A Simple and Practical Model for Mean Size-Density Trajectories of Tree Stands

Masato SHIBUYA*

SHIBUYA, M.: A simple and practical model for mean size-density trajectories derived from TADAKI's model by reparameterizing of the density term is proposed and examined for fitnesses to actual trajectories of monospecific, even-aged tree stands. The model is recognized to be a special case of NAITO's model. Various mean volume-density trajectories of coniferous stands are simulated by the model very closely, and fitness of the model comparison to those of NAITO's model. The present model did not deviate from an actual trajectory in the later self-thinning stages as found in TADAKI's model, and it is considered to be appropriate for the mean size-density trajectories. It was confirmed by examinations of ecological meanings of parameters that the model consisted of a full-density line of a species concerned and the term which describes the approaching process to that line according to the stand initial state. Also, the model can be manipulated like the C-D rule according to the density of the stand initial state when the full-density line already is fixed.

1. Introduction

Stand dynamics of monospecific, even-aged tree stands were described as mean size-density trajectories frequently (TADAKI and SHIDEI, 1959; ANDO, 1962; HOZUMI, 1977). The mean size-density trajectory is a very important factor for stand yield estimations (ANDO, 1968). TADAKI's model (TADAKI, 1963, 1964) has been often employed for mean volume (w)-density (N) relationships of tree stands in forestry studies as follows:

\[ 1/N = Aw + B \]  

(1)

where \( A \) and \( B \) are constants determined by the trajectory. HOZUMI (1977) pointed out that this model approximates the mean volume-density relationship in the earlier stages of the self-thinning of a stand. However, it increasingly deviates from that in the later self-thinning stages. This deviation is considered to be peculiar to the model because its gradient is different from the full-density line on which the trajectory travels in the later self-thinning stages. Because the trajectory approximated by TADAKI's model was assumed to shift to the full-density line when it came to that line (TADAKI, 1963), this disadvantage of the model was not conspicuous.

In the recent ten or more years, MINOWA (1983), HAYASHI (1985) and SMITH and HANN (1986) derived mean size-density trajectories that moved vertically on the log (mean size)-log (density) plot in the initial
stand development stages and approached the full-density line very smoothly with self-thinning. These models seemed to be more appropriate than TADAKI's model, although parameter determinations for them are more or less difficult. It is desired that the model for the mean size-density trajectory easily determines parameters and applies them to harvest estimations.

In this paper, a simple and practical model for mean size-density trajectories of monospecific, even-aged tree stands is proposed and fitted to actual data obtained from various studies, and it is examined on ecological meanings of the parameters.

II. Mean Size-Density Trajectory Data

Mean size-density trajectories of coniferous tree species (Table 1) were employed for simulations by a model proposed later. Although tree species of KHILMI's data (KHILMI, 1957) were not identified, other trajectories consisted of five species of various shade tolerances and stand development stages. All stands had not or hardly had been thinned artificially. Mean size of the stands is mean volume in this paper. Dimensions of tree volume and area are converted into m³ and ha, respectively.

Chamaecyparis obtusa (Sieb. et Zucc.) Endl. data obtained from TAKEUCHI (1980) covered stand ages from 25 to 70 years old. This plantation is located in the Tokyo University Forest in Chiba Prefecture and was investigated for tree diameters on a 0.02 ha-plot. Because stand stock decreased during the 65-70 years old, data for the 70 years old was omitted from the analysis.

An Abies sachalinensis Masters plantation was established in 1929 in the Ikeda District in Hokkaido (Hokkaido Prefectural Forest Office, 1982). Initial stand density was 4,200/ha, and diameter measurements were continued from 1948 at 2-5-year intervals. Stand density decreased from 3,990 to 2,174/ha in the investigating period. This stand was analyzed for frequency distributions of individual volumes by KIKUZAWA (1981).

A Pinus strobus Linn. stand reported by SPURR et al. (1957) was in southeastern Michigan, United States. Many Japanese workers (e.g., TADAKI and SHIDEI, 1959; HOZUMI, 1977; HAYASHI, 1985) studied the mean volume-density relationship of this stand. Stand density after initial mortality was about 11,900/ha. Stand stock and density were investigated nine times during 1916-1955, at 13-52-year-old stand ages. The range of stand density decrease was larger than those of C. obtusa and A. sachalinensis stands (Fig. 1).

KHILMI's data (KHILMI, 1957) was taken from yield tables prepared by Professor TYURIN. These data are supposed to be based on the dynamics of stands that were conducted as cultural works. He indicated stand

Table 1. Results of fittings of the present mean volume-density trajectory model

<table>
<thead>
<tr>
<th>Species</th>
<th>Parameters of the trajectories</th>
<th>$R^2$</th>
<th>Data sources</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chamaecyparis obtusa*</td>
<td>$\log_{10} a = 4.869$</td>
<td>$k = 1.470$</td>
<td>$b = 0.118$</td>
</tr>
<tr>
<td>Abies sachalinensis*</td>
<td>$\log_{10} a = 4.958$</td>
<td>$k = 1.556$</td>
<td>$b = 0.162$</td>
</tr>
<tr>
<td>Pinus strobus*</td>
<td>$\log_{10} a = 4.473$</td>
<td>$k = 1.537$</td>
<td>$b = 0.0115$</td>
</tr>
<tr>
<td>Pine, Site Quality I**</td>
<td>$\log_{10} a = 4.477$</td>
<td>$k = 1.545$</td>
<td>$b = 0.0663$</td>
</tr>
<tr>
<td>Pine, Site Quality IV***</td>
<td>$\log_{10} a = 4.332$</td>
<td>$k = 1.529$</td>
<td>$b = 0.0258$</td>
</tr>
<tr>
<td>Spruce, Site Quality I**</td>
<td>$\log_{10} a = 4.656$</td>
<td>$k = 1.526$</td>
<td>$b = 0.0846$</td>
</tr>
<tr>
<td>Spruce, Site Quality IV**</td>
<td>$\log_{10} a = 4.317$</td>
<td>$k = 1.462$</td>
<td>$b = 0.0337$</td>
</tr>
<tr>
<td>Pinus taeda(1)***</td>
<td>$\log_{10} a = 2.408$</td>
<td>$k = 1.321$</td>
<td>$b = 0.048$</td>
</tr>
<tr>
<td>Pinus taeda(2)***</td>
<td>$\log_{10} a = 2.596$</td>
<td>$k = 1.362$</td>
<td>$b = 0.0146$</td>
</tr>
<tr>
<td>Pinus taeda(3)***</td>
<td>$\log_{10} a = 2.505$</td>
<td>$k = 1.328$</td>
<td>$b = 0.0514$</td>
</tr>
<tr>
<td>Pinus taeda(4)***</td>
<td>$\log_{10} a = 2.274$</td>
<td>$k = 1.230$</td>
<td>$b = 0.0930$</td>
</tr>
<tr>
<td>Picea mariana(1)***</td>
<td>$\log_{10} a = 5.797$</td>
<td>$k = 1.791$</td>
<td>$b = 0.0120$</td>
</tr>
<tr>
<td>Picea mariana(2)***</td>
<td>$\log_{10} a = 5.899$</td>
<td>$k = 1.767$</td>
<td>$b = 0.0446$</td>
</tr>
<tr>
<td>Picea mariana(3)***</td>
<td>$\log_{10} a = 5.850$</td>
<td>$k = 1.843$</td>
<td>$b = 0.0434$</td>
</tr>
</tbody>
</table>

* Artificial stand. ** Stand origin was indistinct. *** Natural stand.
stocks of pine and spruce for six site qualities. Trajectories for Site Qualities I and IV of each tree are tested in this paper (Fig. 2).

Even-aged Pinus taeda Linn. stands regenerated naturally on sites abandoned from agricultural cultivation were surveyed in a succession study of tree communities by Peet and Christensen (1980). These stands are located in North Carolina’s Piedmont area, United States. Investigated plots were established in the early 1930’s and measured for diameter and height of each tree. Initial densities of Stands (1) to (4) in the first survey when stand age was 8 years old were 2,896, 1,065, 578 and 368/ha, respectively (Fig. 3). These plots were remeasured 7 to 9 times until 1978. Mean volume–density trajectories of all stands were more curvilinear than linear in the stages where self-thinning had progressed considerably as the authors pointed out. Especially, the last data for each stand showed an obvious different trend from the
other points, and they were excluded from analyses.

*Picea mariana* (MILL.) B.S.P. stands also invaded naturally after forest fires as even-aged cohorts. CARLETON and WANNAMAKER (1987) reconstructed mean volume-density trajectories through research on tree ages and growth patterns of live and standing dead stems. Although they tried to restore the trajectories exactly, underestimation of density may be inevitable in the earliest growth stage of an old stand. Three stands that had large initial densities were selected for this study from ten of their stands. The authors asserted that these trajectories were rotated sigmoid-shaped (Fig. 4).

### III. Model

A model for mean size-density trajectory is empirically proposed by the reparameterizing of the power term of density in Eq. (1):

\[ w = aN^{-k} - b \]  \hspace{1cm} (2)

where \( a, b, \) and \( k \) are parameters for each trajectory \((a, b, k > 0)\). This model is a kind of reciprocal equation. In this study, the parameters of Eq. (2) are determined by the non-linear least squares method, employing the quasi-Newton method, and the coefficient of determination is the ratio of the explained sum of squares to the total sum of squares.

The trajectory by the model moves vertically in stand establishment stages and along the line expressed by \( w = aN^{-k} \) in stand mature stages. Trajectory shape is similar to MINOWA'S (1983) and HAYASHI'S models (1985). NAITO (1983) also suggested that the same type function as Eq. (2) could be derived when the stand density decrease and the mean size growth followed the RICHARDS function. Furthermore, NAITO (1984) explicitly proposed a more general trajectory model than the present one as follows:

\[ w = (aN^{-\delta} - \gamma)^b \]  \hspace{1cm} (3)

The present model is recognized as a special case of NAITO'S model when the parameter \( \delta \) is assumed to be 1. I examined the fitnesses of both models (Eqs. (2) and (3)) to the trajectory data.

### IV. Results

The results of the fittings of the present model to stand data are given in Table 1 and Figs. 1-4. Coefficients of determination are larger than 0.935 in all stands except for a *C. obtusa* stand, and this model simulated sufficiently these trajectories. NAITO'S model (Eq. (3)) showed larger coefficients of determination than those in the present model for six trajectories out of 14 sets of stand data (Table 2), in spite of repetitive trials of regressions with various initial values of the parameters. NAITO'S model is considered to

<table>
<thead>
<tr>
<th>Species</th>
<th>Parameters of NAITO'S trajectory model</th>
<th>( R^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>Chamaecyparis obtusa</em></td>
<td>40.197, 0.479, -0.039, 5.811</td>
<td>0.8530</td>
</tr>
<tr>
<td><em>Abies sachalinensis</em></td>
<td>124.055, 0.595, 0.703, 1.615</td>
<td>0.9567</td>
</tr>
<tr>
<td><em>Pinus sibiricus</em></td>
<td>105.940, 0.687, 0.080, 2.024</td>
<td>0.9960</td>
</tr>
<tr>
<td><em>Picea mariana (1)</em></td>
<td>145.633, 0.726, 0.183, 1.951</td>
<td>0.99991</td>
</tr>
<tr>
<td><em>Picea mariana (2)</em></td>
<td>42.005, 0.538, 0.267, 2.194</td>
<td>0.9999</td>
</tr>
<tr>
<td><em>Picea mariana (3)</em></td>
<td>62.417, 0.570, 0.191, 2.609</td>
<td>0.9999</td>
</tr>
<tr>
<td><em>Picea mariana (4)</em></td>
<td>57.726, 0.577, 0.121, 2.604</td>
<td>0.99997</td>
</tr>
<tr>
<td><em>Pinus taeda (1)</em></td>
<td>5.511, 0.384, 0.137, 2.754</td>
<td>0.9996</td>
</tr>
<tr>
<td><em>Pinus taeda (2)</em></td>
<td>4.399, 0.274, 0.424, 2.685</td>
<td>0.9773</td>
</tr>
<tr>
<td><em>Pinus taeda (3)</em></td>
<td>6.306, 0.326, 0.579, 2.414</td>
<td>0.9870</td>
</tr>
<tr>
<td><em>Pinus taeda (4)</em></td>
<td>4.801, 0.215, 0.958, 3.138</td>
<td>0.9366</td>
</tr>
<tr>
<td><em>Picea mariana (1)</em></td>
<td>1.038\times10^8, 0.861, -0.243, 7.477</td>
<td>0.9741</td>
</tr>
<tr>
<td><em>Picea mariana (2)</em></td>
<td>5.969\times10^7, 0.776, -0.179, 9.547</td>
<td>0.9569</td>
</tr>
<tr>
<td><em>Picea mariana (3)</em></td>
<td>28.353, 0.368, 0.570, 6.475</td>
<td>0.9272</td>
</tr>
</tbody>
</table>

* Analyzed trajectory data are the same as in Table 1.
be more flexible than the present model, although the results shown in Table 2 indicate that the fitting of Eq. (3) to the mean size–density trajectory of a tree stand is more or less difficult. NAITO (1984) reduced the difficulties of fitting Eq. (3) to trajectories by the assumption of \( \beta = \frac{3}{2} \) based on the \( \frac{3}{2} \) power law of self-thinning (YODA et al., 1963). Consequently, fitnesses of Eq. (2) to the trajectories compares with those of Eq. (3), and Eq. (2) is recognized to be suitable and applicable to mean size–density trajectories. Parameter \( k \) modifies the trajectory shape, and the deviation of the trajectory from actual data in the later self-thinning stages observed in TADAKI’s model as pointed out by HOZUMI (1977) was not found (Fig. 5).

It is notable that parameter values of \( \log_{10} a \) and \( k \) are restricted within narrow ranges, the former between 4.3 and 5.0, and the latter between 1.46 and 1.56, excepting data in irregular trends of \( P. \) taeda and \( P. \) mariana stands. Their ranges are approximately comparable with those of the full-density lines of coniferous tree species (TADAKI, 1969).

V. Discussion

Ecological meanings of the parameters of Eq. (2) are examined here. The gradient of Eq. (2) on the double-logarithmic plot is

\[
\frac{d \log w}{d \log N} = - \frac{k}{1 - (b/a)N^k} \tag{4}
\]

The gradient of a mean size–density trajectory just after stand establishment is hypothesized generally to be a very large negative value excepting for the stand of extremely large density. When \( (d \log w/d \log N) \) approaches negative infinity, \( 1 - (b/a)N^k = 0 \) in Eq. (4). Therefore, \( N = (a/b)^{1/k} \) is satisfied in this case. The stage of stand development when the mean size–density trajectory shifts vertically is defined as the initial state, and density in this state is expressed as \( N_0 \), then \( N_0 = (a/b)^{1/k} \). As understood from this definition, \( N_0 \) is not always equivalent to an actual initial density. Equation (2) is rewritten by substituting \( b = aN_0^{-k} \) as

\[
w = aN^{-k}[1 - (N/N_0)^k] \tag{5}
\]

In the later growth stages, stand density decreased considerably, \( N \) is assumed to be fairly small compared with \( N_0 \). Consequently, \( w \approx aN^{-k} \) is held in these stages. Since TADAKI and SHIDEI (1959) and YODA et al. (1963), a concept that monospecific, even-aged stands with various initial densities and site qualities, tend to converge on the full-density line specific to species after sufficient growth periods, has been supported (HOZUMI, 1977, 1980). When this concept is agreed to, parameters \( a \) and \( k \) in the trajectory model are regarded as the intercept and the thinning exponent of the full-density line, respectively. Therefore, \( a \) and \( k \) are determined by species, and \( N_0 \) is determined by each trajectory, theoretically. On the right side of Eq. (5), the term \( aN^{-k} \) expresses the full-density state for the species concerned, and the term \( 1 - (N/N_0)^k \) determines the approaching process to the full-density line according to the initial state of the stand.

It is natural that the ranges of \( \log_{10} a \) and \( k \) in Table 1 are moderate for coniferous tree species, excluding stands with trajectories of irregular tendencies. Site quality is presumed not to affect the position of the full-density line (YODA et al., 1963); however, trajectories of the poorer site qualities located themselves below those of the more favorable sites in the later growth stages (Fig. 2). More careful examinations are needed about this problem. Parameters \( k \) for \( P. \) taeda and \( P. \) mariana trajectories are different from the 1.5 assumed by the \( \frac{3}{2} \) power law of self-thinning (YODA et al., 1963); nevertheless, parameters \( a \) and \( k \) are approximately similar within each species (Table 1).

Each parameter influences the trajectory in different ways (Fig. 6). Increased parameter \( a \) shifts the
trajectory upward. Parameter $k$ decides the trajectory shape. When density of the initial state $N_0$ increased, the trajectory moves downward parallel along the full-density line. If $k$ is fixed, we can treat this trajectory model like the C-D curve (Shinozaki and Kira, 1961). It must be examined whether or not shapes of mean size-density trajectories with various initial densities are considered to be the same within a species. However, statistical analyses on mean size–density trajectories of pure tree stands by Puettmann et al. (1993) concluded that trajectory shapes were not affected by stand initial density and stand origin within a species. Parameters in Eq. (2) must be determined by non-linear regression, although in the case of a species for which the full-density line is already fixed, the mean size–density trajectory is able to be settled roughly by only the manipulation of the $N_0$-value. Thus the present model is very practical in this characteristic like the C-D curve and easy determinations of the parameters.

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Literature cited


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