A rigorous definition of nonlinear parameter $\gamma$ and effective area $A_{\text{eff}}$ for photonic crystal optical waveguides

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1. INTRODUCTION

Nonlinear optical effects, including second-order and third-order nonlinearities, are promising optical effects for the high-speed and low-energy operation of optical communication devices. Second-order nonlinearities, such as second-harmonic generation (SHG) [12] and Pockels effect [3,4] in the photonic crystal (PC) structure have been reported. Third-order nonlinearities, such as self-phase modulation (SPM) [5–14], cross-phase modulation (XPM) [8,9,12–14], two-photon absorption (TPA) [8–10,12–14], three-photon absorption (ThPA) [8,12,14,16], and four-wave mixing (FWM) [8–10,13–15] have received attention as being key for all-optical signal processing. For the evaluation of third-order nonlinear phenomena in optical waveguides, a nonlinear parameter $\gamma$, which is defined in the nonlinear Schrödinger equation (NLSE), has been widely used [8,14,16]. $\gamma$ is mainly related to SPM, whereas some of the third-order nonlinear phenomena, including XPM, ThPA and FWM, have also been evaluated by $\gamma$ [12,14,16,17]. There are two ways to evaluate $\gamma$: (i) calculate the nonlinear phase shift by directly solving the nonlinear Maxwell’s equation self-consistently using the nonlinear mode solver [18,19] and (ii) calculate the effective area, $A_{\text{eff}}$, of the “linear” optical waveguide and use the definition $\gamma = k_0p/\sqrt{A_{\text{eff}}}$ [14], where $k_0$ is the free-space wavenumber and $n_2$ is the nonlinear refractive index. The method (i) seems to be physically more rigorous than method (ii) and has been used to evaluate the nonlinear phase shift of uniform and periodic optical waveguides. However, iteration is necessary to obtain converged solutions, resulting in huge computational cost. Therefore, method (ii) has been widely used to evaluate $\gamma$ because only linear guided mode analysis is necessary. This method has been used for uniform structures, such as optical fibers and Si-nanowire waveguides, and has also been adapted to non-uniform structures, such as photonic crystal waveguides. In PC waveguides, because the waveguide structure varies along the propagation direction in one period of the waveguide, $A_{\text{eff}}$ is also varied along the propagation direction. To evaluate $\gamma$ of PC waveguides, variations of $A_{\text{eff}}$ along the propagation direction have to be carefully treated. Many definitions of $\gamma$ for PC waveguides have been reported, as shown in Table 1 (Methods B, C, and D). The methods are different each other in terms of the definition of $A_{\text{eff}}$ and the use of slowdown factor $S$ [20]. In [12], the weakly guiding approximation was used for $A_{\text{eff}}$ (Method B). In [15], $A_{\text{eff}}$ is calculated by the modal volume [21,22], which was originally defined for the resonator analysis and has no direct relation to NLSE (Method C). In [23] (Method D-1) and [24] (Method D-2), so-called slowdown factor $S$ is multiplied to $\gamma$ obtained by Methods B and C. Because the $\gamma$ values calculated by these methods are different, it is not certain whether these assumptions are appropriate for HIC periodic waveguides. Hence, a rigorous formalism is required to accurately evaluate $\gamma$, especially for PC waveguides.
A. Definition of the proposed and previous definitions of vectorial formulation for HIC periodic waveguides. We first describe nonlinear mode solver and the experiment for all three waveguide and dispersion engineered PC waveguides using the three-dimensional we compare the values of form [16] and apply equations of [16] to the case of periodic optical waveguides. Here, we review the derivation of γ presented in Section 3, it is demonstrated that γ values calculated by the proposed definition agree well with those obtained by the nonlinear mode solver and the experiment for all three waveguide structures (weakly guiding, strongly guiding, and strongly guiding periodic waveguides), showing the validity of the definition.

2. THEORY AND EVALUATION FORMULA

Here, we review the derivation of γ from NLSE in full-vectorial form [16] and apply equations of [16] to the case of periodic optical waveguides. Additionally, conventionally used γ definitions for periodic optical waveguides are briefly summarized.

A. Definition of γ by Vectorial-Based NLSE.

In this paper, we propose a new definition of γ based on the vectorial formulation for HIC periodic waveguides. We first describe the proposed and previous definitions of γ briefly in Section 2. Next, we compare the values of γ obtained by the proposed definition and conventional definitions for an optical fiber, a Si-nanowire waveguide, and dispersion engineered PC waveguides using the three-dimensional (3D) finite element method (FEM) [19]. From the numerical results presented in Section 3, it is demonstrated that γ values calculated by the proposed definition agree well with those obtained by the nonlinear model solver and the experiment for all three waveguide structures (weakly guiding, strongly guiding, and strongly guiding periodic waveguides), showing the validity of the definition.

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\[
\frac{\partial A}{\partial z} + \beta_n \frac{\partial A}{\partial t} - \frac{1}{2} \beta_n \frac{\partial^2 A}{\partial t^2} = -j \gamma |A|^2 A, \tag{3}
\]

where \(\beta_n\) is the nth order differential value of β. The nonlinear parameter γ is defined as

\[
\gamma = \frac{k_0 \varepsilon_0}{\mu_0} \int \int \int \text{Re} \left[ e \times \mathbf{h}^* \right] \cdot \mathbf{i} \, dxdy, \tag{4}
\]

where \(\varepsilon_0\) and \(\mu_0\) are the permittivity and permeability in vacuum, \(n_0\) is the refractive index, \(e\) denotes the complex conjugate, and \(i\) is the unit vector directed along the z-direction. Re[ ] stands for the real part. As seen in Eq. (3), γ indicates the amount of nonlinear phase shift per propagation length and per propagation power. If a waveguide contains only one nonlinear material that has a large nonlinear refractive index \(n_2\), Eq. (4) can be rewritten with the effective area \(A_{\text{eff}}\) as

\[
\gamma = \frac{k_0 n_2}{A_{\text{eff}}}, \tag{5}
\]

\[
A_{\text{eff}} = \frac{\mu_0}{\varepsilon_0} \int \int |\text{Re} \left[ e \times \mathbf{h}^* \right] \cdot \mathbf{i}|^2 dxdy, \tag{6}
\]

where the description "NL" denotes the area of the nonlinear material.

We now consider non-uniform periodic waveguides, including PC waveguides, which have variations of the transverse field distributions \(e\) and \(h\) along the z-direction. Therefore, the nonlinear parameter γ should be expressed as a function of \(z\). Figures 1 (a) and (b) show the schematics of a line-defect PC waveguide and the electric field distribution |\(e|\) in the \(xz\)-plane and \(xy\)-plane, respectively. As seen in Figs. 1 (a) and (b), \(|e| \) distributions at \(z = \text{constant}\) and \(z = a/2\) are remarkably different, where \(a\) is one period length of the PC waveguides. Figures 1 (c) and (d) show the \(A_{\text{eff}}\) and γ of the PC waveguides shown in Fig. 1(a) calculated by (6) and (5) as a function of \(z\), respectively. The dependence of \(A_{\text{eff}}\) and γ is clearly seen, and careful treatment of the nonlinear phase shift variations is required. In the PC waveguides, we can consider that the nonlinear phase shift is periodically modulated along the z-direction. From the physical meaning of γ, the total amount of the nonlinear phase shift in one period of PC waveguides can be expressed as

\[
\Delta \varphi = P \int_{z=a}^{z=a+a} \gamma(z) \, dz = P a \gamma^{\text{PC}}, \tag{7}
\]

where \(P\gamma^{\text{PC}}\) is equivalent to the mean value of \(\gamma(z)\) [25]. Using Eq. (4) as \(\gamma(z)\), we find the new definition of the nonlinear parameter for HIC periodic optical waveguides as

\[
\gamma^{\text{PC}} = \frac{1}{a} \int_{z=a}^{z=a+a} \left[ \frac{k_0 \varepsilon_0}{\mu_0} \int \int \text{Re} \left[ e \times \mathbf{h}^* \right] \cdot \mathbf{i} \, dxdy \right] \, dz
\]

\[
= a \int_{z=a}^{z=a+a} \left[ \frac{k_0 \varepsilon_0}{\mu_0} \int \int \text{Re} \left[ e \times \mathbf{h}^* \right] \cdot \mathbf{i} \, dxdy \right] \, dz
\]
where we use the relationship
\[ a = \int_{z}^{z+\alpha} dz \]
and the optical power is assumed constant. If the waveguide contains only one nonlinear material that has a large nonlinear refractive index \( n_2 \), Eq. (8) can be rewritten as
\[ \gamma_{PC} = \frac{k_0 n_2}{\mathcal{A}_{eff}}, \]
For uniform structures, Eqs. (8), (10), and (11) are reduced to Eqs. (4), (5), and (6), respectively. Eq. (8) seems to be the most rigorous evaluation formula for the \( \gamma \) of PC waveguides. If we apply the weakly guiding approximation to Eq. (8), it is reduced to one of the definitions of \( \gamma \) in previous works \[12,23\], as shown in following Section 2-B.

**B. \( \gamma \) Calculated by \( \mathcal{A}_{eff} \) with the Weakly Guiding Approximation.**

Applying the weakly guiding approximation to Eq. (6), we can obtain the well-known definition of the nonlinear parameter and effective area as \[14\]
\[ \gamma = \frac{k_0 n_2}{\mathcal{A}_{eff}}, \]

\[ \tilde{\mathcal{A}}_{eff} = \frac{\int_{z}^{z+\alpha} \int_{\mathcal{V}_{clad}} |\mathbf{e}|^2 dx dy dz}{\int_{z}^{z+\alpha} \int_{\mathcal{V}_{core}} |\mathbf{e}|^2 dx dy dz}, \]
where a tilde denotes applying the weakly guiding approximation. We also apply the weakly guiding approximation to Eq. (11) to obtain the previously reported nonlinear parameter and effective area as follows \[12,23\]:
\[ \gamma_{PC} = \frac{k_0 n_2}{\mathcal{A}_{PC}^{eff}}, \]

\[ \tilde{\mathcal{A}}_{PC}^{eff} = \frac{1}{a} \left[ \frac{\int_{z}^{z+\alpha} \int_{\mathcal{V}_{clad}} |\mathbf{e}|^2 dx dy dz}{\int_{z}^{z+\alpha} \int_{\mathcal{V}_{core}} |\mathbf{e}|^2 dx dy dz} \right]. \]

For uniform structures, Eqs. (14) and (15) are reduced to Eqs. (12) and (13), respectively.

**C. \( \gamma \) Calculated by the Modal Volume.**

In some works, the modal volume \( V_{eff} \) for optical resonators is used to estimate the \( \mathcal{A}_{eff} \) of PC waveguides \[15,24\]. The nonlinear parameter using \( V_{eff} \) is given as \[15\]
\[ \gamma_{V}^{eff} = \frac{k_0 n_2}{\mathcal{A}_{V}^{eff}}. \]

**D. Effective Nonlinear Parameter \( \gamma^{eff} \) with Slowdown Factor.**

In addition to the above expressions for the nonlinear parameter, other works mention that the slowdown factor \( S \) \[20\], defined as the ratio of the phase velocity over the group velocity, which is calculated as \( n_g/n_0 \), where \( n_g \) is group index, is required to estimate \( \gamma \) of PC waveguides \[11,23,24,26\]. In \[20\], it was pointed out that the value of \( \gamma \) increased with \( n_g \), so \( S \) \((1 < S < 2)\) should be multiplied to \( \gamma \) for periodic optical waveguides, leading to the definition of the effective nonlinear parameter as
\[ \gamma_{PC}^{eff} = \frac{k_0 n_2}{\mathcal{A}_{PC}^{eff}} S^{\frac{1}{2}}, \]
Table 2. Summary of $\gamma$ and $A_{\text{eff}}$ obtained by various methods for optical fibers

<table>
<thead>
<tr>
<th>Calculation Method</th>
<th>Value of $\gamma$ [km W$^{-1}$] $[A_{\text{eff}}$ [µm$^2$]]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method A (proposed)</td>
<td>1.16* (76.7) 1.70* (62.0)</td>
</tr>
<tr>
<td>Method B</td>
<td>1.16* (77.1) 1.69* (62.3)</td>
</tr>
<tr>
<td>Method C</td>
<td>2.30* (38.8) 3.23* (32.6)</td>
</tr>
<tr>
<td>Method E</td>
<td>1.16* (76.7) 1.70* (62.0)</td>
</tr>
<tr>
<td>Method F (measured) [27]</td>
<td>1.09 - - -</td>
</tr>
</tbody>
</table>

* $n_2$ is assumed as a range of 2.2×10$^{-20}$ m$^2$W$^{-1}$

\[ \gamma_{\text{eff}} = k_c n_2^2 S^\xi. \]  

In Eqs. (18) and (19), Eqs. (15) and (17) are used for $A_{\text{eff}}$, and $\xi$ is usually 2 [11,23,24,26]. In this paper, $\xi = 2$ unless otherwise noted.

E. Calculation by Nonlinear Modal Analysis.

We proposed a method to directly estimate the nonlinear parameter using the 3D-FEM for nonlinear periodic optical waveguides, based on the self-consistent algorithm [19]. The nonlinear parameter and effective area obtained by the nonlinear modal analysis are defined as

\[ \gamma_{\text{NLmode}} = \frac{\beta_n - \beta_l}{P}, \]  

\[ A_{\text{eff,mode}} = \frac{k_c n_2^2}{\gamma_{\text{NLmode}}}, \]

where $\beta_l$ is the linear propagation constant, and $\beta_n$ is the nonlinear propagation constant for optical power $P$. For sufficiently small $P$, the nonlinear phase shift is proportional to $P$, and we can obtain constant $\gamma$. Although this method seems to give a physically correct $\gamma$, iterations are necessary to obtain converged solutions.

F. Estimation of $\gamma$ by Experimental Value.

To estimate the nonlinear parameter and effective area from experimental results, conventionally, the following equation has been used [14]:

\[ \gamma_{\text{measured}} = \frac{\Delta \phi_{\text{max}}}{P_m L_{\text{eff}}}, \]

where $\Delta \phi_{\text{max}}$ is the observed maximum nonlinear phase shift by SPM, $P_m$ is the input optical power, and $L_{\text{eff}}$ is the effective length given as $[1-\exp(-L\alpha)]$ with the attenuation coefficient $\alpha$ and propagation length $L$. The number of observed peaks $m$ by SPM-induced spectral broadening is related to $\Delta \phi_{\text{max}} = (m - 0.5)\pi$ [5].

In the following, we compare the $\gamma$ values of various waveguides calculated by the above five methods, namely, Method A in Section 2-A, Method B in Section 2-B, Method C in Section 2-C, Methods D-1 and D-2 in Section 2-D, and Method E in Section 2-E with measured values and show that our proposed Method A and the nonlinear modal analysis in Section 2-E give reasonable results.

3. NUMERICAL ANALYSIS AND DISCUSSION

We demonstrate the validity of our proposed method by calculating the nonlinear parameter $\gamma$ of an optical fiber, a Si-nanowire, and dispersion engineered PC waveguides. To avoid confusion, we call each method “Method A, B, C, D-1, D-2, E and F”, and the descriptions and equations are summarized in Table 1. The 3D finite-element mode-solver [19] developed in our laboratory is used. For Methods A, B, C, D-1, and D-2, the analysis can also be done by using commercial software. Although the optical fiber and Si-wire waveguide are uniform along the

<table>
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<th>Calculation Method</th>
<th>Value of $\gamma$ [km W$^{-1}$] $[A_{\text{eff}}$ [µm$^2$]]</th>
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</thead>
<tbody>
<tr>
<td>Method A (proposed)</td>
<td>190~910* (0.067)</td>
</tr>
<tr>
<td>Method B</td>
<td>63~300* (0.20)</td>
</tr>
<tr>
<td>Method C</td>
<td>220~1100* (0.076)</td>
</tr>
<tr>
<td>Method E</td>
<td>190~910* (0.067)</td>
</tr>
<tr>
<td>Method F (measured) [7]</td>
<td>200~2100* -</td>
</tr>
</tbody>
</table>

* $n_2$ is assumed as a range of 3.0×10$^{-19}$ to 1.45×10$^{-18}$ m$^2$W$^{-1}$
The structural parameters are the following: the core width is 2s = 0.18μm and the height of the waveguide is 2h = 0.5μm, respectively. The values calculated by Method A and E agree well with those by Method B because the optical fiber is a weakly guiding structure. The calculated A_eff values by Methods A, B and E are approximately 80 μm^2 at λ_0 = 2π/k_0 = 1.550 μm, which is the A_eff value of the standard step-index fiber. However, the calculated A_eff value by Method C is half of those by Methods A, B and E. The values calculated by Methods A, B and E agree well with the measured value [27], indicating the validity of Methods A and E. Here, Method D is omitted because S is almost 1.

Next, we examine the Si-nanowire waveguide, as shown in Fig. 3. The structural parameters are the following: the core width is σ = 470 nm, the core height is h = 226 nm and the refractive indices of Si (core), SiO2 (lower cladding), and air (upper cladding) are n_{core} = 3.5, n_{clad} = 1.45, and n_{air} = 1.0. We assume a range of n_2 for Si from 3.0×10^{-18} to 14.5×10^{-18} m^2 W^{-1} [10], and n_2 of the cladding materials is neglected. In Table 3, the γ and A_eff values obtained by various methods are summarized. The values calculated by Method A differ from those by Method B because the waveguide is strongly guiding (Δ = 41%) and the weakly guiding approximation cannot hold. In contrast to the case of the optical fiber, the value calculated by Method C is close to that by Method A for the Si-nanowire. Therefore, Method B is not appropriate for strongly guiding structures, and the γ values calculated by Methods A, C and E agree well with the measured value [7]. Here, Method D is also omitted because S ≈ 1.

Finally, we investigate the dispersion engineered PC waveguide [28–30], as shown in Fig. 4(a). The waveguide is designed to enhance the nonlinearity by causing flat-n_2 dispersion. The structural parameters are as follows: the lattice constant is a = 440 nm, the hole radius is r = 0.35a, the thickness of the waveguide is t = 0.5a, the amount of the shift of the third-row holes along the z-direction is s = 0.18a or 0.20a, and the effective indices of the Si (core) and air (cladding) are n_{core} = 3.5 and n_{air} = 1.0, respectively. We assume a range of n_2 for Si from 3.0×10^{-18} to 14.5×10^{-18} m^2 W^{-1} [10], and n_2 of the cladding material is neglected. Figures 4(b) and (c) show the effective index A_eff and the group index n_{eff} as a function of aλ. By introducing hole shift, a flat-n_2 band is formed, as reported in [28]. The purple solid lines in the graphs denote the flat band region, in which the value of n_{eff} is assumed as a range of 3.0×10^{-18} to 14.5×10^{-18} m^2 W^{-1}.
changes ±10%. In Table 4, the γ, γ_eff, and A_eff obtained by various methods of the PC waveguide for s = 0.20a are summarized. There are two frequencies for n_g = 36 when s = 0.20, as shown in Fig. 4 (c). For each frequency, γ has its range determined by the range of n_g. The maximum and minimum γ values are employed for the range of γ in Table 4.

The calculated γ value by Method A is approximately 200 times larger than that by Method B and 30 times larger than that by Method C. As in the previous two examples, the results obtained by Methods A and E agree well, and they are consistent with the measured values. Methods B and C are not appropriate for evaluating the γ of PC waveguides.

In the following we discuss the results obtained by Methods D-1 and D-2 because the slowdown factor is very large (S = 10.29) and γ_eff is 100 (∼S) times larger than γ. Figure 5 shows that γ/γ_eff and γ_eff/γ as a function of n_eff obtained by various methods of dispersion engineered PC waveguides for s = 0.18a (n_g = 51) and 0.20a (n_g = 36). In this graph, numerical values only for the flat band region are displayed (corresponding to the purple solid lines in Fig. 4). Methods A, D-1, D-2, E, and F have roughly n^2_eff dependence, which implies that Method B and C have no dependence on n_eff because γ_eff dependence is not introduced. As seen in Fig. 5, the calculated values obtained by Methods A, E and D-2 roughly agree with the measured values. γ_eff calculated by Method D-1 (A_eff is obtained by Method B) is relatively smaller than the measured values. In Method D, ξ = 2 is conventionally used to estimate the maximum nonlinearity where 1 < ξ < 2 is a fitting parameter. When ξ ≈ 1.5, the calculated γ_eff values obtained by Method D-2 fit the γ values obtained Methods A and E.

Although Method D-2 gives reasonable γ_eff compared with the measured values for dispersion engineered PC waveguides, the value of ξ is usually unknown and requires fitting from measured values. Contrary to this, our proposed Methods A and E give reasonable γ without fitting parameters, such as slowdown factor S. Furthermore, from the results in this section, only Methods A and E give reasonable γ compared with the experiments for all three waveguide structures, showing the validity and versatility of the proposed γ and A_eff definitions. Additionally, Method A does not require the iteration of Method E, and the required computational cost is lower than that of Method E.

4. CONCLUSION

In this paper, we have proposed a definition for the nonlinear parameter γ for HC periodic optical waveguides, and the γ values obtained by the proposed and conventional definitions are compared with those obtained by the nonlinear guided mode analysis and the experiment for three waveguide structures. Our proposed γ (Method A) obtained from the derivation of vectorial-based NLSE is rigorous due to the exclusion of the weakly guiding approximation and an appropriate consideration of the variations of A_eff along the propagation direction. For the dispersion engineered PC waveguides analyzed here, the value of γ obtained by using the weakly guiding approximation (Method B) is 200 times smaller than that of the proposed definition. The results obtained by the proposed definition and the nonlinear guided mode analysis agree well, and the proposed method does not require iteration because only linear guided mode analysis is necessary. Furthermore, our proposed γ is also in good agreement with the experimental values for all three waveguides, as shown in Section 3. These results indicate that our proposed γ and A_eff are indispensable for HC periodic waveguides and strongly encourage the use of Eq. (8) or Eqs. (10) and (11) for the appropriate calculation of nonlinear characteristics in future research.

References


