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# THE PRIMITIVE EQUATIONS IN THE SCALING INVARIANT SPACE $L^\infty(L^1)$

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ABSTRACT. Consider the primitive equations on  $\mathbb{R}^2 \times (z_0, z_1)$  with initial data  $a$  of the form  $a = a_1 + a_2$ , where  $a_1 \in BUC_\sigma(\mathbb{R}^2; L^1(z_0, z_1))$  and  $a_2 \in L^\infty_\sigma(\mathbb{R}^2; L^1(z_0, z_1))$  and where  $BUC_\sigma(L^1)$  and  $L^\infty_\sigma(L^1)$  denote the space of all solenoidal, bounded uniformly continuous and all solenoidal, bounded functions on  $\mathbb{R}^2$ , respectively, which take values in  $L^1(z_0, z_1)$ . These spaces are scaling invariant and represent the anisotropic character of these equations. It is shown that, if  $\|a_2\|_{L^\infty_\sigma(L^1)}$  is sufficiently small, then this set of equations has a unique, local, mild solution. If in addition  $a$  is periodic in the horizontal variables, then this solution is a strong one and extends to a unique, global, strong solution. The primitive equations are thus strongly and globally well-posed for these data. The approach depends crucially on mapping properties of the hydrostatic Stokes semigroup in the  $L^\infty(L^1)$ -setting and can thus be seen as the counterpart of the classical iteration schemes for the Navier-Stokes equations for the situation of the primitive equations.

## 1. INTRODUCTION

The primitive equations for ocean and atmospheric dynamics serve as a fundamental model for many geophysical flows. This set of equations describing the conservation of momentum and mass of a fluid, assuming hydrostatic balance of the pressure, is given in the isothermal setting by

$$(1.1) \quad \begin{cases} \partial_t v + u \cdot \nabla v - \Delta v + \nabla_H \pi = 0, & \text{in } \Omega \times (0, T), \\ \partial_z \pi = 0, & \text{in } \Omega \times (0, T), \\ \operatorname{div} u = 0, & \text{in } \Omega \times (0, T), \\ v(0) = a. \end{cases}$$

Here  $\Omega := \mathbb{R}^2 \times J$ , where  $J = (z_0, z_1)$  is an interval. Denoting the horizontal coordinates by  $x, y \in \mathbb{R}^2$  and the vertical one by  $z \in (z_0, z_1)$ , we use the notation  $\nabla_H = (\partial_x, \partial_y)^T$ , whereas  $\Delta$  denotes the three dimensional Laplacian and  $\nabla$  and  $\operatorname{div}$  the three dimensional gradient and divergence operators. The velocity  $u$  of the fluid is described by  $u = (v, w)$ , where  $v = (v_1, v_2)$  denotes the horizontal component and  $w$  the vertical one.

In the literature various sets of boundary conditions are considered such as Neumann, Dirichlet and mixed boundary conditions. In this article we choose Neumann boundary conditions for  $v$ , i.e.

$$(1.2) \quad \begin{cases} \frac{\partial}{\partial z} v = 0, & \text{on } \partial\Omega \times (0, T), \\ w = 0, & \text{on } \partial\Omega \times (0, T). \end{cases}$$

The rigorous analysis of the primitive equations started with the pioneering work of Lions, Temam and Wang [17–19], who proved the existence of a global weak solution for this set of equations for initial data  $a \in L^2$ . The uniqueness problem for weak solutions remains an open problem until today.

The existence of a local, strong solution for this equation with data  $a \in H^1$  was proved by Guillén-González, Masmoudi and Rodríguez-Bellido in [7].

In 2007, Cao and Titi [2] proved a breakthrough result for this set of equation which says, roughly speaking, that there exists a unique, global strong solution to the primitive equations for *arbitrarily large* initial data  $a \in H^1$ . Their proof is based on *a priori*  $H^1$ -bounds for the solution, which in turn are

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obtained by  $L^\infty(L^6)$  energy estimates. Kukavica and Ziane [13] proved global strong well-posedness of the primitive equations with respect to arbitrary large  $H^1$ -data for the case of mixed Dirichlet-Neumann boundary conditions. For a different approach see also Kobelkov [11]. We also would like to draw the attention of the reader to the recent survey article by Li and Titi [16] on the primitive equations.

Recently, a new approach to the primitive equations based on the theory of evolution equations has been developed in [9] and [8]. This approach is also valid in the  $L^p$ -setting for all  $1 < p < \infty$  and, using this approach, the authors proved global strong well-posedness of the primitive equations for arbitrary large data in  $H^{2/p,p}$  subject to mixed Dirichlet-Neumann boundary conditions. Taking formally the limit  $p \rightarrow \infty$ , it is now tempting to consider initial data  $a \in L^\infty$  with no differentiability assumption on the initial data. This article aims to find a function space, as large as possible, for the initial data for which the primitive equations are strongly and globally well-posed.

Recent regularity results for weak solutions by Li and Titi [15] and Kukavica, Pei, Rusin and Ziane [12] are also pointing in this direction. More specifically, starting from a weak solution to the primitive equations, these authors investigated regularity criteria for weak solutions for the primitive equations, following hereby in a certain sense the spirit of Serrin's condition in the theory of the Navier-Stokes equations and methods of weak-strong uniqueness. Li and Titi proved in [15] that weak solutions are unique for initial values in  $C^0$  or in  $\{u \in L^6: \partial_z u \in L^2\}$  including a small perturbation belonging to  $L^\infty$ . By the weak-strong uniqueness property, it follows that these weak solutions regularize and become strong solutions.

Our approach to rough initial data results for the primitive equations is very different: it considers the primitive equation as an evolution equation in an anisotropic function space of the form  $L^\infty(\mathbb{R}^2; L^1(J))$ . This space is invariant under the scaling

$$v_\lambda(t, x_1, x_2, x_3) = \lambda v(\lambda^2 t, \lambda(x_1, x_2, x_3)), \quad \lambda > 0.$$

By this we mean that  $\|v_\lambda\|_{L^\infty(\mathbb{R}^2; L^1(\lambda^{-1}J))} = \|v\|_{L^\infty(\mathbb{R}^2; L^1(J))}$  for all  $\lambda > 0$ . Moreover,  $v_\lambda$  is a solution to the primitive equations whenever  $v$  has this property. For further information on the Navier-Stokes equations in critical spaces see [1, 3, 14].

Based on  $L^\infty$ -type estimates for the underlying hydrostatic Stokes semigroup and as well as on its gradient estimates, we develop an iteration scheme yielding first the existence of a unique, local mild solution for initial data of the form  $a = a_1 + a_2$  with

$$a_1 \in BUC_\sigma(\mathbb{R}^2, L^1(J)) \text{ and } a_2 \text{ being a small perturbation in } L^\infty_\sigma(\mathbb{R}^2; L^1(J)).$$

Here  $BUC(\mathbb{R}^2)$  denotes the space of all bounded and uniformly continuous functions on  $\mathbb{R}^2$ . The subscript  $\sigma$  means the subspace of solenoidal fields rigorously defined in the next section. Assuming that  $a_1, a_2$  are periodic with respect to horizontal variables, we are able to prove that the solution regularizes sufficiently and thus, by an appropriate a priori estimate, can be extended to global, strong solution without any restriction on the size of  $a_1$ .

Comparing our assumptions on the initial data with the ones given by Li and Titi [15], observe first that our assumptions are slightly less restrictive compared to the case of continuous initial data, while our assumptions are not comparable to their second case.

Our approach may be viewed as the counterpart of the classical iteration schemes for the Navier-Stokes equations due to Giga [4] and Kato [10]. Note that, in contrast to the case of the Navier-Stokes equations, our iteration scheme presented here combined with a suitable a priori estimate yields the existence of a unique, global strong solutions not only for small data as in the case of the Navier-Stokes equations, but for arbitrary large solenoidal data  $a \in BUC_\sigma(\mathbb{R}^2; L^1(J))$ .

As written above, our approach depends crucially on  $L^\infty(L^1)$ -mapping properties of the underlying hydrostatic Stokes semigroup, including gradient estimates. These are collected in Proposition 5.6 and are of independent interest.

This article is organized as follows. Section 2 presents the two main results of this article. The following Sections 3, 4 and 5 are devoted to anisotropic estimates for fractional derivatives, the heat semigroup as well as for the hydrostatic semigroup. In Section 6 we present a proof of our main results based on an iteration scheme.

## 2. PRELIMINARIES AND MAIN RESULTS

Let  $z_0 \in \mathbb{R}$ ,  $z_1 = z_0 + h$  for some  $h > 0$  and  $J$  be the interval  $J = (z_0, z_1)$ . The incompressibility condition  $\operatorname{div} u = 0$  in  $\Omega \times (0, T)$  implies

$$w(x, y, z) = \int_z^{z_1} \operatorname{div}_H v(x, y, z) dz,$$

where the boundary condition  $w = 0$  on  $\partial\Omega$  was taken into account. Also,  $w = 0$  on  $\partial\Omega$  implies

$$\operatorname{div}_H \bar{v} = 0 \text{ in } \mathbb{R}^2,$$

where  $\bar{v}$  denotes the vertical average of  $v$ , i.e.

$$\bar{v}(x, y) = \frac{1}{z_1 - z_0} \int_{z_0}^{z_1} v(x, y, z) dz.$$

The linearization of equation (1.1) are the *hydrostatic Stokes equations*, which are given by

$$(2.1) \quad \begin{cases} \partial_t v - \Delta v + \nabla_H \pi = 0, & \text{in } \Omega \times (0, T), \\ \operatorname{div}_H \bar{v} = 0, & \text{in } \Omega \times (0, T), \\ v(0) = a & \text{in } \Omega. \end{cases}$$

The name 'hydrostatic Stokes equations' is motivated by the assumption of a hydrostatic balance when deriving the full primitive equations. The equations (2.1) are supplemented by Neumann boundary conditions (1.2) for  $v$ .

For a function  $f : \mathbb{R}^2 \times J \rightarrow \mathbb{C}$ , we define for  $1 \leq p, q < \infty$  the  $L^q(\mathbb{R}^2, L^p(J))$ -norm of  $f$  by

$$\|f\|_{L^q(\mathbb{R}^2; L^p(J))} := \left( \int_{\mathbb{R}^2} \left( \int_J |f(x', x_3)|^q dx_3 \right)^{q/p} dx' \right)^{1/q},$$

where we use the shorthand notation  $L^q(L^p)$  for the spaces and  $\|\cdot\|_{q,p}$  for the norms. The usual modifications hold for the cases  $p = \infty$  or  $q = \infty$ . The space  $L^p(\mathbb{R}^2; L^q(J))$  consisting of all measurable function  $f$  with  $\|f\|_{p,q} < \infty$  and equipped with the above norm becomes a Banach space.

Following [9], we introduce the hydrostatic Helmholtz projection as follows. For a function  $f = (f_1, f_2) : \mathbb{R}^2 \times J \rightarrow \mathbb{C}$ , we define the hydrostatic Helmholtz projection by

$$\mathbb{P}f := f + \nabla_H (-\Delta)^{-1} \operatorname{div}_H \bar{f}.$$

The solenoidal subspace  $L^\infty_\sigma(\mathbb{R}^2; L^p(J))$  is then defined for  $1 \leq p \leq \infty$  as the closed subspace of  $L^\infty(\mathbb{R}^2; L^p(J))$  given by

$$L^\infty_\sigma(\mathbb{R}^2; L^p(J)) := \{v \in L^\infty(\mathbb{R}^2; L^p(J)) : \int_{\mathbb{R}^2} \bar{v} \nabla_H \varphi dx = 0 \text{ for all } \varphi \in \widehat{W}^{1,1}(\mathbb{R}^2)\},$$

where  $\widehat{W}^{1,1}(\mathbb{R}^2)$  denotes the homogeneous Sobolev space of the form  $\widehat{W}^{1,1}(\mathbb{R}^2) = \{\varphi \in L^1_{loc}(\mathbb{R}^2) : \nabla_H \varphi \in L^1(\mathbb{R}^2)\}$ , that is, the condition  $\operatorname{div}_H \bar{v} = 0$  is understood in the sense of distributions.

If  $a \in L^\infty_\sigma(\mathbb{R}^2; L^p(J))$  for some  $1 \leq p \leq \infty$ , then the solution of equation (2.1) can be represented as  $v(t) = S(t)a$ , where  $S$  denotes the *hydrostatic Stokes semigroup* on  $L^\infty_\sigma(\mathbb{R}^2; L^p(J))$ . The latter semigroup may be represented as follows: consider the one-dimensional heat equation

$$u_t - u_{zz} = 0, \quad u(0) = u_0,$$

in  $J \times (0, \infty)$ , where  $J = (z_0, z_1)$ , subject to the boundary conditions

$$(2.2a) \quad u_z(z_1) = 0, \quad u_z(z_0) = 0 \quad \text{or}$$

$$(2.2b) \quad u_z(z_1) = 0, \quad u(z_0) = 0.$$

Given  $u_0 \in L^p(J)$  for some  $p \in [1, \infty]$ , its solution  $u$  is given by  $u(t) = e^{t\Delta_N} u_0$  for (2.2a) and by  $e^{t\Delta_{DN}} u_0$  for (2.2b), where  $e^{t\Delta_N}$  and  $e^{t\Delta_{DN}}$  denote the analytic semigroups on  $L^p(J)$  generated by the Laplacian subject to Neumann or Dirichlet-Neumann boundary conditions, respectively. For  $a \in L^\infty_\sigma(\mathbb{R}^2; L^p(J))$ , the solution of (2.1) is thus given by  $v(t) = S(t)a$ , where

$$S(t) = e^{t\Delta_H} \otimes e^{t\Delta_N}, \quad t > 0,$$

and where  $e^{t\Delta_H}$  denotes the heat semigroup on  $L^\infty(\mathbb{R}^2)$ . The semigroup  $S$  is not strongly continuous on  $L^\infty(\mathbb{R}^2; L^p(J))$ , however, its restriction to

$$BUC_\sigma(L^p) := BUC(\mathbb{R}^2; L^p(J)) \cap L^\infty_\sigma(L^p)$$

defines for  $1 \leq p < \infty$  a bounded analytic  $C_0$ -semigroup on this space.

Our first main result concerns the existence of a unique, mild solution to the primitive equations with initial value  $a$ , i.e. a function  $v$  satisfying

$$(2.3) \quad v(t) = S(t)a - \int_0^t S(t-s) \mathbb{P} \nabla \cdot (u(s) \otimes v(s)) ds, \quad 0 < t < T,$$

where  $w(s) = \int_z^{z_1} \operatorname{div}_H v(s) dx_3$  and  $u(s) = (v(s), w(s))$  for all  $s \in [0, t]$ , and  $\nabla \cdot (u \otimes v) = u \cdot \nabla v$  since  $\operatorname{div} u = 0$ .

**Theorem 2.1.** *Assume that  $a$  is of the form  $a = a_1 + a_2$  with  $a_1 \in BUC_\sigma(L^1)$  and  $a_2 \in L^\infty_\sigma(L^1)$ . Then there exists a constant  $\varepsilon_0 > 0$  such that if  $\|a_2\|_{L^\infty_\sigma(L^1)} < \varepsilon_0$ , there exists  $T > 0$  such that (1.1) subject to (1.2) admits a unique, local mild solution  $v$  with*

$$v \in C((0, T); BUC_\sigma(L^1(J)))$$

satisfying

$$\overline{\lim}_{t \rightarrow 0} t^{1/2} \|\nabla v(t)\|_{L^\infty(L^1)} \leq C \|a_2\|_{L^\infty_\sigma(L^1)}$$

for some constant  $C > 0$  independent of  $a_1$  and  $a_2$ .

In particular, one has  $v - Sa_2 \in C([0, T]; BUC_\sigma(L^1(J)))$ , where  $Sa_2 \in C((0, T); BUC_\sigma(L^1(J))) \cap L^\infty(0, T; BUC_\sigma(L^1(J)))$ . The mild solution constructed in Theorem 2.1 exists at least for some non-trivial time interval  $[0, T)$  where  $T$  depends on  $a$ . We are further able to estimate the existence time  $T > 0$  from below in terms of the following  $\|\cdot\|$ -norm defined for  $a \in L^\infty_\sigma(\mathbb{R}^2; L^1(J))$  by

$$\|a\| := [a]_\mu + \|a\|_{\infty, 1}, \quad \text{where} \quad [a]_\mu := \sup_{0 < t < 1} t^\mu \|\nabla S(t)a\|_{\infty, 1},$$

for some  $\mu \in [0, 1/2)$

**Proposition 2.2** (Estimate on life span).

*Assume that  $a \in BUC_\sigma(L^1)$ , i.e.  $a_2 = 0$ , satisfies  $[a]_\mu < \infty$  for some  $\mu \in [0, 1/2)$ . Then there exists a unique, local mild solution  $v \in C([0, T], BUC_\sigma(L^1))$  of (1.1) and (1.2). Moreover, there exists  $c_*$ , depending only on  $\mu$ , such that*

$$1/T \leq \min(c_* \|a\|, 1)^{2/(1/2-\mu)}$$

*This estimate for  $1/T$  remains valid for  $a \in L^\infty_\sigma(L^1)$  if  $C([0, T], BUC_\sigma(L^1))$  is being replaced by  $L^\infty((0, T), L^\infty(L^1))$ .*

Using  $L^\infty(L^p)$ - $L^\infty(L^1)$  smoothing properties of the semigroup and assuming initial data in  $L^\infty(L^p)$  for  $p > 1$ , one can control the  $L^\infty(L^p)$ -norms of  $v(t), \nabla v(t)$  by the corresponding  $L^\infty(L^1)$ -norms thus improving the results of Theorem 2.1.

**Proposition 2.3** (Local existence for  $p > 1$ ).

*Let  $T > 0$  be as in Theorem 2.1. If in addition to the assumptions of Theorem 2.1 the initial data  $a$  satisfies*

- a)  $a \in L^\infty_\sigma(L^p)$  for some  $p \in (1, \infty]$ , then  $t^{1/2-1/2p}v, t^{1-1/2p}\nabla v \in L^\infty(0, T; L^\infty_\sigma(L^p))$ ;
- b)  $a \in BUC_\sigma(L^p)$  for some  $p \in (1, \infty]$ , then  $t^{1/2-1/2p}v, t^{1-1/2p}\nabla v \in C([0, T], BUC_\sigma(L^p))$ ;
- c)  $a \in BUC_\sigma(BUC)$ , then  $t^{1/2}v, t\nabla v \in C([0, T], BUC_\sigma(BUC))$ .

The following theorem on the global strong well-posedness of the primitive equation with rough and arbitrary data is the second main result of this article.

**Theorem 2.4** (Global existence). *Suppose in addition to the assumptions of Theorem 2.1 that  $a$  is periodic with respect to the horizontal variables. Then, for any  $T^* > 0$  the unique, mild solution  $v$  obtained in Theorem 2.1 can be extended to a unique, strong solution of (1.1) on the interval  $(0, T^*)$ .*

### 3. INTERPOLATION INEQUALITIES FOR FRACTIONAL DERIVATIVES

In this section we consider the Caputo fractional derivative of a function  $f \in L^\infty(J; \mathbb{C})$ , where  $J = (z_0, z_1)$  for  $z_0 \in \mathbb{R}$  and  $z_1 = z_0 + h$  for some  $h > 0$ . To this end, we denote for  $\alpha > 0$  by  $I_{z_0}^\alpha$  the Riemann-Liouville operator of the form

$$(I_{z_0}^\alpha f)(z) = \frac{1}{\Gamma(\alpha)} \int_{z_0}^z (z - \zeta)^{\alpha-1} f(\zeta) d\zeta, \quad z \in \bar{J},$$

where  $\Gamma$  denotes the usual Gamma function, i.e.  $\Gamma(\alpha) = \int_0^\infty e^{-\zeta} \zeta^{\alpha-1} d\zeta$ . Considering the zero extension of  $f$  to  $\mathbb{R}$  still denoted by  $f$  and the zero extension of  $z$  from  $(0, h)$  by zero onto  $\mathbb{R}$  denoted by  $z_+$ , the Riemann-Liouville operator is defined as convolution

$$I_{z_0}^\alpha f = \frac{z_+^{\alpha-1}}{\Gamma(\alpha)} * f, \quad f \in L^\infty(J).$$

Then  $I_{z_0}^\alpha f$  is the  $\alpha$ -times integral of  $f$  from  $z_0$  whenever  $\alpha > 0$  and  $I_{z_0}^{\alpha_1 + \alpha_2} = I_{z_0}^{\alpha_1} I_{z_0}^{\alpha_2}$  for all  $\alpha_1, \alpha_2 > 0$ , and we set  $I_{z_0}^0 f = f$ .

The Caputo derivative  $\partial_z^\alpha$  for  $\alpha \in (0, 1)$  is defined by

$$(\partial_z^\alpha f)(z) := (I_{z_0}^{1-\alpha}(\partial_z f))(z), \quad z \in \bar{J},$$

where  $\partial_z f = \partial f / \partial z$ . This formula is well-defined provided  $f \in W^{1,1}(J)$  defining then  $\partial_z^\alpha f$  as an integrable function or for  $f \in W^{1,\infty}(J)$  and thus in particular for Lipschitz continuous  $f$ . Indeed, by the Hausdorff-Young inequality for convolutions we have

$$(3.1) \quad \|\partial_z^\alpha f\|_{L^1(z_0, z_0+\mu)} = \int_{z_0}^{z_0+\mu} |\partial_z^\alpha f(z)| dz \leq \frac{\mu^{1-\alpha}}{\Gamma(2-\alpha)} \|\partial_z f\|_{L^1(z_0, z_0+\mu)}$$

for  $\mu \in (0, h)$ , since  $\int_0^\mu z^{-\alpha} dz = \mu^{1-\alpha} / (1-\alpha)$ . Here we identified  $\partial_z f$  with  $\partial_z f \chi_{(z_0, z_0+\mu)}$  and  $z_+$  with  $z \chi_{(0, \mu)}$  denoting by  $\chi_{(z_0, z_0+\mu)}$  and  $\chi_{(0, \mu)}$  the corresponding characteristic functions.

We next state an interpolation inequality for  $\|\partial_z^\alpha f\|_1 = \|\partial_z^\alpha f\|_{L^1(J)}$ .

**Lemma 3.1.** *If  $\alpha \in (0, 1)$ , then*

$$(3.2) \quad \|\partial_z^\alpha f\|_1 \leq \frac{2}{\Gamma(1-\alpha)} \|f\|_1^{1-\alpha} \|\partial_z f\|_1^\alpha$$

*holds for all  $f \in W^{1,1}(J)$  satisfying  $f(z_0) = 0$ .*

*Proof.* We may assume that  $\|\partial_z f\|_1 \neq 0$  and  $\|f\|_1 \neq 0$ . Given  $\mu \in (0, h)$  and  $z \in (z_0 + \mu, z_1]$ , we subdivide the integral into two parts and integrate by parts to obtain

$$\begin{aligned} (\partial_z^\alpha f)(z) &= \frac{1}{\Gamma(1-\alpha)} \left( \int_{z_0}^{z-\mu} + \int_{z-\mu}^z \right) (z-\zeta)^{-\alpha} \partial_\zeta f(\zeta) d\zeta \\ &= \frac{1}{\Gamma(1-\alpha)} \left( \int_{z-\mu}^z (z-\zeta)^{-\alpha} \partial_\zeta f(\zeta) d\zeta + \alpha \int_{z_0}^{z-\mu} (z-\zeta)^{-\alpha-1} f(\zeta) d\zeta + \mu^{-\alpha} f(z-\mu) - (z-z_0)^{-\alpha} f(z_0) \right). \end{aligned}$$

Since  $f(z_0) = 0$ , applying the Hausdorff-Young inequality yields

$$(3.3) \quad \int_{z_0+\mu}^{z_1} |\partial_z^\alpha f(z)| dz \leq \frac{\mu^{1-\alpha}}{\Gamma(2-\alpha)} \|\partial_z f\|_1 + \frac{1}{\Gamma(1-\alpha)} \mu^{-\alpha} \|f\|_1 + \frac{\mu^{-\alpha}}{\Gamma(1-\alpha)} \|f\|_1.$$

Combining (3.1) and (3.3), we get

$$(3.4) \quad \|\partial_z^\alpha f\|_1 \leq \frac{2\mu^{1-\alpha}}{\Gamma(2-\alpha)} \|\partial_z f\|_1 + \frac{2\mu^{-\alpha}}{\Gamma(1-\alpha)} \|f\|_1,$$

and since by Poincaré's inequality

$$\|f\|_1 \leq h \|\partial_z f\|_1,$$

we obtain the desired estimate by setting  $\mu = \|f\|_1 / \|\partial_z f\|_1$  in (3.4).  $\square$

**Remark 3.2.** The estimate (3.2) remains valid if the  $L^1$  norm is replaced by the  $L^p$  norm for some  $p \in [1, \infty]$ , and we obtain in this case with essentially the same proof

$$\|\partial_z^\alpha f\|_p \leq \frac{2}{\Gamma(1-\alpha)} \|f\|_p^{1-\alpha} \|\partial_z f\|_p^\alpha,$$

where  $\|f\|_p = \|f\|_{L^p(J)}$ .

We next derive an interpolation inequality for the horizontal Laplace operator in the space  $L^\infty(L^1)$ .

**Lemma 3.3.** *Let  $\alpha \in (0, 1)$ . Then there exists a constant  $C > 0$ , depending only on  $\alpha$ , such that*

$$(3.5) \quad \left\| \nabla_H (-\Delta_H)^{-\alpha/2} f \right\|_{\infty,1} \leq C \|f\|_{\infty,1}^\alpha \|\nabla_H f\|_{\infty,1}^{1-\alpha}$$

for all  $f \in L^\infty(\mathbb{R}^2; L^1(J))$  with  $\nabla_H f \in L^\infty(\mathbb{R}^2; L^1(J))$ .

*Proof.* We may assume that  $\|\nabla_H f\|_{\infty,1} \neq 0$  and  $\|f\|_{\infty,1} \neq 0$ . Denoting by  $G_t$  the 2-dimensional Gauss kernel, i.e.,  $G_t(x) = (4\pi t)^{-1} \exp(-|x|^2/4t)$  for  $x \in \mathbb{R}^2$  and  $t > 0$  and setting  $e^{t\Delta_H} f = G_t *_H f$ , where  $*_H$  denotes convolution in the horizontal variables, only, and the negative fractional powers of  $-\Delta_H$  are defined as

$$(3.6) \quad (-\Delta_H)^{-\alpha/2} f = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty s^{\frac{\alpha}{2}-1} e^{s\Delta_H} f \, ds,$$

For  $\mu \in (0, \infty)$  we obtain thus

$$|\nabla_H (-\Delta_H)^{-\alpha/2} f| \leq \frac{1}{\Gamma(\alpha/2)} \left( \int_\mu^\infty s^{\frac{\alpha}{2}-1} |\nabla_H e^{s\Delta_H} f| \, ds + \int_0^\mu s^{\frac{\alpha}{2}-1} |e^{s\Delta_H} \nabla_H f| \, ds \right).$$

We now employ the estimates

$$|\nabla_H e^{s\Delta_H} f| = |(\nabla_H G_s) *_H f| \leq |\nabla_H G_s| *_H |f|, \quad |e^{s\Delta_H} \nabla_H f| \leq |G_s| *_H |\nabla_H f|.$$

By a direct calculation,

$$\left| \partial_i e^{-|x|^2/4t} \right| \leq \sup_{x \in \mathbb{R}^2, t > 0} \left( \frac{|x|}{2t^{1/2}} e^{-|x|^2/8t} \right) \cdot t^{-1/2} e^{-|x|^2/8t} = C_0 t^{-1/2} e^{-|x|^2/8t}, \quad x \in \mathbb{R}^2, t > 0,$$

with  $C_0 = \sup\{z e^{-z^2/2} : z > 0\} < \infty$ . Thus

$$(3.7) \quad |\partial_i G_t(x)| \leq C t^{-1/2} G_{2t}(x), \quad x \in \mathbb{R}^2, t > 0, i = 1, 2.$$

and an application of Fubini's theorem yields

$$\begin{aligned} \int_J |\nabla_H (-\Delta_H)^{-\alpha/2} f|(\cdot, z) \, dz &\leq C_2 \int_\mu^\infty s^{\frac{\alpha}{2}-1-\frac{1}{2}} |G_{2s}| *_H \left( \int_J |f(\cdot, z)| \, dz \right) \, ds \\ &\quad + C_3 \int_0^\mu s^{\frac{\alpha}{2}-1} |G_s| *_H \left( \int_J |\nabla_H f(\cdot, z)| \, dz \right) \, ds. \end{aligned}$$

Since  $\|G_s *_H g\|_\infty \leq \|g\|_\infty$  for  $\partial_z g = 0$ , we obtain

$$\|\nabla_H (-\Delta_H)^{-\alpha/2} f\|_{\infty,1} = C_4 \mu^{\frac{\alpha}{2}-\frac{1}{2}} \|f\|_{\infty,1} + C_5 \mu^{\frac{\alpha}{2}} \|\nabla_H f\|_{\infty,1},$$

where the constants  $C_j$  depends only on  $\alpha$ . Choosing  $\mu = (\|f\|_{\infty,1} / \|\nabla_H f\|_{\infty,1})^2$ , we obtained the desired estimate.  $\square$

**Remark 3.4.** The above proof shows that (3.5) remains true if the  $\|\cdot\|_{\infty,1}$ -norm is replaced by the  $\|\cdot\|_{\infty,p}$ -norm for any  $p \in [1, \infty)$ .

## 4. ANISOTROPIC ESTIMATES FOR THE HEAT SEMIGROUP

In this section we derive various estimates on the semigroups  $e^{t\Delta_N}$  and  $e^{t\Delta_{DN}}$  introduced in Section 2. We denote by  $e^{t\Delta^*}$  one of these semigroups and start with a regularizing decay estimate for  $e^{t\Delta^*}$ .

**Lemma 4.1.** *Given  $\alpha \in (0, 1)$ , there exists a constant  $C > 0$  such that*

$$\|e^{t\Delta^*} \partial_z I_{z_0}^\alpha f\|_1 \leq Ct^{-(1-\alpha)/2} \|f\|_1, \quad t > 0$$

for all  $f \in L^1(J)$  satisfying  $I_{z_0}^\alpha f(z_1) = 0$  with  $z_1 = z_0 + h$ .

*Proof.* Note that by duality

$$\|e^{t\Delta^*} \partial_z I_{z_0}^\alpha f\|_1 = \sup \{ \langle e^{t\Delta^*} \partial_z I_{z_0}^\alpha f, \varphi \rangle \mid \varphi \in C_c^\infty(J), \|\varphi\|_\infty \leq 1 \}$$

where  $\langle \varphi, \psi \rangle = \int_J \varphi \psi dz$ . Note that

$$\langle e^{t\Delta^*} \partial_z I_{z_0}^\alpha f, \varphi \rangle = \langle \partial_z I_{z_0}^\alpha f, e^{t\Delta^*} \varphi \rangle = - \langle I_{z_0}^\alpha f, \partial_z e^{t\Delta^*} \varphi \rangle,$$

where in the last identity we used the fact that  $(I_{z_0}^\alpha f)(z_1) = 0$  and  $(I_{z_0}^\alpha f)(z_0) = 0$ ; the latter is trivial by definition. Since

$$\langle I_{z_0}^\alpha f, \psi \rangle = \langle f, \bar{I}_{z_1}^\alpha \psi \rangle$$

with

$$\bar{I}_{z_1}^\alpha \psi(z) = \frac{1}{\Gamma(\alpha)} \int_z^{z_1} (\xi - z)^{\alpha-1} \psi(\xi) d\xi,$$

we conclude that

$$\langle e^{t\Delta^*} \partial_z I_{z_0}^\alpha f, \varphi \rangle = - \langle f, \bar{I}_{z_1}^\alpha \partial_z e^{t\Delta^*} \varphi \rangle.$$

Since  $\bar{I}_{z_1}^\alpha \partial_z$  resembles the Caputo derivative and  $\partial_z e^{t\Delta} \varphi(z_1, t) = 0$  by (2.2a) or (2.2b), we are able to adapt Lemma 3.1 to obtain

$$\|\bar{I}_{z_1}^\alpha \partial_z e^{t\Delta^*} \varphi\|_\infty \leq \frac{2}{\Gamma(\alpha)} \|e^{t\Delta^*} \varphi\|_\infty^\alpha \|\partial_z e^{t\Delta^*} \varphi\|_\infty^{1-\alpha}.$$

Notice that  $\|e^{t\Delta^*} \varphi\|_\infty \leq \|\varphi\|_\infty$  for all  $t > 0$  and

$$\|\partial_z e^{t\Delta^*} \varphi\|_\infty \leq C_0 t^{-1/2} \|\varphi\|_\infty \quad \text{for all } t > 0$$

for some constant  $C_0 > 0$ . This can be seen by extending the problem to the whole space problem by extending  $\varphi$  periodically in a suitable way to  $\mathbb{R}$  to obtain  $e^{t\Delta^*} \varphi = G_t * \tilde{\varphi}$ , where  $\tilde{\varphi}$  is the extension of  $\varphi$ . Thus

$$\|\bar{I}_{z_1}^\alpha \partial_z e^{t\Delta^*} \varphi\|_\infty \leq C_1 t^{-(1-\alpha)/2} \|\varphi\|_\infty, \quad t > 0,$$

with  $C_1$  depending on  $\alpha$ , only. We thus conclude that

$$|\langle e^{t\Delta^*} \partial_z I_{z_0}^\alpha f, \varphi \rangle| \leq \|f\|_1 \|\bar{I}_{z_1}^\alpha \partial_z e^{t\Delta^*} \varphi\|_\infty \leq C_1 t^{-(1-\alpha)/2} \|f\|_1 \|\varphi\|_\infty, \quad t > 0$$

which gives the desired estimate.  $\square$

In the following, we derive further regularity estimates for the heat semigroup  $e^{t\Delta}$  on  $L^\infty(\mathbb{R}^d)$  for  $d \geq 1$ . We denote by  $R_i = \partial_i (-\Delta)^{-1/2}$  the  $i$ -th Riesz transform, where  $\partial_i = \partial/\partial x_i$  for all  $1 \leq i \leq d$ .

**Lemma 4.2.** *Given  $\alpha \in (0, 1)$  and  $d \in \mathbb{N}$ , there exists a constant  $C > 0$  such that*

(i) *for all  $x \in \mathbb{R}^d$ , all  $t > 0$  and all  $f \in L^\infty(\mathbb{R}^d)$*

$$\left| e^{t\Delta} (-\Delta)^{\alpha/2} f(x) \right| \leq t^{-\alpha/2} (H_t * |f|)(x)$$

where  $H_t \in L^1(\mathbb{R}^d)$  such that  $\|H_t\|_1 \leq C_\alpha$  with a constant  $C_\alpha$  independent of  $t > 0$ . In particular,

$$\|e^{t\Delta} (-\Delta)^{\alpha/2} f\|_\infty \leq C_\alpha t^{-\alpha/2} \|f\|_\infty, \quad t > 0.$$



(ii) for all  $x \in \mathbb{R}^d$ , all  $t > 0$  and all  $f \in L^\infty(\mathbb{R}^d)$

$$\left| e^{t\Delta} R_i R_j (-\Delta)^{-\alpha/2} f(x) \right| \leq t^{-\alpha/2} (\tilde{H}_t * |f|)(x),$$

with  $\tilde{H}_t$  having the same properties as  $H_t$ . In particular,

$$\left\| e^{t\Delta} R_i R_j (-\Delta)^{\alpha/2} f \right\|_\infty \leq \tilde{C}_\alpha t^{-\alpha/2} \|f\|_\infty, \quad t > 0.$$

(iii) for all  $x \in \mathbb{R}^d$ , all  $t > 0$  and all  $f \in L^\infty(\mathbb{R}^d)$

$$\left| e^{t\Delta} R_i R_j \partial_k f(x) \right| \leq t^{-1/2} (\check{H}_t * |f|)(x),$$

with  $\check{H}_t$  having the same property as  $H_t$ . In particular,

$$\left\| e^{t\Delta} R_i R_j (-\Delta)^{\alpha/2} f \right\|_\infty \leq \check{C} t^{-1/2} \|f\|_\infty, \quad t > 0.$$

*Proof.* i) Using the Bochner representation formula for fractional powers of the Laplacian (see e.g. [20] p. 260)

$$(4.1) \quad (-\Delta)^{\alpha/2} f = \frac{1}{|\Gamma(-\alpha/2)|} \int_0^\infty s^{-(1+\alpha/2)} (e^{s\Delta} - 1) f ds,$$

we obtain

$$\begin{aligned} e^{t\Delta} (-\Delta)^{\alpha/2} f &= \frac{1}{|\Gamma(-\alpha/2)|} \int_0^\infty s^{-(1+\alpha/2)} (G_{t+s} - G_t) * f ds \\ &= \frac{1}{|\Gamma(-\alpha/2)|} \int_t^\infty (s-t)^{-(1+\alpha/2)} (G_s - G_t) * f ds \\ &= \frac{t^{-\alpha/2}}{|\Gamma(-\alpha/2)|} \int_1^\infty (u-1)^{-(1+\alpha/2)} (G_{tu} - G_t) * f ds, \quad t > 0. \end{aligned}$$

Therefore

$$|e^{t\Delta} (-\Delta)^{\alpha/2} f|(x) \leq t^{-\alpha/2} (H_t * |f|)(x), \quad t > 0,$$

where

$$H_t = \frac{1}{|\Gamma(-\alpha/2)|} \int_1^\infty |G_{tu} - G_t| (u-1)^{-(1+\alpha/2)} du$$

satisfies

$$\|H_t\|_1 \leq \frac{2}{|\Gamma(-\alpha/2)|} \int_1^\infty (u-1)^{-(1+\alpha/2)} du = C_\alpha < \infty.$$

ii). Observe that  $e^{t\Delta} R_i R_j (-\Delta)^{\alpha/2} f = \partial_i \partial_j e^{t\Delta} (-\Delta)^{-(1-\alpha/2)} f$  for all  $1 \leq i, j \leq d$ . The representation formula (3.6) implies

$$\begin{aligned} \partial_i \partial_j e^{t\Delta} (-\Delta)^{-(1-\alpha/2)} f &= \frac{1}{\Gamma(1-\alpha/2)} \int_0^\infty s^{-\alpha/2} (\partial_i \partial_j G_{t+s}) * f ds \\ &= \frac{1}{\Gamma(1-\alpha/2)} \int_t^\infty (s-t)^{-\alpha/2} (\partial_i \partial_j G_s) * f ds, \quad t > 0, \end{aligned}$$

where we used the identities

$$(\partial_i \partial_j e^{t\Delta}) e^{s\Delta} f = (\partial_i \partial_j G_t) * (G_s * f) = \partial_i \partial_j (G_t * G_s * f) = (\partial_i \partial_j G_{t+s}) * f.$$

A calculation similar to the one given in the proof of Lemma 3.3 yields  $|\partial_i \partial_j G_t| \leq C t^{-1} G_t$  for all  $t > 0$  and proceeding as above we obtain

$$|e^{t\Delta} R_i R_j (-\Delta)^{\alpha/2} f|(x) \leq t^{-\alpha/2} (\tilde{H}_t * |f|)(x), \quad t > 0,$$

where

$$\tilde{H}_t = \frac{1}{\Gamma(1-\alpha/2)} \int_1^\infty G_{2tu} (u-1)^{-\alpha/2} u^{-1} du$$

satisfies  $\|\tilde{H}_t\|_1 = \frac{1}{\Gamma(1-\alpha/2)} \int_1^\infty (u-1)^{-\alpha/2} u^{-1} du = \tilde{C}_\alpha < \infty$ .

iii) As above we have  $e^{t\Delta} R_i R_j \partial_k f = \partial_i \partial_j \partial_k e^{t\Delta} (-\Delta)^{-1} f$  and using the estimate  $|\partial_i \partial_j \partial_k G_t| \leq Ct^{-3/2} G_{2t}$  for  $t > 0$  yields the desired result by the same arguments.  $\square$

**Remark 4.3.** The assertions of Lemma 4.1 and Lemma 4.2 remain true for the case  $\alpha = 0$  by interpreting  $(-\Delta)^0$  and  $I_{z_0}^0$  as identity operators. For  $\alpha = 1$  the assertion of Lemma 4.1 remains true as well since then  $\partial_z I_{z_0}^\alpha f = f$  if  $f(z_0) = 0$ . However, the assertion of Lemma 4.2 is no longer true for  $\alpha = 1$ , since this would imply the boundedness of the Riesz transforms on  $L^\infty(\mathbb{R}^d)$  or  $L^1(\mathbb{R}^d)$ .

**Remark 4.4.** The assertions of Lemma 4.1 remains valid if the  $L^1$ -norm is replaced by the  $L^p$ -norm for any  $p \in [1, \infty]$ . In fact, we obtain

$$\|e^{t\Delta^*} \partial_z I_{z_0}^\alpha f\|_p \leq Ct^{-(1-\alpha)/2} \|f\|_p, \quad t > 0,$$

for all  $f \in L^p(J)$  satisfying  $(I_{z_0}^\alpha f)(z_1) = 0$  with  $z_1 = z_0 + h$ . The proof parallels then the one given above provided we have the estimates

$$\|e^{t\Delta^*} \varphi\|_p \leq \|\varphi\|_p, \quad \|\partial_z e^{t\Delta^*} \varphi\|_p \leq C_0 t^{-1/2} \|\varphi\|_p, \quad t > 0.$$

at hand. These estimates are well known and we give an elementary and self-contained proof of them by a periodization method, which is explained in the next section, see Lemma 5.5.

## 5. ESTIMATES FOR THE HYDROSTATIC STOKES SEMIGROUP

Consider the hydrostatic Stokes semigroup  $S$  on  $L^\infty(L^p(J))$  for  $p \in [1, \infty]$  as introduced in Section 2 and given by

$$S(t) = e^{t\Delta_H} \otimes e^{t\Delta_N}, \quad t > 0.$$

This semigroup admits an extension to the space  $L^\infty(L^p)$  for all  $p \in [1, \infty]$ , which we denote by  $S_\infty$ .

The following estimates with respect to the  $\|\cdot\|_{\infty,1}$ -norm for the semigroup  $S_\infty$  and hence in particular for the hydrostatic Stokes semigroup  $S$  will be of crucial importance in our iteration argument in the subsequent Section 6.

**Lemma 5.1.** *Let  $\alpha \in (0, 1)$ . Then there is a constant  $C$  such that*

(i) *for all  $f \in L^\infty(L^1)$  and all  $t > 0$*

$$\|\nabla S_\infty(t) f\|_{\infty,1} \leq Ct^{-1/2} \|f\|_{\infty,1}, \quad \|S_\infty(t) \nabla_H \cdot f\|_{\infty,1} \leq Ct^{-1/2} \|f\|_{\infty,1},$$

(ii) *for all  $f \in L^\infty(L^1)$  with  $I_{z_0}^\alpha f = 0$  at  $z = z_1$  and all  $t > 0$*

$$\|S_\infty(t) \partial_z I_{z_0}^\alpha f\|_{\infty,1} \leq Ct^{-(1-\alpha)/2} \|f\|_{\infty,1}.$$

*Moreover, for all  $f \in L^\infty(L^1)$  and all  $t > 0$*

$$\|S_\infty(t) \partial_z f\|_{\infty,1} \leq Ct^{-1/2} \|f\|_{\infty,1}.$$

(iii) *for all  $f \in L^\infty(L^1)$  and all  $t > 0$*

$$\left\| S_\infty(t) \mathbb{P}(-\Delta_H)^{\alpha/2} f \right\|_{\infty,1} \leq Ct^{-\alpha/2} \|f\|_{\infty,1}.$$

(iv) *for all  $f \in L^\infty(L^1)$  and all  $t > 0$*

$$\|S_\infty(t) \mathbb{P} \nabla_H \cdot f\|_{\infty,1} \leq Ct^{-1/2} \|f\|_{\infty,1}.$$

**Remark 5.2.** We note that assertion ii) remains true also for the cases  $\alpha = 0$  and  $\alpha = 1$  by Remark 4.3. The later implies that  $S_\infty$  is a bounded semigroup in  $L^\infty(L^1)$ .

*Proof.* i) This assertion follows from the estimates

$$\|\partial_z e^{t\Delta^*} f\|_1 \leq Ct^{-1/2} \|f\|_1, \quad t > 0 \text{ and } \|e^{t\Delta^*} f\|_1 \leq \|f\|_1, \quad t > 0,$$

and from the pointwise estimates

$$|\nabla_H e^{t\Delta_H} f| \leq Ct^{-1/2} G_{2t} * |f|, \quad |e^{t\Delta_H} f| \leq G_t * |f|,$$

compare (3.7), as well as  $e^{t\Delta_H} \partial_{x_i} f = \partial_{x_i} e^{t\Delta_H} f$  for  $i = 1, 2$  as in the proof of Proposition 5.6.

ii) Since

$$|e^{t\Delta_H} g|(x') \leq (G_t * |g|)(x'), \quad t > 0,$$

Fubini's theorem implies

$$\|e^{t\Delta_H} e^{t\Delta_N} \partial_z I_{z_0}^\alpha f(x', \cdot)\|_{L^1(J)} \leq G_t * \|e^{t\Delta_N} \partial_z I_{z_0}^\alpha f(x', \cdot)\|_{L^1(J)}, \quad t > 0$$

for almost all  $x' \in \mathbb{R}^2$ . By Lemma 4.1

$$\|e^{t\Delta_N} \partial_z I_{z_0}^\alpha f(x', \cdot)\|_{L^1(J)} \leq Ct^{-(1-\alpha)/2} \|f(x', \cdot)\|_{L^1(J)}, \quad t > 0,$$

which allows us to conclude that

$$\|S_\infty(t) \partial_z I_{z_0}^\alpha f\|_{\infty,1} \leq Ct^{-(1-\alpha)/2} \|G_t\|_1 \|f\|_{\infty,1} = Ct^{-(1-\alpha)/2} \|f\|_{\infty,1}, \quad t > 0.$$

The proof is also valid for the case  $\alpha = 0$  yielding  $\|S_\infty(t) \partial_z f\|_{\infty,1} \leq Ct^{-1/2} \|f\|_{\infty,1}$  for all  $t > 0$ .

(iii) We verify by Lemma 4.2 i) and ii) that

$$\begin{aligned} \|S_\infty(t) \mathbb{P}(-\Delta_H)^{\alpha/2} f(x', \cdot)\|_{L^1(J)} &\leq \|e^{t\Delta_H} e^{t\Delta_N} (-\Delta_H)^{\alpha/2} f(x', \cdot)\|_{L^1(J)} \\ &\quad + \sum_{1 \leq i, j \leq 2} \|e^{t\Delta_H} e^{t\Delta_N} R_i R_j (-\Delta_H)^{\alpha/2} \bar{f}\|_{L^1(J)} \\ &\leq t^{-\alpha/2} \left( \|H_t * |f|(x', \cdot)\|_{L^1(J)} + h(\tilde{H}_t * \bar{f})(x') \right), \quad t > 0, \end{aligned}$$

for almost all  $x' \in \mathbb{R}^2$  since  $\bar{f}$  is independent of  $z$ . By Fubini's theorem

$$\int_J H_t * |f|(x', z) dz = \left( H_t * \int_J |f(\cdot, z)| dz \right)(x'), \quad \text{a.a. } x' \in \mathbb{R}^2,$$

which allows us to conclude that

$$\begin{aligned} \|S_\infty(t) \mathbb{P}(-\Delta_H)^{\alpha/2} f\|_{\infty,1} &\leq t^{-\alpha/2} \left( \|H_t\|_{L^1(\mathbb{R}^2)} + \|\tilde{H}_t\|_{L^1(\mathbb{R}^2)} \right) \|f\|_{\infty,1} \\ &\leq (C_\alpha + \tilde{C}_\alpha) t^{-\alpha/2} \|f\|_{\infty,1}, \quad t > 0. \end{aligned}$$

(iv) As above we have

$$\|S_\infty(t) \mathbb{P} \nabla_H \cdot f\|_{L^1(J)} \leq \|\nabla_H e^{t\Delta_H} f\|_{L^1(J)} + \sum_{1 \leq i, j \leq 2} \|e^{t\Delta_H} R_i R_j \nabla_H \cdot \bar{f}\|_{L^1(J)}.$$

The first term was already estimated and the second one is treated in the same way as in iii).  $\square$

In order to extend the above estimates to  $L^\infty(L^q)$  space, it is convenient to investigate the periodic heat semigroup.

**Lemma 5.3.** *Let  $\mathbb{T} = \mathbb{R}/\omega_0\mathbb{Z}$  for some  $\omega_0 > 0$  and  $f \in L^1(\mathbb{T})$ . Then*

$$(G_t * f)(z) = \int_0^{\omega_0} E_t(z-y) f(y) dy, \quad t > 0, z \in \mathbb{T},$$

where  $E_t(z) = \sum_{k=-\infty}^{\infty} G_t(z - k\omega_0)$  for  $z \in \mathbb{T}$ . In particular,

$$\|G_t * f\|_{L^p(\mathbb{T})} \leq \|f\|_{L^p(\mathbb{T})}, \quad t > 0.$$

If for all  $p \in [1, \infty]$ .

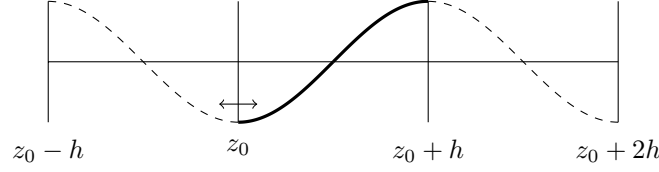


FIGURE 1. Periodic extension by even reflexion

*Proof.* The above representation for  $G_t * f$  follows by noting that

$$(G_t * f)(z) = \sum_{k=-\infty}^{\infty} \int_{k\omega_0}^{(k+1)\omega_0} G_t(z-y)f(y)dy, \quad t > 0, z \in T,$$

and

$$\int_{k\omega_0}^{(k+1)\omega_0} G_t(z-y)f(y)dy = \int_0^{\omega_0} G_t(z-y-k\omega_0)f(y+k\omega_0)dy, \quad t > 0$$

where  $f(y+k\omega_0) = f(y)$  for all  $k \in \mathbb{Z}$  by periodicity. The estimate claimed follows by Young's inequality since  $\int_0^{\omega_0} E_t(z-y)dz = \int_{-\infty}^{\infty} G_t(z-y)dz = 1$  for all  $t > 0$  and since  $E_t \geq 0$  for all  $t > 0$ .  $\square$

**Lemma 5.4** (Derivative estimate for the periodic heat semigroup). *Under the assumption of Lemma 5.3, there exists a constant  $C > 0$ , independent of  $\omega_0$ , such that*

$$|\partial_z(G_t * f)(z)| \leq Ct^{-1/2} \int_0^{\omega_0} E_{2t}(z-y)|f(y)|dy, \quad t > 0, z \in \mathbb{T},$$

In particular,

$$\|\partial_z(G_t * f)\|_{L^p(\mathbb{T})} \leq Ct^{-1/2}\|f\|_{L^p(\mathbb{T})}, \quad t > 0,$$

for all  $p \in [1, \infty]$ .

*Proof.* By (3.7)

$$|\partial_z G_t(z)| \leq Ct^{-1/2}G_{2t}(z), \quad t > 0, z \in \mathbb{T},$$

which implies the first assertion. The second one follows by Young's inequality.  $\square$

**Lemma 5.5** (Periodization). *Given  $1 \leq p \leq \infty$ , there exists a constant  $C > 0$  such that*

$$\|e^{t\Delta_*} f\|_{L^p(J)} \leq \|f\|_{L^p(J)} \quad \text{and} \quad \|\partial_z e^{t\Delta_*} f\|_{L^p(J)} \leq Ct^{-1/2}\|f\|_{L^p(J)}, \quad t > 0, f \in L^p(J).$$

*Proof.* Consider the case, where  $\Delta_* = \Delta_N$ . We first extend  $f \in L^p(J)$  to  $(z_0 - h, z_0)$  by even extension, i.e., by setting  $f(z) = f(-z)$  for  $z \in (z_0 - h, z_0)$ . and extend then  $f$  to a periodic function  $f_{per}$  with period  $\omega_0 = 2h$  by  $f_{per}(z) = f(z - k\omega_0)$  for  $z \in (k\omega_0, (k+1)\omega_0)$  and  $k \in \mathbb{Z}$ , see Figure 1. It then follows that

$$e^{t\Delta_N} f = e^{t\Delta} f_{per}|_J,$$

and  $\|f\|_{L^p(J)} = \|f_{per}\|_{L^p(-h,h)}/2$ . The desired estimates follow then from Lemma 5.3 and Lemma 5.4.  $\square$

These estimates are important in order to extend Lemma 5.1 to the situation of  $L^\infty(L^q)$  spaces.

**Proposition 5.6.** *Let  $S_\infty$  be the semigroup on  $L^\infty(\mathbb{R}^2; L^q(J))$  given by  $S_\infty(t) = e^{t\Delta_H} \otimes e^{t\Delta_N}$ . Furthermore let  $\alpha \in [0, 1)$  and  $q \in [1, \infty]$ . Then there exists a constant  $C > 0$  such that*

(i) for all  $f \in L^\infty(L^q)$

$$\|\nabla S_\infty(t)f\|_{\infty,q} \leq Ct^{-1/2}\|f\|_{\infty,q}, \quad \|S_\infty(t)\nabla_H \cdot f\|_{\infty,q} \leq Ct^{-1/2}\|f\|_{\infty,q}, \quad t > 0.$$

(ii) for all  $f \in L^\infty(L^q)$  satisfying  $I_{z_0}^\alpha f = 0$  at  $z = z_1$

$$\|S_\infty(t)\partial_z I_{z_0}^\alpha f\|_{\infty,q} \leq Ct^{-(1-\alpha)/2}\|f\|_{\infty,q}, \quad t > 0.$$

(iii) for all  $f \in L^\infty(L^q)$

$$\|S_\infty(t)\mathbb{P}(-\Delta_H)^{\alpha/2}f\|_{\infty,q} \leq Ct^{-\alpha/2}\|f\|_{\infty,q}, \quad t > 0.$$

**Remark 5.7.** Note that also the following  $L^\infty(L^1)$ - $L^\infty(L^q)$  smoothing holds for  $q \in [1, \infty]$

$$\|S_\infty(t)f\|_{\infty,q} \leq Ct^{-(1-1/q)}\|f\|_{\infty,1}, \quad t > 0.$$

We also remark that, if the case  $\alpha = 0$  is considered in assertion ii), the term  $I_{z_0}^0$  is interpreted as the identity operator and there is no restriction for  $f$  other than  $f \in L^\infty(L^q)$ .

*Proof.* i) We first prove that  $\|\partial_z S_\infty(t)f\|_{\infty,q} \leq Ct^{-1/2}\|f\|_{\infty,q}$  for all  $t > 0$ . By Lemma 5.5

$$\|\partial_z S_\infty(t)f(x', \cdot)\|_{L^q(J)} \leq Ct^{-1/2}\|e^{t\Delta_H}f(x', \cdot)\|_{L^q(J)}$$

for almost all  $x' \in \mathbb{R}^2$ . By Minkowski's inequality and due to the positivity of  $e^{t\Delta_H}$

$$\|e^{t\Delta_H}f(x', \cdot)\|_{L^q(J)} \leq e^{t\Delta_H}\|f(x', \cdot)\|_{L^q(J)},$$

and thus

$$\|\partial_z S_\infty(t)f\|_{\infty,q} \leq Ct^{-1/2} \text{ess.sup}_{x'}(e^{t\Delta_H}\|f(x', \cdot)\|_{L^q(J)}) \leq Ct^{-1/2}\|f\|_{\infty,q}, \quad t > 0.$$

We next prove that  $\|\nabla_H S_\infty(t)f\|_{\infty,q} \leq Ct^{-1/2}\|f\|_{\infty,q}$  for all  $t > 0$ . To this end, we estimate

$$\|e^{t\Delta_N}\nabla_H e^{t\Delta_H}f(x', \cdot)\|_{L^q(J)} \leq \|\nabla_H e^{t\Delta_H}f(x', \cdot)\|_{L^q(J)}.$$

As in Lemma 5.4, we observe that

$$|\nabla_H e^{t\Delta_H}f(x', z)| \leq Ct^{-1/2}(G_{2t} *_H |f|)(x', z),$$

and applying Minkowski's inequality yields

$$\|\nabla_H e^{t\Delta_H}f(x', \cdot)\|_{L^q(J)} \leq Ct^{-1/2}(G_{2t} *_H \|f(x', \cdot)\|_{L^q(J)}).$$

We thus conclude that

$$\|\nabla_H S_\infty(t)f\|_{\infty,q} \leq Ct^{-1/2}\|f\|_{\infty,q}, \quad t > 0.$$

The second part of i) follows from  $S(t)\partial_{x_i}f = \partial_{x_i}S(t)f$  for  $i = 1, 2$ . The remaining assertions ii) and iii) follow from the  $L^q$  versions of Lemma 5.1 and Remark 4.4.  $\square$

## 6. PROOF OF MAIN RESULTS

In the following, we construct a solution of the integral equation (2.3). We start by estimating the integral term for functions with vanishing vertical average.

**Lemma 6.1.** *For all  $\alpha \in (0, 1)$  there exists a constant  $C > 0$  such that*

$$\|S(t)\mathbb{P}\nabla \cdot (\tilde{u} \otimes v)\|_{\infty,1} \leq Ct^{-(1-\alpha)/2}(\|\nabla\tilde{v}\|_{\infty,1}\|v\|_{\infty,1} + \|\tilde{v}\|_{\infty,1}\|\nabla v\|_{\infty,1})^{1-\alpha}(\|\nabla v\|_{\infty,1}\|\nabla\tilde{v}\|_{\infty,1})^\alpha.$$

for all  $v \in L_\sigma^\infty(L^1)$  satisfying  $\bar{v} = 0$  and all  $\tilde{u} = (\tilde{v}, \tilde{w})$  with  $\tilde{v} \in L_\sigma^\infty(L^1)$  satisfying  $\bar{\tilde{v}} = 0$  and  $\tilde{w} = \int_z^{z_1} \text{div}_H \tilde{v} dx_3$ . The statement still holds true for  $\alpha = 0$ .

*Proof.* We first note that

$$\nabla \cdot (\tilde{u} \otimes v) = \nabla_H \cdot (\tilde{v} \otimes v) + \partial_z(\tilde{w}v).$$

Since  $\operatorname{div}_H \bar{v} = 0$  we obtain  $\tilde{w} = 0$  at  $z = z_0$  and since  $\tilde{w} = 0$  at  $z = z_1$  by definition, we see that  $\overline{\partial_z(\tilde{w}v)} = 0$ . Hence,

$$\mathbb{P}\nabla \cdot (\tilde{u} \otimes v) = \mathbb{P}\nabla_H \cdot (\tilde{v} \otimes v) + \partial_z(\tilde{w}v).$$

The cases  $\alpha = 0$  is now straight forward using Lemma 5.1 (ii), (iv). Consider now the case  $\alpha \in (0, 1)$ . Noting that  $(-\Delta_H)^{(1-\alpha)/2}, (-\Delta_H)^{-(1-\alpha)/2}$  and  $\nabla_H$  commute, we write

$$\begin{aligned} S(t)\mathbb{P}\nabla \cdot (\tilde{u} \otimes v) &= S(t)\mathbb{P}(-\Delta_H)^{(1-\alpha)/2}\nabla_H \cdot (-\Delta_H)^{-(1-\alpha)/2}(\tilde{v} \otimes v) + S(t)\partial_z I_{z_0}^\alpha I_{z_0}^{1-\alpha} \partial_z(\tilde{w}v), \quad t > 0, \\ &=: I + II. \end{aligned}$$

Applying Lemma 5.1 iii) and Lemma 3.3 yields

$$\|I\|_{\infty,1} \leq Ct^{-(1-\alpha)/2} \left\| \nabla_H \cdot (-\Delta_H)^{-(1-\alpha)/2} \tilde{v} \otimes v \right\|_{\infty,1} \leq Ct^{-(1-\alpha)/2} \|\nabla_H(\tilde{v} \otimes v)\|_{\infty,1}^\alpha \|\tilde{v} \otimes v\|_{\infty,1}^{1-\alpha}, \quad t > 0.$$

Since  $\bar{v} = 0$  and  $\bar{\tilde{v}} = 0$ , we obtain the estimates

$$\begin{aligned} \|\nabla(\tilde{v} \otimes v)\|_{\infty,1} &\leq \|\nabla\tilde{v}\|_{\infty,1} \|v\|_{\infty,\infty} + \|\tilde{v}\|_{\infty,\infty} \|\nabla v\|_{\infty,1}, \\ \|\tilde{v} \otimes v\|_{\infty,1} &\leq \|\tilde{v}\|_{\infty,1} \|v\|_{\infty,\infty} + \|\tilde{v}\|_{\infty,\infty} \|v\|_{\infty,1}, \\ \|v\|_{\infty,\infty} &\leq \|\partial_z v\|_{\infty,1}, \\ \|\tilde{v}\|_{\infty,\infty} &\leq \|\partial_z \tilde{v}\|_{\infty,1} \end{aligned}$$

and the term  $\|I\|_{\infty,1}$  can be thus estimated as claimed.

In order to estimate  $\|II\|_{\infty,1}$  we observe that Lemma 5.1 (ii) and Lemma 3.1 yield

$$\|II\|_{\infty,1} \leq Ct^{-(1-\alpha)/2} \|\partial_z^\alpha(\tilde{w}v)\|_{\infty,1} \leq Ct^{-(1-\alpha)/2} \|\tilde{w}v\|_{\infty,1}^{1-\alpha} \|\partial_z(\tilde{w}v)\|_{\infty,1}^\alpha, \quad t > 0.$$

Here we invoked the fact that

$$I_{z_0}^\alpha (I_{z_0}^{1-\alpha} \partial_z(\tilde{w}v))(z_1) = (\tilde{w}v)(z_1) = 0.$$

Since

$$\|\tilde{w}\|_{\infty,\infty} \leq C \|\partial_z \tilde{w}\|_{\infty,1} \leq C \|\nabla_H \tilde{v}\|_{\infty,1}$$

we are able to estimate  $\|II\|_{\infty,1}$  in the same way as  $I$ . This completes the proof.  $\square$

Our next step consists in proving a similar estimate for the above integral term, however, without assuming that the vertical average of the functions involved is vanishing. To this end, we set

$$\|v\|_{1,\infty,1} := \|v\|_{\infty,1} + \|\nabla v\|_{\infty,1}.$$

**Proposition 6.2.** *There exists a constant  $C > 0$  such that for all  $\alpha \in (0, 1)$*

$$\|S(t)\mathbb{P}\nabla \cdot (\tilde{u} \otimes v)\|_{\infty,1} \leq Ct^{-(1-\alpha)/2} (\|\tilde{v}\|_{1,\infty,1} \|v\|_{\infty,1} + \|v\|_{1,\infty,1} \|\tilde{v}\|_{\infty,1})^{1-\alpha} (\|\tilde{v}\|_{1,\infty,1} \|v\|_{1,\infty,1})^\alpha.$$

for all  $v \in L_\sigma^\infty(L^1)$  satisfying  $\nabla v \in L^\infty(L^1)$  and all  $\tilde{u} = (\tilde{v}, \tilde{w})$  with  $\tilde{v} \in L_\sigma^\infty(L^1)$  and  $\nabla \tilde{v} \in L^\infty(L^1)$  and  $\tilde{w} = \int_z^{z_1} \operatorname{div}_H \tilde{v} \, dx_3$ . The statement still holds true for  $\alpha = 0$ .

*Proof.* We argue similarly as in the proof of Lemma 6.1. In order to estimate  $\|v\|_{\infty,\infty}$  we write

$$\|v\|_{\infty,\infty} \leq \|v - \bar{v}\|_{\infty,\infty} + \|\bar{v}\|_{\infty,\infty}.$$

Observing that

$$\|v - \bar{v}\|_{\infty,\infty} \leq \|\partial_z v\|_{\infty,1}, \quad \|\bar{v}\|_{\infty,\infty} \leq \|v\|_{\infty,1}$$

we conclude that

$$\|v\|_{\infty,\infty} \leq \|\partial_z v\|_{\infty,1} + \|v\|_{\infty,1}.$$

Thus

$$\begin{aligned} \|\nabla(\tilde{v} \otimes v)\|_{\infty,1} &\leq \|\tilde{v}\|_{1,\infty,1} \|v\|_{1,\infty,1} \\ \|\tilde{v} \otimes v\|_{\infty,1} &\leq \|\tilde{v}\|_{\infty,1} \|v\|_{1,\infty,1} + \|v\|_{\infty,1} \|\tilde{v}\|_{1,\infty,1} \end{aligned}$$

and the desired estimate follows as in the proof of Lemma 6.1.  $\square$

We now give a proof of our main results.

*Proof of Theorem 2.1.* Consider the recursively defined sequence  $(v_m)$  defined for  $t \geq 0$  by

$$\begin{aligned} v_{m+1}(t) &:= S(t)a - \int_0^t S(t-s)\mathbb{P}\nabla \cdot (u_m(s) \otimes v_m(s))ds, \quad m \in \mathbb{N} \\ v_0(t) &:= S(t)a. \end{aligned}$$

Applying Lemma 5.1 (i), (ii) with  $\alpha = 0$ , see Remark 5.2, we obtain

$$\begin{aligned} (6.1) \quad \|v_{m+1}(t)\|_{\infty,1} &\leq \|S(t)a\|_{\infty,1} + C_1 \int_0^t (t-s)^{-1/2} \|u_m(s) \otimes v_m(s)\|_{\infty,1} ds \\ &\leq \|S(t)a\|_{\infty,1} + C_1 \int_0^t (t-s)^{-1/2} \|u_m(s)\|_{\infty,\infty} \|v_m(s)\|_{\infty,1} ds \\ &\leq \|S(t)a\|_{\infty,1} + C_1 \int_0^t (t-s)^{-1/2} \|v_m(s)\|_{1,\infty,1} \|v_m(s)\|_{\infty,1} ds, \end{aligned}$$

with all constants  $C_1, C_2, C_3, C_4 > 0$  here and below being independent of  $v_m, u_m$  and  $t$ . We now estimate  $\|\nabla v_{m+1}(t)\|_{\infty,1}$  by Proposition 6.2. Since

$$\nabla S(t-s) = \nabla S\left(\frac{t-s}{2}\right)S\left(\frac{t-s}{2}\right)$$

Lemma 5.1 (i) and Proposition 6.2 with  $\alpha = 1/2$  yield

$$(6.2) \quad \|\nabla v_{m+1}(t)\|_{\infty,1} \leq \|\nabla S(t)a\|_{\infty,1} + C_2 \int_0^t (t-s)^{-1/2} (t-s)^{-1/4} \|v_m(s)\|_{1,\infty,1}^{3/2} \|v_m(s)\|_{\infty,1}^{1/2} ds, \quad t > 0.$$

Note that in the above estimate we may also take any  $\alpha \in (0, 1)$ . For  $m \in \mathbb{N} \cup \{0\}$  we now set

$$\begin{aligned} K_m(t) &:= \sup_{0 < \tau < t} \tau^{1/2} \|v_m(\tau)\|_{1,\infty,1}, \\ H_m(t) &:= \sup_{0 < \tau < t} \|v_m(\tau)\|_{\infty,1}, \\ M_m(t) &:= \sup_{0 < \tau < t} \tau^{1/2} \|v_m(\tau)\|_{\infty,1}. \end{aligned}$$

Estimate (6.1) combined with  $\|S(t)a\|_{\infty,1} \leq \|a\|_{\infty,1}$  for all  $t > 0$  yields

$$(6.3) \quad H_{m+1}(t) \leq \|a\|_{\infty,1} + C_1 K_m(t) H_m(t), \quad t > 0.$$

Multiplying (6.2) by  $t^{1/2}$  yields

$$(6.4) \quad \sup_{0 < \tau < t} \tau^{1/2} \|\nabla v_{m+1}(\tau)\|_{\infty,1} \leq \sup_{0 < \tau < t} \tau^{1/2} \|\nabla S(\tau)a\|_{\infty,1} + C_3 K_m(t)^{3/2} H_m(t)^{1/2}, \quad t > 0,$$

and by multiplying (6.1) with  $t^{1/2}$ , we obtain

$$(6.5) \quad M_{m+1}(t) \leq \sup_{0 < \tau < t} \tau^{1/2} \|S(\tau)a\|_{\infty,1} + C_4 t^{1/2} K_m(t) H_m(t),$$

provided  $t \leq T$  for some  $T \leq 1$ . Adding (6.4) and (6.5) yields

$$(5.9) \quad K_{m+1}(t) \leq K_0(t) + C_3 K_m(t)^{3/2} H_m(t)^{1/2} + C_4 t^{1/2} K_m(t) H_m(t), \quad m \geq 0$$

with

$$K_0(t) = \sup_{0 < \tau < t} \tau^{1/2} \|S(\tau)a\|_{1,\infty,1}.$$

If  $a_1 \in BUC_\sigma(L^1)$ , then

$$\tau^{1/2} \|\nabla S(\tau)a_1\|_{\infty,1} \rightarrow 0 \quad \text{as } \tau \searrow 0, \quad \text{and } \tau^{1/2} \|\nabla S(\tau)a_2\|_{\infty,1} \leq C \|a_2\|_{\infty,1}$$

by Lemma 5.1 (i). Thus, by (6.3) and (5.9), the sequences  $(H_m)$  and  $(K_m)$  fulfill the assumptions of the following Lemma 6.3 provided  $t$  is small enough, say  $t \leq t_0$  since  $K_0(t) \rightarrow 0$  as  $t \searrow 0$ , and  $\|a_2\|_{\infty,1}$  is sufficiently small. Thus the sequences  $\{H_m\}$  and  $\{K_m\}$  are uniformly bounded.

It is not difficult to prove that  $(v_m - Sa_2)$  is a Cauchy sequence in  $C([0, t_0], BUC_\sigma(L^1))$  and that  $(t^{1/2}\nabla(v_m - Sa_2))$  is a Cauchy sequence in  $C([0, t_0], L^\infty(L^1))$ . We thus obtain  $v$  as the limit of  $(v_m)$  satisfying the desired estimate.

The proof of the uniqueness follows a similar line of arguments. Let  $v, \tilde{v}$  be two solutions, then

$$(v - \tilde{v})(t) = \int_0^t S(t-s) \mathbb{P}\nabla \cdot (u(s) \otimes (v - \tilde{v})(s) + (u - \tilde{u}) \otimes \tilde{v}(s)) ds, \quad t > 0,$$

and one obtains as above using Lemma 5.1 (i) and Proposition 6.2 with  $\alpha = 1/2$ , and setting

$$K(v)(t) := \sup_{0 < \tau < t} \tau^{1/2} \|v(\tau)\|_{1,\infty,1} \quad \text{and} \quad H(v)(t) := \sup_{0 < \tau < t} \|v(\tau)\|_{\infty,1}$$

that for  $N(v)(t) := \max\{K(v)(t), H(v)(t)\}$  one has

$$\begin{aligned} N(v) &\leq C(K(v)H(v - \tilde{v}) + K(v - \tilde{v})H(v))^{1/2} (K(v)K(v - \tilde{v}))^{1/2} \\ &\quad + C(K(\tilde{v})H(v - \tilde{v}) + K(v - \tilde{v})H(\tilde{v}))^{1/2} (K(\tilde{v})K(v - \tilde{v}))^{1/2}. \end{aligned}$$

Hence one obtains

$$(6.6) \quad N(v - \tilde{v}) \leq N(v - \tilde{v})C \left\{ (K(v) + H(v))^{1/2} K(v)^{1/2} + (K(\tilde{v}) + H(\tilde{v}))^{1/2} K(\tilde{v})^{1/2} \right\}.$$

By assumption, if  $t$  is small we have

$$K(\tilde{v})(t), K(v)(t) \leq C \|a_2\|_{\infty,1}, \quad \text{and} \quad H(\tilde{v}), H(v)(t) \leq C \|a\|_{\infty,1}.$$

Thus supposing that  $t$  and  $\|a_2\|_{\infty,1} \cdot \|a\|_{\infty,1}$  are small enough, one has

$$C \{ (K(v) + H(v))K(v) + (K(\tilde{v}) + H(\tilde{v}))K(\tilde{v}) \} < 1,$$

and therefore by (6.6) one has  $K(v - \tilde{v}) = 0$  on  $(0, T_0)$  and  $H(v - \tilde{v}) = 0$  on  $[0, T_0]$  for some  $0 < T_0 \leq T$ . Iterating this argument it follows that the solutions are unique on  $[0, T]$ .  $\square$

**Lemma 6.3.** *Let  $A, \varepsilon > 0$  be constants and assume that  $\{H_m\} \subset \mathbb{R}$  and  $\{K_m\} \subset \mathbb{R}$  are sequences satisfying*

$$\begin{aligned} H_0 &\leq A, & H_{m+1} &\leq A + C_1 H_m K_m, \\ K_0 &\leq \varepsilon, & K_{m+1} &\leq \varepsilon + C_2 K_m^{3/2} H_m^{1/2} + (4A)^{-1} K_m H_m, \end{aligned}$$

for all  $m \geq 0$ , where  $C_1 > 0$  and  $C_2 > 0$  are constants independent of  $m$ . Then there exists  $\varepsilon_0 = \varepsilon_0(C_1, C_2, A) > 0$  such that  $\{K_m\}$  and  $\{H_m\}$  are bounded sequences provided that  $\varepsilon \leq \varepsilon_0$ .

*Proof.* Note first that if  $K_m \leq 1/(2C_1)$  for  $m \leq m_0$ , then  $H_m \leq 2A$  for all  $m \leq m_0 + 1$ . Next, we choose  $\varepsilon$  small enough so that the graphs of  $y = x$  and  $y = \varepsilon + \sqrt{2AC_2}x^{3/2} + x/2$  have an intersection. Denote by  $x_0(\varepsilon)$  the abscissa of the intersection point closest to  $x = 0$ . Clearly  $x_0(\varepsilon) \searrow 0$  as  $\varepsilon \rightarrow 0$ .

Choose now  $\varepsilon_0$  so small that  $x_0(\varepsilon_0) < 1/(2C_1)$ . Then,  $K_m \leq x_0(\varepsilon)$  and  $H_m \leq 2A$  for all  $m \geq 1$  provided  $\varepsilon \leq \varepsilon_0$ . Indeed, we proved this by induction. The estimate is trivial for  $m = 1$ . Assume that  $K_m \leq x_0(\varepsilon)$ ,  $H_m \leq 2A$  for all  $m \leq m_0$ . Since  $x_0(\varepsilon) < 1/(2C_1)$ , the inequality for  $H_m$  implies  $H_{m+1} \leq 2A$  and the inequality for  $K_m$  implies  $K_{m+1} \leq x_0(\varepsilon)$  by the choice of  $x_0(\varepsilon)$  since  $H_m \leq 2A$ . We thus conclude that  $K_m \leq x_0(\varepsilon)$  and  $H_m \leq 2A$ .  $\square$

The solution  $v$  constructed in Theorem 2.1 exists at least for some nontrivial time interval  $[0, T)$ , where  $T > 0$  depends on  $a$ . Given  $a \in BUC_\sigma(L^p)$  for some  $p > 1$  we are unfortunately unable to estimate  $T$  from below by terms involving the norm of  $a$ , only. However, in Proposition 2.3 we claim that  $v \in C([0, T], BUC_\sigma(L^p))$  for all  $p \in (1, \infty)$  in the same time interval.



*Proof of Proposition 2.3.* We estimate the integral equation (2.3) by writing  $S(t) = S(\frac{t}{2})S(\frac{t}{2})$  and then using the  $L^p$ - $L^1$ -estimate from Remark 5.7 and Proposition 6.2 with  $\alpha = 0$  to obtain

$$\begin{aligned} \|v(t)\|_{\infty,p} &\leq \|S(t)a\|_{\infty,p} + C \int_0^t (t-s)^{-(1-1/p)/2} (t-s)^{-1/2} \|v(s)\|_{1,\infty,1} \|v(s)\|_{\infty,1} ds, \\ &\leq \|S(t)a\|_{\infty,p} + C \int_0^t (t-s)^{-(1-1/2p)} s^{-1/2} s^{1/2} \|v(s)\|_{1,\infty,1} \|v(s)\|_{\infty,1} ds, \quad t > 0. \end{aligned}$$

Since  $a \in L^\infty_\sigma(L^p)$  and  $v \in L^\infty(0, T; L^\infty_\sigma(L^1))$  by Theorem 2.1 we see that  $t^{1/2-1/2p}v$  is in  $L^\infty(0, T; L^\infty_\sigma(L^p))$ .

First note that

$$\nabla S(t) = \nabla S(\frac{t}{3})S(\frac{t}{3})S(\frac{t}{3}), \quad t > 0.$$

Differentiating (2.3), applying first Proposition 5.6 (i), second using the  $L^p$ - $L^1$ -estimate from Remark 5.7 and third, applying Proposition 6.2 with  $\alpha \in (0, 1)$  arbitrary yields

$$\|\nabla v(t)\|_{\infty,p} \leq \|\nabla S(t)a\|_{\infty,p} + C \int_0^t (t-s)^{-(1-1/2p+(1-\alpha)/2)} s^{-(1+\alpha)/2} s^{(1+\alpha)/2} \|v(s)\|_{1,\infty,1}^{1+\alpha} \|v(s)\|_{\infty,1}^{1-\alpha} ds.$$

This yields the desired bound for  $t^{1-2/p}\nabla v$ . The continuity of  $v$  follows from strong continuity of  $S$ .  $\square$

*Proof of Proposition 2.2.* We argue similarly as in the proof of Theorem 2.1. Setting

$$L_m(t) := \sup_{0 < \tau < t} \tau^\mu \|v_m(\tau)\|_{1,\infty,1}, \quad 0 < t < T,$$

we obtain by (6.1) for  $m \geq 0$  and  $t \in (0, T)$

$$H_{m+1}(t) \leq \|a\|_{\infty,1} + C_1 t^{1/2-\mu} L_m(t) H_m(t)$$

instead of (6.3). Similarly, instead of (5.9), we now obtain

$$L_{m+1}(t) \leq \|a\|_{\infty,1} + [a]_\mu + C_3 t^{(1/2-\mu)/2} L_m^{3/2}(t) H_m(t)^{1/2} + C_4 t^{1/2} L_m(t) H_m(t).$$

It follows that if  $T$  fulfills  $1/T \geq \min(c_* \|a\|, 1)^{2/(1/2-\mu)}$  for some  $c_*$  independent of  $a$ , then  $(L_m)$  and  $(H_m)$  are bounded sequences for  $t \in [0, T]$ . Moreover,  $\{v_m\}$  is a Cauchy sequence in  $C([0, T], BUC_\sigma(L^1))$ , which is proved as before.  $\square$

Since  $e^{t\Delta_H}$  as well as  $\mathbb{P}$  and the nonlinearity leave horizontal periodicity invariant we obtain the following.

**Lemma 6.4.** *If  $a$  is in addition to the assumption of Theorem 2.1 periodic with respect to the horizontal variables, so is the solution  $v(t)$  for  $t > 0$ .*

*Proof of Theorem 2.4.* In order to extend the local solutions to a global one we make use of the regularization of the solution. By Theorem 2.1  $v(t_0), \nabla v(t_0) \in BUC(L^1)$  for  $t_0 > 0$ , in particular  $v(t_0) \in BUC(W^{1,1})$ , and since  $W^{1,1}(J) \hookrightarrow L^p(J)$  for all  $p \in [1, \infty]$ , one has  $v(t_0) \in BUC(L^p)$ . Now, by Proposition 2.3  $v(t_1), \nabla v(t_1) \in BUC(L^p)$  for  $t_1 > t_0$ , and in particular, using Lemma 6.4, where w.l.o.g. the period is 1, one has for the restriction

$$v(t_1)|_{[0,1]^2 \times J} \in \{v \in W^{1,p}([0,1]^2 \times J) \mid v \text{ periodic in } x, y, \operatorname{div}_H \bar{v} = 0\} \quad 0 < t_1 < T.$$

Using this for  $p \geq 2$  as new initial value, it follows using e.g. [6] that  $v$  extends to a global strong solution.  $\square$

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