$$\boldsymbol{X}(t,u) = (t\cos(t), t\sin(t), 5t) + u\left(\frac{\cos(t) - t\sin(t)}{\sqrt{t^2+26}}, \frac{\sin(t) + t\cos(t)}{\sqrt{t^2+26}}, \frac{5}{\sqrt{t^2+26}}\right).$$

This surface is a ruled surface such that the base curve is $\boldsymbol{\gamma}(t) = (t\cos(t), t\sin(t), 5t)$. Thus, $\boldsymbol{\gamma}$ is a regular curve on the surface $M = \operatorname{Im}\boldsymbol{X}$. It follows that

$$\begin{aligned}
\dot{\boldsymbol{\gamma}}(t) &= (\ -2\sin(t) - t\cos(t)\ ,\ 2\cos(t) - t\sin(t)\ ,\ 0\ ) \\
\frac{\partial \boldsymbol{X}}{\partial t} \times \frac{\partial \boldsymbol{X}}{\partial u} &= \frac{\alpha(t) + \beta(t,u)}{(t^4 + 29t^2 + 104)^{7/2}}\Big(\ \left(t^2+52\right)\sin(t) + t\left(t^2+27\right)\cos(t)\ , \\
&\qquad t\left(t^2+27\right)\sin(t) - \left(t^2+52\right)\cos(t)\ ,\ 5t\ \Big),
\end{aligned}$$

where

$$\begin{aligned}
\alpha(t) &= t^{12} + 87t^{10} + 2835t^8 + 42485t^6 + 294840t^4 + 940992t^2 + 1124864, \\
\beta(t,u) &= u\left(31460t\sqrt{t^4+29t^2+104} - 5t^7\sqrt{t^4+29t^2+104}\right. \\
&\qquad \left. -320t^5\sqrt{t^4+29t^2+104} - 5850t^3\sqrt{t^4+29t^2+104}\ \right).
\end{aligned}$$

In this case the Darboux frame of $\boldsymbol{\gamma}$ is

$$\begin{aligned}
\boldsymbol{n}_\gamma(t) &= \frac{\boldsymbol{X}_t \times \boldsymbol{X}_u}{\|\boldsymbol{X}_t \times \boldsymbol{X}_u\|} \\
&= \frac{1}{t^6 + 55t^4 + 858t^2 + 2704}\Big(\ \left(t^2+52\right)\sin(t) + t\left(t^2+27\right)\cos(t)\ , \\
&\qquad t\left(t^2+27\right)\sin(t) + \left(-t^2-52\right)\cos(t)\ ,\ 5t\ \Big), \\
\boldsymbol{t}(t) &= \frac{\dot{\boldsymbol{\gamma}}(t)}{\left\|\dot{\boldsymbol{\gamma}}(t)\right\|} = \left(\frac{\cos(t) - t\sin(t)}{\sqrt{t^2+26}}, \frac{\sin(t) + t\cos(t)}{\sqrt{t^2+26}}, \frac{5}{\sqrt{t^2+26}}\right), \\
\boldsymbol{b}(t) &= \boldsymbol{n}_\gamma(t) \times \boldsymbol{t}(t).
\end{aligned}$$

Since $\ddot{\boldsymbol{\gamma}}(t) = (-2\sin(t) - t\cos(t), 2\cos(t) - t\sin(t), 0)$, we have

$$\kappa_g(t) = \frac{\det\left(\dot{\boldsymbol{\gamma}}(t), \ddot{\boldsymbol{\gamma}}(t), \boldsymbol{n}_\gamma(t)\right)}{\|\dot{\boldsymbol{\gamma}}(t)\|^3} = 0,$$

so that $\boldsymbol{\gamma}$ is a geodesic of M. Moreover, we have

$$\kappa_n(t) = \frac{\langle \ddot{\boldsymbol{\gamma}}(t), \boldsymbol{n}_\gamma(t)\rangle}{\|\dot{\boldsymbol{\gamma}}(t)\|^2} = -\frac{1}{\left(t^2+26\right)^2},\ \kappa_n'(t) = \frac{4t}{\left(t^2+26\right)^3},$$

$$\tau_g(t) = \frac{\det\left(\dot{\boldsymbol{\gamma}}(t), \boldsymbol{n}_{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{n}}_{\boldsymbol{\gamma}}(t)\right)}{\|\dot{\boldsymbol{\gamma}}(t)\|^2} = \frac{5\left(t^2+6\right)}{\left(t^2+26\right)\left(t^4+29t^2+104\right)^2},$$

$$\tau_g'(t) = -\frac{20t\left(2t^6+83t^4+950t^2+3484\right)}{\left(t^2+26\right)^2\left(t^4+29t^2+104\right)^3}.$$

We can draw the pictures of γ and M in Fig. 3,4,5.

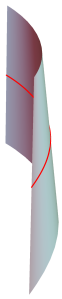

Figure 3: M and γ

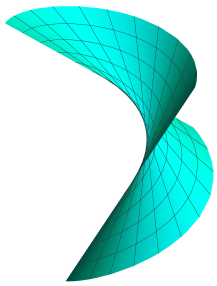

Figure 4: $ND_{\boldsymbol{\gamma}}$

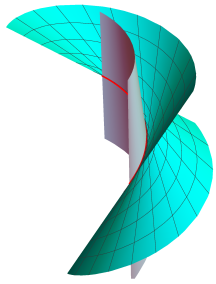

Figure 5: M and $ND_{\boldsymbol{\gamma}}$

7.2 Lines of curvature

We consider lines of curvature. Let $\gamma : I \longrightarrow M$ be a unit speed line of curvature of M with $\kappa_g(s) \neq 0$. Since $\tau_g(s) \equiv 0$, we have

$$ND_{\boldsymbol{\gamma}}(s,u) = \boldsymbol{\gamma}(s) + u\boldsymbol{n}_{\boldsymbol{\gamma}}(s)$$

and $\delta_r(s) = \kappa_n(s)$. Moreover, if $\kappa_n(s) \neq 0$, then $\sigma_r(s) = -\dfrac{\kappa_n'}{\kappa_n^2}(s)$, $\sigma_r{}'(s) = \dfrac{2{\kappa_n'}^2 - \kappa_n''\kappa_n}{\kappa_n^3}(s)$. Then we have the following proposition as a corollary of Theorem 3.3.

Proposition 7.3 *Let $\boldsymbol{\gamma} : I \longrightarrow M$ be a line of curvature on a surface M with $\kappa_g(s) \neq 0$. Then we have the following*:
1) *$ND_{\boldsymbol{\gamma}}$ is a regular surface at (s_0, u_0) if $u_0\kappa_n(s_0) \neq \pm 1$.*
2) *$ND_{\boldsymbol{\gamma}}$ is locally diffeomorphic to the cuspidal edge at (s_0, u_0) if $\kappa_n(s_0) \neq 0$ and $u_0 = \pm\dfrac{1}{\kappa_n(s_0)}$.*
3) *$ND_{\boldsymbol{\gamma}}$ is locally diffeomorphic to the swallowtail at (s_0, u_0) if $\kappa_n(s_0) \neq 0, \kappa_n'(s_0) = 0, \kappa_n''(s_0) \neq 0$ and $u_0 = \pm\dfrac{1}{\kappa_n(s_0)}$.*

Example 7.4 We now consider an ellipsoid as an example. The following parametrization of an ellipsoid is known. Let $\boldsymbol{X}(u,v) = (x(u,v), y(u,v), z(u,v))$ be a surface in $\mathbb{R}^3$ defined by

$$x(u,v) = \pm\sqrt{\frac{a(a-u)(a-v)}{(a-b)(a-c)}}, y(u,v) = \pm\sqrt{\frac{b(b-u)(b-v)}{(b-c)(b-a)}}, z(u,v) = \pm\sqrt{\frac{c(c-u)(c-v)}{(c-a)(c-b)}},$$

where $0 < c \leq v \leq b \leq u \leq a$. We can easily show that the image of $\boldsymbol{X}$ is an ellipsoid defined by $\dfrac{x^2}{a} + \dfrac{y^2}{b} + \dfrac{z^2}{c} = 1$. Moreover, it is classically known that each u-curve and v-curve are

lines of curvature, respectively. However, we do not know the references, so that we can show as follows: For a u-curve defined by $\boldsymbol{\gamma}(u) = \boldsymbol{X}(u, v_0) = (x(u, v_0), y(u, v_0), z(u, v_0))$, the unit normal vector along $\boldsymbol{\gamma}$ is

$$\boldsymbol{n}_{\boldsymbol{\gamma}}(u) = \frac{\boldsymbol{X}_u \times \boldsymbol{X}_v}{\|\boldsymbol{X}_u \times \boldsymbol{X}_v\|}(u, v_0) = \begin{pmatrix} -\dfrac{y_u(u, v_0)z_u(u, v_0)(b-c)(u-v_0)}{\omega_u(u, v_0)(u-b)(u-c)(v_0-b)(v_0-c)} \\ -\dfrac{z_u(u, v_0)x_u(u, v_0)(c-a)(u-v_0)}{\omega_u(u, v_0)(u-a)(u-c)(v_0-a)(v_0-c)} \\ -\dfrac{x_u(u, v_0)y_u(u, v_0)(a-b)(u-v_0)}{\omega_u(u, v_0)(u-a)(u-b)(v_0-a)(v_0-b)} \end{pmatrix},$$

where

$$\omega(u, v) = \sqrt{-\frac{uv(u-v)^2}{(a-u)(a-v)(u-b)(b-v)(u-c)(c-v)}}.$$

Since

$$\dot{\boldsymbol{\gamma}}(u) = \left(\frac{a(a-v_0)}{2x_u(u, v_0)(a-b)(a-c)}, \frac{b(b-v_0)}{2y_u(u, v_0)(b-a)(b-c)}, \frac{c(c-v_0)}{2z_u(u, v_0)(c-a)(c-b)} \right),$$

$$\ddot{\boldsymbol{\gamma}}(u) = \left(\frac{a^2(a-v_0)^2}{4x_u(u, v_0)^{3/2}(a-b)^2(a-c)^2}, \frac{b^2(b-v_0)^2}{4y_u(u, v_0)^{3/2}(b-a)^2(b-c)^2}, \frac{c^2(c-v_0)^2}{4z_u(u, v_0)^{3/2}(c-a)^2(c-b)^2} \right),$$

we have $\tau_g(u) = \dfrac{\det(\dot{\boldsymbol{\gamma}}(u), \boldsymbol{n}_{\boldsymbol{\gamma}}(u), \dot{\boldsymbol{n}}_{\boldsymbol{\gamma}}(u))}{\|\dot{\boldsymbol{\gamma}}(u)\|^2} = 0$. This means that each u-curve is a line of curvature. By the same calculation as the above, we can show that each v-curve is a line of curvature.

Suppose that $a = 20, b = 10, c = 5$. Then we can draw the pictures of the ellipsoid, the u-curve for $v = 7$ and the normal developable surface along the u-curve with $v = 7$.

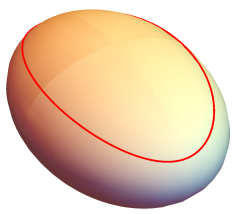

Figure 6: The ellipsoid and the u-curve

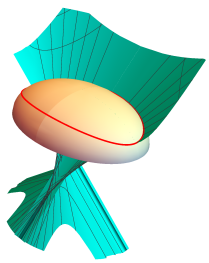

Figure 7: The ellipsoid and the normal developable along the u-curve

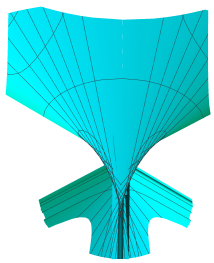

Figure 8: The normal developable along the u-curve

We can observe that there appears the swallowtail on the normal developable surface. We also have

$$\kappa_n(u) = \frac{\langle \ddot{\boldsymbol{\gamma}}(u), \boldsymbol{n}_{\boldsymbol{\gamma}}(u) \rangle}{\|\dot{\boldsymbol{\gamma}}(u)\|^2} = -\frac{bcx_u(u, v_0)(u-v_0)}{4uy_u(u, v_0)z_u(u, v_0)(a-u)(a-v_0)(b-c)}$$

and show that the singular point is the swallowtail. We can also show that the normal developable surface along a v-curve has the swallowtail. Since the director of the normal developable

surface along a line of curvature $\boldsymbol{\gamma}$ is $\boldsymbol{n}_{\boldsymbol{\gamma}}$, if we consider all u-curves and v-curves, then we have the evolute as the trajectory of the singular value sets of corresponding normal developable surfaces. We can draw the picture of the trajectory of the singular value sets of normal developable surfaces along u-curves in Fig. 9, which gives one of the branch of the evolute of the ellipsoid. The other branch of the evolute is depicted in Fig. 10, which is given by the normal developable suffices along v-curves.

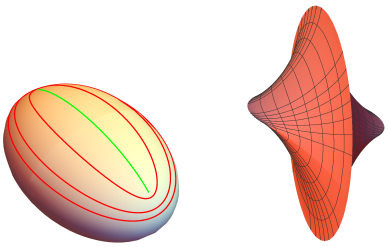

Figure 9: u-curves and the trajectory of the singular values

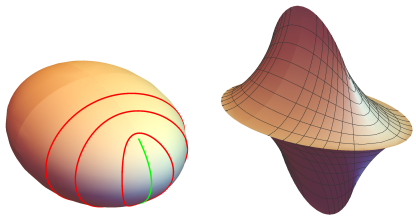

Figure 10: v-curves and trajectory of the singular values

The picture of the union of two branches of the evolute is drawn in Fig. 11. It is known that the purse $\boldsymbol{PS}$ appears as a singular point of the evolute of the ellipsoid [5]. Here

$$\boldsymbol{PS} = \{(3u^2 + uv, 3v^2 + wu, w) \in \mathbb{R}^3 \mid w^2 = 36uv\ \}.$$

Moreover, the purse corresponds to an umbilical point of the ellipsoid. We can observe the purse in Fig.12.

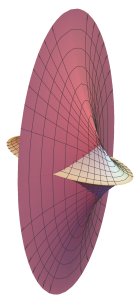

Figure 11: The union of the Fig.9 and Fig. 10

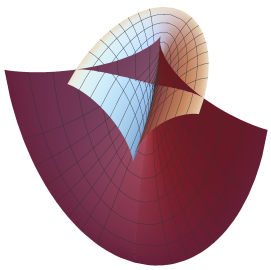

Figure 12: The purse of the evolute

8 Curves on a surface of revolution

In this section we consider curves on surfaces of revolution. A *surface of revolution* is defined to be

$$\boldsymbol{X}(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$$

for $(u, v) \in U \subset \mathbb{R}^2$. We assume that $f(u) \neq 0$. It is easy to show that the unit normal vector field alone $M = \boldsymbol{X}(U)$ is

$$\boldsymbol{n}(u, v) = \left(-\frac{f(u)g'(u)\cos v}{\sqrt{f(u)^2\left(f'(u)^2 + g'(u)^2\right)}}, -\frac{f(u)g'(u)\sin v}{\sqrt{f(u)^2\left(f'(u)^2 + g'(u)^2\right)}}, \frac{f(u)f'(u)}{\sqrt{f(u)^2\left(f'(u)^2 + g'(u)^2\right)}} \right).$$

For a curve

$$\boldsymbol{\gamma}(t) = (\ f(u(t))\cos v(t)\ ,\ f(u(t))\sin v(t)\ ,\ g(u(t))\ )$$

on M, the Darboux frame is given by

$$\boldsymbol{n}_{\boldsymbol{\gamma}}(t) = \frac{1}{\sqrt{f^2 h}}\left(-fg'\cos t_v\ ,\ \ -fg'\sin t_v\ ,\ ff'\right)$$

$$\boldsymbol{t}(t) = \frac{1}{\sqrt{f^2\dot{v}^2 + h\dot{u}^2}}\left(f'\dot{u}\cos t_v - f\dot{v}\sin t_v\ ,\ f'\dot{u}\sin t_v + f\dot{v}\cos t_v\ ,\ g'\dot{u}\right)$$

$$\boldsymbol{b}(t) = \frac{1}{\sqrt{h\left(f^2\dot{v}^2 + h\dot{u}^2\right)}}\left(-h\dot{u}\sin t_v - ff'\dot{v}\cos t_v\ ,\ h\dot{u}\cos t_v - ff'\dot{v}\sin t_v\ ,\ \ -fg'\dot{v}\right),$$

where

$$f' = \frac{d}{du}f\ ,\ \ g' = \frac{d}{du}g\ ,\ h(t) = f'(u(t))^2 + g'(u(t))^2, \dot{u}(t) = \frac{d}{dt}u(t),\ \dot{v}(t) = \frac{d}{dt}v(t)\ ,\ t_v = v(t).$$

Moreover, we can calculate that

$$\begin{aligned}
\kappa_g(t) &= \frac{\det\left(\dot{\boldsymbol{\gamma}}(t), \ddot{\boldsymbol{\gamma}}(t), \boldsymbol{n}(t)\right)}{\|\dot{\boldsymbol{\gamma}}(t)\|^3} \\
&= \frac{1}{\sqrt{h\left(f^2\dot{v}^2 + h\dot{u}^2\right)^3}}\Big(2\dot{u}^2\dot{v}f'^3 + ff'^2\left(\dot{u}\ddot{v} - \ddot{u}\dot{v}\right) + \dot{v}f'\left(f\left(f\dot{v}^2 - \dot{u}^2 f''\right) + 2\dot{u}^2 g'^2\right), \\
&\qquad\qquad + fg'\left(-\dot{u}^2\dot{v}g'' - \ddot{u}\dot{v}g' + \dot{u}\ddot{v}g'\right)\Big) \\
\kappa_n(t) &= \frac{\langle\ddot{\boldsymbol{\gamma}}(t), \boldsymbol{n}(t)\rangle}{\|\dot{\boldsymbol{\gamma}}(t)\|^2} = \frac{\dot{u}^2\left(f'g'' - f''g'\right) + f\dot{v}^2 g'}{\sqrt{h}\left(\dot{u}^2 f'^2 + f^2\dot{v}^2 + \dot{u}^2 g'^2\right)}, \\
\tau_g(t) &= \frac{\det\left(\dot{\boldsymbol{\gamma}}(t), \boldsymbol{n}(t), \dot{\boldsymbol{n}}_{\boldsymbol{\gamma}}(t)\right)}{\|\dot{\boldsymbol{\gamma}}(t)\|^2} = \frac{\dot{u}\dot{v}\left(ff''g' - ff'g'' + f'^2 g' + g'^3\right)}{h\left(\dot{u}^2 f'^2 + f^2\dot{v}^2 + \dot{u}^2 g'^2\right)}.
\end{aligned}$$

For a meridian curve $\boldsymbol{\gamma}(u) = \boldsymbol{X}(u, v_0) = (f(u)\cos v_0, f(u)\sin v_0, g(u))$, we have

$$\kappa_g(u) \equiv 0\ ,\ \tau_g(u) \equiv 0\ ,\ \kappa_n(u) = \frac{f'g'' - f''g'}{\left(f'^2 + g'^2\right)^{3/2}}.$$

In this case the normal developable surface along γ is a normal slice of M (cf. Fig.13 and 14).

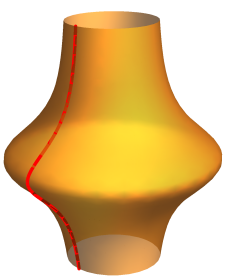

Figure 13: A meridian of M

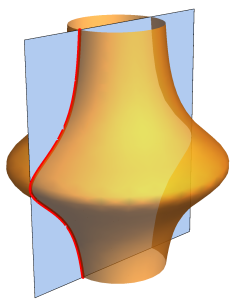

Figure 14: The normal slice of M

For a parallel circle $\boldsymbol{\gamma}(v) = \boldsymbol{X}(u_0, v) = (f(u_0)\cos v, f(u_0)\sin v, g(u_0))$, we have

$$\kappa_g(v) = \frac{10v}{(v^2+2)^2} \ , \ \tau_g(v) = 0 \ , \ \kappa_n(v) = 1.$$

Since $\boldsymbol{n}_{\boldsymbol{\gamma}}(v) = -\boldsymbol{\gamma}(v)$, the normal developable surface along $\boldsymbol{\gamma}$ is $ND_{\boldsymbol{\gamma}}(v, w) = w\boldsymbol{\gamma}(v)$. It is the circular cone through the origin (cf. Fig. 15, 16 and 17).

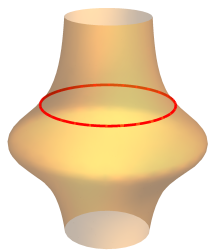

Figure 15: A parallel circle

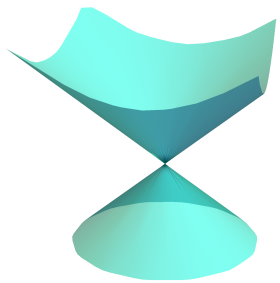

Figure 16: The circular cone

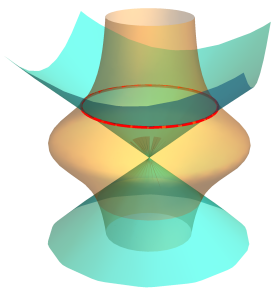

Figure 17: The union of Fig.15 and 16

References

[1] J. W. Bruce and P. J. Giblin, *Curves and singularities (second edition).* Cambridge University press (1992)

[2] P. Hartman and L. Nirenberg, *On spherical image maps whose Jacobians do not change sign.* Amer. J. Math. **81** (1959), 901–920

[3] S. Izumiya and N. Takeuchi, *Geometry of ruled surfaces*, Applicable math. in the golden age (2003) 305-338.

[4] S. Izumiya and S. Otani, *Flat approximations of surfaces along curves*, DEMONSTRATIO MATHEMATICA **XLVIII** (2015), DOI: 10.1515/dema-2015-0016

[5] I. R. Porteous, *Geometric differentiation for the intelligence of curves and surfaces (second edition).* Cambridge University press (2001)

[6] I. Vaisman, *A first course in differential geometry.* Pure and applied Mathematics, A series of Monograph and Textbooks, MARCEL DEKKER, (1984)

Shyuichi Izumiya, Department of Mathematics, Hokkaido University, Sapporo 060-0810,Japan
e-mail:`izumiya@math.sci.hokudai.ac.jp`

Satoshi Hananoi, Fudooka high school, 1-7-45, Fudooka, Kazo, Saitama 347-8513, Japan
e-mail:`noi.87noi@gmail.com`