ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE GENERALIZED KdV-BURGERS EQUATION

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Abstract

We study the asymptotic behavior of global solutions to the initial value problem for the generalized KdV-Burgers equation. One can expect that the solution to this equation converges to a self-similar solution to the Burgers equation, due to earlier works related to this problem. Actually, we obtain the optimal asymptotic rate similar to those results and the second asymptotic profile for the generalized KdV-Burgers equation.

Keywords: Korteweg-de Vries-Burgers equation, asymptotic behavior, second asymptotic profile.

1 Introduction

In this paper, we consider the asymptotic behavior of global solutions to the following generalized Korteweg-de Vries-Burgers equation (we call it generalized KdV-Burgers equation for short):

$$\begin{gathered} u_t + (f(u))_x + k u_{xxx} = u_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}, \end{gathered} \tag{1.1}$$

where $u_0 \in L^1(\mathbb{R})$, $f(u) = (b/2)u^2 + (c/3)u^3$ and $b, c, k \in \mathbb{R}$. The subscripts t and x denote the partial derivatives with respect to t and x, respectively. The aim of our study is to obtain an asymptotic profile of the solution $u(x,t)$ and to examine the optimality of its asymptotic rate.

First of all, we recall known results concerning this problem. When $k = 0$, (1.1) becomes the generalized Burgers equation:

$$\begin{gathered} u_t + (f(u))_x = u_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{gathered} \tag{1.2}$$

It was shown in Matsumura and Nishihara [12] that the solution of (1.2) converges to a nonlinear diffusion wave defined by

$$\chi(x,t) \equiv \frac{1}{\sqrt{1+t}} \chi_* \left(\frac{x}{\sqrt{1+t}} \right), \quad t \ge 0, \quad x \in \mathbb{R}, \tag{1.3}$$

where

$$\chi_*(x) \equiv \frac{1}{b} \frac{(e^{b\delta/2} - 1) e^{-x^2/4}}{\sqrt{\pi} + (e^{b\delta/2} - 1) \int_{x/2}^{\infty} e^{-y^2} dy}, \quad \delta \equiv \int_{\mathbb{R}} u_0(x) dx, \quad b \neq 0. \tag{1.4}$$

Note that $\chi(x,t)$ is a solution of the Burgers equation

$$\chi_t + \left(\frac{b}{2} \chi^2 \right)_x = \chi_{xx}, \tag{1.5}$$

satisfying

$$\int_{\mathbb{R}} \chi(x,0) dx = \delta.$$

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Moreover, if $u_0 \in L^1_1(\mathbb{R}) \cap H^1(\mathbb{R})$ and $\|u_0\|_{L^1_1} + \|u_0\|_{H^1}$ is sufficiently small, then the optimal asymptotic rate to the nonlinear diffusion wave is obtained by Kato [9] by constructing the second asymptotic profile $V_1(x,t)$ which is the leading term of $u - \chi$. Here $L^1_\beta(\mathbb{R})$ is a subset of $L^1(\mathbb{R})$ whose elements satisfy $\|u_0\|_{L^1_\beta} \equiv \int_{\mathbb{R}} |u_0(x)|(1+|x|)^\beta dx < \infty$. Indeed, the following decay estimate is established:

$$\|u(\cdot,t) - \chi(\cdot,t) - V_1(\cdot,t)\|_{L^\infty} \le C(\|u_0\|_{L^1_1} + \|u_0\|_{H^1})(1+t)^{-1}, \quad t \ge 1, \tag{1.6}$$

where

$$V_1(x,t) \equiv -\frac{cd}{12\sqrt{\pi}} V_*\left(\frac{x}{\sqrt{1+t}}\right)(1+t)^{-1}\log(1+t), \quad t \ge 0, \quad x \in \mathbb{R}, \tag{1.7}$$

$$V_*(x) \equiv (b\chi_*(x) - x)e^{-x^2/4}\eta_*(x) = 2\frac{d}{dx}(\eta_*(x)e^{-x^2/4}), \tag{1.8}$$

$$\eta_*(x) \equiv \exp\left(\frac{b}{2}\int_{-\infty}^x \chi_*(y)dy\right), \quad d \equiv \int_{\mathbb{R}} (\eta_*(y))^{-1}(\chi_*(y))^3 dy. \tag{1.9}$$

From (1.6), the triangle inequality and (1.7), we see that the original solution $u(x,t)$ tends to the nonlinear diffusion wave $\chi(x,t)$ at the rate of $t^{-1}\log t$, and in addition, if $\delta \ne 0$ and $c \ne 0$, then this asymptotic rate is optimal with respect to the time decaying order. Also we see that $u - \chi$ tends to the second asymptotic profile $V_1(x,t)$ at the rate of t^{-1}.

Next, we consider the case where $b = k = 1$ and $c = 0$ in (1.1):

$$\begin{gathered} u_t + uu_x + u_{xxx} = u_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{gathered} \tag{1.10}$$

This equation is called the KdV-Burgers equation. It was shown in Kaikina and Ruiz-Paredes [5] that if $u_0 \in L^1_1(\mathbb{R}) \cap H^s(\mathbb{R})$ with $s > -1/2$, the following estimate

$$\|u(\cdot,t) - \chi(\cdot,t-1) - V_2(\cdot,t)\|_{L^\infty} \le Ct^{-1}\sqrt{\log t} \tag{1.11}$$

holds for sufficiently large t, where

$$V_2(x,t) \equiv -\frac{d}{32\sqrt{\pi}} V_*\left(\frac{x}{\sqrt{t}}\right) t^{-1}\log t$$

with $V_*(x)$ being defined by (1.8). We see from this result that the solution of (1.10) also tends to the nonlinear diffusion wave $\chi(x,t)$ at the rate of $t^{-1}\log t$ and this rate is optimal. On the other hand, the asymptotic rate given by (1.11) is rougher than (1.6) by $\sqrt{\log t}$, although they mentioned in [5] that the term $\sqrt{\log t}$ in the estimate (1.11) could be removed by more delicate consideration, without any proof.

In this paper, we consider (1.1) for all $b, c, k \in \mathbb{R}$ with $b \ne 0$, and obtain the following result:

Theorem 1.1 (Main Theorem)**.** *Assume that $u_0 \in L^1(\mathbb{R}) \cap H^3(\mathbb{R})$ and $\|u_0\|_{L^1} + \|u_0\|_{H^3}$ is sufficiently small. Then (1.1) has a unique global solution $u(x,t)$ satisfying $u \in C^0([0,\infty); H^3)$ and $\partial_x u \in L^2(0,\infty; H^3)$. Moreover if $u_0 \in L^1_1(\mathbb{R}) \cap H^3(\mathbb{R})$ and $\|u_0\|_{L^1_1} + \|u_0\|_{H^3}$ is sufficiently small, then we have*

$$\|u(\cdot,t) - \chi(\cdot,t) - V(\cdot,t)\|_{L^\infty} \le C(\|u_0\|_{L^1_1} + \|u_0\|_{H^3})(1+t)^{-1}, \quad t \ge 1, \tag{1.12}$$

where $\chi(x,t)$ is defined by (1.3), while $V(x,t)$ is defined by

$$V(x,t) \equiv -\frac{d}{4\sqrt{\pi}}\left(\frac{b^2k}{8} + \frac{c}{3}\right) V_*\left(\frac{x}{\sqrt{1+t}}\right)(1+t)^{-1}\log(1+t), \quad t \ge 0, \quad x \in \mathbb{R}, \tag{1.13}$$

with $V_*(x)$ being defined by (1.8).

REMARK 1.2. From (1.12), the triangle inequality and (1.13), if $\delta \ne 0$ and $(b^2k)/8 + c/3 \ne 0$, we see that the original solution $u(x,t)$ tends to the nonlinear diffusion wave $\chi(x,t)$ at the rate of $t^{-1}\log t$. Actually, we have

$$\tilde{C}(1+t)^{-1}\log(1+t) \le \|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} \le C(1+t)^{-1}\log(1+t)$$

holds for sufficiently large t. Therefore, this asymptotic rate $t^{-1}\log t$ is optimal with respect to the time decaying order. On the other hand, if $(b^2k)/8 + c/3 = 0$, then we find the asymptotic rate to the nonlinear diffusion wave is t^{-1}, because $V(x,t)$ vanishes identically.

REMARK 1.3. It seems that our regularity assumption on the initial data is stronger than previous works [5] and [9]. However, even if we assume the same regularity as those works, since the solution of (1.1) becomes smooth $u(x,t) \in C^\infty((0,\infty);H^\infty(\mathbb{R}))$ by virtue of the smoothing effect of the parabolic type equations, we may assume that the initial data $u_0 \in L^1(\mathbb{R}) \cap H^3(\mathbb{R})$ by changing the initial time. Taking this fact into account, we can say that (1.11) is improved, and that the results due to Kato [9] and Kaikina and Ruiz-Paredes [5] are unified.

This paper is organized as follows. In Section 2, we prove the L^p-decay estimates of solutions to (1.1). In Section 3, we prepare a couple of lemmas for an auxiliary problems. Finally, we give the proof of our main theorem in Section 4. In the proof of the main theorem, in order to estimate the dispersion term in the integral equation (4.11) below, we rewrite this term by making the integration by parts. It is the main novelty of this paper since we can not estimate this term in the same way as previous works.

2 Decay Estimates for Solutions to (1.1)

In this section, we shall derive decay estimates for solutions to (1.1). First, we introduce the Green function associated with the linear part of the equation in (1.1). Here and later, for $f, g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, we denote the Fourier transform of f and the inverse Fourier transform of g as follows:

$$\begin{aligned}\hat{f}(\xi) \equiv& F[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)dx, \\ \check{g}(x) \equiv& F^{-1}[g](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} g(\xi)d\xi.\end{aligned}$$

Then the Green function is defined by

$$S(x,t) \equiv F^{-1}[e^{-t|\xi|^2 + itk\xi^3}](x).$$

By a direct calculation, we can show the estimates of derivatives of $S(x,t)$. For the proof, see Lemma A.1. and Lemma A.2. in [8].

Lemma 2.1. *Let l be a non-negative integer. Then, for $p \in [2,\infty]$, we have*

$$\|\partial_x^l S(\cdot,t)\|_{L^1} \le Ct^{-l/2}(1+t^{-1/4}), \quad t > 0, \tag{2.1}$$

$$\|\partial_x^l S(\cdot,t)\|_{L^p} \le Ct^{-(1/2)(1-1/p)-l/2}, \quad t > 0. \tag{2.2}$$

Moreover, for the convolution $S(t) * f$, we obtain the following estimate:

Lemma 2.2. *Let m be a positive integer. Suppose $f \in H^m(\mathbb{R}) \cap L^1(\mathbb{R})$. Then the estimate*

$$\|\partial_x^l (S(t) * f)\|_{L^2} \le C(1+t)^{-1/4-l/2}\|f\|_{L^1} + e^{-t}\|\partial_x^l f\|_{L^2}, \quad t \ge 0 \tag{2.3}$$

holds for any integer $0 \le l \le m$.

Proof. By using Plancherel's theorem, we have

$$\begin{aligned}\|\partial_x^l (S * f)\|_{L^2}^2 &= \|e^{-t|\xi|^2+itk\xi^3}(i\xi)^l \hat{f}(\xi)\|_{L^2}^2 = \int_{\mathbb{R}} e^{-2t|\xi|^2}|(i\xi)^l \hat{f}(\xi)|^2 d\xi \\ &= \left(\int_{|\xi|\ge 1} + \int_{|\xi|\le 1}\right) e^{-2t|\xi|^2}|(i\xi)^l \hat{f}(\xi)|^2 d\xi \\ &\equiv I_1 + I_2.\end{aligned}$$

First, we evaluate I_1. By Plancherel's theorem, we have

$$I_1 \le e^{-2t} \int_{|\xi|\ge 1} |(i\xi)^l \hat{f}(\xi)|^2 d\xi \le e^{-2t} \int_{\mathbb{R}} |\widehat{\partial_x^l f}(\xi)|^2 d\xi = e^{-2t}\|\partial_x^l f\|_{L^2}^2. \tag{2.4}$$

Next, we evaluate I_2. Since $|\hat{f}(\xi)| \le C\|f\|_{L^1}$ for all $\xi \in \mathbb{R}$, we have

$$\begin{aligned}I_2 &= \int_{|\xi|\le 1} e^{-2t|\xi|^2}|(i\xi)^l|^2 |\hat{f}(\xi)|^2 d\xi \le C\left(\sup_{|\xi|\le 1}|\hat{f}(\xi)|\right)^2 \int_{|\xi|\le 1} e^{-2t|\xi|^2}|\xi|^{2l} d\xi \\ &\le C\|f\|_{L^1}^2 \int_0^1 e^{-2t\xi^2}\xi^{2l} d\xi \le C(1+t)^{-1/2-l}\|f\|_{L^1}^2.\end{aligned} \tag{2.5}$$

From (2.4) and (2.5) we obtain (2.3). $\square$

Now we turn back to (1.1). Local existence and uniqueness of the solution to (1.1) can be shown by the standard argument (see e.g. [8], [10], [12]). Moreover, one can obtain the global solution satisfying

$$\|u(\cdot,t)\|_{H^3}^2 + \int_0^t \|\partial_x u(\cdot,s)\|_{H^3}^2 ds \le C\|u_0\|_{H^3}^2, \quad t \ge 0. \tag{2.6}$$

Furthermore, the solution satisfies the following decay estimates:

Lemma 2.3. *Assume that $u_0 \in L^1(\mathbb{R}) \cap H^3(\mathbb{R})$ and $\|u_0\|_{L^1} + \|u_0\|_{H^3}$ is sufficiently small. Then the solution $u(x,t)$ to (1.1) satisfies*

$$\|\partial_x^l u(\cdot,t)\|_{L^1} \le C(\|u_0\|_{L^1} + \|u_0\|_{H^3})t^{-l/2}(1+t^{-1/4}), \quad t>0, \tag{2.7}$$

$$\|\partial_x^l u(\cdot,t)\|_{L^2} \le C(\|u_0\|_{L^1} + \|u_0\|_{H^3})(1+t)^{-1/4-l/2}, \quad t\ge 0 \tag{2.8}$$

for $l = 0,1,2,3$.

Proof. We consider the following integral equation associated with the initial value problem (1.1):

$$\begin{aligned} u(t) =& S(t) * u_0 - \int_0^t S(t-s) * (f(u)_x)(s)ds \\ =& S(t) * u_0 - \int_0^t (\partial_x S(t-s)) * (f(u))(s)ds \\ =& S(t) * u_0 - \frac{b}{2}\int_0^t (\partial_x S(t-s)) * u^2(s)ds - \frac{c}{3}\int_0^t (\partial_x S(t-s)) * u^3(s)ds \\ \equiv& I_1 + I_2 + I_3. \end{aligned} \tag{2.9}$$

First, we shall prove (2.8). If we set

$$M(T) \equiv \sup_{0\le t\le T} \sum_{n=0}^{3} (1+t)^{1/4+n/2}\|\partial_x^n u(\cdot,t)\|_{L^2}, \tag{2.10}$$

then from the Sobolev inequality

$$\|f\|_{L^\infty} \le C\|f\|_{L^2}^{1/2}\|f'\|_{L^2}^{1/2}, \quad f \in H^1(\mathbb{R}),$$

we have

$$\|\partial_x^l u(\cdot,t)\|_{L^\infty} \le M(T)(1+t)^{-1/2-l/2}, \quad l = 0,1,2. \tag{2.11}$$

Here and later, $M(T)$ are assumed to be small. Before evaluating I_1 , I_2 and I_3 , we prepare the following estimates for $l = 0,1,2,3$:

$$\|\partial_x^l (u^2(\cdot,t))\|_{L^1} \le C(1+t)^{-1/2-l/2}M(T)^2, \tag{2.12}$$

$$\|\partial_x^l (u^3(\cdot,t))\|_{L^1} \le C(1+t)^{-1-l/2}M(T)^3. \tag{2.13}$$

Let $l = 0,1,2,3$ and $0 \le t \le T$. We have from (2.10) and (2.11)

$$\begin{aligned} \|\partial_x^l (u^2(\cdot,t))\|_{L^1} \le& C\sum_{m=0}^{l} \|\partial_x^m u(\cdot,t)\|_{L^2}\|\partial_x^{l-m} u(\cdot,t)\|_{L^2} \\ \le& C(1+t)^{-1/2-l/2}M(T)^2, \end{aligned}$$

and

$$\begin{aligned} \|\partial_x^l (u^3(\cdot,t))\|_{L^1} \le& C\|\partial_x^l u(\cdot,t)\|_{L^2}\|u(\cdot,t)\|_{L^2}\|u(\cdot,t)\|_{L^\infty} \\ &+ C\sum_{m=0}^{l-1}\sum_{n=0}^{l-m} \|\partial_x^m u(\cdot,t)\|_{L^\infty}\|\partial_x^n u(\cdot,t)\|_{L^2}\|\partial_x^{l-m-n} u(\cdot,t)\|_{L^2} \\ \le& C(1+t)^{-1/4-l/2}(1+t)^{-1/4}(1+t)^{-1/2}M(T)^3 \\ &+ C\sum_{m=0}^{l-1}\sum_{n=0}^{l-m} (1+t)^{-1/2-m/2}(1+t)^{-1/4-n/2}(1+t)^{-1/4-(l-m-n)/2}M(T)^3 \\ \le& C(1+t)^{-1-l/2}M(T)^3. \end{aligned}$$

Thus we get (2.12) and (2.13).

By Lemma 2.2, we get

$$\|\partial_x^l I_1(\cdot,t)\|_{L^2} \le C(1+t)^{-1/4-l/2}(\|u_0\|_{L^1}+\|u_0\|_{H^l}), \quad t\ge 0. \tag{2.14}$$

From Young's inequality, Lemma 2.1 and (2.12), we have

$$\begin{aligned}
\|\partial_x^l I_2(\cdot,t)\|_{L^2} \le & C\int_0^{t/2}\|(\partial_x^{l+1}S(t-s))*u^2(s)\|_{L^2}ds + C\int_{t/2}^t \|(\partial_x S(t-s))*\partial_x^l(u^2)(s)\|_{L^2}ds \\
\le & C\int_0^{t/2}\|\partial_x^{l+1}S(t-s)\|_{L^2}\|u^2(\cdot,s)\|_{L^1}ds + C\int_{t/2}^t \|\partial_x S(t-s)\|_{L^2}\|\partial_x^l(u^2(\cdot,s))\|_{L^1}ds \\
\le & C\int_0^{t/2}(t-s)^{-1/4-(l+1)/2}(1+s)^{-1/2}M(T)^2ds + C\int_{t/2}^t (t-s)^{-3/4}(1+s)^{-1/2-l/2}M(T)^2ds \\
\le & C(1+t)^{-1/4-l/2}M(T)^2, \quad t\ge 1.
\end{aligned} \tag{2.15}$$

Similarly, we have from Young's inequality, Lemma 2.1 and (2.13)

$$\begin{aligned}
\|\partial_x^l I_3(\cdot,t)\|_{L^2} \le & C\int_0^{t/2}\|(\partial_x^{l+1}S(t-s))*u^3(s)\|_{L^2}ds + C\int_{t/2}^t \|(\partial_x S(t-s))*\partial_x^l(u^3)(s)\|_{L^2}ds \\
\le & C\int_0^{t/2}\|\partial_x^{l+1}S(t-s)\|_{L^2}\|u^3(\cdot,s)\|_{L^1}ds + C\int_{t/2}^t \|\partial_x S(t-s)\|_{L^2}\|\partial_x^l(u^3(\cdot,s))\|_{L^1}ds \\
\le & C\int_0^{t/2}(t-s)^{-1/4-(l+1)/2}(1+s)^{-1}M(T)^3ds + C\int_{t/2}^t (t-s)^{-3/4}(1+s)^{-1-l/2}M(T)^3ds \\
\le & C(1+t)^{-3/4-l/2}\log(1+t)M(T)^3, \quad t\ge 1.
\end{aligned} \tag{2.16}$$

Therefore, from (2.9), (2.14) through (2.16), we have

$$\|\partial_x^l u(\cdot,t)\|_{L^2} \le C(1+t)^{-1/4-l/2}(\|u_0\|_{L^1}+\|u_0\|_{H^3}+M(T)^2), \quad 1\le t\le T. \tag{2.17}$$

For $0\le t\le 1$, from (2.6), we see that (2.17) is also valid. Thus, we get

$$M(T)\le C(\|u_0\|_{L^1}+\|u_0\|_{H^3}+M(T)^2).$$

Since $\|u_0\|_{L^1}+\|u_0\|_{H^3}$ is small, we obtain the desired estimate

$$M(T)\le C(\|u_0\|_{L^1}+\|u_0\|_{H^3}). \tag{2.18}$$

This completes the proof of (2.8).

Next, we shall prove (2.7). By Young's inequality and Lemma 2.1, we have

$$\|\partial_x^l I_1(\cdot,t)\|_{L^1} \le Ct^{-l/2}(1+t^{-1/4})\|u_0\|_{L^1}, \quad t>0. \tag{2.19}$$

Moreover, we have from Young's inequality and Lemma 2.1, (2.8), (2.12), (2.13) and (2.18).

$$\begin{aligned}
\|\partial_x^l I_2(\cdot,t)\|_{L^1} \le & C\int_0^{t/2}\|(\partial_x^{l+1}S(t-s))*u^2(s)\|_{L^1}ds + C\int_{t/2}^t \|(\partial_x S(t-s))*\partial_x^l(u^2)(s)\|_{L^1}ds \\
\le & C\int_0^{t/2}\|\partial_x^{l+1}S(t-s)\|_{L^1}\|u^2(\cdot,s)\|_{L^1}ds + C\int_{t/2}^t \|\partial_x S(t-s)\|_{L^1}\|\partial_x^l(u^2(\cdot,s))\|_{L^1}ds \\
\le & C\int_0^{t/2}(t-s)^{-1/2-l/2}(1+(t-s)^{-1/4})(1+s)^{-1/2}(\|u_0\|_{L^1}+\|u_0\|_{H^3})^2ds \\
& + C\int_{t/2}^t (t-s)^{-1/2}(1+(t-s)^{-1/4})(1+s)^{-1/2-l/2}(\|u_0\|_{L^1}+\|u_0\|_{H^3})^2ds \\
\le & Ct^{-1/2-l/2}(1+t^{-1/4})t^{1/2}(\|u_0\|_{L^1}+\|u_0\|_{H^3}) \\
& + C(1+t)^{-1/2-l/2}(t^{1/2}+t^{1/4})(\|u_0\|_{L^1}+\|u_0\|_{H^3}) \\
\le & C(\|u_0\|_{L^1}+\|u_0\|_{H^3})t^{-l/2}(1+t^{-1/4}), \quad t>0,
\end{aligned} \tag{2.20}$$

and

$$
\begin{aligned}
\|\partial_x^l I_3(\cdot,t)\|_{L^1} \leq & C\int_0^{t/2} \|(\partial_x^l S(t-s)) * \partial_x(u^3)(s)\|_{L^1} ds + C\int_{t/2}^t \|(\partial_x S(t-s)) * \partial_x^l(u^3)(s)\|_{L^1} ds \\
\leq & C\int_0^{t/2} \|\partial_x^l S(t-s)\|_{L^1} \|\partial_x(u^3(\cdot,s))\|_{L^1} ds + C\int_{t/2}^t \|\partial_x S(t-s)\|_{L^1} \|\partial_x^l(u^3(\cdot,s))\|_{L^1} ds \\
\leq & C\int_0^{t/2} (t-s)^{-l/2}(1+(t-s)^{-1/4})(1+s)^{-3/2}(\|u_0\|_{L^1}+\|u_0\|_{H^3})^3 ds \\
& + C\int_{t/2}^t (t-s)^{-1/2}(1+(t-s)^{-1/4})(1+s)^{-1-l/2}(\|u_0\|_{L^1}+\|u_0\|_{H^3})^3 ds \\
\leq & Ct^{-l/2}(1+t^{-1/4})(\|u_0\|_{L^1}+\|u_0\|_{H^3}) \\
& + C(1+t)^{-1-l/2}(t^{1/2}+t^{1/4})(\|u_0\|_{L^1}+\|u_0\|_{H^3}) \\
\leq & C(\|u_0\|_{L^1}+\|u_0\|_{H^3})t^{-l/2}(1+t^{-1/4}), \quad t>0.
\end{aligned}
\tag{2.21}
$$

Therefore, summing up (2.9) and (2.19) through (2.21), we get (2.7). □

3 Basic Lemmas and Auxiliary Problem

In order to show basic estimates for auxiliary problems, we prepare a couple of lemmas. First, we treat the nonlinear diffusion wave $\chi(x,t)$ defined by (1.3), and the heat kernel

$$
G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.
$$

A direct calculation yields

$$
|\chi(x,t)| \leq C|\delta|(1+t)^{-1/2} e^{-x^2/4(1+t)}, \quad t \geq 0,\ x \in \mathbb{R}. \tag{3.1}
$$

Moreover, we can estimate derivatives of $\chi(x,t)$ and $G(x,t)$ (for the proof, see e.g. Lemma 4.1 of [12]).

Lemma 3.1. *Let α and β be non-negative integers. Then, for $p \in [1,\infty]$, we have*

$$
\|\partial_x^\alpha \partial_t^\beta \chi(\cdot,t)\|_{L^p} \leq C|\delta|(1+t)^{-(1/2)(1-1/p)-\alpha/2-\beta}, \quad t \geq 0, \tag{3.2}
$$

$$
\|\partial_x^\alpha \partial_t^\beta G(\cdot,t)\|_{L^p} \leq Ct^{-(1/2)(1-1/p)-\alpha/2-\beta}, \quad t > 0. \tag{3.3}
$$

Next, for the latter sake, we define

$$
\eta_1(x,t) \equiv \eta_*\left(\frac{x}{\sqrt{1+t}}\right) = \exp\left(\frac{b}{2}\int_{-\infty}^x \chi(y,t)dy\right), \tag{3.4}
$$

$$
\eta_2(x,t) \equiv (\eta_1(x,t))^{-1}. \tag{3.5}
$$

For these functions, we can easily show

$$
\min\{1, e^{b\delta/2}\} \leq \eta_1(x,t) \leq \max\{1, e^{b\delta/2}\}, \tag{3.6}
$$

$$
\min\{1, e^{-b\delta/2}\} \leq \eta_2(x,t) \leq \max\{1, e^{-b\delta/2}\}. \tag{3.7}
$$

Moreover, we have the following estimates by using Lemma 3.1 (for the proof, see Corollary 2.3 of [9]).

Lemma 3.2. *Let l be a positive integer and $p \in [1,\infty]$. For $i = 1,2$, if $|\delta| \leq 1$, then we have*

$$
\|\partial_x^l \eta_i(\cdot,t)\|_{L^p} \leq C|\delta|(1+t)^{-(1/2)(1-1/p)-l/2+1/2}, \quad t \geq 0. \tag{3.8}
$$

In order to prove the main theorem, we introduce an auxiliary problem. We set $\psi(x,t) = u(x,t) - \chi(x,t)$, where $u(x,t)$ is the original solution to (1.1) and $\chi(x,t)$ is the nonlinear diffusion wave defined by (1.3). Then, $\psi(x,t)$ satisfies the following equation:

$$
\psi_t + (b\chi\psi)_x + \left(\frac{b}{2}\psi^2\right)_x + \left(\frac{c}{3}(\psi+\chi)^3\right)_x + k\psi_{xxx} + k\chi_{xxx} - \psi_{xx} = 0.
$$

Based on the observation given in the paper [5] and [9], one may guess that the main term of asymptotic expansion of ψ at $t \to \infty$ is governed by the solution to the following equation:

$$v_t + (b\chi v)_x + \left(\frac{c}{3}\chi^3\right)_x + k\chi_{xxx} - v_{xx} = 0.$$

This observation leads to the following auxiliary problem:

$$\begin{aligned} z_t + (b\chi z)_x - z_{xx} &= \partial_x \lambda(x,t), \quad t > 0, \quad x \in \mathbb{R}, \\ z(x,0) &= z_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{3.9}$$

where $\lambda(x,t)$ is a given regular function decaying fast enough at spatial infinity. The explicit representation formula (3.11) below plays an important roles in the proof of the main theorem, especially in the proofs of Proposition 4.2 and Proposition 4.3 below. If we set

$$U[h](x,t,s) \equiv \int_{\mathbb{R}} \partial_x(G(x-y,t-s)\eta_1(x,t))\eta_2(y,s)\left(\int_{-\infty}^{y} h(\xi)d\xi\right)dy, \quad 0 \le s < t, \quad x \in \mathbb{R}, \tag{3.10}$$

then we have:

Lemma 3.3. *The solution of* (3.9) *is given by*

$$z(x,t) = U[z_0](x,t,0) + \int_0^t U[\partial_x \lambda(s)](x,t,s)ds, \quad t > 0, \quad x \in \mathbb{R}. \tag{3.11}$$

Proof. We set

$$r(x,t) = \int_{-\infty}^{x} z(y,t)dy,$$

and integrate both sides of the equation (3.9). Then, we get

$$\begin{aligned} r_t + b\chi r_x - r_{xx} &= \lambda, \quad t > 0, \quad x \in \mathbb{R}, \\ r(x,0) &= \int_{-\infty}^{x} z_0(y)dy. \end{aligned} \tag{3.12}$$

Multiplying $\eta_2(x,t)$ both sides of (3.12), we have

$$\eta_2 r_t - 2(\partial_x \eta_2) r_x - \eta_2 r_{xx} = \eta_2 \lambda, \tag{3.13}$$

since $\partial_x \eta_2(x,t) = -(b/2)\chi(x,t)\eta_2(x,t)$. Now, we put $E(x,t) = \eta_2(x,t)r(x,t)$. Since $\partial_t \eta_2 - \partial_x^2 \eta_2 = 0$, (3.13) leads to

$$E_t - E_{xx} = \eta_2 \lambda.$$

Therefore, we obtain

$$E(x,t) = \int_{\mathbb{R}} G(x-y,t)E(y,0)dy + \int_0^t \int_{\mathbb{R}} G(x-y,t-s)\eta_2(y,s)\lambda(y,s)dyds,$$

or

$$\eta_2(x,t)r(x,t) = \int_{\mathbb{R}} G(x-y,t)\eta_2(y,0)r(y,0)dy + \int_0^t \int_{\mathbb{R}} G(x-y,t-s)\eta_2(y,s)\lambda(y,s)dyds.$$

Since $\eta_1 = \eta_2^{-1}$, we have

$$\begin{aligned} \int_{-\infty}^{x} z(y,t)dy =& \eta_1(x,t) \int_{\mathbb{R}} G(x-y,t)\eta_2(y,0)\left(\int_{-\infty}^{y} z_0(\xi)d\xi\right)dy \\ &+ \eta_1(x,t) \int_0^t \int_{\mathbb{R}} G(x-y,t-s)\eta_2(y,s)\lambda(y,s)dyds. \end{aligned}$$

Thus we get (3.11). □

For the first term and the second term of (3.11), the following estimates are established (for the proof, see Corollary 3.4 and Lemma 3.5 in [9]).

Lemma 3.4. *Let m be a positive integer. Assume that $|\delta| \leq 1$, $z_0 \in H^m(\mathbb{R}) \cap L^1_1(\mathbb{R})$ and $\int_{\mathbb{R}} z_0(x)dx = 0$. Then the estimate*

$$\|\partial_x^l U[z_0](\cdot, t, 0)\|_{L^2} \leq C(\|z_0\|_{H^l} + \|z_0\|_{L^1_1})(1+t)^{-3/4-l/2}, \quad t > 0 \tag{3.14}$$

holds for any integer $0 \leq l \leq m$.

Lemma 3.5. *Let m be a positive integer. Assume that $|\delta| \leq 1$ and $\lambda \in C^0(0,\infty; H^m) \cap C^0(0,\infty; W^{m,1})$. Then the estimate*

$$\begin{aligned}\left\|\partial_x^l \int_0^t U[\partial_x \lambda(s)](\cdot, t, s)ds\right\|_{L^2} \leq & C\int_0^{t/2}(1+t-s)^{-3/4-l/2}\|\lambda(\cdot, s)\|_{L^1}ds \\ & + C\sum_{n=0}^{l}\int_{t/2}^t (1+t-s)^{-3/4}(1+s)^{-(l-n)/2}\|\partial_x^n \lambda(\cdot, s)\|_{L^1}ds \\ & + C\sum_{n=0}^{l}\left(\int_0^t e^{-(t-s)}(1+s)^{-(l-n)}\|\partial_x^n \lambda(\cdot, s)\|_{L^2}^2 ds\right)^{1/2}\end{aligned} \tag{3.15}$$

holds for any integer $0 \leq l \leq m$.

4 Proof of the Main Theorem

In this section, we shall prove our main theorem. First, we consider

$$\begin{aligned} v_t + (b\chi v)_x + \left(\frac{c}{3}\chi^3\right)_x + k\chi_{xxx} - v_{xx} = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ v(x, 0) = 0, \quad x \in \mathbb{R}. \end{aligned} \tag{4.1}$$

The solution of this problem satisfies the following estimates.

Lemma 4.1. *Let l be a non-negative integer. Assume that $|\delta| \leq 1$. Then we have*

$$\|\partial_x^l v(\cdot, t)\|_{L^2} \leq C|\delta|(1+t)^{-3/4-l/2}\log(2+t), \quad t \geq 0, \tag{4.2}$$

where $v(x,t)$ is the solution to (4.1)*. In particular, we get*

$$\|\partial_x^l v(\cdot, t)\|_{L^\infty} \leq C|\delta|(1+t)^{-1-l/2}\log(2+t), \quad t \geq 0. \tag{4.3}$$

Proof. Applying Lemma 3.3, the solution $v(x,t)$ to (4.1) is given by

$$v(x,t) = \int_0^t U\left[\left(\partial_x\left(-\frac{c}{3}\chi^3\right) - k\chi_{xxx}\right)(s)\right](x, t, s)ds.$$

By Lemma 3.5, we have

$$\begin{aligned}\|\partial_x^l v(\cdot, t)\|_{L^2} \leq & C\int_0^{t/2}(1+t-s)^{-3/4-l/2}(\|\chi^3(\cdot, s)\|_{L^1} + \|\chi_{xx}(\cdot, s)\|_{L^1})ds \\ & + C\sum_{m=0}^{l}\int_{t/2}^t(1+t-s)^{-3/4}(1+s)^{-(l-m)/2}(\|\partial_x^m(\chi^3(\cdot, s))\|_{L^1} + \|\partial_x^{m+2}\chi(\cdot, s)\|_{L^1})ds \\ & + C\sum_{m=0}^{l}\left(\int_0^t e^{-(t-s)}(1+s)^{-(l-m)}(\|\partial_x^m(\chi^3(\cdot, s))\|_{L^2}^2 + \|\partial_x^{m+2}\chi(\cdot, s)\|_{L^2}^2)ds\right)^{1/2} \\ \equiv & I_1 + I_2 + I_3.\end{aligned} \tag{4.4}$$

For $p \in [1, \infty]$, we obtain

$$\begin{aligned}\|\partial_x^m(\chi^3(\cdot, s))\|_{L^p} \leq & C\sum_{n=0}^{m}\sum_{i=0}^{m-n}\|\partial_x^n\chi(\cdot, s)\|_{L^\infty}\|\partial_x^i\chi(\cdot, s)\|_{L^\infty}\|\partial_x^{m-n-i}\chi(\cdot, s)\|_{L^p} \\ \leq & C|\delta|^3\sum_{n=0}^{m}\sum_{i=0}^{m-n}(1+s)^{-1/2-n/2}(1+s)^{-1/2-i/2}(1+s)^{-1/2+1/2p-(m-n-i)/2} \\ \leq & C|\delta|^3(1+s)^{-(3-1/p+m)/2},\end{aligned} \tag{4.5}$$

where we used Lemma 3.1. Thus we get

$$
\begin{aligned}
I_1 \le & C\int_0^{t/2}(1+t-s)^{-3/4-l/2}(|\delta|^3(1+s)^{-1}+|\delta|(1+s)^{-1})ds \\
\le & C|\delta|(1+t)^{-3/4-l/2}\log(2+t).
\end{aligned} \tag{4.6}
$$

Moreover, we have from (4.5) and Lemma 3.1

$$
\begin{aligned}
I_2 \le & C\sum_{m=0}^{l}\int_{t/2}^{t}(1+t-s)^{-3/4}(1+s)^{-(l-m)/2}(|\delta|^3(1+s)^{-1-m/2}+|\delta|(1+s)^{-1-m/2})ds \\
\le & C|\delta|\int_{t/2}^{t}(1+t-s)^{-3/4}(1+s)^{-1-l/2}ds \\
\le & C|\delta|(1+t)^{-3/4-l/2}
\end{aligned} \tag{4.7}
$$

and

$$
\begin{aligned}
I_3 \le & C\sum_{m=0}^{l}\left(\int_0^t e^{-(t-s)}(1+s)^{-(l-m)}(|\delta|^6(1+s)^{-5/2-m}+|\delta|^2(1+s)^{-5/2-m})ds\right)^{1/2} \\
\le & C|\delta|\left(\int_0^t e^{-(t-s)}(1+s)^{-5/2-l}ds\right)^{1/2} \\
\le & C|\delta|(1+t)^{-5/4-l/2}.
\end{aligned} \tag{4.8}
$$

Summing up (4.4), (4.6), (4.7) and (4.8), we obtain (4.2). □

Our first step to prove the main theorem is to show the following proposition.

Proposition 4.2. *If $u_0 \in L^1_1(\mathbb{R})\cap H^3(\mathbb{R})$ and $\|u_0\|_{L^1_1}+\|u_0\|_{H^3}$ is sufficiently small, then the estimate*

$$
\|\partial_x^l(u(\cdot,t)-\chi(\cdot,t)-v(\cdot,t))\|_{L^2}\le C(\|u_0\|_{L^1_1}+\|u_0\|_{H^3})(1+t)^{-3/4-l/2},\quad t\ge 0 \tag{4.9}
$$

holds for $l=0,1$, where $\chi(x,t)$ is defined by (1.3), while $v(x,t)$ is the solution to (4.1). In particular, we get

$$
\|u(\cdot,t)-\chi(\cdot,t)-v(\cdot,t)\|_{L^\infty}\le C(\|u_0\|_{L^1_1}+\|u_0\|_{H^3})(1+t)^{-1},\quad t\ge 0. \tag{4.10}
$$

Proof. We set

$$
w(x,t)=u(x,t)-\chi(x,t)-v(x,t).
$$

Then $w(x,t)$ satisfies the following equation:

$$
\begin{aligned}
& w_t+(b\chi w)_x-w_{xx}=g(w,\chi,v)_x-kw_{xxx}-kv_{xxx},\quad t>0,\quad x\in\mathbb{R}, \\
& w(x,0)=w_0,
\end{aligned}
$$

where we have set

$$
g(w,\chi,v)=-\frac{b}{2}(w+v)^2-\frac{c}{3}\Big(w^3+v^3+3(w+v)(w+\chi)(\chi+v)\Big),
$$

$$
w_0(x)=u_0(x)-\chi(x,0).
$$

By the assumption on the initial data, (1.4) and (3.2), we get $w_0\in L^1_1(\mathbb{R})\cap H^3(\mathbb{R})$ and $\int_{\mathbb{R}} w_0(x)dx=0$. From Lemma 3.3, we obtain

$$
\begin{aligned}
w(x,t)= & U[w_0](x,t,0)+\int_0^t U[\partial_x g(w,\chi,v)(s)](x,t,s)ds-k\int_0^t U[(w_{xxx}+v_{xxx})(s)](x,t,s)ds \\
\equiv & I_1+I_2+I_3.
\end{aligned} \tag{4.11}
$$

Now, we define $N(T)$ by

$$
N(T)\equiv\sup_{0\le t\le T}\sum_{n=0}^{1}(1+t)^{3/4+n/2}\|\partial_x^n w(\cdot,t)\|_{L^2}. \tag{4.12}
$$

Then, from the Sobolev inequality, we have

$$\|w(\cdot,t)\|_{L^\infty} \le N(T)(1+t)^{-1}. \tag{4.13}$$

Before evaluating I_1, I_2 and I_3, we prepare the following estimates for $l=0,1$:

$$\|\partial_x^l g(\cdot,t)\|_{L^1} \le C(1+t)^{-3/2-l/2}((|\delta|\log(2+t))^2 + N(T)^2), \tag{4.14}$$

$$\|\partial_x^l g(\cdot,t)\|_{L^2} \le C(1+t)^{-7/4-l/2}((|\delta|\log(2+t))^2 + N(T)^2). \tag{4.15}$$

We shall prove only (4.14), since we can prove (4.15) in the same way. Here and later, $|\delta|$ and $N(T)$ are assumed to be small. We put $h_1(x,t) = w(x,t) + v(x,t)$, $h_2(x,t) = w(x,t) + \chi(x,t)$ and $h_3(x,t) = \chi(x,t) + v(x,t)$. Let $m = 0,1$ and $0 \le t \le T$. Then we have from Lemma 4.1 and (4.12)

$$\|\partial_x^m h_1(\cdot,t)\|_{L^2} \le C(1+t)^{-3/4-m/2}(|\delta|\log(2+t) + N(T)). \tag{4.16}$$

In particular, from the Sobolev inequality, we get

$$\|h_1(\cdot,t)\|_{L^\infty} \le C(1+t)^{-1}(|\delta|\log(2+t) + N(T)). \tag{4.17}$$

Moreover, we get from Lemma 3.1 and (4.12)

$$\|\partial_x^m h_2(\cdot,t)\|_{L^2} \le C(1+t)^{-1/4-m/2}(|\delta| + N(T)). \tag{4.18}$$

In particular, we get

$$\|h_2(\cdot,t)\|_{L^\infty} \le C(1+t)^{-1/2}(|\delta| + N(T)). \tag{4.19}$$

Moreover, from Lemma 3.1 and (4.3), we have

$$\|\partial_x^m h_3(\cdot,t)\|_{L^\infty} \le C|\delta|(1+t)^{-1/2-m/2}. \tag{4.20}$$

Hence, for $l = 0,1$, we have from Lemma 4.1 (4.12), (4.13), (4.16), (4.18) and (4.20)

$$\begin{aligned} \|\partial_x^l((w+v)^2(\cdot,t))\|_{L^1} \le& C\sum_{m=0}^{l} \|\partial_x^m h_1(\cdot,t)\|_{L^2}\|\partial_x^{l-m} h_1(\cdot,t)\|_{L^2} \\ \le& C(1+t)^{-3/2-l/2}((|\delta|\log(2+t))^2 + N(T)^2), \end{aligned} \tag{4.21}$$

$$\begin{aligned} \|\partial_x^l(w^3(\cdot,t))\|_{L^1} \le& C\|\partial_x^l w(\cdot,t)\|_{L^2}\|w(\cdot,t)\|_{L^2}\|w(\cdot,t)\|_{L^\infty} \\ &+ C\sum_{m=0}^{l-1}\sum_{n=0}^{l-m} \|\partial_x^m w(\cdot,t)\|_{L^\infty}\|\partial_x^n w(\cdot,t)\|_{L^2}\|\partial_x^{l-m-n} w(\cdot,t)\|_{L^2} \\ \le& C(1+t)^{-3/4-l/2}(1+t)^{-3/4}(1+t)^{-1}N(T)^3 \\ &+ C\sum_{m=0}^{l-1}\sum_{n=0}^{l-m}(1+t)^{-1-m/2}(1+t)^{-3/4-n/2}(1+t)^{-3/4-l/2+m/2+n/2}N(T)^3 \\ \le& C(1+t)^{-5/2-l/2}N(T)^3, \end{aligned} \tag{4.22}$$

$$\begin{aligned} \|\partial_x^l(v^3(\cdot,t))\|_{L^1} \le& C\sum_{m=0}^{l}\sum_{n=0}^{l-m} \|\partial_x^m v(\cdot,t)\|_{L^\infty}\|\partial_x^n v(\cdot,t)\|_{L^2}\|\partial_x^{l-m-n} v(\cdot,t)\|_{L^2} \\ \le& C\sum_{m=0}^{l}\sum_{n=0}^{l-m}(1+t)^{-1-m/2}(1+t)^{-3/4-n/2}(1+t)^{-3/4-l/2+m/2+n/2}(|\delta|\log(2+t))^3 \\ \le& C(1+t)^{-5/2-l/2}(|\delta|\log(2+t))^3 \\ \le& C(1+t)^{-3/2-l/2}(|\delta|\log(2+t))^2 \end{aligned} \tag{4.23}$$

and

$$
\begin{aligned}
\|\partial_x^l(h_1h_2h_3)(\cdot,t)\|_{L^1} \leq & C\sum_{m=0}^{l}\sum_{n=0}^{l-m}\|\partial_x^m h_1(\cdot,t)\|_{L^2}\|\partial_x^n h_2(\cdot,t)\|_{L^2}\|\partial_x^{l-m-n}h_3(\cdot,t)\|_{L^\infty} \\
\leq & C\sum_{m=0}^{l}\sum_{n=0}^{l-m}(1+t)^{-3/4-m/2}(|\delta|\log(2+t)+N(T)) \\
& \times(1+t)^{-1/4-n/2}(|\delta|+N(T))(1+t)^{-1/2-l/2+m/2+n/2}|\delta| \\
\leq & C(1+t)^{-3/2-l/2}(|\delta|\log(2+t)+N(T))(|\delta|+N(T)) \\
\leq & C(1+t)^{-3/2-l/2}((|\delta|\log(2+t))^2+N(T)^2).
\end{aligned}
\tag{4.24}
$$

We note that the second term in (4.22) does not appear for $l=0$. Summing up (4.21) through (4.24), we obtain (4.14).

Now, we start with evaluation of I_1, I_2 and I_3. By using Lemma 3.4 and $|\delta|\leq\|u_0\|_{L^1_1}$, we get

$$
\|\partial_x^l I_1(\cdot,t)\|_{L^2}\leq C(\|u_0\|_{H^l}+\|u_0\|_{L^1_1})(1+t)^{-3/4-l/2},\quad l=0,1,2,3. \tag{4.25}
$$

From Lemma 3.5, for $l=0,1$, we have

$$
\begin{aligned}
\|\partial_x^l I_2(\cdot,t)\|_{L^2} \leq & C\int_0^{t/2}(1+t-s)^{-3/4-l/2}\|g(\cdot,s)\|_{L^1}ds \\
& +C\sum_{m=0}^{l}\int_{t/2}^{t}(1+t-s)^{-3/4}(1+s)^{-(l-m)/2}\|\partial_x^m g(\cdot,s)\|_{L^1}ds \\
& +C\sum_{m=0}^{l}\left(\int_0^t e^{-(t-s)}(1+s)^{-(l-m)}\|\partial_x^m g(\cdot,s)\|_{L^2}^2 ds\right)^{1/2} \\
\equiv & I_{2.1}+I_{2.2}+I_{2.3}.
\end{aligned}
\tag{4.26}
$$

We have from (4.14) and (4.15)

$$
\begin{aligned}
I_{2.1} \leq & C\int_0^{t/2}(1+t-s)^{-3/4-l/2}(1+s)^{-3/2}((|\delta|\log(2+s))^2+N(T)^2)ds \\
\leq & C(1+t)^{-3/4-l/2}(|\delta|^2+N(T)^2),
\end{aligned}
\tag{4.27}
$$

$$
\begin{aligned}
I_{2.2} \leq & C\sum_{m=0}^{l}\int_{t/2}^{t}(1+t-s)^{-3/4}(1+s)^{-3/2-l/2}((|\delta|\log(2+s))^2+N(T)^2)ds \\
\leq & C(1+t)^{-3/2-l/2}((|\delta|\log(2+t))^2+N(T)^2)\int_{t/2}^{t}(1+t-s)^{-3/4}ds \\
\leq & C(1+t)^{-5/4-l/2}((|\delta|\log(2+t))^2+N(T)^2)
\end{aligned}
\tag{4.28}
$$

and

$$
\begin{aligned}
I_{2.3} \leq & C\sum_{m=0}^{l}\left(\int_0^t e^{-(t-s)}(1+s)^{-7/2-l}((|\delta|\log(2+s))^4+N(T)^4)ds\right)^{1/2} \\
\leq & C((|\delta|\log(2+t))^2+N(T)^2)\left(\int_0^t e^{-(t-s)}(1+s)^{-7/2-l}ds\right)^{1/2} \\
\leq & C(1+t)^{-7/4-l/2}((|\delta|\log(2+t))^2+N(T)^2).
\end{aligned}
\tag{4.29}
$$

Summarizing (4.26) through (4.29), we obtain

$$
\|\partial_x^l I_2(\cdot,t)\|_{L^2}\leq C(|\delta|^2+N(T)^2)(1+t)^{-(3/4+l/2)}. \tag{4.30}
$$

Finally, we evaluate I_3. At first, since $\psi=u-\chi=w+v$, from Lemma 2.3, Lemma 3.1 and Lemma 4.1, we have

$$
\|\partial_x^l\psi(\cdot,t)\|_{L^1}\leq C(\|u_0\|_{L^1}+\|u_0\|_{H^3})t^{-l/2}(1+t^{-1/4}),\quad t>0,\quad l=0,1,2,3, \tag{4.31}
$$

$$
\|\partial_x^l\psi(\cdot,t)\|_{L^2}\leq C(\|u_0\|_{L^1}+\|u_0\|_{H^3})(1+t)^{-1/4-l/2},\quad t\geq 0,\quad l=0,1,2,3, \tag{4.32}
$$

$$
\|\partial_x^l\psi(\cdot,t)\|_{L^2}\leq C(|\delta|\log(2+t)+N(T))(1+t)^{-3/4-l/2},\quad l=0,1. \tag{4.33}
$$

From the definition of I_3, it follows that

$$\begin{aligned}
\partial_x^l I_3(x,t) =& -k\partial_x^l \int_0^t U[\psi_{xxx}(s)](x,t,s)ds \\
=& -k\int_0^t \int_{\mathbb{R}} \partial_x^{l+1}(G(x-y,t-s)\eta_1(x,t))\eta_2(y,s)\psi_{yy}(y,s)dyds \\
=& -k\sum_{n=0}^{l+1}\binom{l+1}{n}\partial_x^{l+1-n}\eta_1(x,t)\partial_x^n J(x,t),
\end{aligned}$$

where we put

$$J(x,t) = \int_0^t \int_{\mathbb{R}} G(x-y,t-s)\eta_2(y,s)\psi_{yy}(y,s)dyds.$$

Therefore, from Lemma 3.2, we have

$$\|\partial_x^l I_3(\cdot,t)\|_{L^2} \le C\sum_{n=0}^{l+1}(1+t)^{-(l+1-n)/2}\|\partial_x^n J(\cdot,t)\|_{L^2}. \tag{4.34}$$

By making the integration by parts, we have

$$\begin{aligned}
J(x,t) =& \int_0^t \int_{\mathbb{R}} \partial_x^2 G(x-y,t-s)\eta_2(y,s)\psi(y,s)dyds - 2\int_0^t \int_{\mathbb{R}} \partial_x G(x-y,t-s)\partial_y\eta_2(y,s)\psi(y,s)dyds \\
&+ \int_0^t \int_{\mathbb{R}} G(x-y,t-s)\partial_y^2\eta_2(y,s)\psi(y,s)dyds.
\end{aligned}$$

Then, it follows that

$$\|\partial_x^n J(\cdot,t)\|_{L^2} \le C\sum_{r=0}^{2}\left\|\partial_x^n \int_0^t \int_{\mathbb{R}} \partial_x^{2-r}G(x-y,t-s)\partial_y^r\eta_2(y,s)\psi(y,s)dyds\right\|_{L^2} \equiv C\sum_{r=0}^{2}J_r. \tag{4.35}$$

First, we shall evaluate J_r for $r=0,1$. By Plancherel's Theorem, we have

$$\begin{aligned}
J_r \le& \left\|(i\xi)^{n+2-r}\int_0^t e^{-(t-s)|\xi|^2}F[(\partial_x^r\eta_2)\psi](\xi,s)ds\right\|_{L^2(|\xi|\le 1)} \\
&+ \left\|(i\xi)^{n+2-r}\int_0^t e^{-(t-s)|\xi|^2}F[(\partial_x^r\eta_2)\psi](\xi,s)ds\right\|_{L^2(|\xi|\ge 1)} \\
\equiv& J_{r.1}+J_{r.2}
\end{aligned} \tag{4.36}$$

and

$$\begin{aligned}
J_{r.1} \le& \left(\int_0^{t/2} + \int_{t/2}^t\right)\|(i\xi)^{n+2-r}e^{-(t-s)|\xi|^2}F[(\partial_x^r\eta_2)\psi](\xi,s)\|_{L^2(|\xi|\le 1)}ds \\
\equiv& J_{r.1.1}+J_{r.1.2}.
\end{aligned} \tag{4.37}$$

Since

$$\int_{|\xi|\le 1}|\xi|^j e^{-2(t-s)|\xi|^2}d\xi \le C(1+t-s)^{-j/2-1/2}, \quad j\ge 0, \tag{4.38}$$

and (4.31), (3.7) and Lemma 3.2, we have

$$\begin{aligned}
J_{r.1.1} \le& C\int_0^{t/2}\sup_{|\xi|\le 1}|F[(\partial_x^r\eta_2)\psi](\xi,s)|\left(\int_{|\xi|\le 1}|\xi|^{2n+4-2r}e^{-2(t-s)|\xi|^2}d\xi\right)^{1/2}ds \\
\le& C\int_0^{t/2}(1+t-s)^{-n/2-5/4+r/2}\|((\partial_x^r\eta_2)\psi)(\cdot,s)\|_{L^1}ds \\
\le& C(1+t)^{-n/2-5/4+r/2}\int_0^{t/2}(1+s)^{-r/2}(\|u_0\|_{L^1}+\|u_0\|_{H^3})(1+s^{-1/4})ds \\
\le& C(\|u_0\|_{L^1}+\|u_0\|_{H^3})(1+t)^{-n/2-1/4}.
\end{aligned} \tag{4.39}$$

While, we have from Lemma 3.2, (3.7), (4.31), (4.32) and (4.38)

$$
\begin{aligned}
J_{r.1.2} \leq & C\int_{t/2}^{t} \sup_{|\xi|\leq 1} |(i\xi)^{n+1-r}F[(\partial_x^r\eta_2)\psi](\xi,s)| \left(\int_{|\xi|\leq 1} |\xi|^2 e^{-2(t-s)|\xi|^2} d\xi\right)^{1/2} ds \\
\leq & C\int_{t/2}^{t} (1+t-s)^{-3/4} \|\partial_x^{n+1-r}((\partial_x^r\eta_2)\psi)(\cdot,s)\|_{L^1} ds \\
\leq & C\sum_{m=0}^{n+1-r}\int_{t/2}^{t} (1+t-s)^{-3/4} \|\partial_x^{n+1-m}\eta_2(\cdot,s)\partial_x^m\psi(\cdot,s)\|_{L^1} ds \\
\leq & C\sum_{m=0}^{n+1-r}\int_{t/2}^{t} (1+t-s)^{-3/4}(1+s)^{-n/2-1/2+m/2}(\|u_0\|_{L^1}+\|u_0\|_{H^3})s^{-m/2}(1+s^{-1/4}) ds \\
\leq & C(\|u_0\|_{L^1}+\|u_0\|_{H^3})(1+t)^{-n/2-1/4}, \quad t\geq 1.
\end{aligned}
\tag{4.40}
$$

For $|\xi|\geq 1$, by using the Schwarz inequality, we have

$$
\begin{aligned}
& \left|(i\xi)^{n+2-r}\int_0^t e^{-(t-s)|\xi|^2} F[(\partial_x^r\eta_2)\psi](\xi,s) ds\right| \\
\leq & C\int_0^t |\xi| e^{-(t-s)|\xi|^2} |(i\xi)^{n+1-r}F[(\partial_x^r\eta_2)\psi](\xi,s)| ds \\
\leq & C\left(\int_0^t |\xi|^2 e^{-(t-s)|\xi|^2} ds\right)^{1/2}\left(\int_0^t e^{-(t-s)|\xi|^2}|(i\xi)^{n+1-r}F[(\partial_x^r\eta_2)\psi](\xi,s)|^2 ds\right)^{1/2} \\
\leq & C\left(\int_0^t e^{-(t-s)|\xi|^2}|(i\xi)^{n+1-r}F[(\partial_x^r\eta_2)\psi](\xi,s)|^2 ds\right)^{1/2}.
\end{aligned}
$$

Therefore we have from Lemma 3.2 and (4.32)

$$
\begin{aligned}
J_{r.2} \leq & C\left(\int_{|\xi|\geq 1}\int_0^t e^{-(t-s)|\xi|^2}|(i\xi)^{n+1-r}F[(\partial_x^r\eta_2)\psi](\xi,s)|^2 ds d\xi\right)^{1/2} \\
\leq & C\left(\int_0^t e^{-(t-s)}\int_{|\xi|\geq 1}|(i\xi)^{n+1-r}F[(\partial_x^r\eta_2)\psi](\xi,s)|^2 d\xi ds\right)^{1/2} \\
\leq & C\sum_{m=0}^{n+1-r}\left(\int_0^t e^{-(t-s)}\|\partial_x^{n+1-m}\eta_2(\cdot,s)\partial_x^m\psi(\cdot,s)\|_{L^2}^2 ds\right)^{1/2} \\
\leq & C\sum_{m=0}^{n+1-r}\left(\int_0^t e^{-(t-s)}|\delta|^2(1+s)^{-n-1+m}(\|u_0\|_{L^1}+\|u_0\|_{H^3})^2(1+s)^{-1/2-m} ds\right)^{1/2} \\
\leq & C(\|u_0\|_{L^1}+\|u_0\|_{H^3})\left(\int_0^t e^{-(t-s)}(1+s)^{-3/2-n} ds\right)^{1/2} \\
\leq & C(\|u_0\|_{L^1}+\|u_0\|_{H^3})(1+t)^{-n/2-3/4}.
\end{aligned}
\tag{4.41}
$$

From (4.36), (4.37), (4.39), (4.40) and (4.41), we obtain

$$
J_r \leq C(\|u_0\|_{L^1}+\|u_0\|_{H^3})(1+t)^{-n/2-1/4}, \quad t\geq 1, \quad r=0,1. \tag{4.42}
$$

Next, we evaluate J_2. For $n=0$, we have from Lemma 2.1 with $k=0$

$$
\begin{aligned}
J_2 \leq & C\int_0^t \|e^{(t-s)\Delta}[((\partial_x^2\eta_2)\psi)(s)](\cdot)\|_{L^2} ds \\
\leq & C\int_0^t \big((1+t-s)^{-1/4}\|((\partial_x^2\eta_2)\psi)(\cdot,s)\|_{L^1} + e^{-(t-s)}\|((\partial_x^2\eta_2)\psi)(\cdot,s)\|_{L^2}\big) ds \\
\equiv & J_{2.0.1}+J_{2.0.2}.
\end{aligned}
\tag{4.43}
$$

From Lemma 3.2 and (4.33), we have

$$\begin{aligned}J_{2.0.1} =&C\bigg(\int_0^{t/2}+\int_{t/2}^{t}\bigg)(1+t-s)^{-1/4}\|((\partial_x^2\eta_2)\psi)(\cdot,s)\|_{L^1}ds\\ \le&C(1+t)^{-1/4}\int_0^{t/2}|\delta|(1+s)^{-3/4}(|\delta|\log(2+s)+N(T))(1+s)^{-3/4}ds\\ &+C\int_{t/2}^{t}(1+t-s)^{-1/4}|\delta|(1+s)^{-3/4}(|\delta|\log(2+s)+N(T))(1+s)^{-3/4}ds\\ \le&C(1+t)^{-1/4}(|\delta|^2+N(T)^2)+C(1+t)^{-3/4}(|\delta|^2\log(2+t)+|\delta|^2+N(T)^2)\\ \le&C(1+t)^{-1/4}(|\delta|^2+N(T)^2)\end{aligned} \tag{4.44}$$

and

$$\begin{aligned}J_{2.0.2} \le&C\int_0^t e^{-(t-s)}|\delta|(1+s)^{-1}(|\delta|\log(2+s)+N(T))(1+s)^{-3/4}ds\\ \le&C(|\delta|^2\log(2+t)+|\delta|^2+N(T)^2)\int_0^t e^{-(t-s)}(1+s)^{-7/4}ds\\ \le&C(1+s)^{-7/4}(|\delta|^2\log(2+t)+|\delta|^2+N(T)^2).\end{aligned} \tag{4.45}$$

Therefore, we obtain from (4.43) through (4.45)

$$J_2 \le C(1+t)^{-1/4}(|\delta|^2+N(T)^2), \quad n=0. \tag{4.46}$$

In the following, let $n \ge 1$. By using Plancherel's theorem, we have

$$\begin{aligned}J_2 \le&\bigg\|(i\xi)^n\int_0^t e^{-(t-s)|\xi|^2}F[(\partial_x^2\eta_2)\psi](\xi,s)ds\bigg\|_{L^2(|\xi|\le1)}\\ &+\bigg\|(i\xi)^n\int_0^t e^{-(t-s)|\xi|^2}F[(\partial_x^2\eta_2)\psi](\xi,s)ds\bigg\|_{L^2(|\xi|\ge1)}\\ \equiv&J_{2.1}+J_{2.2}.\end{aligned} \tag{4.47}$$

First, we estimate $J_{2.1}$. It follows that

$$\begin{aligned}J_{2.1} \le&\bigg(\int_0^{t/2}+\int_{t/2}^{t}\bigg)\|(i\xi)^n e^{-(t-s)|\xi|^2}F[(\partial_x^2\eta_2)\psi](\xi,s)\|_{L^2(|\xi|\le1)}ds\\ \equiv&J_{2.1.1}+J_{2.1.2}.\end{aligned} \tag{4.48}$$

From Lemma 3.2, (4.33) and (4.38), we have

$$\begin{aligned}J_{2.1.1} \le&C\int_0^{t/2}\sup_{|\xi|\le1}|F[(\partial_x^2\eta_2)\psi](\xi,s)|\bigg(\int_{|\xi|\le1}|\xi|^{2n}e^{-2(t-s)|\xi|^2}d\xi\bigg)^{1/2}ds\\ \le&C\int_0^{t/2}(1+t-s)^{-n/2-1/4}\|((\partial_x^2\eta_2)\psi)(\cdot,s)\|_{L^1}ds\\ \le&C(1+t)^{-n/2-1/4}\int_0^{t/2}|\delta|(1+s)^{-3/4}(|\delta|\log(2+s)+N(T))(1+s)^{-3/4}ds\\ \le&C(1+t)^{-n/2-1/4}(|\delta|^2+N(T)^2)\end{aligned} \tag{4.49}$$

and

$$\begin{aligned}
J_{2.1.2} \leq & C\int_{t/2}^{t} \sup_{|\xi|\leq 1} |(i\xi)^{n-1}F[(\partial_x^2\eta_2)\psi](\xi,s)|\left(\int_{|\xi|\leq 1}|\xi|^2 e^{-2(t-s)|\xi|^2}d\xi\right)^{1/2} ds \\
\leq & C\int_{t/2}^{t}(1+t-s)^{-3/4}\|\partial_x^{n-1}((\partial_x^2\eta_2)\psi)(\cdot,s)\|_{L^1}ds \\
\leq & C\sum_{m=0}^{n-1}\int_{t/2}^{t}(1+t-s)^{-3/4}\|\partial_x^{n+1-m}\eta_2(\cdot,s)\partial_x^m\psi(\cdot,s)\|_{L^1}ds \\
\leq & C\sum_{m=0}^{n-1}\int_{t/2}^{t}(1+t-s)^{-3/4}|\delta|(1+s)^{-n/2-1/2+m/2+1/4} \\
& \qquad \times(|\delta|\log(2+s)+N(T))(1+s)^{-m/2-3/4}ds \\
\leq & C(1+t)^{-n/2-1}(|\delta|^2\log(2+t)+|\delta|^2+N(T)^2)\int_{t/2}^{t}(1+t-s)^{-3/4}ds \\
\leq & C(1+t)^{-n/2-3/4}(|\delta|^2\log(2+t)+|\delta|^2+N(T)^2).
\end{aligned} \tag{4.50}$$

In the same way as (4.41), we have from Lemma 3.2 and (4.33)

$$\begin{aligned}
J_{2.2} \leq & C\left(\int_{|\xi|\geq 1}\int_0^t e^{-(t-s)|\xi|^2}|(i\xi)^{n-1}F[(\partial_x^2\eta_2)\psi](\xi,s)|^2 ds d\xi\right)^{1/2} \\
\leq & C\left(\int_0^t e^{-(t-s)}\int_{|\xi|\geq 1}|(i\xi)^{n-1}F[(\partial_x^2\eta_2)\psi](\xi,s)|^2 d\xi ds\right)^{1/2} \\
\leq & C\sum_{m=0}^{n-1}\left(\int_0^t e^{-(t-s)}\|\partial_x^{n+1-m}\eta_2(\cdot,s)\partial_x^m\psi(\cdot,s)\|_{L^2}^2 ds\right)^{1/2} \\
\leq & C\sum_{m=0}^{n-1}\left(\int_0^t e^{-(t-s)}|\delta|^2(1+s)^{-n-1+m}(|\delta|\log(2+s)+N(T))^2(1+s)^{-3/2-m}ds\right)^{1/2} \\
\leq & C(|\delta|^2\log(2+t)+|\delta|^2+N(T)^2)\left(\int_0^t e^{-(t-s)}(1+s)^{-5/2-n}ds\right)^{1/2} \\
\leq & C(1+t)^{-n/2-5/4}(|\delta|^2\log(2+t)+|\delta|^2+N(T)^2).
\end{aligned} \tag{4.51}$$

Therefore, summing up (4.46) through (4.51), for $n \geq 0$, we obtain

$$J_2 \leq C(1+t)^{-n/2-1/4}(|\delta|^2+N(T)^2). \tag{4.52}$$

Therefore, from (4.34), (4.35), (4.42) and (4.52), we have

$$\|\partial_x^l I_3(\cdot,t)\|_{L^2} \leq C(\|u_0\|_{L^1}+\|u_0\|_{H^3}+|\delta|^2+N(T)^2)(1+t)^{-(3/4+l/2)}, \quad t\geq 1, \quad l=0,1. \tag{4.53}$$

Since $|\delta| \leq \|u_0\|_{L^1_1}$, summarizing (4.11), (4.25), (4.30) and (4.53), we obtain

$$\|\partial_x^l w(\cdot,t)\|_{L^2} \leq C(1+t)^{-3/4-l/2}(\|u_0\|_{L^1_1}+\|u_0\|_{H^3}+N(T)^2), \quad 1\leq t\leq T, \quad l=0,1. \tag{4.54}$$

For $0 \leq t \leq 1$, from (2.6), (3.2) and (4.2), we obtain

$$\begin{aligned}
\|\partial_x^l w(\cdot,t)\|_{L^2} \leq & C\|\partial_x^l u(\cdot,t)\|_{L^2}+C\|\partial_x^l\chi(\cdot,t)\|_{L^2}+C\|\partial_x^l v(\cdot,t)\|_{L^2} \\
\leq & C\|u_0\|_{H^3}+C|\delta|, \quad 0\leq t\leq 1, \quad l=0,1.
\end{aligned} \tag{4.55}$$

Finally, combining (4.54) and (4.55), we get

$$(1+t)^{3/4+l/2}\|\partial_x^l w(\cdot,t)\|_{L^2} \leq C(\|u_0\|_{L^1_1}+\|u_0\|_{H^3}+N(T)^2), \quad 0\leq t\leq T, \quad l=0,1.$$

Since $\|u_0\|_{L^1_1}+\|u_0\|_{H^3}$ is small, we obtain the desired estimate

$$N(T) \leq C(\|u_0\|_{L^1_1}+\|u_0\|_{H^3}).$$

This completes the proof. $\square$

In order to complete the proof of the main theorem, it is sufficent to show Proposition 4.3 below. Here we need to improve the proof of Lemma 3 in [5] to avoid the factor $\sqrt{\log t}$.

Proposition 4.3. *Assume that* $|\delta| \le 1$. *Then the estimate*

$$\|v(\cdot,t) - V(\cdot,t)\|_{L^\infty} \le C|\delta|(1+t)^{-1}, \ \ t \ge 1 \tag{4.56}$$

holds. Here $v(x,t)$ *is the solution to* (4.1) *and* $V(x,t)$ *is defined by* (1.13).

Proof. We set

$$\begin{aligned}\Lambda(x,y,t,s) &\equiv \partial_x (G(x-y,t-s)\eta_1(x,t)) \\ &= \partial_x G(x-y,t-s)\eta_1(x,t) + \frac{b}{2}\chi(x,t)G(x-y,t-s)\eta_1(x,t), \\ F(y,s) &\equiv \eta_2(y,s)(-(c/3)\chi(y,s)^3 - k\chi_{yy}(y,s)).\end{aligned} \tag{4.57}$$

By Lemma 3.3 and (3.10), we have

$$\begin{aligned}v(x,t) &= \int_0^t \int_{\mathbb{R}} \Lambda(x,y,t,s)F(y,s)dyds \\ &= \int_{t/2}^t \int_{\mathbb{R}} \Lambda(x,y,t,s)F(y,s)dyds + \int_0^{t/2} \int_{\mathbb{R}} \Lambda(x,y,t,s)F(y,s)dyds \\ &\equiv J_1 + J_2.\end{aligned} \tag{4.58}$$

First, we evaluate J_1. We shall show

$$\|J_1(\cdot,t)\|_{L^\infty} \le C|\delta|(1+t)^{-1}. \tag{4.59}$$

It follows that from (4.57)

$$\begin{aligned}J_1 &= -\frac{c}{3}\eta_1(x,t)\int_{t/2}^t \int_{\mathbb{R}} \left(\partial_x G(x-y,t-s) + \frac{b}{2}\chi(x,t)G(x-y,t-s)\right)\eta_2(y,s)\chi(y,s)^3 dyds \\ &\quad - k\eta_1(x,t)\int_{t/2}^t \int_{\mathbb{R}} \left(\partial_x G(x-y,t-s) + \frac{b}{2}\chi(x,t)G(x-y,t-s)\right)\eta_2(y,s)\chi_{yy}(y,s)dyds \\ &\equiv J_{1.1} + J_{1.2}.\end{aligned}$$

For $J_{1.1}$, by making the integration by parts, we have

$$\begin{aligned}J_{1.1}(x,t) &= -\frac{c}{3}\eta_1(x,t)\int_{t/2}^t \int_{\mathbb{R}} G(x-y,t-s)\left(\partial_y(\eta_2(y,s)\chi(y,s)^3) + \frac{b}{2}\chi(x,t)(\eta_2(y,s)\chi(y,s)^3)\right)dyds \\ &= -\frac{c}{3}\eta_1(x,t)\int_{t/2}^t \int_{\mathbb{R}} G(x-y,t-s)\Big(-\frac{b}{2}\chi(y,s)^4\eta_2(y,s) \\ &\qquad + 3\eta_2(y,s)\chi(y,s)^2\chi_y(y,s) + \frac{b}{2}\chi(x,t)(\eta_2(y,s)\chi(y,s)^3)\Big)dyds.\end{aligned}$$

Therefore, from Lemma 3.1, we obtain

$$\begin{aligned}\|J_{1.1}(\cdot,t)\|_{L^\infty} &\le \int_{t/2}^t \|G(\cdot,t-s)\|_{L^1}(\|\chi(\cdot,s)\|_{L^\infty}^4 + \|\chi(\cdot,s)\|_{L^\infty}^2\|\chi_x(\cdot,s)\|_{L^\infty} + \|\chi(\cdot,t)\|_{L^\infty}\|\chi(\cdot,s)\|_{L^\infty}^3)ds \\ &\le C|\delta|^3\int_{t/2}^t ((1+s)^{-2} + (1+t)^{-1/2}(1+s)^{-3/2})ds \\ &\le C|\delta|^3(1+t)^{-1}.\end{aligned} \tag{4.60}$$

For $J_{1.2}$, by making the integration by parts, we have

$$\begin{aligned}J_{1.2}(x,t) &= -k\eta_1(x,t)\int_{t/2}^t \int_{\mathbb{R}} G(x-y,t-s)\left(\partial_y(\eta_2(y,s)\chi_{yy}(y,s)) + \frac{b}{2}\chi(x,t)(\eta_2(y,s)\chi_{yy}(y,s))\right)dyds. \\ &= -k\eta_1(x,t)\int_{t/2}^t \int_{\mathbb{R}} G(x-y,t-s)\Big(-\frac{b}{2}\chi(y,s)\chi_{yy}(y,s)\eta_2(y,s) \\ &\qquad + \eta_2(y,s)\chi_{yyy}(y,s) + \frac{b}{2}\chi(x,t)(\eta_2(y,s)\chi_{yy}(y,s))\Big)dyds.\end{aligned}$$

Therefore, from Lemma 3.1, we have

$$\begin{aligned}\|J_{1.2}(\cdot,t)\|_{L^\infty} &\le \int_{t/2}^{t} \|G(\cdot,t-s)\|_{L^1}(\|\chi(\cdot,s)\|_{L^\infty}\|\chi_{xx}(\cdot,s)\|_{L^\infty}+\|\chi_{xxx}(\cdot,s)\|_{L^\infty}+\|\chi(\cdot,t)\|_{L^\infty}\|\chi_{xx}(\cdot,s)\|_{L^\infty})ds\\ &\le C|\delta|\int_{t/2}^{t}((1+s)^{-2}+(1+t)^{-1/2}(1+s)^{-3/2})ds\\ &\le C|\delta|(1+t)^{-1}.\end{aligned} \tag{4.61}$$

Hence, from (4.60) and (4.61), we have (4.59).

Next, we evaluate J_2. Spliting the y-integral at $y=0$ and making the integration by parts, we obtain

$$\begin{aligned}J_2 &= \int_0^{t/2}\int_{\mathbb{R}} \Lambda(x,y,t,s)F(y,s)dyds\\ &= \int_0^{t/2}\int_0^{\infty} \Lambda_y(x,y,t,s)\int_y^{\infty} F(q,s)dqdyds - \int_0^{t/2}\int_{-\infty}^{0} \Lambda_y(x,y,t,s)\int_{-\infty}^{y} F(q,s)dqdyds\\ &\quad+\int_0^{t/2} \Lambda(x,0,t,s)\int_{\mathbb{R}} F(q,s)dqds\\ &\equiv J_3+J_4+J_5.\end{aligned} \tag{4.62}$$

First, we note that Lemma 3.1 yields

$$\begin{aligned}&\sup_{0\le s\le t/2}\sup_{x\in\mathbb{R}}\sup_{y\in\mathbb{R}}|\Lambda_y(x,y,t,s)|\\ &\le C\sup_{0\le s\le t/2}(\|\partial_x^2 G(\cdot,t-s)\|_{L^\infty}+\|\chi(\cdot,t)\|_{L^\infty}\|\partial_x G(\cdot,t-s)\|_{L^\infty})\\ &\le C\sup_{0\le s\le t/2}((t-s)^{-3/2}+(1+t)^{-1/2}(t-s)^{-1})\le Ct^{-3/2},\end{aligned} \tag{4.63}$$

since

$$\Lambda_y(x,y,t,s) = -\eta_1(x,t)\bigg(\partial_x^2 G(x-y,t-s)+\frac{b}{2}\chi(x,t)\partial_x G(x-y,t-s)\bigg).$$

By making the integration by parts, we have

$$\begin{aligned}\int_y^{\infty}\eta_2(q,s)\chi_{qq}(q,s)dq &= -\eta_2(y,s)\chi_y(y,s)+\frac{b}{4}\int_y^{\infty}\eta_2(q,s)(\chi(q,s)^2)_q dq\\ &= -\eta_2(y,s)\chi_y(y,s)-\frac{b}{4}\eta_2(y,s)\chi(y,s)^2+\frac{b^2}{8}\int_y^{\infty}\eta_2(q,s)\chi(q,s)^3dq.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_y^{\infty} F(q,s)dq &= \int_y^{\infty}\eta_2(q,s)\bigg(-\frac{c}{3}\chi(q,s)^3-k\chi_{qq}(q,s)\bigg)dq\\ &= k\eta_2(y,s)\bigg(\chi_y(y,s)+\frac{b}{4}\chi(y,s)^2\bigg)-\bigg(\frac{b^2k}{8}+\frac{c}{3}\bigg)\int_y^{\infty}\eta_2(q,s)\chi(q,s)^3dq.\end{aligned} \tag{4.64}$$

Similarly, we obtain

$$\int_{-\infty}^{y} F(q,s)dq = -k\eta_2(y,s)\bigg(\chi_y(y,s)+\frac{b}{4}\chi(y,s)^2\bigg)-\bigg(\frac{b^2k}{8}+\frac{c}{3}\bigg)\int_{-\infty}^{y}\eta_2(q,s)\chi(q,s)^3dq, \tag{4.65}$$

because

$$\begin{aligned}\int_{-\infty}^{y}\eta_2(q,s)\chi_{qq}(q,s)dq &= \eta_2(y,s)\chi_y(y,s)+\frac{b}{4}\int_{-\infty}^{y}\eta_2(q,s)(\chi(q,s)^2)_q dq\\ &= \eta_2(y,s)\chi_y(y,s)+\frac{b}{4}\eta_2(y,s)\chi(y,s)^2+\frac{b^2}{8}\int_{-\infty}^{y}\eta_2(q,s)\chi(q,s)^3dq.\end{aligned}$$

From Lemma 3.1, (4.63) and (4.64), we have

$$\begin{aligned}
|J_3(x,t)| \leq & Ct^{-3/2}\int_0^{t/2}\int_0^{\infty}\Big(|\chi_y(y,s)|+|\chi(y,s)|^2+\int_y^{\infty}|\chi(q,s)|^3dq\Big)dyds \\
\leq & Ct^{-3/2}\Big(\int_0^{t/2}\int_{\mathbb{R}}(|\chi_y(y,s)|+|\chi(y,s)|^2)dyds+\int_0^{t/2}\int_0^{\infty}\int_0^{q}|\chi(q,s)|^3dydqds\Big) \\
\leq & Ct^{-3/2}\Big(\int_0^{t/2}(\|\chi_x(\cdot,s)\|_{L^1}+\|\chi(\cdot,s)\|_{L^2}^2)ds+\int_0^{t/2}\int_0^{\infty}q|\chi(q,s)|^3dqds\Big) \\
\leq & Ct^{-3/2}\Big(\int_0^{t/2}(|\delta|(1+s)^{-1/2}+|\delta|^2(1+s)^{-1/2})ds+\int_0^{t/2}\int_{\mathbb{R}}|y||\chi(y,s)|^3dyds\Big) \\
\leq & C|\delta|(1+t)^{-1}+Ct^{-3/2}\int_0^{t/2}\int_{\mathbb{R}}|y||\chi(y,s)|^3dyds.
\end{aligned}$$

By using (3.1), we have

$$\begin{aligned}
\int_{\mathbb{R}}|y||\chi(y,s)|^3dy &\leq C|\delta|^3\int_{\mathbb{R}}(1+s)^{-3/2}e^{-3y^2/4(1+s)}|y|dy \\
&= C|\delta|^3\int_{\mathbb{R}}(1+s)^{-1/2}|y|(1+s)^{-1}e^{-y^2/4(1+s)}dy \\
&\leq C|\delta|^3\|\partial_x G(\cdot,1+s)\|_{L^1} \\
&\leq C|\delta|^3(1+s)^{-1/2}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\|J_3(\cdot,t)\|_{L^\infty} &\leq C|\delta|(1+t)^{-1}+Ct^{-3/2}\int_0^{t/2}|\delta|^3(1+s)^{-1/2}ds \\
&\leq C|\delta|(1+t)^{-1}, \quad t\geq 1.
\end{aligned} \tag{4.66}$$

Similarly, we have from (4.63) and (4.65)

$$\begin{aligned}
|J_4(x,t)| \leq & Ct^{-3/2}\int_0^{t/2}\int_{-\infty}^{0}\Big(|\chi_y(y,s)|+|\chi(y,s)|^2+\int_{-\infty}^{y}|\chi(q,s)|^3dq\Big)dyds \\
\leq & Ct^{-3/2}\Big(\int_0^{t/2}\int_{\mathbb{R}}(|\chi_y(y,s)|+|\chi(y,s)|^2)dyds+\int_0^{t/2}\int_{-\infty}^{0}\int_q^{0}|\chi(q,s)|^3dydqds\Big) \\
\leq & Ct^{-3/2}\Big(\int_0^{t/2}(\|\chi_x(\cdot,s)\|_{L^1}+\|\chi(\cdot,s)\|_{L^2}^2)ds+\int_0^{t/2}\int_{-\infty}^{0}(-q)|\chi(q,s)|^3dqds\Big) \\
\leq & C|\delta|(1+t)^{-1}+Ct^{-3/2}\int_0^{t/2}\int_{\mathbb{R}}|y||\chi(y,s)|^3dyds.
\end{aligned}$$

Therefore, we have

$$\|J_4(\cdot,t)\|_{L^\infty}\leq C|\delta|(1+t)^{-1}, \quad t\geq 1. \tag{4.67}$$

Finally, we evaluate J_5. By the integration by parts, we get

$$\begin{aligned}
\int_{\mathbb{R}}F(q,s)dq &= -\frac{c}{3}\int_{\mathbb{R}}\eta_2(q,s)\chi(q,s)^3dq-k\int_{\mathbb{R}}\eta_2(q,s)\chi_{qq}(q,s)dq \\
&= -\frac{c}{3}\int_{\mathbb{R}}\eta_2(q,s)\chi(q,s)^3dq-\frac{bk}{4}\int_{\mathbb{R}}\eta_2(q,s)(\chi(q,s)^2)_qdq \\
&= -\Big(\frac{c}{3}+\frac{b^2k}{8}\Big)\int_{\mathbb{R}}\eta_2(q,s)\chi(q,s)^3dq.
\end{aligned}$$

From the definition of η_2 and χ, and (1.9), we have

$$\begin{aligned}
\int_{\mathbb{R}}\eta_2(q,s)\chi(q,s)^3dq &= \int_{\mathbb{R}}\eta_*\Big(\frac{q}{\sqrt{1+s}}\Big)^{-1}(1+s)^{-3/2}\chi_*\Big(\frac{q}{\sqrt{1+s}}\Big)^3dq \\
&= (1+s)^{-1}\int_{\mathbb{R}}\eta_*(z)^{-1}\chi_*(z)^3dz \\
&= d(1+s)^{-1}.
\end{aligned}$$

Thus, we obtain

$$\int_{\mathbb{R}} F(q,s)dq = -d\Big(\frac{b^2k}{8}+\frac{c}{3}\Big)(1+s)^{-1}.$$

Therefore, we have

$$\begin{aligned}
J_5 =& -d\Big(\frac{b^2k}{8}+\frac{c}{3}\Big)\eta_1(x,t)\int_0^{t/2}(1+s)^{-1}\Big(\partial_x G(x,t-s)+\frac{b}{2}\chi(x,t)G(x,t-s)\Big)ds\\
=& -d\Big(\frac{b^2k}{8}+\frac{c}{3}\Big)\eta_1(x,t)\int_0^{t/2}(1+s)^{-1}\Big(\partial_x G(x,t-s)-\partial_x G(x,t+1)\\
&+\frac{b}{2}\chi(x,t)(G(x,t-s)-G(x,t+1))\Big)ds\\
&-d\Big(\frac{b^2k}{8}+\frac{c}{3}\Big)\eta_1(x,t)\Big(\partial_x G(x,t+1)+\frac{b}{2}\chi(x,t)G(x,t+1)\Big)\log\Big(1+\frac{t}{2}\Big)\\
\equiv& J_{5.1}+J_{5.2}.
\end{aligned}\tag{4.68}$$

Since

$$\partial_x^l G(x,t-s)-\partial_x^l G(x,t+1) = -(1+s)\int_0^1(\partial_t\partial_x^l G)(x,1+t-\theta(1+s))d\theta$$

for $l=0,1$, we have

$$|\partial_x^l G(x,t-s)-\partial_x^l G(x,t+1)| \le C(1+s)(t-s)^{-3/2-l/2}.$$

From (1.9), (3.7) and Lemma 3.1, we obtain

$$|d| \le \int_{\mathbb{R}}|\eta_*(y)|^{-1}|\chi_*(y)|^3dy \le C\|\chi(\cdot,0)\|_{L^3}^3 \le C|\delta|^3.$$

Therefore, we obtain

$$\begin{aligned}
\|J_{5.1}(\cdot,t)\|_{L^\infty} \le& C|\delta|^3\int_0^{t/2}(1+s)^{-1}\Big((1+s)(t-s)^{-2}+(1+t)^{-1/2}(1+s)(t-s)^{-3/2}\Big)ds\\
\le& C|\delta|^3(1+t)^{-1}, \quad t\ge 1.
\end{aligned}\tag{4.69}$$

Finally, by the definition of $V(x,t)$, $J_{5.2}$ can be written as follows:

$$\begin{aligned}
J_{5.2}(x,t) =& -d\Big(\frac{b^2k}{8}+\frac{c}{3}\Big)\eta_1(x,t)\Big(\partial_x G(x,t+1)+\frac{b}{2}\chi(x,t)G(x,t+1)\Big)\log\Big(1+\frac{t}{2}\Big)\\
=& V(x,t)+\frac{d}{4\sqrt{\pi}}\Big(\frac{b^2k}{8}+\frac{c}{3}\Big)V_*\Big(\frac{x}{\sqrt{1+t}}\Big)(1+t)^{-1}\Big(\log(1+t)-\log\Big(1+\frac{t}{2}\Big)\Big).
\end{aligned}$$

Since $V_*(x)$ is bounded, we obtain

$$\begin{aligned}
\|J_{5.2}(\cdot,t)-V(\cdot,t)\|_{L^\infty} \le& C|\delta|^3\Big\|V_*\Big(\frac{\cdot}{\sqrt{1+t}}\Big)\Big\|_{L^\infty}\log\Big(1+\frac{t/2}{1+t/2}\Big)(1+t)^{-1}\\
\le& C|\delta|^3(1+t)^{-1}, \quad t\ge 1.
\end{aligned}\tag{4.70}$$

Therefore, summarizing (4.58), (4.59), (4.62) and (4.66) through (4.70), we obtain (4.56). □

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