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Proceedings of the 36th Sapporo Symposium  
on Partial Differential Equations

T. Ozawa, Y. Giga, T. Sakajo, S. Jimbo, H. Takaoka,  
K. Tsutaya, Y. Tonegawa, and G. Nakamura

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# Proceedings of the 36th Sapporo Symposium on Partial Differential Equations

Edited by

T. Ozawa, Y. Giga, T. Sakajo, S. Jimbo,  
H. Takaoka, K. Tsutaya, Y. Tonegawa,  
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## PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 22 through August 24 in 2011 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Professor Taira Shirota started the symposium more than 30 years ago. Professor Kôji Kubota and Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

T. Ozawa, Y. Giga, T. Sakajo, S. Jimbo,  
H. Takaoka, K. Tsutaya, Y. Tonegawa,  
and G. Nakamura

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# The 36th Sapporo Symposium on Partial Differential Equations

## (第36回偏微分方程式論札幌シンポジウム)

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Program Committee: T. Ozawa, Y. Giga, T. Sakajo, S. Jimbo,  
H. Takaoka, G. Nakamura, Y. Tonegawa, K. Tsutaya

Period (期間) August 22, 2011 - August 24, 2011

Venue (場所) Room 203, Faculty of Science Building #5, Hokkaido University  
北海道大学 理学部 5号館大講義室(203号室)

URL <http://www.math.sci.hokudai.ac.jp/sympo/sapporo/program.html>

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The Boltzmann equation without angular cutoff approximation  
- uniqueness, existence and regularity of solutions

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Some results on nonlinear singular partial differential equations

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14:30-15:00 Slim Ibrahim ( University of Victoria)  
About the local and global wellposedenss of the Navier-Stokes-Maxwell coupled system

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On the term-wise estimates for the norm-inflation solution of the Navier-Stokes flows

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Life span of positive solutions for a semilinear heat equation with non-decaying initial data

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Standing waves for a system of nonlinear Schrödinger equations

12:00-12:30

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**THE BOLTZMANN EQUATION  
WITHOUT ANGULAR CUTOFF APPROXIMATION  
– UNIQUENESS, EXISTENCE AND REGULARITY  
OF SOLUTIONS**

YOSHINORI MORIMOTO  
KYOTO UNIVERSITY

In this talk we consider the Cauchy problem for the non-cutoff Boltzmann equation in the whole space. The uniqueness of solutions is first discussed in different function spaces, according to each of soft and hard potentials. Next we show the local existence of classical solutions without assuming the smallness of initial data, and moreover we establish the global existence of solutions in the framework of a small perturbation of an equilibrium state, together with the regularizing effect and the time decay rate of the perturbation. Here the regularizing effect means that the solution belongs to  $C^\infty$  for the positive time, which is the characteristic that the Boltzmann equation without angular cutoff approximation originally has. The contents of this talk are based on a series of joint-works with R. Alexandre, S. Ukai, C.-J. Xu and T. Yang.

**• Boltzmann equation and assumptions**

Consider the Cauchy problem for the spatially inhomogeneous Boltzmann equation,

$$(1) \quad \begin{cases} \partial_t f + v \cdot \nabla_x f = Q(f, f), & x, v \in \mathbb{R}^3, t > 0, \\ f(0, x, v) = f_0(x, v), \end{cases}$$

where  $f = f(t, x, v)$  is the density distribution function of particles with position  $x \in \mathbb{R}^3$  and velocity  $v \in \mathbb{R}^3$  at time  $t$ . The term appearing in the right hand side of this equation is the so-called quadratic Boltzmann collision operator associated to the Boltzmann bilinear operator

$$Q(f, g)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \{f'_* g' - f_* g\} d\sigma dv_*,$$

where  $f'_* = f(t, x, v'_*)$ ,  $g' = g(t, x, v')$ ,  $f_* = f(t, x, v_*)$ ,  $g = g(t, x, v)$ ,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma,$$

for  $\sigma$  belonging to the unit sphere  $\mathbb{S}^2$ . Notice that the collision operator  $Q(\cdot, \cdot)$  acts only on the velocity variable  $v \in \mathbb{R}^3$ . Those relations between the post and pre collisional velocities follows from the conservations of momentum and kinetic energy in the binary collisions

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

The non-negative cross section  $B(z, \sigma)$  depends only on  $|z|$  and the scalar product  $\frac{z}{|z|} \cdot \sigma$ . In what follows we assume that it takes the form

$$B(|v - v_*|, \cos \theta) = \Phi(|v - v_*|) b(\cos \theta), \quad \cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

where the angular factor  $b(\cos \theta)$  is assumed to have the following singularity;

$$(2) \quad b(\cos \theta) \theta^{2+2s} \rightarrow K \text{ as } \theta \rightarrow +0,$$

for  $0 < s < 1$  and a constant  $K > 0$ , and the kinetic factor  $\Phi = \Phi_\gamma$  is given by

$$(3) \quad \Phi_\gamma(|v - v_*|) = |v - v_*|^\gamma,$$

for some  $\gamma > \max\{-3, -3/2 - 2s\}$ . If the inter-molecule potential satisfies the inverse power law potential  $U(\rho) = \rho^{-(q-1)}$ ,  $q > 2$  (, where  $\rho$  denotes the distance between two interacting molecules), then

$$\Phi(|v - v_*|) = |v - v_*|^{(q-5)/(q-1)} \text{ and}$$

$$b(\cos \theta) \theta^{2+2s} \rightarrow K \text{ as } \theta \rightarrow +0,$$

where  $K > 0$  and  $0 < s = 1/(q-1) < 1$ . Namely, for this physical case, we have

$$\gamma = \frac{q-5}{q-1} = 1 - 4 \frac{1}{q-1} = 1 - 4s$$

which is contained in our assumptions  $0 < s < 1$  and  $\gamma > \max\{-3, -3/2 - 2s\}$ .

### • Uniqueness

We use the usual function spaces as follows: For  $p \geq 1$  and  $\beta \in \mathbb{R}$ , we set

$$\|f\|_{L_\beta^p} = \left( \int_{\mathbb{R}^3} |\langle v \rangle^\beta f(v)|^p dv \right)^{1/p},$$

and for  $s \in \mathbb{R}$

$$\|f\|_{H_\beta^s(\mathbb{R}_v^3)} = \left( \int_{\mathbb{R}^3} |\langle D_v \rangle^s (\langle v \rangle^\beta f(v))|^2 dv \right)^{1/2}.$$

Furthermore

$$\|f\|_{H_\beta^s(\mathbb{R}_{x,v}^6)} = \left( \int_{\mathbb{R}^6} |\langle D_x, D_v \rangle^s (\langle v \rangle^\beta f(x, v))|^2 dv dx \right)^{1/2}.$$

For the uniqueness of solution, we first consider the function space with polynomial decay in the velocity variable. For  $m \in \mathbb{R}$ ,  $\ell \geq 0$ ,  $T > 0$  set

$$\begin{aligned} \mathcal{P}_\ell^m([0, T] \times \mathbb{R}_{x,v}^6) = & \left\{ f \in C^0([0, T]; \mathcal{D}'(\mathbb{R}_{x,v}^6)); \right. \\ & \left. s.t. f \in L^\infty([0, T] \times \mathbb{R}_x^3; H_\ell^m(\mathbb{R}_v^3)) \right\}. \end{aligned}$$

Our first theorem concerns the uniqueness of solution for the case  $\gamma \leq 0$ , which is called *soft potential case* in the classical sense and *Maxwellian molecule type*.

**Theorem 1.** *Assume that  $0 < s < 1$  and  $\max(-3, -3/2 - 2s) < \gamma \leq 0$ . Let  $0 < T < +\infty$  and let  $\ell_0 \geq 14$ . Suppose that the Cauchy problem (1) admits two solutions  $f_1(t), f_2(t) \in \mathcal{P}_{\ell_0}^{2s}([0, T] \times \mathbb{R}_{x,v}^6)$  for the same initial datum  $f_0 \in L^\infty(\mathbb{R}_x^3; H_{\ell_0}^{2s}(\mathbb{R}_v^3))$ . If one solution is non-negative then  $f_1(t) \equiv f_2(t)$ .*

For the uniqueness of solution in the case  $\gamma > 0$  which is called *hard potential case* in the classical sense, we consider the function space with the Maxwellian type exponential decay in the velocity variable. More precisely, for  $m \in \mathbb{R}$ , set

$$\mathcal{M}_0^m(\mathbb{R}_{x,v}^6) = \left\{ g \in \mathcal{D}'(\mathbb{R}_{x,v}^6); \exists \rho_0 > 0 \text{ s.t. } e^{\rho_0 \langle v \rangle^2} g \in L^\infty(\mathbb{R}_x^3; H^m(\mathbb{R}_v^3)) \right\},$$

and for  $T > 0$

$$\begin{aligned} \mathcal{M}^m([0, T] \times \mathbb{R}_{x,v}^6) = & \left\{ f \in C^0([0, T]; \mathcal{D}'(\mathbb{R}_{x,v}^6)); \exists \rho > 0 \right. \\ & \left. s.t. e^{\rho \langle v \rangle^2} f \in L^\infty([0, T] \times \mathbb{R}_x^3; H^m(\mathbb{R}_v^3)) \right\}. \end{aligned}$$

**Theorem 2.** Assume that  $0 < s < 1$  and  $\max(-3, -3/2 - 2s) < \gamma < 2 - 2s$ . Let  $0 < T < +\infty$  and suppose that the Cauchy problem (1) admits two solutions  $f_1(t), f_2(t) \in \mathcal{M}^{2s}([0, T] \times \mathbb{R}_{x,v}^6)$  for the same initial datum  $f_0 \in \mathcal{M}_0^{2s}(\mathbb{R}_{x,v}^6)$ . If one solution is non-negative then  $f_1(t) \equiv f_2(t)$ .

### • Existence

To consider time local solutions without specifying any limit behaviors at the spatial infinity we use the uniformly local Sobolev space with respect to the space variable as follows: For  $k \in \mathbb{N} \cap \{0\}$  we set

$$\begin{aligned} H_{ul}^k(\mathbb{R}^6) &= \{g \in \mathscr{D}'(\mathbb{R}^6); \|g\|_{H_{ul}^k} \\ &= \sum_{|\alpha+\beta| \leq k} \sup_{a \in \mathbb{R}^3} \iint |\phi(x-a) \partial_\beta^\alpha g(x, v)|^2 dx dv < \infty\}, \\ \phi &\in C_0^\infty(\mathbb{R}^3), \quad \phi(x) = 1(|x| < 1), = 0(|x| > 2), \\ \partial_\beta^\alpha &= \partial_x^\alpha \partial_v^\beta \quad (\alpha, \beta \in \mathbb{N}^3). \end{aligned}$$

Similar to Theorem 2 we define the following function spaces with the Maxwellian type exponential decay in the velocity variable. For  $k \in \mathbb{N}$ , our function space of initial data is

$$\mathcal{E}_0^k(\mathbb{R}^6) = \left\{ g \in \mathscr{D}'(\mathbb{R}_{x,v}^6); \exists \rho_0 > 0 \text{ s.t. } e^{\rho_0 \langle v \rangle^2} g \in H_{ul}^k(\mathbb{R}_{x,v}^6) \right\},$$

while the function space of solutions is, for  $T > 0$ ,

$$\begin{aligned} \mathcal{E}^k([0, T] \times \mathbb{R}_{x,v}^6) &= \left\{ f \in C^0([0, T]; \mathscr{D}'(\mathbb{R}_{x,v}^6)); \exists \rho > 0 \right. \\ &\quad \left. \text{s.t. } e^{\rho \langle v \rangle^2} f \in C^0([0, T]; H_{ul}^k(\mathbb{R}_{x,v}^6)) \right\}. \end{aligned}$$

**Theorem 3.** Assume that  $0 < s < 1/2$ ,  $\gamma > -3/2$ , and  $2s + \gamma < 1$ . If the initial datum  $f_0$  is non-negative and belongs to the function space  $\mathcal{E}_0^{k_0}(\mathbb{R}^6)$  for some  $4 \leq k_0 \in \mathbb{N}$ , then there exists  $T_* > 0$  such that the Cauchy problem (1) admits a non-negative unique solution  $f$  in the function space  $\mathcal{E}^{k_0}([0, T_*] \times \mathbb{R}^6)$ .

In order to obtain time global solutions we consider the perturbation around a normalized Maxwellian distribution

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}},$$

by setting  $f = \mu + \sqrt{\mu}g$ . Since  $Q(\mu, \mu) = 0$ , we have

$$Q(\mu + \sqrt{\mu}g, \mu + \sqrt{\mu}g) = Q(\mu, \sqrt{\mu}g) + Q(\sqrt{\mu}g, \mu) + Q(\sqrt{\mu}g, \sqrt{\mu}g).$$

Denote

$$\Gamma(g, h) = \mu^{-1/2} Q(\sqrt{\mu}g, \sqrt{\mu}h).$$

Then the linearized Boltzmann operator takes the form

$$\mathcal{L}g = \mathcal{L}_1 g + \mathcal{L}_2 g = -\Gamma(\sqrt{\mu}, g) - \Gamma(g, \sqrt{\mu}).$$

Now the original problem (1) is reduced to the Cauchy problem for the perturbation  $g$

$$(4) \quad \begin{cases} g_t + v \cdot \nabla_x g + \mathcal{L}g = \Gamma(g, g), & t > 0, \\ g|_{t=0} = g_0. \end{cases}$$

According to each of  $\gamma + 2s > 0$  and  $\gamma + 2s \leq 0$  we have the following two theorems.

**Theorem 4.** Assume that  $0 < s < 1$  and  $\gamma + 2s > 0$ . Let  $g_0 \in H_\ell^k(\mathbb{R}^6)$  for some  $k \geq 6$ ,  $\ell > 3/2 + 2s + \gamma$ . There exists  $\varepsilon_0 > 0$ , such that if  $\|g_0\|_{H_\ell^k(\mathbb{R}^6)} \leq \varepsilon_0$ , then the Cauchy problem (4) admits a global solution

$$g \in L^\infty([0, +\infty[; H_\ell^k(\mathbb{R}^6)) \cap C^\infty((0, \infty[ \times \mathbb{R}^6)).$$

Furthermore, if  $f_0(x, v) = \mu + \sqrt{\mu}g_0(x, v) \geq 0$  then  $f(t, x, v) = \mu + \sqrt{\mu}g(t, x, v) \geq 0$ .

In the case  $\gamma + 2s \leq 0$  we introduce the weighted Sobolev space whose weight-order changes, corresponding to the number of derivatives w.r.t.  $v$  variables. For  $k \in \mathbb{N}$ ,  $\ell \in \mathbb{R}$  we set

$$\begin{aligned} \tilde{\mathcal{H}}_\ell^k(\mathbb{R}^6) &= \left\{ f \in \mathcal{S}'(\mathbb{R}_{x,v}^6); \|f\|_{\tilde{\mathcal{H}}_\ell^k(\mathbb{R}^6)}^2 = \sum_{|\alpha|+|\beta| \leq k} \|\tilde{W}_{\ell-|\beta|} \partial_\beta^\alpha f\|_{L^2(\mathbb{R}^6)}^2 < +\infty \right\}. \\ \tilde{W} &= \langle v \rangle^{2s+\gamma}, \quad \tilde{W}_\ell = \langle v \rangle^{2s+\gamma\ell}, \quad \partial_\beta^\alpha = \partial_x^\alpha \partial_v^\beta \quad (\alpha, \beta \in \mathbb{N}^3). \end{aligned}$$

**Theorem 5.** Assume that  $0 < s < 1$ ,  $\gamma + 2s \leq 0$  and  $\gamma > \max\{-3, -2s - 3/2\}$ . Let  $g_0 \in \tilde{\mathcal{H}}_\ell^N(\mathbb{R}^6)$  for  $N \geq 5$ ,  $\ell \geq N$ . There exists  $\varepsilon_0 > 0$ , such that if  $\|g_0\|_{\tilde{\mathcal{H}}_\ell^N(\mathbb{R}^6)} \leq \varepsilon_0$ , then the Cauchy problem (4) admits a global solution

$$g \in L^\infty([0, +\infty[; \tilde{\mathcal{H}}_\ell^N(\mathbb{R}^6)) \cap C^\infty((0, \infty[ \times \mathbb{R}^6)).$$

Furthermore, if  $f_0(x, v) = \mu + \sqrt{\mu}g_0(x, v) \geq 0$  then  $f(t, x, v) = \mu + \sqrt{\mu}g(t, x, v) \geq 0$ .

The time decay rate of the perturbation  $g$  and the regularity theorem of solutions will be detailed in the talk.

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# Some Results on Nonlinear Singular Partial Differential Equations <sup>☆</sup>

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## Abstract

Let  $B$  be a manifold with conical singularities, i.e.  $B$  is paracompact, dimension of  $B = n+1$ , and with conical points  $B_0 = \{b_0, b_1, \dots, b_M\} \subset \partial B$ , and  $B \setminus B_0$  is  $C^\infty$  smooth.

Let  $X$  a closed compact  $C^\infty$  manifold,  $X^\Delta = \overline{\mathbb{R}_+} \times X / (\{0\} \times X)$ , is a local model interpreted as a cone with the base  $X$ , and  $X^\wedge = \mathbb{R}_+ \times X$  is an open stretched cone with the base  $X$ . For every  $b \in B_0$ , there is an open neighborhood  $U$  in  $B$ , such that there is a homeomorphism  $\varphi : U \rightarrow X^\Delta$  for some closed compact  $C^\infty$  manifold  $X = X(b)$ , and  $\varphi$  restricts a diffeomorphism  $\varphi' : U \setminus \{b\} \rightarrow X^\wedge$ .

The stretched manifold  $\mathbb{B}$  of  $B$  is defined as a  $C^\infty$  manifold with compact  $C^\infty$  boundary  $\partial\mathbb{B} \cong \bigcup_{b \in B_0} X(b)$ , with diffeomorphism:  $B \setminus B_0 \cong \mathbb{B} \setminus \partial\mathbb{B} = \text{int}(\mathbb{B})$ , defined as

$$U_1 \setminus B_0 \cong V_1 \setminus \partial\mathbb{B} : \{\tilde{x} = (t, x) \in V_1 \mid \left| \frac{\tilde{x}}{|\tilde{x}|} - \tilde{x}_0 \right| < \epsilon_1 \text{ with } |\tilde{x}_0| = 1\},$$

where  $U_1 \subset B$ , near points of  $B_0$ ,  $V_1 \subset \mathbb{B}$ . Thus  $\mathbb{B} = [0, 1) \times X_b$  (here we can suppose  $\epsilon_1 \approx 1$ ),  $\partial\mathbb{B} = \{0\} \times X_b$ .

Let  $g_X(t)$  be an  $t$ -dependent family of Riemannian metric on a closed compact  $C^\infty$  manifold  $X$ , which is infinitely differentiable in  $t \in \mathbb{R}_+$ . Then

$$g := \left( \frac{dt}{t} \right)^2 + g_X(t),$$

is a Riemannian metric on  $X^\wedge$ . In this case the gradient  $\nabla_{\mathbb{B}} = (t\partial_t, \partial_{x_1}, \dots, \partial_{x_n})$  and the Laplacian (Fuchsian type)  $\Delta_{\mathbb{B}} = (t\partial_t)^2 + \partial_{x_1}^2 + \dots + \partial_{x_n}^2$ , which is

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totally characteristic degenerate elliptic operator on the boundary  $t = 0$ . Thus the problem for a standard operator defined on  $B$  near conical point has been transposed as the problem for a singular (e.g. Fuchsian type) operator defined on  $\mathbb{B}$ .

In this talk, we would talk some recent results on

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- (I) Generalized Cauchy-Kowalevski Theorem and Summability of Formal Solutions;
  - (II) Boundary-value problem for semilinear degenerate elliptic equations on conical singular manifolds

# ABOUT THE LOCAL AND GLOBAL WELLPOSEDNESS OF THE NAVIER-STOKES-MAXWELL COUPLED SYSTEM

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## 1. INTRODUCTION

This is the summary of joint works with S. Keraani (University of Lille 1, France) [8], and more recently with N. Masmoudi (Courant Institute, NYU, USA) [9].

We investigate the wellposedness of solutions of a full Magneto-Hydro-Dynamic system (MHD) in the space dimension two and three. The full MHD system is a coupling of a forced Navier-Stokes equations with Maxwell equations. It reads as follows

$$(1.1) \quad \begin{cases} \partial_t v + v \cdot \nabla v - \nu \Delta v + \nabla p = j \times B \\ \partial_t E - \nabla \times B = -j \\ \partial_t B + \nabla \times E = 0 \\ \operatorname{div} v = \operatorname{div} B = 0 \\ \sigma(E + v \times B) = j \end{cases}$$

with the initial data

$$v|_{t=0} = v_0, \quad B|_{t=0} = B_0, \quad E|_{t=0} = E_0.$$

Here,  $v, E, B : \mathbb{R}_t^+ \times \mathbb{R}_x^d \rightarrow \mathbb{R}^3$  are vector fields defined on  $\mathbb{R}^d$  ( $d = 2$  or  $3$ ). The vector field  $v = (v_1, \dots, v_d)$  represents the velocity of the fluid, and the positive constants  $\nu$  and  $\sigma$  are its viscosity and resistivity, respectively. The scalar function  $p$  stands for the pressure. The vector fields  $E$  and  $B$  are the electric and magnetic fields of the fluid, respectively. The last equation in the system expresses Ohm's law for the electric current  $j$ . The force term  $j \times B$  in the Navier-Stokes equations comes from Lorentz force under a quasi-neutrality assumption of the net charge carried by the fluid. Note that the pressure  $p$  can be recovered from  $v$  and  $j \times B$  via an explicit Calderón-Zygmund type operator (for example, see [4]). The second equation in (1.1) is the Ampère-Maxwell equation for an electric field  $E$ . The third equation is nothing but Faraday's law. For a detailed introduction to the MHD, we refer to Davidson [6] and Biskamp [1].

Our main goal is to solve the system of equation (1.1). Before going any further, we first recall a few fundamental known results for the standard Navier-Stokes equations.

The incompressible Navier-Stokes equations are

$$(1.2) \quad \frac{\partial v}{\partial t} + v \cdot \nabla v - \nu \Delta v + \nabla p = 0, \quad \nabla \cdot v = 0.$$

From the one hand, multiplying (1.2) by  $v$  and integrating in space formally gives the energy identity

$$(1.3) \quad \frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \nu \|\nabla v\|_{L^2}^2 = 0,$$

which shows that the viscosity dissipates the energy. Given an  $L^2$  initial data, J. Leray [11] constructed a global weak solution  $v \in L^\infty((0, \infty), L^2(\mathbb{R}^d)) \cap L^2((0, \infty), \dot{H}^1(\mathbb{R}^d))$  satisfying the energy inequality.

On the other hand, applying Leray's projection  $\mathcal{P}$  to (1.2) the pressure disappears and the solution of (1.2) can be written in the integral form (mild solution)

$$v(t) = e^{\nu t \Delta} v_0 - \int_0^t e^{\nu(t-t')\Delta} \mathcal{P} \nabla(v \otimes v)(t') dt'.$$

With  $\dot{H}^{\frac{d}{2}-1}$  initial data, Fujita and Kato [7] constructed a unique (in  $\mathcal{C}_t(\dot{H}^{\frac{d}{2}-1}) \cap L_t^2(\dot{H}^{\frac{d}{2}})$ ) local mild solution which is global if the data is small. Moreover,  $u \in L_t^4(\dot{H}^{\frac{d-1}{2}})$ . When the space dimension is two, the two above solutions coincide and therefore we have both uniqueness and regularity of the solution. However, when the space dimension is three, the questions of the uniqueness of Leray's solutions and the global regularity of mild solutions remain outstanding open problems in contemporary Mathematics.

Similarly, for the full MHD system (1.1), one can formally get the following energy identity

$$\frac{1}{2} \frac{d}{dt} [\|v\|_{L^2}^2 + \|B\|_{L^2}^2 + \|E\|_{L^2}^2] + \|j\|_{L^2}^2 + \nu \|\nabla v\|_{L^2}^2 = 0.$$

showing that both the viscosity and the resistivity effects dissipate the energy. Therefore, one may wonder to extend Leray's result of the existence of global weak solutions to (1.1). Unfortunately, a such result remain an interesting open problem in both space dimension two and three. Indeed, one cannot have compactness for the term  $E \times B$  in the Lorentz force. Recently, Masmoudi [12] constructed, in dimension two, a unique global strong solutions to (1.1) starting from initial data  $(v^0, E^0, B^0) \in L^2(\mathbb{R}^2) \times (H^s(\mathbb{R}^2))^2$  with  $s > 0$ . This extra regularity on the electromagnetic field was needed to get an  $L_t^1(L_x^\infty)$  bound of the velocity field through a standard bi-dimensional logarithmic estimate. One of our main goals in this work is to reduce as much as possible the regularity required on the electromagnetic field.

Notice that in dimension two, if the electromagnetic field is just an  $L^2$  function, then the term  $E \times B$  in the Lorentz force is just integrable, and therefore one cannot gain regularity using the parabolic regularization of the Stokes operator. In addition, to define (in the distributional sense) the trilinear term  $(v \times B) \times B$ , the vector field should (heuristically) be a bounded function. Our first result then requires less regularity on the electromagnetic field than Masmoudi's one. The regularity we impose is the minimal that enables us to have parabolic regularization. However, our condition on the velocity vector field is a little bit more restrictive in order for us to define the trilinear term. To be more precise, let us first set the functional spaces we work in.

**Definition 1.1.** • For  $s \in \mathbb{R}$ , define the space  $\dot{H}_{\log}^s$  as the set of tempered distributions that satisfy

$$\|\psi\|_{\dot{H}_{\log}^s}^2 := \sum_{q \leq 0} 2^{2qs} \|\Delta_q \psi\|_{L^2}^2 + \sum_{q > 0} q 2^{2qs} \|\Delta_q \psi\|_{L^2}^2 < \infty.$$

Here,  $\Delta_q$  stands for the dyadic localization operator in the frequency space.

- Recall the Besov space  $\dot{B}_{p,q}^s$  defined by

$$\|u\|_{\dot{B}_{p,q}^s}^q = \sum_{j \in \mathbb{Z}} 2^{qjs} \|\Delta_j u\|_{L^p}^q$$

- For every  $r \in [1, +\infty]$  the space  $\tilde{L}_T^r L_{\log}^2$  is endowed with the norm

$$\|\phi\|_{\tilde{L}_T^r L_{\log}^2}^2 := \sum_{q \leq 0} \|\Delta_q \phi\|_{L_T^r L^2}^2 + \sum_{q > 0} q \|\Delta_q \phi\|_{L_T^r L^2}^2.$$

Our main Theorem is

**Theorem 1.2.** Let  $d = 2$ , and set

$$\mathcal{X}_2 := \dot{B}_{2,1}^0(\mathbb{R}^2) \times L_{\log}^2(\mathbb{R}^2) \times L_{\log}^2(\mathbb{R}^2).$$

There exists a small constant  $\delta > 0$  such that to any initial data  $(v^0, E^0, B^0) \in \mathcal{X}_2$  satisfying

$$\|(v^0, E^0, B^0)\|_{\mathcal{X}_2} \leq \delta,$$

corresponds a unique global solution  $(v, E, B)$  of (1.1)

$$v \in \mathcal{C}(\mathbb{R}^+; \dot{B}_{2,1}^0) \cap \tilde{L}^2(\mathbb{R}^+; \dot{B}_{2,1}^1), \quad E, B \in \tilde{L}^\infty(\mathbb{R}^+; L_{\log}^2).$$

The proof of the above result goes by compactness. It basically relies on two major steps. The first one is a microlocal refinement of the parabolic regularization given by the following Lemma

**Lemma 1.3.** *Let  $u$  be a smooth divergence free vectors fields solving*

$$\partial_t u - \Delta u + \nabla p = F \times B, \quad u|_{t=0} = 0,$$

on some time interval  $[0, T]$ . Then,

$$(1.4) \quad \|u\|_{\tilde{L}_T^2 \dot{B}_{2,1}^1} \lesssim \|F\|_{L_T^2 L_{\log}^2} \left( \|B\|_{\tilde{L}_T^\infty L_{\log}^2} + \|B\|_{L_T^2 L_H^2} \right).$$

The norm in  $L_H^2$  is given by

$$\|\psi\|_{L_H^2}^2 := \sum_{q \leq 0} 2^{2q} \|\Delta_q \psi\|_{L^2}^2 + \sum_{q \geq 0} \|\Delta_q \psi\|_{L^2}^2 < \infty.$$

To use this Lemma, one has to prove a decay (in time) of the magnetic field. This is given by the following result

**Lemma 1.4.** *Let  $(E, B)$  be a smooth solution of*

$$\begin{aligned} \partial_t E - \operatorname{curl} B + E &= f \times g, \\ \partial_t B + \operatorname{curl} E &= 0 \end{aligned}$$

on some interval  $[0, T]$ . Then, we have

$$\|E\|_{\tilde{L}_T^\infty L_{\log}^2} + \|E\|_{L_T^2 L_{\log}^2} + \|B\|_{\tilde{L}_T^\infty L_{\log}^2} \lesssim \|(E_0, B_0)\|_{L_{\log}^2} + \|f \times g\|_{L_T^2 L_{\log}^2},$$

and

(1.5)

$$\|B\|_{L_T^2 L_H^2} \lesssim \|(E_0, B_0)\|_{L^2} + \|g\|_{L_T^\infty L^2} \|f\|_{\tilde{L}_T^2 \dot{B}_{2,1}^1} + \left( \|\nabla f\|_{L_T^2 L^2} + \|f\|_{L_T^\infty L^2} \right) \|g\|_{L_T^2 L_H^2}.$$

Observe that the factor in front of  $\|g\|_{L_T^2 L_H^2}$  in (1.5) can be made arbitrary small when the data is small. Thus, the term in (1.5) containing  $\|g\|_{L_T^2 L_H^2}$  can be absorbed in the left hand side giving an *à priori* bound on  $\|g\|_{L_T^2 L_H^2}$ .

The proof of this lemma uses the representation of solutions of the damped wave equation satisfied by  $B$ . For high frequencies, such an equation behaves like the damped wave and therefore solutions decay exponentially in time. However, for low frequencies, it behaves like the heat equation, and the decay rate is rather weak.

In the three dimensional case, we construct mild solution using a fixed point argument based on the following nonlinear estimates

**Lemma 1.5.** *There exists a constant  $C > 0$  such that*

$$\begin{aligned} \|FB\|_{\tilde{L}_T^2 \dot{B}_{2,1}^{-\frac{1}{2}}} &\leq C \|F\|_{L^2 \dot{H}^{\frac{1}{2}}} \|B\|_{\tilde{L}_T^\infty \dot{H}^{\frac{1}{2}}}, \\ \|vB\|_{L_T^2 \dot{H}^{\frac{1}{2}}} &\leq C \|v\|_{\tilde{L}_T^2 \dot{B}_{2,1}^{\frac{3}{2}}} \|B\|_{\tilde{L}_T^\infty \dot{H}^{\frac{1}{2}}}, \\ \|uv\|_{L_T^1 \dot{B}_{2,1}^{\frac{3}{2}}} &\leq C \|u\|_{\tilde{L}_T^2 \dot{B}_{2,1}^{\frac{3}{2}}} \|v\|_{\tilde{L}_T^2 \dot{B}_{2,1}^{\frac{3}{2}}}, \end{aligned}$$

for all smooth functions  $F, B, v$  and  $u$  defined on some interval  $[0, T]$ .

With a more careful study of the full MHD system, we recently obtained with N. Masmoudi [9] the following result about the local and global wellposedness of (1.1).

**Theorem 1.6.** *Let  $d = 2, 3$ , and set*

$$\mathcal{X}_d := \dot{H}^{\frac{d}{2}-1} \times \dot{H}_{\log}^{\frac{d}{2}-1}(\mathbb{R}^d) \times \dot{H}_{\log}^{\frac{d}{2}-1}(\mathbb{R}^d).$$

*For any  $\Gamma^0 := (v^0, E^0, B^0) \in \mathcal{X}_d$ , there exists  $T > 0$  and a unique mild solution  $\Gamma$  of (1.1) with initial data  $\Gamma^0$  and*

$$v \in \mathcal{C}((0, T); \dot{H}^{\frac{d}{2}-1}) \cap \tilde{L}^2((0, T); \dot{H}^{\frac{d}{2}}) + \tilde{L}^2((0, T); \dot{B}_{2,1}^{\frac{d}{2}}), \quad E, B \in \tilde{L}^\infty(\mathbb{R}^+; \dot{H}_{\log}^{\frac{d}{2}-1}).$$

*Moreover, the solution is global (i.e.  $T = \infty$ ) if the initial data is sufficiently small.*

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# HOMOGENIZATION OF HELE-SHAW-TYPE PROBLEMS IN RANDOM AND PERIODIC MEDIA

Norbert Pozar

We consider an inhomogeneous Hele-Shaw-type problem in random and periodic media. The aim of this talk is to present a recent result on the asymptotic convergence of the solution as time  $t \rightarrow \infty$  to a symmetric self-similar solution of the homogeneous Hele-Shaw problem with a point source.

Let  $n \geq 2$  be a dimension and let us fix the sets  $K \subset \Omega_0 \subset \mathbb{R}^n$ , such that  $0 \in \text{int } K$ ,  $K$  is a nonempty compact set and  $\Omega_0$  is a bounded open set. Moreover, we assume that  $K$  and  $\Omega_0$  have smooth boundaries.

The (exterior) Hele-Shaw-type problem in random media can be formulated as a nonlocal free boundary problem for the *pressure*  $v(x, t, \omega) : \mathbb{R}^n \times [0, \infty) \times A \rightarrow \mathbb{R}$ , formally

$$\begin{cases} -\Delta v = 0 & \text{in } \{v > 0\} \setminus K, \\ v = 1 & \text{on } K, \\ v_t = g(x, \omega)|Dv|^2 & \text{on } \partial\{v > 0\}, \\ v(\cdot, 0, \omega) = v_0 & \text{on } \Omega_0 \setminus K, \end{cases} \quad (\text{HS})$$

where the initial data  $v_0(x) : \mathbb{R}^n \rightarrow \mathbb{R}$  is the harmonic function in  $\Omega_0 \setminus K$  with  $v_0 = 1$  on  $K$  and  $v = 0$  on  $\Omega_0^c$ . Moreover,  $Dv$  and  $\Delta v$  are, respectively, the gradient and the Laplacian of  $v$  with respect to the space variable  $x$ , and  $v_t$  is the partial derivative of  $v$  with respect to the time variable  $t$ . One can observe that if the *free boundary*,  $\partial\{v > 0\}$ , is a smooth curve and the derivatives of  $v$  are continuous, the normal velocity  $V$  of the free boundary can be expressed as  $V = \frac{v_t}{|Dv|} = g(x, \omega)|Dv|$ .

In this model, the random medium is described by the function  $g(x, \omega) : \mathbb{R}^n \times A \rightarrow \mathbb{R}$ , where  $(A, \mathcal{F}, \mu)$  is a probability space. To guarantee the well-posedness of (HS),  $g(x, \omega)$  has to be a continuous function in  $x$  for a.e.  $\omega \in A$ , satisfying

$$0 < m \leq g(x, \omega) \leq M \quad \text{for all } x \in \mathbb{R}^n, \text{ a.e. } \omega \in A, \quad (1)$$

where  $m$  and  $M$  are positive constants. In order to observe some averaging behavior as  $t \rightarrow \infty$ , we assume that  $g$  is *stationary ergodic*. In other words, we assume that we have a group  $\{\tau_x\}_{x \in \mathbb{R}^n}$  of measure preserving transformations  $\tau_x : A \rightarrow A$  such that

$$g(x + x', \omega) = g(x', \tau_x \omega) \quad \text{for all } x, x' \in \mathbb{R}^n \text{ and a.e. } \omega \in A., \quad (2)$$

i.e.  $g$  is *stationary*. Furthermore, we require that  $\{\tau_x\}_{x \in \mathbb{R}^n}$  is *ergodic*, that is, if  $B \subset A$  such that  $\tau_x(B) = B$  for all  $x \in \mathbb{R}^n$ , then  $\mu(B) = 0$  or 1. For a more detailed discussion on stationary ergodic media, see for instance [5, 19].

The classical (homogeneous) Hele-Shaw problem, (HS) with  $g \equiv 1$ , was introduced in [10] as a model of a slow flowing viscous fluid injected in between two parallel plates small distance apart that form the so-called Hele-Shaw cell. This problem naturally generalizes to all dimensions  $n \geq 1$ .

The Hele-Shaw-type problem (HS) considered here describes a pressure-driven motion of a fluid in an inhomogeneous random medium that influences the velocity law of the fluid at the free boundary. The set  $K$  represents a source where the fluid is pumped in under a constant pressure and the fluid initially fills the set  $\Omega_0$ . The expansion of this wet region as the fluid flows is then captured by the evolution of the positivity set  $\{v > 0\}$ . Free boundary problems with similar velocity laws have various applications in the plastics industry [22, 24, 28], in electromechanical machining [18] and serve also as a model of a flow in porous media, to name just a few. In fact, Hele-Shaw problem can be thought of as a quasi-stationary limit of the one-phase Stefan problem with a similar boundary velocity law, modelling a heat transfer, see [2, 16, 21, 27].

Before proceeding onto the discussion of the asymptotic behavior of (HS), we need to clarify the notion of solutions. Due to possible topological changes of the interface, pinching, cusps and other singularity formation in finite time even with smooth initial data, it is necessary to consider solutions in a weak sense.

The notion of *weak solutions* for (HS) was introduced in [8] (see also [9]) using the natural monotonicity of the expansion of  $\{v > 0\}$ , by considering the function  $u(x, t) : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$  defined by

$$u(x, t) = \int_0^t v(x, s) \, ds,$$

instead of  $v$ . This transformation was first used for the porous dam problem in [3] (see also [4]) and for the one-phase Stefan problem in [7]. For more

details on the variational inequality framework, see also the survey [26]. The function  $u$  formally solves the Euler-Lagrange equation

$$\begin{cases} -\Delta u = -\frac{1}{g(x, \omega)} \chi_{\mathbb{R}^n \setminus \Omega_0} & \text{in } \{u > 0\}, \\ u = |Du| = 0 & \text{on } \partial\{u > 0\}, \\ u = t & \text{on } K \end{cases} \quad (3)$$

of some obstacle problem. To be more specific, we have the following definition.

**Definition 1.** *The function  $u(x, t)$  is called the weak solution of problem (HS) if for every  $t \geq 0$ ,  $w = u(\cdot, t)$  solves the obstacle problem*

$$\begin{cases} w \in \mathcal{K}_t, \\ \int_{\mathbb{R}^n} Dw \cdot D(\varphi - w) dx \geq \int_{\mathbb{R}^n \setminus \Omega_0} -\frac{1}{g(x)} (\varphi - w) dx \quad \text{for all } \varphi \in \mathcal{K}_t, \end{cases} \quad (4)$$

where

$$\mathcal{K}_t = \{\varphi \in H_0^1(\mathbb{R}^n), \varphi \geq 0 \text{ in } \Omega, \varphi = t \text{ on } K\}.$$

The classical theory of obstacle problems applies and yields well-posedness, comparison principle and regularity for  $u$ , see [8, 25].

Since (HS) satisfies a comparison principle, the notion of *viscosity solutions* offers an alternative definition of weak solutions. They were first introduced for (HS) in [11] (see also [13, 15]), where the comparison principle and well-posedness were also established. It is in fact necessary to consider merely semicontinuous viscosity solutions because a solution of (HS) might jump in time if a topological change of the support occurs.

In this talk, we are concerned with the *homogenization* of the solutions for large times in the sense of their asymptotic convergence to a certain solution of the homogeneous ( $g \equiv 1$ ) problem (HS). In order to formulate the main result, Theorem 3, we need to introduce the natural scaling of the solutions. Let us define the rescaled solution

$$\begin{aligned} v^\lambda(x, t) &= \lambda^{(n-2)/n} v(\lambda^{1/n} x, \lambda t), & \text{if } n \geq 3, \\ v^\lambda(x, t) &= \log \mathcal{R}(\lambda) v(\mathcal{R}(\lambda)x, \lambda t), & \text{if } n = 2, \end{aligned}$$

for any  $\lambda > 1$  and  $\mathcal{R} > 0$  is the unique solution of

$$\mathcal{R}^2 \log \mathcal{R} = \lambda.$$

It is easy to see that the function  $v^\lambda$  formally satisfies a rescaled version of the problem (HS) with the free boundary condition

$$\begin{aligned} v_t^\lambda &= g(\lambda^{1/n}x) |Dv^\lambda|^2 && \text{if } n \geq 3 \\ v_t^\lambda &= g(\log \mathcal{R}(\lambda)x) |Dv^\lambda|^2 && \text{if } n = 2. \end{aligned}$$

on  $\partial\{v^\lambda > 0\}$ .

Now as we send  $\lambda \rightarrow \infty$  (which for a fixed  $t$  corresponds to a long-time behavior of  $v$ ), we expect to see some averaging effect since  $g$  is assumed to be stationary ergodic. This is captured by the following lemma.

**Lemma 2** (cf. [13, Lemma 4.1]). *For  $g$  that satisfies the assumptions above, there exists a constant, denoted  $\langle \frac{1}{g} \rangle$ , such that if  $\Omega \subset \mathbb{R}^n$  is a bounded measurable set and if  $\{u^\varepsilon\}_{\varepsilon>0} \subset L^2(\Omega)$  is a collection of functions such that  $u^\varepsilon \rightarrow u$  strongly in  $L^2(\Omega)$  as  $\varepsilon \rightarrow 0$ , then*

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \frac{1}{g(x/\varepsilon, \omega)} u^\varepsilon(x) dx = \int_{\Omega} \left\langle \frac{1}{g} \right\rangle u(x) dx \quad \text{a.e. } \omega \quad (5)$$

At this point we are ready to state the main result presented here.

**Theorem 3** ([20, Theorem 7.1]). *There exists a constant  $C$  depending only on  $K$  and  $n$  such that, for a.e.  $\omega \in A$ , the rescaled viscosity solution  $v^\lambda$  converges locally uniformly to  $\bar{v}$  in  $(\mathbb{R}^n \setminus \{0\}) \times [0, \infty)$  as  $\lambda \rightarrow \infty$ , where  $\bar{v}$  is the unique radially symmetric self-similar solution of the homogeneous Hele-Shaw problem with a point source, formally*

$$\begin{cases} -\Delta \bar{v} = C\delta & \text{in } \{\bar{v} > 0\}, \\ \bar{v}_t = \frac{1}{\left\langle \frac{1}{g} \right\rangle} |D\bar{v}|^2 & \text{on } \partial\{\bar{v} > 0\}. \end{cases} \quad (6)$$

Here  $\delta$  is the Dirac  $\delta$ -function with mass at the origin.

Furthermore the free boundary  $\partial\{v^\lambda > 0\}$  converges locally uniformly to the free boundary  $\partial\{\bar{v} > 0\}$  with respect to the Hausdorff distance.

The homogenization of the Hele-Shaw-type problem (HS) was studied recently in [13] (see also [14] for a similar result on the one phase Stefan problem). In this setting, the free boundary condition in (HS) is set to depend on a new parameter  $\varepsilon > 0$  as  $g^\varepsilon(x, \omega) := f\left(\frac{x}{\varepsilon}, \omega\right)$ ,

$$v_t^\varepsilon = f\left(\frac{x}{\varepsilon}, \omega\right) |Dv^\varepsilon|^2, \quad \text{on } \partial\{v^\varepsilon > 0\}. \quad (\text{HS}^\varepsilon)$$

The parameter  $\varepsilon$  represents the “scale” of the oscillations of the medium. For a given  $\varepsilon$ , the problem has the unique solution  $v^\varepsilon$ . When  $\varepsilon \rightarrow 0$  we expect an averaging effect of the medium, due to our assumption of stationarity and ergodicity of  $g^\varepsilon$ . In fact, it was shown in [13] that the weak solutions  $u^\varepsilon$  converge locally uniformly to the weak solution  $u$  of (HS) with  $g \equiv \frac{1}{\langle 1/f \rangle}$ , the constant from Lemma 2. Furthermore, the free boundary of  $v^\varepsilon$  converges locally uniformly to the free boundary of  $v$  as  $\varepsilon \rightarrow 0$  with respect to the Hausdorff distance. Their main tool was a new result on the correspondence between the two notions of weak solutions.

A similar result to Theorem 3 was previously obtained in [23] for weak solutions of the homogeneous Hele-Shaw problem ( $g \equiv 1$ ). In the current situation, however, the velocity law of the free boundary depends on the position and therefore the techniques from [23] can only provide lower and upper bounds on the free boundary radius. This requires us to use a more refined method to prove the convergence of the solution to the self-similar asymptotic profile. We combine the strengths of two notions of solutions of (HS) – viscosity and weak – using their correspondence obtained in [13].

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# GRADIENT ESTIMATES AND EXISTENCE OF MEAN CURVATURE FLOW WITH TRANSPORT TERM

KEISUKE TAKASAO

**ABSTRACT.** In this talk we consider a hypersurface of the graph of the mean curvature flow with transport term. The existence of the mean curvature flow with transport term was proved by Liu, Sato and Tonegawa [9] by using geometric measure theory. We give a proof of the gradient estimates and the short time existence for the mean curvature flow with transport term by applying the backward heat kernel [8].

## 1. INTRODUCTION

A family of hypersurfaces  $\{\Gamma(t)\}_{0 \leq t < \infty}$  in  $\mathbb{R}^n$  moves by mean curvature if the velocity of  $\{\Gamma(t)\}_{0 \leq t < \infty}$  is

$$V_\Gamma = H\nu \quad \text{on } \Gamma(t), \quad t \geq 0.$$

Here  $\nu = (\nu^1, \nu^2, \dots, \nu^n)$  is the unit normal vector and  $H$  is the mean curvature of  $\Gamma(t)$ .

Brakke proved the existence of the generalized evolution  $\{\Gamma(t)\}_{0 < t < \infty}$  by using varifold methods from geometric measure theory [1]. Ecker and Huisken studied the interior regularity estimates for the mean curvature flow [4, 5, 6]. In [2] and [7], they proved the existence of the viscosity solutions of mean curvature flow by using the level set method. Colding and Minicozzi proved the sharp estimates of the interior gradient and the area for the graph of the mean curvature flow [3].

In this talk we consider the family of hypersurfaces  $\{\Gamma(t)\}_{0 \leq t < \infty}$  in  $\mathbb{R}^n$  whose velocity is

$$(1.1) \quad V_\Gamma = (F \cdot \nu)\nu + H\nu \quad \text{on } \Gamma(t), \quad t \geq 0.$$

Here  $F$  is the transport term. We assume that  $\nu^n > 0$  on  $\Gamma(t)$  for  $t \geq 0$ . From the assumption there exists  $u = u(x, t)$  such that  $\Gamma(t) = \{(x, u(x, t)) \mid x \in \mathbb{R}^{n-1}\}$  for  $t \geq 0$ .

The main results are related to the pioneering work by Liu, Sato and Tonegawa [9]. They proved the existence of the generalized evolution  $\{\Gamma(t)\}_{0 < t < \infty}$  in dimension  $n = 2, 3$  by using geometric measure theory by Brakke [1].

Our purpose is to give a simple proof of the gradient estimate of  $u$ , and to prove the short time existence of the graph  $\Gamma(t)$  for any dimension.

## 2. MAIN RESULTS

Let  $n \geq 2$ ,  $\Omega = (\mathbb{R}/\mathbb{Z})^{n-1} \simeq [0, 1)^{n-1}$  and  $F : \Omega \times \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}^n$  be a  $C^1$  vector valued function. We consider the mean curvature flow with transport term:

$$(2.1) \quad \begin{cases} \frac{\partial_t u}{v} = H + F(x, u, t) \cdot \nu, & (x, t) \in \Omega \times (0, \infty), \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases}$$

here  $H = \operatorname{div} \left( \frac{du}{v} \right)$ ,  $du = (\partial_{x_1} u, \partial_{x_2} u, \dots, \partial_{x_{n-1}} u)$ ,  $v = (1 + |du|^2)^{\frac{1}{2}}$  and  $\nu = (\nu^1, \nu^2, \dots, \nu^n) = \frac{(-du, 1)}{v}$ . We remark that we may obtain this PDE from (1.1). Let  $G = \sup_{(x,y) \in \Omega \times \mathbb{R}, t \in [0, 1]} (|F|^2 +$

$|DF|)$  and  $v_0 = \max_{x \in \Omega} v(x, 0)$ , here  $DF = (dF, \partial_{x_n} F)$ . We denote  $Q_T = \Omega \times (0, T)$  and  $Q_T^\varepsilon = \Omega \times (\varepsilon, T)$ . The first main result is the following:

**Theorem 2.1.** Let  $u \in C([0, 1]; C^2(\Omega)) \cap C^1((0, 1); C(\Omega))$  be a solution of (2.1). Assume that  $F \in C^1(\Omega \times \mathbb{R} \times [0, 1]; \mathbb{R}^n)$  and  $G < \infty$ . Then there exists  $T > 0$  such that

$$(2.2) \quad v(x, t) \leq 2v_0^2, \quad (x, t) \in Q_T.$$

Furthermore the constant  $T > 0$  is given by

$$T = \min \left\{ \frac{C}{Gv_0^6}, 1 \right\},$$

where  $C > 0$  is a constant depending only on  $n$ .

By Theorem 2.1 we obtain the second main result:

**Theorem 2.2.** Fix  $\alpha \in (0, 1)$ . Assume that

$$K := \max \{ \|DF\|_{L^\infty(Q_T)}, \|\partial_t F\|_{L^\infty(Q_T)}, \sup_{c \in \mathbb{R}} \|F(\cdot, c, \cdot)\|_{C^{\alpha, \frac{\alpha}{2}}(Q_T)} \} < \infty$$

and  $u_0$  is a Lipschitz function, namely there exists  $L > 0$  such that  $|u_0(x) - u_0(y)| < L|x - y|$  for any  $x, y \in \Omega$ . Then there exists a unique solution  $u \in C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T) \cap C(\overline{Q_T})$  of (2.1). Furthermore there exists  $C > 0$  depending only on  $n, \alpha, L, K$  and  $\varepsilon > 0$  such that

$$(2.3) \quad \|u\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T^\varepsilon)} < C.$$

### 3. OUTLINE OF THE PROOF

First we define the backward heat kernel.

**Definition 3.1.** For  $s, t > 0$  ( $s > t$ ) and  $X, Y \in \mathbb{R}^n$  we define  $\rho = \rho_{(Y,s)}(X, t)$  by

$$\rho_{(Y,s)}(X, t) = \frac{1}{(4\pi(s-t))^{\frac{n-1}{2}}} \exp \left( -\frac{|X - Y|^2}{4(s-t)} \right).$$

We remark that for continuous function  $g$  and  $x, y \in \mathbb{R}^{n-1}$  we have

$$(3.1) \quad \lim_{t \nearrow s} \int_{\Gamma(t)} g(\cdot, u(\cdot, t), t) \rho_{(y,u(y,s),s)}(\cdot, u(\cdot, t), t) d\mathcal{H}^{n-1} = g(y, u(y, s), s).$$

The following lemma, for the purpose of showing Theorem 2.1, is given by the modification of the proof of Huisken's monotonicity formula [8].

**Lemma 3.2.** Assume that  $u$  satisfies (2.1) and  $\Gamma(t)$  is the surface of (2.1) extended periodically to all of  $x \in \mathbb{R}^{n-1}$ . Let  $g = g(x, t) : \mathbb{R}^{n-1} \times [0, \infty) \rightarrow [0, \infty)$  be a non-negative  $C^{2,1}$  function. Then we have

$$(3.2) \quad \begin{aligned} \frac{d}{dt} \int_{\Gamma(t)} g \rho d\mathcal{H}^{n-1} &\leq \int_{\Gamma(t)} \rho \partial_t g - \rho \Delta_{\Gamma(t)} g + \rho (dg \cdot \bar{\nu}) \frac{\partial_t u}{v} \\ &\quad + \frac{1}{4} g \rho f^2(u) d\mathcal{H}^{n-1}, \end{aligned}$$

here  $\Delta_{\Gamma(t)}$  is the Laplace-Beltrami operator on  $\Gamma(t)$ ,  $\bar{\nu} = \frac{-du}{v}$  and  $f(u) = F(x, u(x, t), t) \cdot \nu$ .

We choose  $(y, s) \in \overline{\Omega} \times [0, T]$  such that  $v_\infty = v(y, s)$ . From (3.2) we obtain

$$(3.3) \quad \frac{d}{dt} \int_{\Gamma(t)} v \rho_\infty d\mathcal{H}^{n-1} \leq CGv_\infty^4,$$

here  $\rho_\infty = \rho_{(y, u(y, s), s)}$  and  $C = C(n) > 0$ . On the other hand, we have

$$(3.4) \quad \left| \int_{\Gamma(t)} v \rho_\infty d\mathcal{H}^{n-1} \right|_{t=0} \leq \int_{\mathbb{R}^{n-1}} \rho_\infty(x, 0) v_0^2 dx \leq v_0^2,$$

here  $v_0 = \max_{x \in \Omega} v(x, 0)$  and  $d\mathcal{H}^{n-1} = v dx$ . Hence by (3.1), (3.3) and (3.4) we obtain

$$(3.5) \quad v_\infty - v_0^2 \leq \int_0^s \frac{d}{dt} \left( \int_{\Gamma(t)} v \rho_\infty d\mathcal{H}^{n-1} \right) dt \leq C_0 G s v_\infty^4,$$

here  $C_0 = C_0(n) > 0$ . By (3.5) if  $s \leq T := \frac{1}{2^6 C_0 G v_0^6}$ , then we obtain

$$v_\infty \leq 2v_0^2.$$

Thus Theorem 2.1 is proved.

From Theorem 2.1, the equation (2.1) is uniformly elliptic. Hence standard Schauder estimates imply Theorem 2.2.

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# Numerical Study of Two-Dimensional Turbulence on a Rotating Sphere<sup>1</sup>

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## Abstract

Viscous flow motion on a rotating sphere is discussed with an attention focused into the stability and bifurcation structure of steady solutions, and the formation of zonal<sup>3</sup> flow and its long time asymptotic behaviors.

## 1 Introduction

Flow motions governed by the Navier-Stokes equations have long been studied in relation to fluid phenomena of human sizes, and extensive experimental studies have provided guidelines for theoretical analysis. Nowadays we have detailed experimental knowledge on a set of basic fluid motions in simple but important configurations. However, as the subject of research extended to large-scale flows on the earth and other planets, experimental observation in controlled experiments became difficult or impossible, and we have to perform theoretical or numerical research of a set of basic flows, for example, on a rotating sphere, often by making use of large scale numerical calculation. Below we present a piece of such research, focusing our attention on the behavior of two-dimensional(2D) viscous flows on a rotating sphere.

The 2D sphere,  $S^2$ , is one of the typical 2D boundaryless compact sets, along with 2D torus,  $T^2$  on which detailed flow studies have been made since the proposal of Kolmogorov in 1959 to investigate a simplest model to see the origin of turbulent disturbance. The 2D flow on a 2D torus under a periodic (in one of the coordinates) external force field is now called Kolmogorov flow, in which several flow properties have been found including instability and bifurcation of the basic flows, detailed structures of chaotic behaviors (Inubushi, Kobayashi, Takehiro and MY [1]) and a symmetry restoration of unstable steady solutions in the inviscid limit (Okamoto et al. [2], Kim et al. [3]). Below we will see that some of these properties are shared with 2D flows on  $S^2$ , but that the rotation has a strong influence on dynamical properties of flow field.

From a geophysical point of view, the most distinguished property of the flow on a rotating sphere may be a formation of zonal band structure

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<sup>3</sup>Here "zonal" means "in the east-west direction".

as observed on Jupiter. Actually many solar planets more or less shows the zonal flows although the width of the zonal band depends on planets. The mechanism of the zonal band formation is, however, not well understood. Here we discuss the zonal flow formation in the framework of 2D Navier-Stokes flows on a rotating sphere, and show that the zonal jet flows emerges in this simplest model, which, however, is not suitable for description of the real zonal flows on the solar planets.

## 2 Steady Solutions in Non-Rotating and Rotating Cases

We consider 2D flow fields on a rotating unit sphere governed by the Navier Stokes equation for incompressible fluid,

$$\frac{\partial \zeta}{\partial t} + \left( \frac{\partial \psi}{\partial \phi} \frac{\partial \zeta}{\partial \mu} - \frac{\partial \zeta}{\partial \phi} \frac{\partial \psi}{\partial \mu} \right) + 2\Omega \frac{\partial \psi}{\partial \phi} = F + \frac{1}{R} (\nabla^2 + 2) \zeta. \quad (1)$$

Here,  $\phi$  is the longitude,  $\mu$  the sine of latitude,  $t$  the time, and  $\psi$  the stream function such that  $(u_\phi, u_\mu) = (-\sqrt{1-\mu^2} \partial \psi / \partial \mu, (1/\sqrt{1-\mu^2}) \partial \psi / \partial \phi)$ , and  $\zeta \equiv \nabla^2 \psi$  the vorticity, where  $\nabla^2$  is the horizontal Laplacian on the sphere.  $\Omega$  a dimensionless constant rotation rate of the sphere,  $R$  the Reynolds number, and  $F = F(\phi, \mu, t)$  the vorticity forcing function. This 2D equation is obtained by taking the limit of vanishing depth of thin fluid layer on the sphere under the assumption that the velocity is proportional to the radius. The viscous terms conserves the total angular momentum of the system as expected (Silberman [14]).

A simplest zonal flow solution  $\psi_l = AY_l^0$  is realized by assuming a forcing term  $F = A(l(l+1)(2-l(l+1))/R)Y_l^0$ , where  $A$  is a constant<sup>4</sup>. The steady solution  $\psi_1$  only expresses a uniform rotation which is invariant due to the angular momentum conservation. In the case of  $l=2$ , the steady solution  $\psi_2$  (Fig.1) expresses zonal jets toward different directions in the northern and the southern hemispheres. We can prove that for any solution

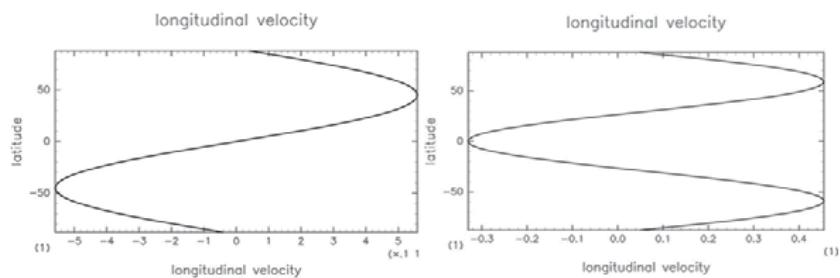


Figure 1: Zonal flow profiles of  $\psi_2$  (left) and  $\psi_3$  (right).

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<sup>4</sup> $Y_l^m$  denotes a spherical harmonic function.

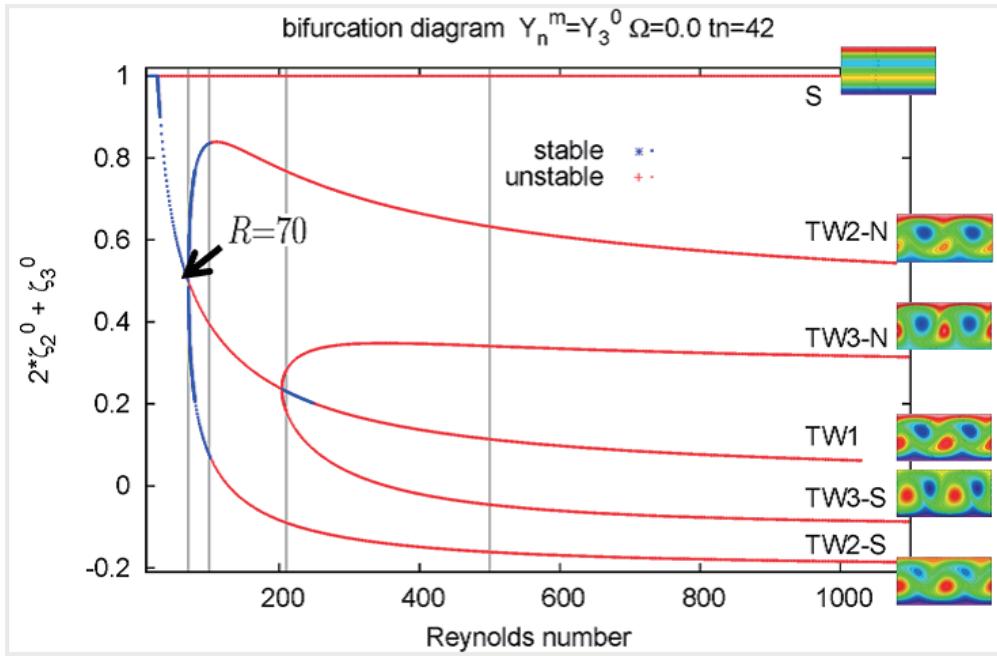


Figure 2: Bifurcation diagram in non-rotating case. Flow patterns for the bifurcating solutions are also shown (Sasaki, Takehiro and MY [8])

$\psi(t) = \psi_2 + \psi_{rest}(t)$ ,  $\|\psi_{rest}(t)\|_2 \rightarrow 0$  ( $t \rightarrow \infty$ ), the *global asymptotic stability* of  $\psi_2$ , similarly to the case of the Kolmogorov flow on  $T^2$ .

In contrast, in the case of  $l = 3$ , the zonal flow  $\psi_3$  (Fig.1) becomes unstable as the Reynolds number increases. The bifurcation diagram from the solution  $\psi_3$  for the case of  $\Omega = 0$  is shown in Fig.2, where we take the Reynolds number as a bifurcation parameter. We see that all the solutions found numerically are unstable at sufficiently large Reynolds numbers. The zonal solution  $\psi_3$  (S) is uniform in the longitudinal direction, but every bifurcating solution has some longitudinal structure and its flow pattern is not symmetric with respect to any meridional line, nor to the equator. Instead a

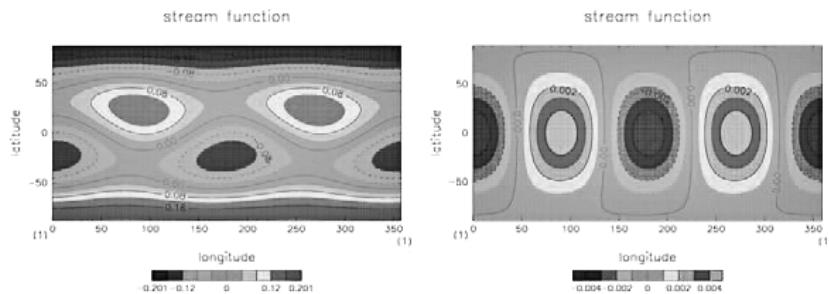


Figure 3: Symmetry restoration of the solution TW1. The flow pattern at  $R = 30$  (left) is not symmetric with respect to the equator, but is nearly symmetric at  $R = 10^6$  (right). (Sasaki, Takehiro and MY [8])

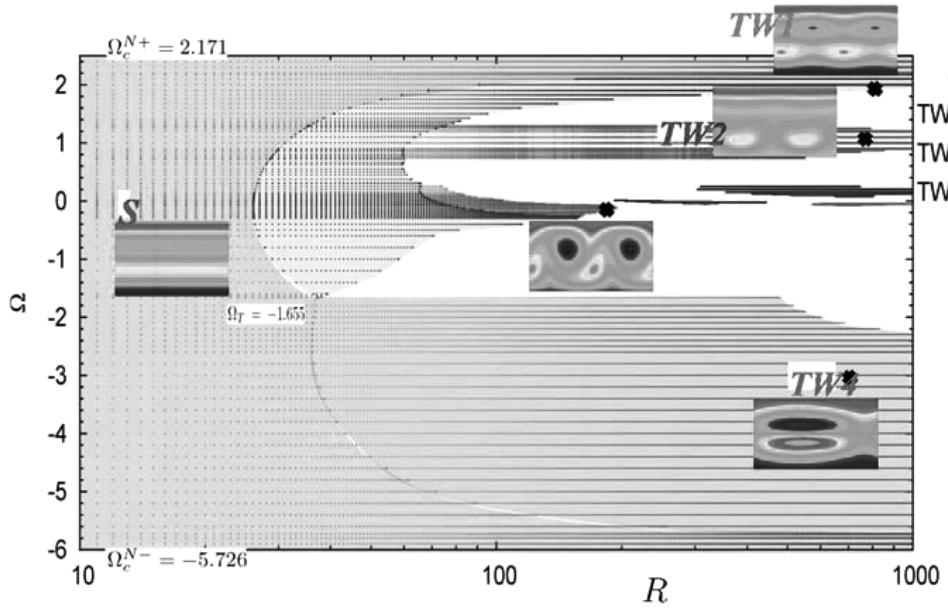


Figure 4: Stable solutions for  $\Omega \neq 0$ . Flow patterns of the stable solutions are also shown. There is no stable steady solution in white region. (Sasaki, Takehiro and MY [8])

solution (TW1) is invariant to  $\pi/2$ -translation in the longitudinal direction followed by the mirror transformation with respect to the equator, while other solutions are not. It is interesting that the (unstable) solution recovers the mirror symmetry when the Reynolds number is increased (Fig.3), as has been observed in the planar Kolmogorov flows (Okamoto et al.[2], Kim et al. [3]).

The rotation of the sphere has a significant effect on the bifurcation structure as seen in Fig.4 which shows stable solutions in the case of  $l = 3$  for  $-6 \leq \Omega \leq 2.5$  and  $0 \leq R \leq 1000$ . We find that for  $\Omega > 2.171$  and

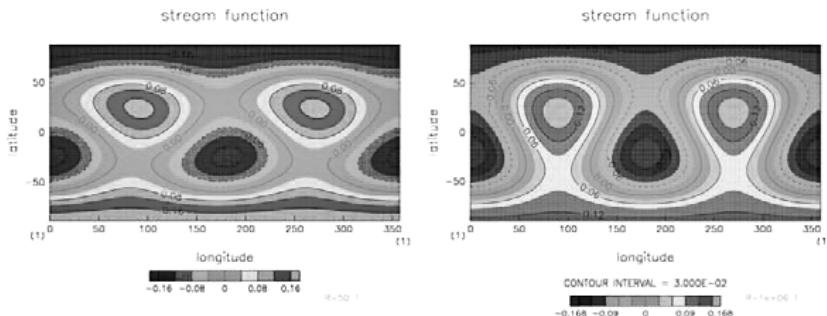


Figure 5: Flow patterns at  $R = 50$  (left) and  $R = 10^6$  (right) in rotating case ( $\Omega = 1.0$ ). The vortices keep the staggered configuration at higher Reynolds number. (Sasaki, Takehiro and MY [8])

$\Omega < -5.726$ , the zonal flow  $\phi_3$  is linearly stable at least up to  $R = 1000$ , suggesting that it is linearly stable for an arbitrary Reynolds number. Thus the rotation of the sphere greatly enhances the stability of the zonal flow (S). We remark that the symmetry recovering with increasing  $R$  is not well observed in the rotating case (Fig.5), where the staggered configuration of positive and negative vortices remains at high Reynolds numbers.

### 3 Time Development in Non-Forced Case

2D Flow patterns on a rotating sphere has attracted much attention because it is expected to be a simplest model of large scale flows in planetary atmosphere.

In a non-rotating planar case, while inviscid 3D flow conserves the energy which tends to cascades toward small scale motions, inviscid 2D flows conserves both the energy and the enstrophy  $\|\zeta\|_2^2$  and then the energy tends to cascade toward large scale motions, forming large coherent vortices in the viscous case (inverse cascade). Similarly on a non-rotating sphere, there appears large coherent vortices in 2D flow fields in the course of time development.

However, the rotation interferes with the inverse cascade of energy. This phenomena was first found by Rhines [4] who performed a numerical experiment of 2D turbulence with the "β-plane" approximation which takes a partial account of the effect of rotation. He then found a multiple zonal-band structure with alternating westward and eastward jets. Many succeeding studies have confirmed the emergence of the multiple zonal-band structure on both a  $\beta$  plane and a two-dimensional sphere by introducing several types of the energy injection [5, 6]. The multiple zonal-band structure suggests many fascinating problems such as the mechanism of energy's concentration to zonal jets [7, 9], and the asymmetry of the eastward and westward jets'

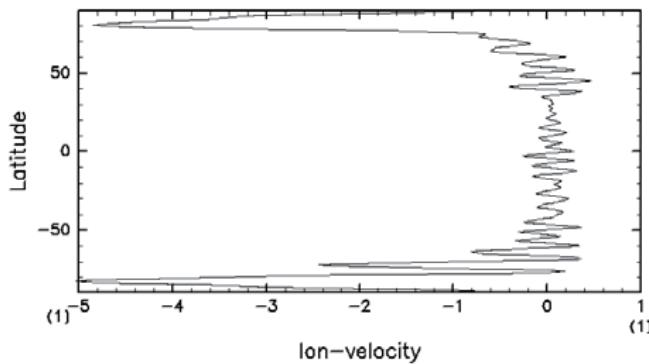


Figure 6: Circumpolar jet formation in non-forced 2D flow fields. The zonal mean of the zonal velocity is shown. Strong westward jets is observed around the north and the south poles.

profiles. However, it does not seem to be clear whether the long-time asymptotic state is actually characterized by such multiple zonal band structure, even in the  $\beta$ -plane approximation.

Recently, Yoden and Yamada [10] investigated the asymptotic states of freely decaying two-dimensional barotropic incompressible flows on a rotating sphere. Interestingly, the asymptotic states are not necessarily characterized by the multiple zonal-band structure but sharp strong westward circumpolar jets along the north- and south-poles become prominent, although there still exists weak multiple zonal band structure in the low and middle latitudes (Fig.6). The scaling laws for this circumpolar jets are obtained by Takehiro, Yamada and Hayashi [11]; when the rotation rate of the sphere  $\Omega$  increases, the strength of the jets increases as  $\Omega^{\frac{1}{4}}$  and the width of the jets decreases as  $\Omega^{-\frac{1}{4}}$ .

## 4 Time Development with Small Scale Forcing

Real planetary flows are believed to have been maintained for a long time by the energy injection, for example, in the form of sunshine or from inside the planet. This should be a motivation of the introduction of a forcing and of the study of long time asymptotic behaviors of the forced flow field.

For 2D incompressible flows on a rotating sphere, Nozawa and Yoden [13] performed a series of numerical simulations, with a Markovian random forcing of 18 cases with different combinations of a rotation rate of the sphere and a forcing wavenumber. There, they showed that the generated flow fields are characterized by a multiple zonal-band structure or a structure with westward circumpolar jets. In contrast, Huang et al. [12] performed simulations with a white noise forcing, and obtained an asymptotic state consisting of only two zonal jets. They then inferred that the Markovian random forcing in Nozawa and Yoden [13] may be regarded as a strong drag with small wavenumber dissipation which maintains the formed multiple zonal-band structure.

The numerical time integration of Nozawa and Yoden [13], however, does not seem to be long enough to obtain long-time asymptotic states, since the observed jets appear to be still changing. Therefore, we reexamine the long-time asymptotic states with a small-scale, homogeneous, isotropic, and Markovian random forcing (Obuse, Takehiro and MY [15]), extending the integration time of numerical simulation to about 100 to 500 times of that of Nozawa and Yoden [13].

The vorticity forcing function  $F$  is taken to be the same as that in Nozawa and Yoden [13]; small-scale, homogeneous, isotropic, Markovian random function is given by

$$F(\phi, \mu, j\Delta t) = R_m F(\phi, \mu, (j-1)\Delta t) + \sqrt{1 - R_m^2} \hat{F}(\phi, \mu, j\Delta t), \quad (2)$$

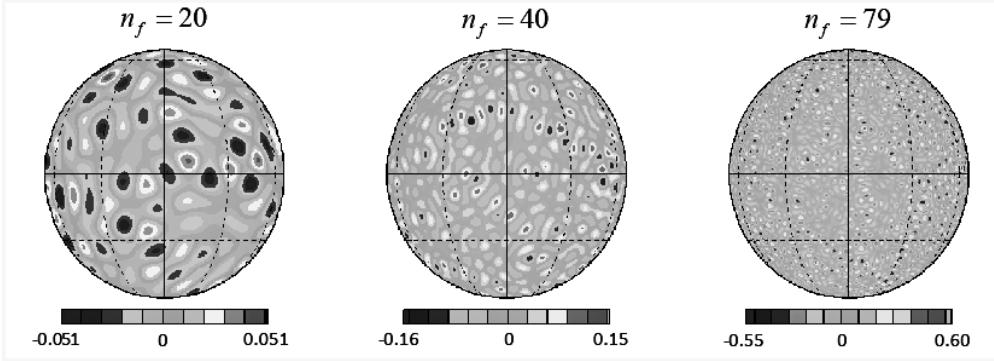


Figure 7: Vorticity forcing field at dimensionless time  $t = 1000$ .  $n_f$  of the left, the middle and the right figures are 20, 40, and 79, respectively. (Obuse, Takehiro and MY [15])

where  $\Delta t$  is the time step interval and  $R_m = 0.982$  is the memory coefficient (Fig.7).  $\hat{F}$  is a random source generated at each time step as

$$\hat{F}(\phi, \mu, j\Delta t) = \sum_{n=n_f-\Delta n}^{n_f+\Delta n} \sum_{\substack{m=-n \\ m \neq 0}}^n \hat{F}_n^m(j) Y_n^m(\phi, \mu), \quad (3)$$

where the phase of  $\hat{F}_n^m$  ( $m \geq 0$ ) are random and uniformly distributed on  $[0, 2\pi]$ . The amplitude of  $\hat{F}_n^m$  are also random with  $\|F\|_2$  being a prescribed value, and  $\Delta n = 2$ . The parameters are set as [13];  $\nu = 3.46 \times 10^{-6}$ ,  $\Omega/\Omega_J = 0.25, 0.5, 1.0, 2.0$ , and  $4.0$ , with  $\Omega_J \equiv 2\pi$ ,  $(n_f, \|F\|_2) = (20, 1.412 \times 10^{-2})$ ,  $(40, 3.929 \times 10^{-2})$ ,  $(79, 1.415 \times 10^{-1})$ . The spectral method with the spherical harmonics,  $\psi(\phi, \mu, t) = \sum_{n=0}^{N_T} \sum_{m=-n}^n \psi_n^m(t) Y_n^m(\phi, \mu)$ , with  $N_T = 199$  is used for numerical integration. The initial velocity field is  $\mathbf{u} = 0$  for every case.

We show in Fig.8 time-development of zonal mean of zonal angular momentum,

$$\overline{[L_{lon}]} \equiv \frac{1}{2\pi} \int_0^{2\pi} u_{lon} \sqrt{1 - \mu^2} d\phi, \quad (4)$$

where  $u_{lon} = -\sqrt{1 - \mu^2} (\partial\psi/\partial\mu)$  is the longitudinal component of velocity.

Nozawa and Yoden [13] performed the time integration up to  $t = 1000$ , and reported that a multiple zonal-band structure appears in the course of time development and then enters a quasi-steady state with little change in its flow pattern. However, in the longer time integration, we find that the zonal jets still merge or disappear. In most cases, two prograde jets merge and a retrograde jet between the two prograde jets disappears. At the final stage of the time integration, a zonal-band structure with only a few broad zonal jets is realized; two jets remain in run 2–6, 8–12, 14and 15, and three

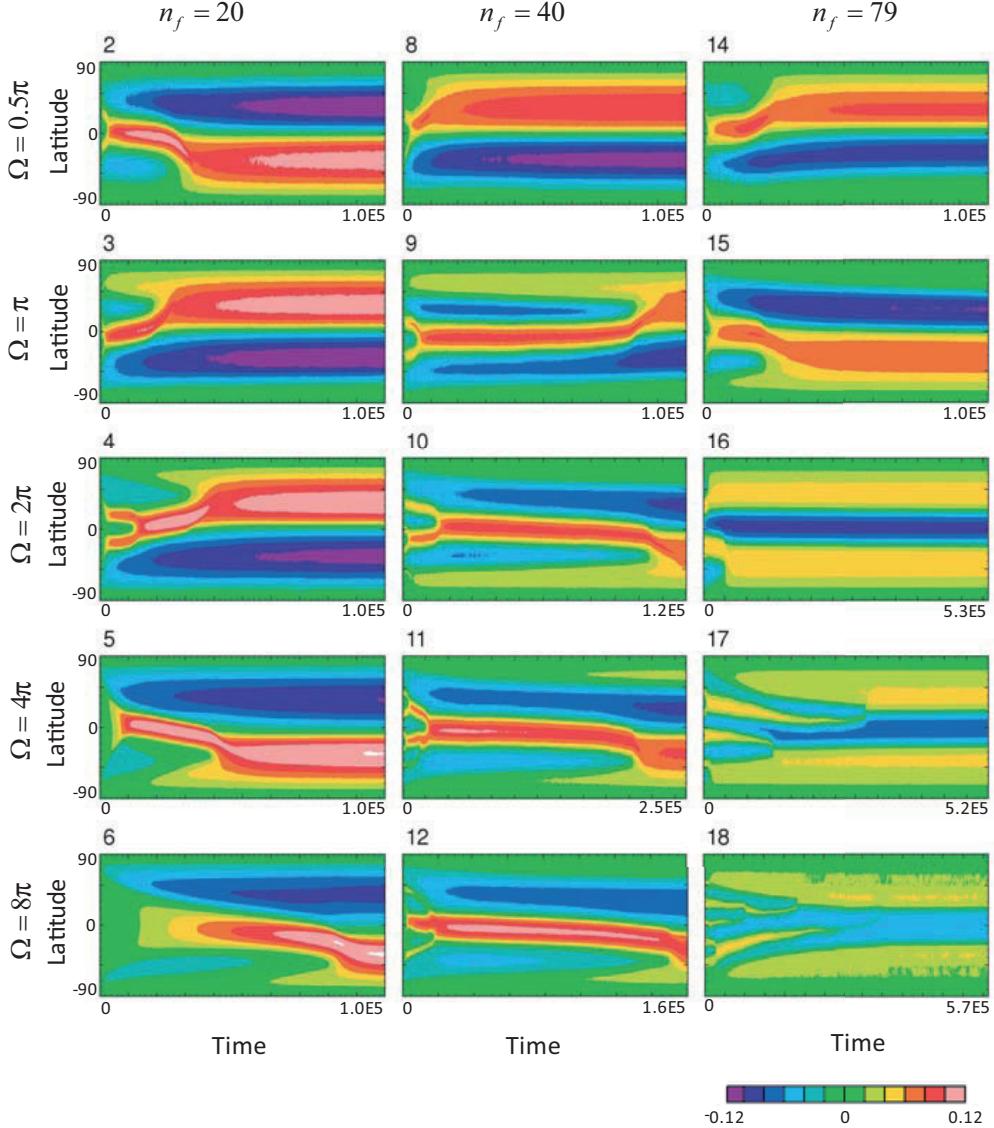


Figure 8: Long-time development of the zonal-mean zonal angular momentum. The temporal integrations have performed  $t = 0 - 1 \times 10^5$  in run 2 – 6, 8, 9, 14, and 15,  $t = 0 - 1.2 \times 10^5$  in run 10,  $t = 0 - 2.5 \times 10^5$  in run 11,  $t = 0 - 1.6 \times 10^5$  in run 12,  $t = 0 - 5.3 \times 10^5$  in run 16,  $t = 0 - 5.2 \times 10^5$  in run 17, and  $t = 0 - 5.7 \times 10^5$  in run 18. (Obuse, Takehiro and MY [15])

jets in run 16–18 (Fig.8). The structure with two broad jets, which consists of a eastward and a westward jets, shows no correlation with whether the eastward jet covers the Northern hemisphere or the Southern hemisphere.

The structure with two broad zonal jets is one of the long-time asymptotic states of the system. The inverse cascade does not proceed any more, and the two zonal jets cannot merge to one zonal jet because of the conservation law of the total angular momentum of the system. Therefore, according to our numerical results, the asymptotic states of the flow in run

$2 - 6, 8 - 12, 14$ , and  $15$  consists of two broad zonal jets dominating over the whole sphere. On the other hand, the final states in run  $16, 17$ , and  $18$  consists of three broad zonal jets. We tested the stability of the 3-jet states by adding some artificial disturbances of limited magunitude, and they were found to return to the 3-jet states, suggesting that the 3-jet states are asymptotic states. It may be interesting to note that if the final states are taken as laminar ones, the 3-jet states are in the stable region of Fig.4, and the 2-jet states are similar to the globally stable solution  $\phi_2$ .

Huang et al. [12] has argued that the inverse energy cascade reaches the 2-jet state when the forcing is white noise, but not definitely when it is a Markovian random forcing. In our case of Markovian random forcing, however, the inverse energy cascade does not stop but proceeds down to lower wavenumbers in the course of long-time evolution. This suggests that, in the forced 2D incompressible flow on a rotating sphere, the inverse energy cascade cannot be arrested irrespective of the kind of the forcing, and the asymptotic states consists of a very small number of zonal jets. This may also imply that a forced 2D incompressible flow on a rotating sphere is not an appropriate model for the dynamics of the planetary atmospheres which show multiple zonal-band structure as seen on the Jupiter, as far as long-time asymptotic states are concerned.

Last but not least, we remark that the mergers/disappearances of the zonal jets seen in the simulations in this paper appears not to be explained by an instability of laminar jets, as the zonal jets have a meridional scale large enough to be linearly stable. This strongly suggests that, although the energy is almost concentrated on the zonal components. the turbulence superimposed on the zonal jets is essential for the mergers/disappearances. This point of view is supported by a weakly nonlinear analysis of the zonal jet on  $\beta$ -plane taking into account a simple background wavy flow, which concludes that the zonal jet becomes unstable and disappears under the effect of the background flow (Obuse, Takehiro and MY [16]).

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# Energetic Variational Approaches for Complex Fluids

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## 1 Complex fluids — fluids with microstructures

The most common origin and manifestation of anomalous phenomena in complex fluids are different “elastic” effects [29]. They can be the elasticity of deformable particles; elastic repulsion between charged liquid crystals, polarized colloids or multi-component phases; elasticity due to microstructures, or bulk elasticity endowed by polymer molecules in viscoelastic complex fluids. These elastic effects can be represented in terms of certain internal variables, for example, the orientational order parameter in liquid crystals (related to their microstructures), the distribution density function in the dumb-bell model for polymeric materials, the electric, magnetic field in electrorheological and magneto-hydrodynamic fluids, the volume fraction in mixture of different materials etc. The different rheological and hydrodynamic properties can be attributed to the special coupling between the kinematic transportation of the internal variable and the induced elastic stress. In our energetic formulation, this contributes to a competition between the kinetic energy and the elastic energy.

In complex fluids, it is the interaction between the (microscopic) elastic properties and the (macroscopic) fluid motions that gives not only the complicated rheological phenomena, but also formidable challenges in analysis and numerical simulations of the materials. In electro- and magneto-rheological fluids, material inhomogeneity and electro-magnetic effects can also lead to viscoelastic phenomena [29, 46, 6, 14]. In particular, how the deformation tensor  $F$  transports in the flow field and how elastic energy described by a functional of  $F$ ,  $W(F)$ , competes with the kinetic energy in the flow play an important role in the study of complex fluids. In principle, the deformation tensor  $F$  carries all the transport/kinematic information of the microstructures, patterns and configurations in complex fluids.

As an example, for an isotropic viscoelastic fluid system, the following action functional summarizes the competition between the kinetic and elastic energy:

$$A(x) = \int_0^T \int_{\Omega_0} \frac{1}{2} \rho |x_t(X, t)|^2 - \lambda W(F) dX dt, \quad (1)$$

where  $v = x_t(X, t)$  is the fluid velocity,  $x(X, t)$  is the mapping between the Lagrange coordinate and the Eulerian coordinate system of the fluid,  $\lambda$  represents the ratio between the kinetic and elastic energies,  $\Omega_0$  is the original domain occupied by the material in the Lagrange coordinate. The fluid incompressibility implies  $J = \det F = 1$ . Using the Least Action Principle, we can derive the momentum transport equation [1, 2, 16, 29]:

$$\rho(v_t + v \cdot \nabla v) + \nabla p = \nabla \cdot \tau + f, \quad (2)$$

where  $p$  is the pressure,  $f$  is the external force density, and  $\tau$  is the extra stress given by  $\tau = \mu D(v) + \lambda(1/J)S(F)F^T$ . Here  $S(F) = [\partial W/\partial F]$  takes the Piola Kirchhoff form. The deformation tensor is transported through:

$$F_t + v \cdot \nabla F = \nabla v \cdot F. \quad (3)$$

These constitute a closed hydrodynamical system describing isotropic viscoelastic fluids. In the absence of the viscosity, the system provide an Eulerian description of incompressible elasticity [23, 38, 36]. In the case of “linear” elasticity, where  $W(F) = \frac{H}{2}|F|^2 = \frac{H}{2}\text{tr}(FF^T)$ , the system describes the infinite Weissenberg number Oldroyd-B viscoelasticity [29, 42].

Note that the viscous stress can be derived by either postulating the dissipation functional and then using the Maximum Dissipation Principle [14, 20, 19, 17] or using the stochastic approach [43, 22]. The later is consistent with the fluctuation-dissipation theorem in thermodynamics [28]. With this, the system satisfies the energy estimate:

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} [\rho(\phi)|v|^2 + \lambda(1/J)W(F)]dx + \int_{\Omega} \mu(\phi)|D(v)|^2dx = 0. \quad (4)$$

While the above viscoelastic models have been applied to various real materials, the study of other elastic complex fluids can be framed in an unified energetic variational approaches. Note that the above isotropic viscoelastic systems only reflect the transport part of the material. For other specific complex fluids, the elastic energy will take other more specific forms, in particular as those for MHD [12] and electro-kinetic fluids [44]. There the energy contributions will also take effects in the microscopic configuration and evolution equations.

## 2 EnVarA — Energetic Variational Approaches

The general energetic variational framework for classical mechanics had been developed by Rayleigh and Onsager in their seminal works published in 1873 [45] and 1936 [40].

In isothermal situations, a dissipative system satisfies the Second law of thermodynamics

$$\frac{d}{dt} E^{total} = -\Delta,$$

where  $E^{total}$  is the total energy, including both the kinetic energy and the internal energy (in this case, we do not need to distinguish the Helmholtz free energy and the internal energy), and  $\Delta$  is the dissipation functional which is equal to entropy production in this situation.

The Least Action Principle, which states that the equation of motion for a Hamiltonian system can be derived from the variation of the action functional with respect to the flow maps, is really the manifestations of the following general rule:

$$\delta E = force \cdot \delta x.$$

It gives a unique procedure to derive the conservative forces for the system.

The Maximum Dissipation Principle, variation of the dissipation functional with respect to the rate (such as velocity), gives the dissipative force for the system:

$$\delta \frac{1}{2} \Delta = force \cdot \delta u.$$

**Basic mechanics.** The good example to illustrate the procedure is the following Hookean spring model:

$$mx_{tt} + \gamma x_t + kx = 0.$$

The equation posses the following energy law:

$$\frac{d}{dt} \left( \frac{1}{2} mx_t^2 + \frac{1}{2} kx^2 \right) = -\gamma x_t^2.$$

From the Hamiltonian, we can obtain the Lagrangian of the system  $\int \frac{1}{2}mx_t^2 - \frac{1}{2}kx^2 dt$ . Employing the Least Action Principle, we arrive at the conservative part of the system.

The dissipative (damping) term is really by the Maximum Dissipation Principle, i.e. the variation of the dissipation with respect to the velocity.

The whole system is really the balance of all the forces. Notice the conservative part of the system really reflect the short time (near initial data) dynamics, the transient behavior of the whole dynamics. The dissipation part reflects the long time, near equilibrium, part of the dynamics. The choice of the dissipation functional, the quadratic form of the velocity, reflects the linear response theory for the near equilibrium dynamics.

**Newtonian fluids.** Next we look at the familiar model of the Navier-Stokes equation for incompressible Newtonian fluids:

$$u_t + u \cdot \nabla u + \nabla p = \mu \Delta u,$$

with incompressible constraint  $\nabla \cdot u = 0$ .

Again the system posses the energy law:

$$\frac{d}{dt} \int \frac{1}{2}|u|^2 dx = - \int \mu |\nabla u|^2 dx.$$

The Hamiltonian part of dynamics, from the Least Action Principle, is the Euler equation, which represents the short time (near initial data) dynamics. The dissipation part from the Maximum Dissipation Principle is the Stokes equation, for the long time dynamics near equilibrium.

**Diffusion equations.** Finally, we want to look at the following parabolic equations:

$$f_t = c\Delta f$$

. We rewrite the system into the following equivalent coupled system:

$$\begin{aligned} f_t + \nabla \cdot (uf) &= 0, \\ \nabla(cf) &= -fu. \end{aligned}$$

The first equation is just the common conservation of mass, which is just a change of variable from the Lagrangian particle coordinate to Eulerian coordinate. Set the left hand side  $p = cf$ , it immediate resembles to the Darcy's law, with compressible equation of states the same as that of ideal gas. Indeed, according to the first law of thermodynamics, the corresponding internal energy density will be  $cf \ln f$ . Moreover,

$$\frac{d}{dt} \int cf \ln f dx = - \int fu^2 dx.$$

In fact, we can see that the force corresponding to the pressure  $\nabla(cf)$  is obtained from the variation of the left hand side of the above equation with respect to the flow map  $x$ , while the dissipative force  $-fu$  is from the variation of the right hand side with respect to the velocity  $u$ .

### 3 Electrorheological fluids and ionic solutions

How microstructures affect the bulk rheology of complex fluids is exemplified by electrorheological fluids. The hydrodynamical properties in a electrokinetic flow is determined by the coupling of the

transport of the concentration of the charges and the induced elastic force (the Lorentz force). The mathematical system can be illustrated in the following system [4, 5, 44, 11, 3, 47, 48]:

$$\rho(u_t + u \cdot \nabla u) + \nabla \pi = \nu \Delta u + (n - p) \nabla V, \quad \nabla \cdot u = 0, \quad (5)$$

$$n_t + u \cdot \nabla n = \nabla \cdot (D_n \nabla n - \mu_n n \nabla V), \quad (6)$$

$$p_t + u \cdot \nabla p = \nabla \cdot (D_p \nabla p + \mu_p p \nabla V), \quad (7)$$

$$\nabla \cdot (\epsilon \nabla V) = n - p. \quad (8)$$

Equations (5) represents the momentum equations where  $u$  is the fluid velocity,  $\pi$  is the pressure,  $\rho$  is the fluid density and  $\nu$  the fluid viscosity. In equation (5),  $(n - p) \nabla V$  is the macroscopic Lorentz (or Coulomb) force. Equations (6), (7) and (8) form the Nernst-Planck-Poisson system of a binary charge system,  $n$  and  $p$  are the densities of diffuse, negative and positive charges respectively.  $D_n, D_p$  are the respective diffusivity constants and  $\mu_n, \mu_p$  are the respective mobility constants.  $D_n, D_p$  and  $\mu_n, \mu_p$  are related by Einstein's relation and the valence of the charged particles.

The elastic energy combines both the electric energy and the entropy (contributes to the diffusion of the charge density). These different energy functions will generate various microstructures, as other elastic complex fluids. The induced Lorentz force is also due to the kinetic transport of the charged particles and electric energy balance.

The above system is very important in understanding the complicated behaviors relevant to electrophysiology.

In [49], we studied the stationary configurations of the stationary Ernest-Planck-Poisson equations and the limiting behavior as the (non-dimensional) Debye constant becomes small. We give the rigorous proof of the different Debye layer configuration between the electrical neutral and non-neutral cases [32]. These special boundary layer properties are crucial in the application [4, 5].

In [50] we reformulated the above hydrodynamical system of ER fluids, using different variational procedures. The method reveals the fundamental structures of the coupling and transport in these systems. We notice the work [41, 39] on the convection-diffusion equations. We obtained the well-posedness results of the system, based on our energetic variational approach. Moreover, combined with our diffusive interface methods, we developed systems/numerical algorithms to model the deformation of the vesicle membranes with preferable charge (ions) selections.

In [53], a hydrodynamical model for non-diluted ER fluids had been established and the numerical results had shown good agreements with the experimental data. We are studying the analytical properties on these system with nonlocal interactions or correlations and extend the theory to the ionic biological fluids and their interaction with proteins (ion channels) [24, 25].

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# On the term-wise estimates for the norm-inflation solution of the Navier-Stokes flows

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## 1 Introduction

This note is based on the paper [13].

We consider the nonstationary incompressible viscous flow of the ideal fluid in the whole space  $\mathbf{R}^3$ . This is mathematically described as the Cauchy problem of the Navier-Stokes equations:

$$(NS) \quad \begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = 0 & \text{in } \mathbf{R}^3 \times (0, T), \\ \nabla \cdot u = 0 & \text{in } \mathbf{R}^3 \times (0, T), \\ u|_{t=0} = u_0 & \text{in } \mathbf{R}^3. \end{cases}$$

This Cauchy problem is called (NS) in here. We define the notations of derivatives as follows:  $u_t := \partial_t u := \partial u / \partial t$ ,  $\partial_j := \partial / \partial x_j$  for  $j = 1, 2, 3$ ,  $\nabla := (\partial_1, \partial_2, \partial_3)$ ,  $\Delta := \sum_{j=1}^3 \partial_j^2$ . Here, for vectors  $a = (a^1, a^2, a^3)$  and  $b = (b^1, b^2, b^3)$ ,  $a \cdot b$  or  $(a, b)$  denotes  $\sum_{j=1}^3 a^j b^j$ . The velocity (vector field) of the fluid  $u = (u^1, u^2, u^3) = (u^1(x, t), u^2(x, t), u^3(x, t))$  and its pressure (scalar)  $p = p(x, t)$  are unknown functions at the place  $x \in \mathbf{R}^3$  and time  $t \in (0, T)$ , while the initial velocity  $u_0 = (u_0^1(x), u_0^2(x), u_0^3(x))$  is given. It is natural to impose the compatibility condition on  $u_0$ , that is,  $\nabla \cdot u_0 = 0$  holds for all  $x \in \mathbf{R}^3$ .

It is a famous open problem whether one can obtain the uniqueness and smoothness of Leray's weak solutions constructed in [11], that is, (NS) admits a time-global unique solution in  $L^2(\mathbf{R}^3)$ . In this note our aim is different to this, so we do not penetrate into its detail.

Bisides, by the Duhamel principle we derive the integral equation from (NS)

$$(INT) \quad u(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-\tau)\Delta} \mathbf{P}(u(\tau), \nabla) u(\tau) d\tau.$$

Here, we denote the heat semigroup  $e^{t\Delta} := G_t *$ , the Gauss kernel  $G_t(x) := \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x|^2}{4t}}$ , convolution with respect to spatial variables  $f * g(x) := \int_{\mathbf{R}^3} f(x-z)g(z)dz$ , the

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Helmholtz projection  $\mathbf{P} := (\delta_{ij} + R_i R_j)_{i,j=1,2,3}$ , Kronecker's delta  $\delta_{ij} = 1$  if  $i = j$ ,  $\delta_{ij} = 0$  if  $i \neq j$ , the Riesz transform  $R_i := \partial_i(-\Delta)^{-1/2} := \mathcal{F}^{-1} \frac{\sqrt{-1}\xi_i}{|\xi|} \mathcal{F}$ . The Fourier transform is defined by  $\mathcal{F}$ , and  $\mathcal{F}^{-1}$  is its inverse.

We call the solution of (INT) a **mild solution**. The formal equivalency between (INT) and (NS) can be justified in the classical sense when  $u$  has a sufficient regularity, provided if  $p$  is under the suitable assumption, for example,

$$p = \sum_{i,j=1}^n R_i R_j u^i u^j. \quad (1.1)$$

We rather discuss (INT) and mild solutions than (NS) and classical solutions. The function space  $C([0, T]; X)$  is the natural one to which mild solutions belong as long as mild solutions exist, when  $u_0 \in X$  with a certain Banach space  $X$ . Mild solutions are usually constructed by the limit of the successive approximation

$$u_1(t) := e^{t\Delta} u_0 \quad \text{and} \quad u_{j+1}(t) := u_j - \mathcal{B}(u_j) \quad \text{for } j \in \mathbb{N}, \quad (1.2)$$

where

$$\mathcal{B}(u, v) := \int_0^t e^{(t-\tau)\Delta} \mathbf{P}(u(\tau), \nabla)v(\tau) d\tau \quad \text{and} \quad \mathcal{B}(u) := \mathcal{B}(u, u). \quad (1.3)$$

We discuss on the ill-posedness of the Navier-Stokes equations in the whole space with initial data in the critical spaces due to the behavior of mild solutions. This note is contributed to understand for such a negative results by Bourgain and Pavlovic [3]. In fact, they showed a lack of equicontinuity of mild solutions within  $\dot{B}_{\infty,\infty}^{-1}$ .

**Theorem 1.1** (Bourgain-Pavlovic [3]). *For  $\delta \in (0, 1)$  and  $T \in (0, 1)$  there exists an initial velocity  $u_0 \in \dot{B}_{\infty,\infty}^{-1}(\mathbf{R}^3)$  such that  $\|u_0\|_{\dot{B}_{\infty,\infty}^{-1}} < \delta$  with  $\nabla \cdot u_0 = 0$ , there exists a mild solution  $u$  in  $C([0, T]; \dot{B}_{\infty,\infty}^{-1})$  and  $\|u(T)\|_{\dot{B}_{\infty,\infty}^{-1}} > 1/\delta$ .*

The definition of function spaces will be denoted in Section 2. One of our purposes is to give a rigorous proof of their assertion. We also state the following results:

**Theorem 1.2** (S. [13]). *For  $T \in (0, 1)$  there exists a  $u_0$  such that  $\|u_j(T)\|_{\dot{B}_{\infty,\infty}^{-1}(\mathbf{R}^3)}$  does not converge.*

The proof of theorems will be given in Section 3. The assertion of Theorem 1.2 does not imply the blow-up of mild solutions. This says that one has to take a subsequence of approximation for the convergence.

We now refer to the motivation of recent works related to the results above. To solve (NS) uniquely and time-globally in 3-dimension, one may consider the following steps: firstly the smooth time-local solution is constructed, secondly the

solution is extended uniquely and time-globally. Along this strategy, Kato and Fujita [4] introduced the notion of mild solutions, and proved that (NS) admits a unique time-local smooth solution, when  $u_0 \in H^{\frac{n}{2}-1}(\mathbf{R}^n)$ .

Some researcher wanted to eliminate the smoothness on the initial data, since the smoothness of the solutions is automatically obtained by the usual smoothing effect of solutions to equations of parabolic type. For this purpose Kato [7] (in the whole space) and Giga and Miyakawa [6] (in a bounded domain) studied the properties of the heat semigroup in the Lebesgue spaces (using  $L^p - L^q$  smoothing estimates), and they proved that (NS) admits a time-local unique smooth solution in  $L^n(\mathbf{R}^n)$  for all  $n \geq 2$ . Note that  $L^n(\mathbf{R}^n)$  is a scaling invariant space (to which self-similar solutions belong) under the parabolic scaling  $u_\lambda(x, t) := \lambda u(\lambda^2 x, \lambda t)$ , that is,  $\|u\|_n = \|u_\lambda\|_n$  for all  $\lambda > 0$ . After their articles, there are a lot of contributions of local well-posedness in several scaling invariant spaces. Actually, Kato and Ponce did it in  $\dot{H}_2^{\frac{n}{2}-1}$  in [8], Kozono and Yamazaki showed it in  $\dot{B}_{p,\infty}^{-1+n/p}$  for  $p \in (n, \infty)$  in [10]. In 2001 Koch and Tataru proved it by [9] in  $BMO^{-1}$ . The function spaces which are concerned are wider and wider:

$$\dot{H}_2^{n/2-1} \subset L^n \subset \dot{B}_{p,\infty}^{-1+n/p} \subset BMO^{-1} = \dot{F}_{\infty,2}^{-1} \subset \dot{F}_{\infty,\infty}^{-1} = \dot{B}_{\infty,\infty}^{-1}$$

for  $p \in (n, \infty)$ . These embeddings are continuous (in the norms). Notice that  $\dot{B}_{\infty,\infty}^{-1}$  is possibly the biggest function space in the following sense:  $\dot{B}_{\infty,\infty}^{-1}$  contains all scaling invariant spaces, and one can solve the linearized problem (the heat equations) in  $\dot{B}_{\infty,\infty}^{-1}$ . This implies that all self-similar solution belongs to  $\dot{B}_{\infty,\infty}^{-1}$ . Therefore, from view point of pure mathematical interests, many researchers tried and still try to investigate (NS) in such function spaces.

## 2 Function spaces

We introduce the function spaces in this section. Let  $n \in \mathbf{N}$ ,  $s \in \mathbf{R}$  and let  $1 \leq p, q \leq \infty$ . The set of test functions is denoted by  $\mathcal{D}$  or,  $C_c^\infty(\mathbf{R}^n)$ . Its topological dual stands for  $\mathcal{D}'$ , which is the set of distributions. The set of rapidly decreasing functions (in the sense of Schwartz) is written as  $\mathcal{S}$ ; the set of tempered distributions is  $\mathcal{S}'$ . For  $p \in [1, \infty]$ ,  $L^p := L^p(\mathbf{R}^n) := \{f \in L_{loc}^1; \|f\|_p < \infty\}$  is the Lebesgue space of  $p$ -th integrable functions with the norm  $\|\cdot\|_p$ . We often omit the notation of the domain ( $\mathbf{R}^n$ ).

To define the Besov spaces we now introduce the Paley-Littlewood decomposition. Let us call  $\{\phi_j\}_{j=-\infty}^\infty$  the Paley-Littlewood decomposition if  $\hat{\phi}_0 \in C_c^\infty(\mathbf{R}^n)$ ,  $\text{supp } \hat{\phi}_0 \subset \{\xi; 1/2 \leq |\xi| \leq 2\}$ ,  $\hat{\phi}_j(\xi) = \hat{\phi}_0(2^{-j}\xi)$  and  $\sum_{j=-\infty}^\infty \hat{\phi}_j(\xi) = 1$  except for  $\xi = 0$ , that is, a dyadic decomposition of the unity in the phase space. Let  $\mathcal{Z}'$  be

the topological dual space of

$$\mathcal{Z} := \left\{ f \in \mathcal{S} ; \partial^\alpha \hat{f}(0) = 0, \forall \alpha \in \mathbf{N}_0^n \right\}.$$

**Definition 2.1.** For  $s \in \mathbf{R}$  and  $1 \leq p, q \leq \infty$  we define the homogeneous Besov space by

$$\begin{aligned} \dot{B}_{p,q}^s &:= \left\{ f \in \mathcal{Z}' ; \|f\|_{\dot{B}_{p,q}^s} < \infty \right\}, \\ \|f\|_{\dot{B}_{p,q}^s} &:= \left[ \sum_{j=-\infty}^{\infty} 2^{jsq} \|\phi_j * f\|_p^q \right]^{1/q} \quad \text{if } q < \infty, \\ \|f\|_{\dot{B}_{p,\infty}^s} &:= \sup_{-\infty \leq j \leq \infty} 2^{js} \|\phi_j * f\|_p \quad \text{if } q = \infty. \end{aligned}$$

**Note.** (1) By the definition of  $\phi_j$  it is clear that  $\|f\|_{\dot{B}_{p,q}^s} = 0$  if  $f \in \mathcal{P} := \{\text{polynomials}\}$ . Thus,  $\|\cdot\|_{\dot{B}_{p,q}^s}$  and  $\|\cdot\|_{\dot{F}_{p,q}^s}$  are seminorms. The quotient spaces divided by polynomials  $\dot{B}_{p,q}^s/\mathcal{P}$  and  $\dot{F}_{p,q}^s/\mathcal{P}$  are Banach spaces. See e.g. [14]

(2)  $\dot{B}_{p,q}^s$  is a subset of  $\mathcal{S}'$  if the exponents satisfy

$$\text{either "s} < n/p\text{" or "s} = n/p \text{ and } q = 1\text{"}. \quad (2.1)$$

Under this conditions, the operators  $\mathcal{F}$ ,  $e^{t\Delta}$ ,  $\mathbf{P}$ ,  $R_i$  can be defined on the homogeneous spaces as the tempered distribution sense. Also, it is natural to select the representative element such that

$$f = \sum_{j=-\infty}^{\infty} \phi_j * f \quad \text{in } \mathcal{S}'. \quad (2.2)$$

See the details in Bourdaud [1] or Kozono and Yamazaki [10]. Throughout of this note, we only treat the homogeneous space under the exponents satisfying (2.1).

In this note we mainly deal with the case  $p = \infty$ . It is well-known that there are several equivalent norms of the Besov norm, for example,

$$\|f\|_{\dot{B}_{\infty,\infty}^{-1}} \sim \|f\| := \sup_{\rho > 0} \sqrt{\rho} \|e^{\rho\Delta} f\|_\infty \quad \text{for } f \in \dot{B}_{\infty,\infty}^{-1}.$$

We rather compute  $\|\cdot\|$ , in what follows, for the sake of simplicity of the dependance of constants.

### 3 Outline of the proofs of Theorems

In this section we refer to the outline of the proofs of Theorem 1.1 and 1.2. Theorems follow from the technique of Bourgain [2] for establishing the similar ill-posedness theorem for the KdV equation. His method is so-called “norm inflation”. Before stating the outline of the proof, we now fix the initial velocity, concretely. In what follows, the initial velocity is fixed to be of the form

$$u_0(x) := \frac{Q}{\sqrt{r}} \sum_{s=1}^r h_s [e_2 \cos(k_s \cdot x) + e_3 \cos(l_s \cdot x)], \quad (3.1)$$

that is to say,

$$u_0 = \left( 0, \frac{Q}{\sqrt{r}} \sum_{s=1}^r h_s \cos(h_s x_1), \frac{Q}{\sqrt{r}} \sum_{s=1}^r h_s \cos(h_s x_1 - x_2) \right)$$

with parameters  $Q > 0$  and large  $r \in \mathbf{N}$ ; other notations are as follows:

$$\begin{aligned} e_2 &:= \vec{e}_2 := (0, 1, 0) & (= v_s), \\ e_3 &:= \vec{e}_3 := (0, 0, 1) & (= v'_s), \\ h_s &:= h(s) := 2^{s(s-1)/2} \gamma^{s-1} \eta & \text{for } s \in \mathbf{N}, \\ k_s &:= (h_s, 0, 0), \\ l_s &:= (h_s, -1, 0) & (= k'_s). \end{aligned}$$

Here  $\gamma, \eta \in \mathbf{N}$  are also parameters;  $v_s, v'_s, k'_s$  are the notation in [3]. The specific time  $T$  when the norm-inflation occurs can be regarded as a parameter, replacing the time variable  $[t \mapsto \lambda t]$  with some  $\lambda > 0$ . Using this scaling argument, we can relax the restriction  $T < 1$ . However, for the sake of simplicity of the proof, and for the readers' convenience,  $T$  remains as a given small number in this paper.

It is clear by definition that  $u_0(x) = (0, u_0^2(x_1), u_0^3(x_1, x_2))$  and  $u_0 \in \dot{B}_{\infty, \infty}^{-1}$  by the simple calculation below. Moreover, since  $u_0$  is a uniformly continuous function, one can get the continuity of mild solutions in time up to the initial time; see e.g. [12]. It should be emphasized that we are able to fix the directions of  $v_s = e_2$  and  $v'_s = e_3$  without loss of generality, since (NS) is invariant under the Galilee transformation. In addition, it should be more emphasized that the selections of  $v_s$  and  $v'_s$  are slightly different to those of [3]; that is a crucial point noticed by Yoneda.

It is easy to see that  $h_s \ll h_{s+1}$  for large  $s$  or  $\gamma$ ; this property is so-called ‘lacunary’, and is benefit to control the effect of the interaction between each frequency. The compatibility condition  $\nabla \cdot u_0 = 0$  is satisfied by  $e_2 \cdot k_s = 0$  and  $e_3 \cdot l_s = 0$ , obviously. It is clear that  $u_0$  is a smooth periodic function. This implies that the mild solution is also periodic with the period  $2\pi$ , regarded as a function on the torus  $(2\pi\mathbf{T})^3$ , as long as the mild solution exists. So, the kinematic energy is bounded by the initial energy  $\frac{1}{2} \|u_0\|_{L^2((2\pi\mathbf{T})^3)}^2$ ; this is huge but finite.

Let  $u_1$  be the first approximation of iteration scheme, that is, the solution to the heat equation with initial datum given by (3.1):

$$u_1(x, t) = \frac{Q}{\sqrt{r}} \sum_{s=1}^r h_s [e_2 e^{-h_s^2 t} \cos(h_s x_1) + e_3 e^{-(h_s^2 + 1)t} \cos(h_s x_1 - x_2)].$$

For  $t > 0$  we obtain that  $u_1(t) := u_1(\cdot, t) \in L^\infty \cap BMO^{-1}$ , even though these norms are large.

It is well-known that one can construct the unique mild solution with initial velocity given by (3.1) in the  $L^\infty$ -framework. Moreover, in [5] one can estimate for the possible existence time  $T_*$  (until when we may construct a mild solution by

iteration scheme in  $C([0, T_*]; L^\infty))$  bounded from below:  $T_* \geq C/\|u_0\|_\infty^2 \sim h_r^{-2}$  with the universal constant  $C > 0$ . Indeed, by  $h_r \gg r$  we see that

$$\|u_0\|_\infty \sim \frac{Q}{\sqrt{r}} \sum_{s=1}^r h_s \sim h_r \gg 1 \quad \text{if } r \gg 1.$$

Therefore,  $T_*$  might be very tiny. However, we observe the Besov norm  $\|\cdot\|$  as

$$\|u_0\| \sim \frac{Q}{\sqrt{r}} \ll 1 \quad \text{if } r \gg 1. \quad (3.2)$$

In fact,

$$\begin{aligned} \|u_0\| &= \sup_{\rho>0} \sqrt{\rho} \|e^{\rho\Delta} u_0\|_\infty \\ &= \sup_{\rho} \sqrt{\rho} \frac{Q}{\sqrt{r}} \sup_x \left| \sum_{s=1}^r h_s [e_2 e^{-h_s^2 \rho} \cos(k_s \cdot x) + e_3 e^{-(h_s^2+1)\rho} \cos(l_s \cdot x)] \right| \\ &\leq C_* \frac{Q}{\sqrt{r}} \end{aligned}$$

with the numerical constant  $C_*$  independent of parameters. Roughly speaking, we derive these estimate replaced from sum by integration.

Now we recall the successive approximation and its modification of convergence version. A mild solution  $u$  is usually constructed as the limit of function series  $\{u_j\}_{j=1}^\infty$  (or, its subsequence if necessary) defined by (1.2). Since  $u_0 \in BUC$ ,  $\{u_j\}_{j=1}^\infty$  is a Cauchy sequence in  $C([0, T_*]; BUC)$  provided  $T_*$  is chosen small enough as  $h_r^{-2}$ . Thus, it has a uniform convergence limit  $u$  as the mild solution in  $[0, T_*]$ . In order to observe the norm inflation of mild solutions, we always concern at  $T > T_*$ . Throughout this paper, we use the standard terminology that the bilinear terms denote by (1.3). Let us put the sequence  $\{v_k\}_{k=1}^\infty$  as

$$\begin{aligned} v_1(t) &:= u_1(t) := e^{t\Delta} u_0, \\ v_{k+1}(t) &:= u_{k+1}(t) - u_k(t) = -\mathcal{B}(u_k) + \mathcal{B}(u_{k-1}) \end{aligned}$$

for  $k \in \mathbb{N}$ . Therefore, we may rewrite  $u_j$  and the mild solution  $u = \lim_{j \rightarrow \infty} u_j$  as

$$u_j(t) = \sum_{k=1}^j v_k(t) \quad \text{and} \quad u(t) = \sum_{k=1}^\infty v_k(t). \quad (3.3)$$

In what follows, we shall calculate  $v_k(t)$  and estimate the Besov norm of them at  $t = T$ . Moreover, we easily notice that

$$v_k = (0, 0, v_k^3(x_1, x_2, t)) \quad \text{for } k \geq 2. \quad (3.4)$$

*proof of Theorem 1.1.* The proof of Theorem 1.1 is carried out by the suitable selection of the parameters  $(Q, r, \gamma, \eta)$  for each  $\delta, T \in (0, 1)$ . We see that  $\|u_0\| < \delta$  as well as

$$\|v_1(T)\| \leq \|G_t\|_1 \|u_0\| \leq C_* \frac{Q}{\sqrt{r}} =: S < \delta. \quad (3.5)$$

Also,  $v_2 = M_2 + R_2$  and  $M_2 := e_3 \frac{Q^2}{4} e^{-t} \sin x_2$  with

$$\|v_2(T)\| \simeq \|M_2(T)\| = C_b Q^2 =: L \geq \frac{2}{\delta}. \quad (3.6)$$

Here  $A \simeq B$  means the almost equal, that is,  $A = B + R$  such that  $|R| < \frac{1}{3}|B|$  for the positive (scalar) valued, and  $\|R\| < \frac{1}{3}\|B\|$  for functions;  $C_b > 0$  is a numerical constant. We may see that  $M_2$  is the major term of  $v_2$  at  $t \simeq T$ . Reversely,  $R_2$  is the collection of the remainder terms of  $v_2$  at  $t \simeq T$ , that is,  $\|R_2\| \leq \frac{1}{3}\|M_2\|$ . It is remarkable that  $M_k(t)$  no longer might be the leading term if we take neither a different norm nor  $t \ll T$ . We further prove that  $v_3 = M_3 + R_3$  with

$$M_3 := -\frac{Q^3}{8\sqrt{r}} t e^{-t} \sum_{s=1}^r h_s e^{-h_s^2 t} \{\cos(h_s x_1 + x_2) + \cos(h_s x_1 - x_2)\} e_3,$$

$$\|R_3(T)\| < \frac{1}{3}\|M_3(T)\| \text{ and}$$

$$\|v_3(T)\| \simeq \|M_3(T)\| \simeq \frac{Q^3 \sqrt{T}}{8\sqrt{2er}} \simeq \frac{Q^2}{4\eta} S \quad (3.7)$$

for  $t \simeq T \simeq \eta^{-2}$ . Moreover, we see that for  $v_4$

$$v_4(T) = M_4(T) + R_4(T), \quad M_4(T) = -KM_2(T), \quad K := \frac{(1 - 3e^{-2})Q^2}{8r\eta^2} > 0.$$

By induction one may also show that  $v_k(T) = M_k(T) + R_k(T)$  with  $\|R_k(T)\| < \frac{1}{3}\|M_k(T)\|$  and

$$M_{2k-1}(T) = (-K)^{k-2} M_3(T) \quad \text{and} \quad M_{2k}(T) = (-K)^{k-1} M_2(T) \quad (3.8)$$

for  $k \geq 2$  with  $t \simeq T \simeq \eta^{-2}$ . Once we obtain these estimates, it follows from (3.3):

$$\|u(T)\| \geq \|v_2(T)\| - \sum_{k=2}^{\infty} \left( \frac{4K}{3} \right)^{k-1} \|v_2(T)\| \geq \frac{L}{2},$$

if  $K < 1/4$ . Here we simply discard the sum of odd numbers, since  $S$  is much smaller than  $L$ . Finally, the choice of parameters yields that  $S \simeq \delta$  and  $L \simeq \frac{2}{\delta}$ .

We refer to the selection of the parameters  $(Q, r, \gamma, \eta)$ . Firstly, we always fix  $\gamma := 3$ . We impose that  $\eta \in \mathbf{N}$  with  $\eta \geq 2$  large such that  $\eta \sim T^{-1/2}$  for  $T \in (0, 1)$ . For any  $\delta \in (0, 1)$ , we fix  $Q > 1$  large such that  $Q > \sqrt{\frac{3}{C_b \delta}}$ . Finally, we choose  $r \in \mathbf{N}$  large such that  $r > 4C_*^2 \delta^{-4}$ ,  $T \sim \eta^{-2} > h_r^{-2}$  and  $K < \frac{1}{4}$ . This completes the proof of Theorem 1.1.  $\square$

*proof of Theorem 1.2.* Let us assume  $T < 1/4$  without loss of generality. We choose the initial datum  $u_0$  given by (3.1). Determine  $\gamma = 3$  and  $r = 2$ . Select  $\eta \in \mathbf{N}$  as

$\eta \geq 2$  and  $\eta \simeq T^{-1/2}$ . Let  $Q$  be taken large such that  $K > 4$ . Since  $\|R_k(T)\| < \frac{1}{3}\|M_k(T)\|$  for  $k \in \mathbf{N}$ , by (3.3) one can easily observe that

$$\|u_{4j+2}(T)\| \simeq \sum_{k=1}^{2j+1} (-K)^{k-1} \|M_2(T)\|$$

for all  $j$ . We, in here, discard the odd numbers and the remainder terms for the sake of simplicity. As conclusion,  $\|u_{4j+2}(T)\|$  tends to infinity as  $j \rightarrow \infty$ . This completes the proof of Theorem 1.2.  $\square$

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# Life span of positive solutions for a semilinear heat equation with non-decaying initial data

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## 1 Introduction

### 1.1 Problem

We study the life span of positive solutions of the Cauchy problem for a semi-linear heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + F(u), & (x, t) \in \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = \phi(x) \geq 0, & x \in \mathbb{R}^n, \end{cases} \quad (1)$$

where  $n \geq 2$ . Let  $\phi$  be a bounded continuous function on  $\mathbb{R}^n$ . Throughout this talk, we assume that  $F(u)$  satisfies

$$F(u) \geq u^p \quad \text{for } u \geq 0, \quad (2)$$

with  $p > 1$ .

In this talk, we show a upper bound on the life span of positive solutions of equation (1) for non-decaying initial data. We define the life span (or blow up time)  $T^*$  as

$$T^* = \sup\{T > 0 \mid (1) \text{ possesses a unique classical solution in } \mathbb{R}^n \times [0, T)\}. \quad (3)$$

### 1.2 Known results for $F(u) = u^p$

Results in [1, 2, 6, 7, 9, 16] are summarized as follows:

- (i) Let  $p \in (1, 1+2/n]$ . Then every nontrivial solution of the equation (1) blows up in finite time.
- (ii) Let  $p \in (1+2/n, \infty)$ . Then the equation (1) has a time-global classical solution for some initial data  $\phi$ .

Especially for non-decaying initial data, it was shown that the solution of the equation (1) blows up in finite time for any  $p > 1$ . This result was proved in [8, 10].

Recently, several studies have been made on the life span of solutions for (1). See [3, 4, 5, 8, 11, 12, 13, 14, 15, 17, 18], and references therein.

Gui and Wang [5] proved the following results when initial data takes the form  $\phi(x) = \lambda\psi(x)$ .

- (i)  $\lim_{\lambda \rightarrow \infty} T^* \cdot \lambda^{p-1} = \frac{1}{p-1} \|\psi\|_{L^\infty(\mathbb{R}^n)}^{1-p}$ .
- (ii) If  $\lim_{|x| \rightarrow \infty} \psi(x) = k$ , then  $\lim_{\lambda \rightarrow 0} T^* \cdot \lambda^{p-1} = \frac{1}{p-1} k^{1-p}$ .

The purpose of this talk is to give a upper bound of the life-span of the solution for the equation (1) with initial data having positive inferior limit at space infinity.

## 2 Main results

### 2.1 Conic neighborhood

In order to state main results, we prepare several notations. For  $\xi' \in \mathbb{S}^{n-1}$ , and  $\delta \in (0, \sqrt{2})$ , we set conic neighborhood  $\Gamma_{\xi'}(\delta)$ :

$$\Gamma_{\xi'}(\delta) = \left\{ \eta \in \mathbb{R}^n \setminus \{0\}; \left| \xi' - \frac{\eta}{|\eta|} \right| < \delta \right\}, \quad (4)$$

and set  $S_{\xi'}(\delta) = \Gamma_{\xi'}(\delta) \cap \mathbb{S}^{n-1}$ . Define

$$\phi_\infty(x') = \liminf_{r \rightarrow +\infty} \phi(rx')$$

for  $x' \in \mathbb{S}^{n-1}$ . We note that  $\phi_\infty \in L^\infty(\mathbb{S}^{n-1})$ .

### 2.2 Main results

Now, we state a main result.

**Theorem 1.** *Let  $n \geq 2$ . Assume that there exist  $\xi' \in \mathbb{S}^{n-1}$  and  $\delta > 0$  such that  $\text{ess.inf}_{x' \in S_{\xi'}(\delta)} \phi_\infty(x') > 0$ . Then the classical solution for (1) blows up in finite time, and the blow up time is estimated as*

$$T^* \leq \frac{1}{p-1} \left( \text{ess.inf}_{x' \in S_{\xi'}(\delta)} \phi_\infty(x') \right)^{1-p}. \quad (5)$$

Once we admit Theorem 1, we can prove the following corollary immediately.

**Corollary 1.** *Suppose that  $\|\phi_\infty\|_{L^\infty(\mathbb{S}^{n-1})} > 0$ . Assume that for arbitrary small  $\varepsilon > 0$  there exist  $\eta' \in \mathbb{S}^{n-1}$  and  $\delta > 0$  such that*

$$\text{ess.inf}_{x' \in S_{\eta'}(\tilde{\delta})} \phi_\infty(x') \geq \|\phi_\infty\|_{L^\infty(\mathbb{S}^{n-1})} - \varepsilon. \quad (6)$$

*Then the classical solution for (1) blows up in finite time, and the blow up time is estimated as*

$$T^* \leq \frac{1}{p-1} \|\phi_\infty\|_{L^\infty(\mathbb{S}^{n-1})}^{1-p}. \quad (7)$$

*Proof of Corollary 1.* For arbitrary small  $\varepsilon > 0$ , we obtain

$$T^* \leq \frac{1}{p-1} (\|\phi_\infty\|_{L^\infty(\mathbb{S}^{n-1})} - \varepsilon)^{1-p} \quad (8)$$

from Theorem 1. Taking  $\varepsilon \rightarrow 0$ , we obtain the desired result.  $\square$

### 3 Outline of proof

#### 3.1 Preliminaries

For  $\xi' \in \mathbb{S}^{n-1}$  and  $\delta > 0$  as in the theorem, we first determine the sequences  $\{a_j\} \subset \mathbb{R}^n$  and  $\{R_j\} \subset (0, \sqrt{2})$ . Let  $\{a_j\} \subset \mathbb{R}^n$  be a sequence satisfying that  $|a_j| \rightarrow \infty$  as  $j \rightarrow \infty$ , and that  $a_j/|a_j| = \xi'$  for any  $j \in \mathbb{N}$ . Put  $R_j = (\delta\sqrt{4 - \delta^2}/2)|a_j|$ .

For  $R_j > 0$ , let  $\rho_{R_j}$  be the first eigenfunction of  $-\Delta$  on  $B_{R_j}(0) = \{x \in \mathbb{R}^n; |x| < R_j\}$  with zero Dirichlet boundary condition under the normalization  $\int_{B_{R_j}(0)} \rho_{R_j}(x) dx = 1$ . Moreover, let  $\mu_{R_j}$  be the corresponding first eigenvalue.

For the solutions for (1), we define

$$w_j(t) = \int_{B_{R_j}(0)} u(x + a_j, t) \rho_{R_j}(x) dx. \quad (9)$$

Here, we introduce the properties of the initial value  $\{w_j(0)\}$ .

**Proposition 1.** (i) *We have*

$$\liminf_{j \rightarrow +\infty} w_j(0) \geq \operatorname{ess.inf}_{x' \in S_{\xi'}(\delta)} \phi_\infty(x'). \quad (10)$$

(ii) *We have*

$$\lim_{j \rightarrow +\infty} \frac{\log(1 - \mu_{R_j} w_j^{1-p}(0))}{-\mu_{R_j} w_j^{1-p}(0)} = 1. \quad (11)$$

*Proof.* (i) Changing the variable and using the relation  $\rho_{\frac{\pi}{2}}(x) = (2R_j/\pi)^n \rho_{R_j}(2R_j x/\pi)$ , we have

$$\begin{aligned} w_j(0) &= \int_{B_{R_j}(0)} \phi(x + a_j) \rho_{R_j}(x) dx \\ &= \left(\frac{2R_j}{\pi}\right)^n \int_{B_{\frac{\pi}{2}}(0)} \phi\left(\frac{2R_j}{\pi}x + a_j\right) \rho_{R_j}\left(\frac{2R_j}{\pi}x\right) dx \\ &= \int_{B_{\frac{\pi}{2}}(0)} \phi\left(\frac{2R_j}{\pi}x + a_j\right) \rho_{\frac{\pi}{2}}(x) dx. \end{aligned} \quad (12)$$

By Fatou's lemma, we obtain

$$\liminf_{j \rightarrow \infty} w_j(0) \geq \int_{B_{\frac{\pi}{2}}(0)} \liminf_{j \rightarrow \infty} \phi \left( \frac{2R_j}{\pi} x + a_j \right) \rho_{\frac{\pi}{2}}(x) dx. \quad (13)$$

In order to complete the proof of Proposition 1, we prepare the following lemma.

**Lemma 1.** *For  $x \in B_{\frac{\pi}{2}}(0)$ , the following properties hold.*

- (i)  $\frac{(2R_j/\pi)x + a_j}{|(2R_j/\pi)x + a_j|} = \frac{(2R_k/\pi)x + a_k}{|(2R_k/\pi)x + a_k|}$  for any  $j, k \in \mathbb{N}$ .
- (ii)  $(2R_j/\pi)x + a_j \in B_{R_j}(a_j) \subset \Gamma_{\xi'}(\delta)$ .
- (iii)  $|(2R_j/\pi)x + a_j| \rightarrow \infty$  as  $j \rightarrow \infty$ .

*Proof of Lemma 1.* See [17].

Using the lemma, we obtain

$$\begin{aligned} & \liminf_{j \rightarrow \infty} w_j(0) \\ & \geq \int_{B_{\frac{\pi}{2}}(0)} \liminf_{j \rightarrow \infty} \phi \left( \frac{2R_j}{\pi} x + a_j \right) \rho_{\frac{\pi}{2}}(x) dx \\ & = \int_{B_{\frac{\pi}{2}}(0)} \liminf_{j \rightarrow \infty} \phi \left( \left| \frac{2R_j}{\pi} x + a_j \right| \cdot \frac{\frac{2R_j}{\pi} x + a_j}{\left| \frac{2R_j}{\pi} x + a_j \right|} \right) \rho_{\frac{\pi}{2}}(x) dx \\ & = \int_{B_{\frac{\pi}{2}}(0)} \liminf_{r \rightarrow \infty} \phi \left( r \cdot \frac{\frac{2R_j}{\pi} x + a_j}{\left| \frac{2R_j}{\pi} x + a_j \right|} \right) \rho_{\frac{\pi}{2}}(x) dx \\ & \geq \underset{x' \in S_{\xi'}(\delta)}{\text{ess.inf}} \phi_{\infty}(x') \int_{B_{\frac{\pi}{2}}(0)} \rho_{\frac{\pi}{2}}(x) dx \\ & = \underset{x' \in S_{\xi'}(\delta)}{\text{ess.inf}} \phi_{\infty}(x'). \end{aligned} \quad (14)$$

(ii) From the fact that

$$0 \leq \limsup_{j \rightarrow \infty} \mu_{R_j} w_j^{1-p}(0) \leq \lim_{j \rightarrow \infty} \mu_{R_j} \cdot \left( \liminf_{j \rightarrow \infty} w_j(0) \right)^{1-p} = 0, \quad (15)$$

we have

$$\lim_{j \rightarrow \infty} \mu_{R_j} w_j^{1-p}(0) = 0. \quad (16)$$

Hence, we obtain (11).  $\square$

### 3.2 Proof of Theorem 1.

First, we shall focus on the upper bound of the life span of  $w_j$ . Multiplying both sides of the equation (1) by  $\rho_{R_j}$  and integrating over  $B_{R_j}(0)$ , we obtain the following ordinary differential inequality of Bernoulli type:

$$\begin{cases} w'_j \geq w_j^p - \mu_{R_j} w_j, & t \in (0, T_{w_j}^*), \\ w_j(0) = \int_{B_{R_j}(0)} \phi(x + a_j) \rho_{R_j}(x) dx, \end{cases} \quad (17)$$

where  $T_{w_j}^*$  is the life span of  $w_j$ . By a simple calculation, the life span  $T_{w_j}^*$  is estimated from above as follows:

$$T_{w_j}^* \leq \frac{\log(1 - \mu_{R_j} w_j^{1-p}(0))}{-(p-1)\mu_{R_j}}. \quad (18)$$

Using (10) and (11), we see that

$$\begin{aligned} \limsup_{j \rightarrow \infty} T_{w_j}^* &\leq \limsup_{j \rightarrow \infty} \frac{\log(1 - \mu_{R_j} w_j^{1-p}(0))}{-(p-1)\mu_{R_j}} \\ &= \limsup_{j \rightarrow \infty} \frac{\log(1 - \mu_{R_j} w_j^{1-p}(0))}{-\mu_{R_j} w_j^{1-p}(0)} \cdot \frac{w_j^{1-p}(0)}{p-1} \\ &= \frac{1}{p-1} \lim_{j \rightarrow \infty} \frac{\log(1 - \mu_{R_j} w_j^{1-p}(0))}{-\mu_{R_j} w_j^{1-p}(0)} \cdot \left( \liminf_{j \rightarrow \infty} w_j(0) \right)^{1-p} \\ &\leq \frac{1}{p-1} \left( \operatorname{ess.inf}_{x' \in S_{\xi'}(\delta)} \phi_\infty(x) \right)^{1-p}. \end{aligned} \quad (19)$$

On the other hand, we have

$$\limsup_{j \rightarrow \infty} T_{w_j}^* \geq T^*. \quad (20)$$

Indeed, for fixed  $j \in \mathbb{N}$  and  $t \in (0, T^*)$ , if  $u(t)$  remains bounded then  $w_j(t)$  is finite. This completes the proof.  $\square$

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# Bifurcation analysis for the Lugiato-Lefever equation

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## 1 Introduction

We show some results on bifurcation analysis and numerical simulation for the Lugiato-Lefever equation (LLE) on a 2-dimensional domain. LLE is a cubic nonlinear Schrödinger equation (NLS) with damping, detuning and driving force. It is a model for describing the evolution of transversal patterns in an optical cavity with a Kerr medium [1]. It is given by

$$\partial_t E = -(1 + i\theta) E + i b^2 \Delta E + i|E|^2 E + E_{\text{in}}, \quad x \in \Omega, t > 0, \quad (1)$$

where  $x = (x_1, \dots, x_d)^T$ ,  $\Omega \subseteq \mathbf{R}^d$ , and  $\Delta = \partial_{x_1}^2 + \dots + \partial_{x_d}^2$  is the Laplacian.

The parameters  $b^2, \theta \in \mathbf{R}$  are diffraction and detuning parameters, respectively.  $E = E(x, t) \in \mathbf{C}$  denotes a slowly varying envelope of electric field.  $E_{\text{in}} \geq 0$  denotes the intensity of spatially homogeneous driving field, and it is a main control parameter.

Numerical simulations suggest that LLE in 1- or 2-dimensional space has solitary wave solutions in a certain range of parameters[2]. See Figure 1.

In contrast to NLS, LLE does not satisfy any conservation law of NLS. Typically, a solitary wave solution of LLE appears as an equilibrium point. Moreover, for 2D LLE, a localized spot can undergo a Hopf bifurcation, and it results in a spatially localized and temporally oscillating solution called oscillon [3]. Such solutions are understood as dissipative structures resulting from the balance between gain and loss of energy. Besides, no explicit analytical solutions are known for LLE. The authors have studied stability and bifurcation of a stationary solution for 1D LLE [4], [5]. They have proved that a small localized roll can bifurcate from homogeneous state in  $\Omega = \mathbf{R}$ , and that two mixed-mode solutions can bifurcate as a secondary bifurcation in  $\Omega = \mathbf{T}^1 \simeq (-1/2, 1/2) \subset \mathbf{R}$ , which are considered to be “germs” of localized structures[4]. In addition, they have proved the Strichartz estimates for the linear damped Schrödinger equation with potential and external forcing and investigate the

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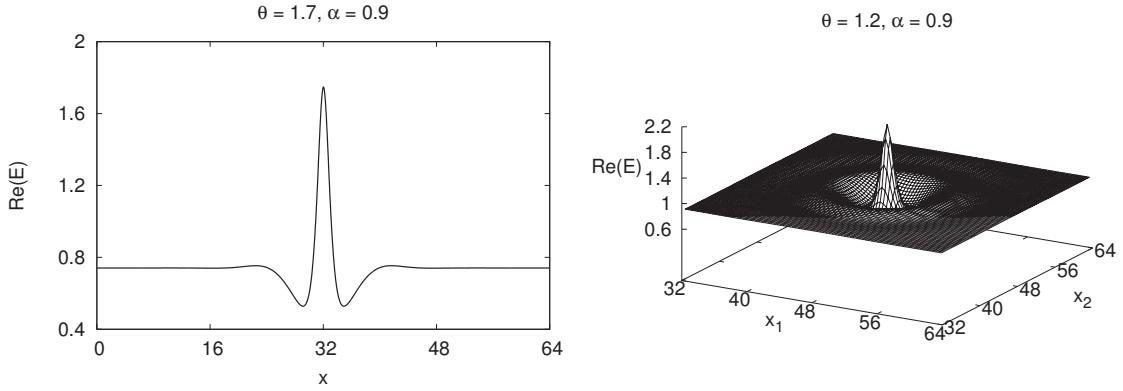


Fig. 1 Stationary solitary waves. (Left)  $\Omega = (0, 64) \subset \mathbf{R}^1, b^2 = 1, \theta = 1.7, E_{\text{in}} \simeq 1.21491$ ; (right)  $\Omega = (0, 96)^2 \subset \mathbf{R}^2, b^2 = 1, \theta = 1.2, E_{\text{in}} \simeq 0.990454$ .

stability of certain stationary solutions for LLE on  $\mathbf{T}^1$  under the initial perturbation within the framework of  $L^2$ [5].

We are interested in oscillons for 2D LLE. As a first step, we study steady-state bifurcation of spatially homogeneous steady state in a mathematically rigorous sense, because it is responsible for the occurrence of stationary localized patterns. For this purpose, we apply the center manifold reduction and group theoretic bifurcation theory [6].

## 2 Preliminaries

Here we note some basic results on spatially homogeneous state. It serves a foundation of bifurcation analysis. See also [2].

The spatially homogeneous steady state  $E_S$  is given implicitly by

$$E_S = \frac{E_{\text{in}}}{1 + i(\theta - \alpha)}, \quad (2)$$

where  $\alpha = |E_S|^2$ . Note that if  $\theta < \sqrt{3}$ , then  $E_{\text{in}} \geq 0$  and  $\alpha \geq 0$  have one-to-one correspondence because they are related by the following equation:

$$E_{\text{in}}^2 = \alpha \{1 + (\theta - \alpha)^2\}. \quad (3)$$

Thus we can regard  $\alpha$  as a bifurcation parameter instead of  $E_{\text{in}}$ . For the convenience of bifurcation analysis, we introduce new unknown functions  $u_1, u_2$  by  $E = E_S(1 + u_1 + iu_2)$ .  $u_1$  and  $u_2$  are real-valued functions satisfying

$$\begin{cases} \partial_t u_1 = -u_1 + (\theta - \alpha)u_2 - b^2 \Delta u_2 - \alpha \{2u_1u_2 + u_2(u_1^2 + u_2^2)\}, \\ \partial_t u_2 = (3\alpha - \theta)u_1 - u_2 + b^2 \Delta u_1 + \alpha \{3u_1^2 + u_2^2 + u_1(u_1^2 + u_2^2)\}. \end{cases} \quad (4)$$

$E_S$  corresponds to the trivial equilibrium point of this system.

Linear stability analysis reveals that the trivial equilibrium of (4) is linearly stable for  $\alpha < 1$  and it loses stability at  $\alpha = 1$ .

### 3 Discussion

First, we study (4) on  $\Omega = (-1/2, 1/2)^2 \subset \mathbf{R}^2$  with periodic boundary conditions. It is the same setting as numerical simulation. Taking into account of the symmetry of the system, we can obtain a catalog of model-independent bifurcation behaviors. Then we apply the center manifold reduction near the modulational instability of homogeneous state. The symmetry of LLE restricts the function form of the vector field on the center manifold. It helps to study model-specific behaviors. The symmetry of LLE is described by  $\Gamma = \mathbf{D}_4 \ltimes \mathbf{T}^2$ , where  $\ltimes$  means semidirect product of groups.  $\mathbf{D}_4$  is generated by the reflection across  $x_2 = x_1$  and  $\pi/2$ -rotation about the origin.  $\mathbf{T}^2$  is a group of translations modulo 1.

Suppose that  $\mathbf{k} = (l, n) \in \mathbf{Z}^2$  is a critical wave vector of modulational instability. We may assume  $l \geq n \geq 0$  without loss of generality. Since we consider single-mode bifurcations, we assume that the solution  $(l, n) \in \mathbf{Z}^2$  to  $l^2 + n^2 = k$  for a given  $k \in \mathbf{N}$  is unique in the above sense. There are three cases to be distinguished: i)  $l = n > 0$ ; ii)  $l > n = 0$ ; iii)  $l > n > 0$ . In the first two cases, the center manifold is 4-dimensional, while that is 8-dimensional in the third case.

We classify the possible bifurcations for (4) in a small neighborhood of the bifurcation point in a mathematically rigorous sense. Especially, it turns out that all bifurcating solutions are unstable near the bifurcation point. However, numerical simulations suggest that there exist stable patterns. In order to capture them, we have to study bifurcations with higher codimension.

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# Reaction-diffusion approximation and related topics

Hirokazu Ninomiya\*

The dynamics in nature are quite complicated. One of the reasons is the coexistence of spatio-temporal stages of phenomena. A system can include the several scales, for the simplest case. For example, it includes fast reaction terms when compared with the other terms. To decompose two scales, a singular limit analysis is a powerful tool. This type of the singular limit is called a *fast reaction limit* or a *reaction-diffusion approximation*. The interest of this study is two-fold : on the one hand, we derive the limit problem for reaction-diffusion systems with fast large terms, on the other hand we provide approximations of non-reaction-diffusion systems by means of reaction-diffusion systems.

In this talk we consider the following reaction-diffusion system with a small parameter  $\varepsilon$ :

$$\mathbf{u}_t^\varepsilon = D\Delta\mathbf{u}^\varepsilon + \mathbf{F}(\mathbf{u}^\varepsilon) + \frac{1}{\varepsilon}\mathbf{G}(\mathbf{u}^\varepsilon), \quad (1)$$

where  $\mathbf{u}^\varepsilon \in \mathbb{R}^m$  and  $\mathbf{F}, \mathbf{G}$  are smooth functions from  $\mathbb{R}^m$  to  $\mathbb{R}^m$ ,  $D$  is a diagonal matrix with positive (or nonnegative) components. Multiplying  $\varepsilon$  to the above equation, we have

$$\varepsilon\mathbf{u}_t^\varepsilon = \varepsilon D\Delta\mathbf{u}^\varepsilon + \varepsilon\mathbf{F}(\mathbf{u}^\varepsilon) + \mathbf{G}(\mathbf{u}^\varepsilon).$$

We may expect

$$\mathbf{G}(\lim_{\varepsilon \rightarrow 0} \mathbf{u}^\varepsilon) = 0.$$

This suggests that the solution converges to the equilibria of the fast reaction system  $\varepsilon\mathbf{u}_t = \mathbf{G}(\mathbf{u})$ . The null set of  $\mathbf{G}$  is denoted by  $\mathcal{E}$ , i.e.,

$$\mathcal{E} := \{\mathbf{u} \in \mathbb{R}^m \mid \mathbf{G}(\mathbf{u}) = 0\}.$$

If  $\mathcal{E}$  consists of discrete points, the limit problem possesses the transition layers (for example, see [2]).

In this talk we consider when  $\mathcal{E}$  is a curve or a two dimensional surface. However the situation may change by the singularity of  $\mathcal{E}$ . For example, let us consider the following

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two-component problem [3], which we will refer to as Problem  $(P^\varepsilon)$ ,

$$(P^\varepsilon) \quad \begin{cases} u_{1t} = d_1 \Delta u_1 + f(u_1) - \frac{1}{\varepsilon} s_1 u_1 u_2, & \text{in } \Omega \times \mathbb{R}^+, \\ u_{2t} = d_2 \Delta u_2 + g(u_2) - \frac{1}{\varepsilon} s_2 u_1 u_2, & \text{in } \Omega \times \mathbb{R}^+, \end{cases}$$

with the Neumann boundary conditions and the initial conditions where  $\Omega$  is a smooth domain of  $\mathbb{R}^N$ , and  $\varepsilon, s_1, s_2, d_1, d_2, \lambda$  and  $\mu$  are positive constants,  $f(s) = \lambda s(1-s)$ ,  $g(s) = \mu s(1-s)$ . In this case  $\mathcal{E}$  consists of two segments:

$$\{(u_1, u_2) \mid u_1 \geq 0, u_2 = 0\} \cup \{(u_1, u_2) \mid u_1 = 0, u_2 \geq 0\}.$$

Since  $\mathcal{E}$  is not continuously differentiable at the origin, the flux of the limit problem becomes discontinuous and then the limit problem turns out to be a two-phase Stefan problem without latent heat. See [3] for the Neumann boundary conditions and [1] for the inhomogeneous Dirichlet boundary conditions. To create the latent heat (transition layer) we need to introduce the new variable and three-component reaction-diffusion system converges to the two-phase Stefan problem with positive latent heat [6]. However since there are different functions  $\mathbf{G}$  with the same null set  $\mathcal{E}$ ,  $\mathcal{E}$  do not characterize all the information of the limit problem.

When  $\mathcal{E}$  is continuously differentiable, we can show that some types of nonlinear diffusion can be approximated by the reaction-diffusion system, see [10, 11, 16]. This means that the rather complicated diffusion process can be realized by the usual random movement together with a reaction mechanism. I will also focus on the relationship between  $\mathcal{E}$  and the limit problem and summarize the recent research in this topics.

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# Standing waves for a system of nonlinear Schrödinger equations \*

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## 1 Introduction

We consider the following system of nonlinear Schrödinger equations

$$\begin{cases} i\partial_t u_1 = -\Delta u_1 - \kappa|u_1|u_1 - \gamma\overline{u_1}u_2 \\ i\partial_t u_2 = -2\Delta u_2 - 2|u_2|u_2 - \gamma u_1^2 \end{cases} \quad (1)$$

for  $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ , where  $u_1$  and  $u_2$  are complex-valued functions of  $(t, x)$ ,  $\kappa \in \mathbb{R}$  and  $\gamma > 0$  are constants, and  $N \leq 3$ . System (1) is related to the Raman amplification in a plasma (see [2, 3, 4, 5]). A similar system also appears as an optics model with quadratic nonlinearity (see [8, 9, 14]).

We regard  $L^2(\mathbb{R}^N, \mathbb{C})$  as a real Hilbert space with the inner product

$$(u, v)_{L^2} = \Re \int_{\mathbb{R}^N} u(x)\overline{v(x)} dx,$$

and define the inner products of real Hilbert spaces  $H = L^2(\mathbb{R}^N, \mathbb{C})^2$  and  $X = H^1(\mathbb{R}^N, \mathbb{C})^2$  by

$$(\vec{u}, \vec{v})_H = (u_1, v_1)_{L^2} + (u_2, v_2)_{L^2}, \quad (\vec{u}, \vec{v})_X = (\vec{u}, \vec{v})_H + (\nabla \vec{u}, \nabla \vec{v})_H.$$

Here and hereafter, we use the vectorial notation  $\vec{u} = (u_1, u_2)$ .

The energy  $E$  and the charge  $Q$  are defined by

$$\begin{aligned} E(\vec{u}) &= \frac{1}{2}\|\nabla \vec{u}\|_H^2 - \frac{\kappa}{3}\|u_1\|_{L^3}^3 - \frac{1}{3}\|u_2\|_{L^3}^3 - \frac{\gamma}{2}\Re \int_{\mathbb{R}^N} u_1^2 \overline{u_2} dx, \\ Q(\vec{u}) &= \frac{1}{2}\|\vec{u}\|_H^2. \end{aligned}$$

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\*This talk is based on a joint work [6] with Mathieu Colin (Université Bordeaux 1).

For  $\theta \in \mathbb{R}$ , we define  $G(\theta)$  and  $J$  by

$$G(\theta)\vec{u} = (e^{i\theta}u_1, e^{2i\theta}u_2), \quad J\vec{u} = (iu_1, 2iu_2), \quad \vec{u} \in X,$$

and  $\langle G(\theta)\vec{f}, \vec{u} \rangle = \langle \vec{f}, G(-\theta)\vec{u} \rangle$ ,  $\langle J\vec{f}, \vec{u} \rangle = -\langle \vec{f}, J\vec{u} \rangle$  for  $\vec{f} \in X^*$  and  $\vec{u} \in X$ , where  $X^*$  is the dual space of  $X$ . For  $y \in \mathbb{R}^N$ , we define  $\tau_y\vec{u}(x) = \vec{u}(x - y)$  for  $\vec{u} \in X$  and  $x \in \mathbb{R}^N$ . Then, (1) is written as

$$\partial_t \vec{u}(t) = -JE'(\vec{u}(t)) \quad \text{in } X^*,$$

and  $E(G(\theta)\tau_y\vec{u}) = E(\vec{u})$  for all  $\theta \in \mathbb{R}$ ,  $y \in \mathbb{R}^N$  and  $\vec{u} \in X$ .

By the standard theory (see, e.g., [1, Chapter 4]), we see that the Cauchy problem for (1) is globally well-posed in  $X$ , and the energy and the charge are conserved.

For  $\omega > 0$ , we define the action  $S_\omega$  by

$$S_\omega(\vec{v}) = E(\vec{v}) + \omega Q(\vec{v}), \quad \vec{v} \in X. \quad (2)$$

The Euler-Lagrange equation  $S'_\omega(\vec{\phi}) = 0$  is written as

$$\begin{cases} -\Delta\phi_1 + \omega\phi_1 = \kappa|\phi_1|\phi_1 + \gamma\overline{\phi_1}\phi_2 \\ -\Delta\phi_2 + \omega\phi_2 = |\phi_2|\phi_2 + (\gamma/2)\phi_1^2 \end{cases} \quad (3)$$

and if  $\vec{\phi} \in X$  satisfies  $S'_\omega(\vec{\phi}) = 0$ , then  $G(\omega t)\vec{\phi}$  is a solution of (1).

**Definition 1.** We say that a standing wave solution  $G(\omega t)\vec{\phi}$  of (1) is *stable* if for all  $\varepsilon > 0$  there exists  $\delta > 0$  with the following property. If  $u_0 \in X$  satisfies  $\|\vec{u}_0 - \vec{\phi}\|_X < \delta$ , then the solution  $\vec{u}(t)$  of (1) with  $\vec{u}(0) = \vec{u}_0$  exists for all  $t \geq 0$ , and satisfies

$$\inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^N} \|\vec{u}(t) - G(\theta)\tau_y\vec{\phi}\|_X < \varepsilon$$

for all  $t \geq 0$ . Otherwise,  $G(\omega t)\vec{\phi}$  is called *unstable*.

Let  $\varphi_\omega \in H^1(\mathbb{R}^N)$  be a unique positive radial solution of

$$-\Delta\varphi + \omega\varphi - \varphi^2 = 0, \quad x \in \mathbb{R}^N. \quad (4)$$

The following result was essentially obtained by [4, 5].

**Theorem 1.** Let  $N \leq 3$ ,  $\kappa \in \mathbb{R}$ ,  $\gamma > 0$ ,  $\omega > 0$ , and let  $\varphi_\omega$  be the positive radial solution of (4). Then, the semi-trivial standing wave solution  $(0, e^{2i\omega t}\varphi_\omega)$  of (1) is stable if  $0 < \gamma < 1$ , and it is unstable if  $\gamma > 1$ .

We remark that the stability property of the semi-trivial standing wave of (1) is independent of  $\kappa$  for the case  $\gamma \neq 1$ . On the other hand, we will see that the sign of  $\kappa$  plays an important role for the case  $\gamma = 1$  (see Theorems 4 and 5 below).

## 2 Main Results

We look for solutions of (3) of the form  $\vec{\phi} = (\alpha\varphi_\omega, \beta\varphi_\omega)$  with  $(\alpha, \beta) \in ]0, \infty[^2$ , where  $\varphi_\omega$  is the positive radial solution of (4). If  $(\alpha, \beta) \in ]0, \infty[^2$  satisfies

$$\kappa\alpha + \gamma\beta = 1, \quad \gamma\alpha^2 + 2\beta^2 = 2\beta, \quad (5)$$

then  $(\alpha\varphi_\omega, \beta\varphi_\omega)$  is a solution of (3). For  $\kappa \in \mathbb{R}$  and  $\gamma > 0$ , we define

$$\mathcal{S}_{\kappa, \gamma} = \{(\alpha, \beta) \in ]0, \infty[^2 : \kappa\alpha + \gamma\beta = 1, \gamma\alpha^2 + 2\beta^2 = 2\beta\}.$$

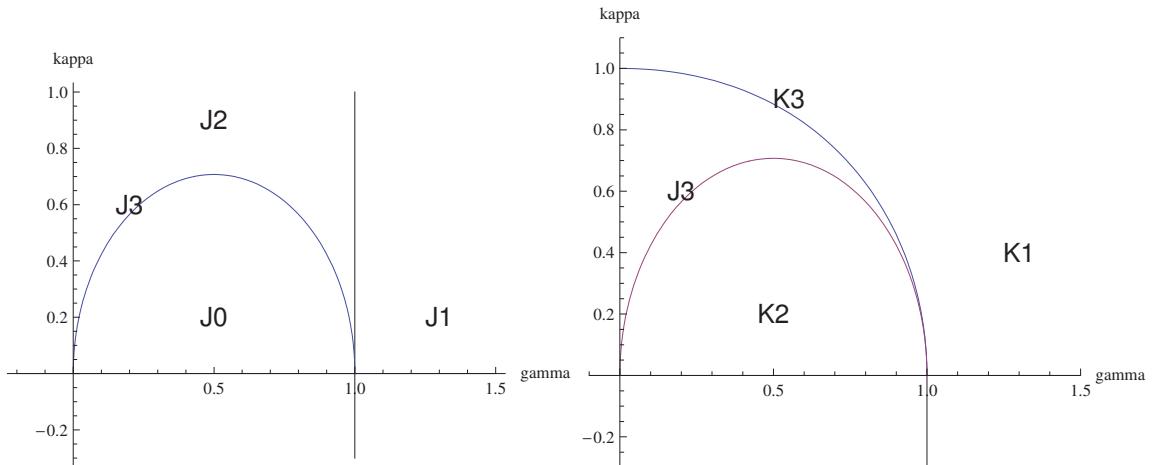
Note that  $\gamma x^2 + 2y^2 = 2y$  is an ellipse with vertices  $(x, y) = (0, 0), (0, 1), (\pm 1/\sqrt{2\gamma}, 1/2)$ , and that  $\mathcal{S}_{\kappa, \gamma} \subset \{(x, y) : 0 < y < 1\}$ .

To determine the structure of the set  $\mathcal{S}_{\kappa, \gamma}$ , we define

$$\begin{aligned}\alpha_{\pm} &= \frac{(2 - \gamma)\kappa \pm \gamma\sqrt{\kappa^2 + 2\gamma(\gamma - 1)}}{2\kappa^2 + \gamma^3}, \\ \beta_{\pm} &= \frac{\kappa^2 + \gamma^2 \pm \kappa\sqrt{\kappa^2 + 2\gamma(\gamma - 1)}}{2\kappa^2 + \gamma^3}, \\ \alpha_0 &= \frac{(2 - \gamma)\kappa}{2\kappa^2 + \gamma^3}, \quad \beta_0 = \frac{\kappa^2 + \gamma^2}{2\kappa^2 + \gamma^3}.\end{aligned}$$

We also divide the parameter domain  $\mathcal{D} = \{(\kappa, \gamma) : \kappa \in \mathbb{R}, \gamma > 0\}$  into the following sets.

$$\begin{aligned}\mathcal{J}_1 &= \{(\kappa, \gamma) : \kappa \leq 0, \gamma > 1\} \cup \{(\kappa, \gamma) : \kappa > 0, \gamma \geq 1\}, \\ \mathcal{J}_2 &= \{(\kappa, \gamma) : 0 < \gamma < 1, \kappa > \sqrt{2\gamma(1 - \gamma)}\}, \\ \mathcal{J}_3 &= \{(\kappa, \gamma) : 0 < \gamma < 1, \kappa = \sqrt{2\gamma(1 - \gamma)}\}, \\ \mathcal{J}_0 &= \{(\kappa, \gamma) : \kappa \in \mathbb{R}, \gamma > 0\} \setminus (\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3).\end{aligned}$$



Note that for  $0 < \kappa \leq 1/\sqrt{2}$ , the equation  $2\gamma(1 - \gamma) = \kappa^2$  has solutions  $\gamma = \gamma_{\pm} := (1 \pm \sqrt{1 - 2\kappa^2})/2$ . By elementary computations, we obtain the following.

**Proposition 1.** (0) If  $(\kappa, \gamma) \in \mathcal{J}_0$ , then  $\mathcal{S}_{\kappa, \gamma}$  is empty.

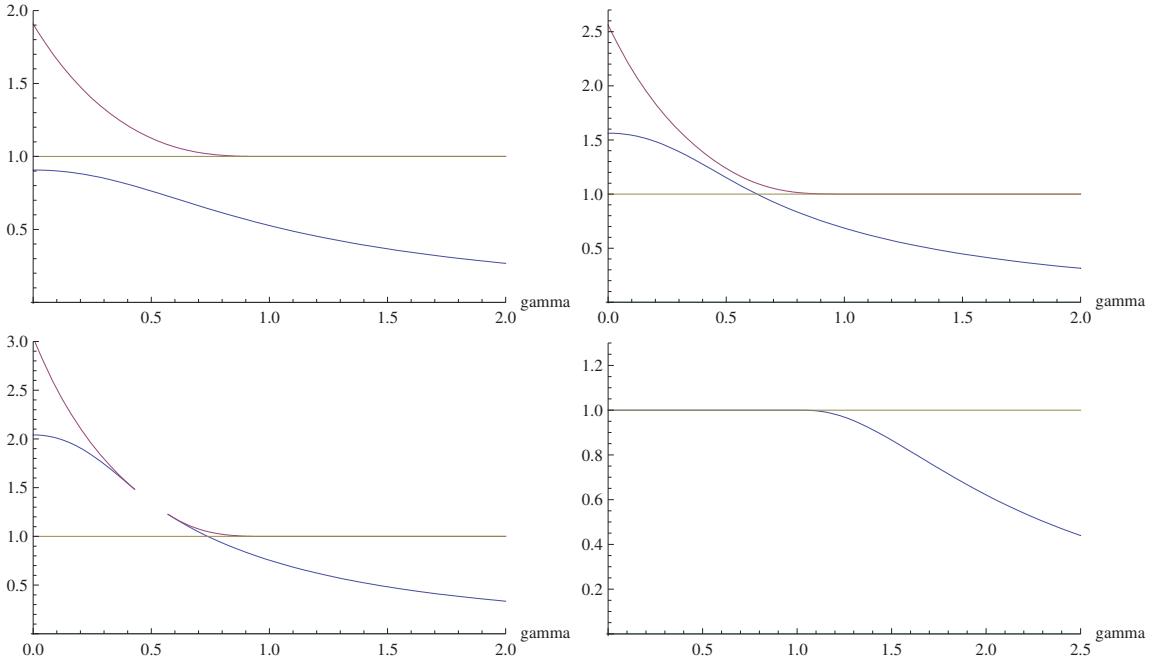
- (1) If  $(\kappa, \gamma) \in \mathcal{J}_1$ , then  $\mathcal{S}_{\kappa, \gamma} = \{(\alpha_+, \beta_-)\}$ .
- (2) If  $(\kappa, \gamma) \in \mathcal{J}_2$ , then  $\mathcal{S}_{\kappa, \gamma} = \{(\alpha_+, \beta_-), (\alpha_-, \beta_+)\}$ .
- (3) If  $(\kappa, \gamma) \in \mathcal{J}_3$ , then  $\mathcal{S}_{\kappa, \gamma} = \{(\alpha_0, \beta_0)\}$ .

**Remark 1.** (1) When  $\kappa \leq 0$ ,  $(\alpha_+, \beta_-) \rightarrow (0, 1)$  as  $\gamma \rightarrow 1 + 0$ . That is, the branch  $\{(\alpha_+ \varphi_\omega, \beta_- \varphi_\omega) : \gamma > 1\}$  of positive solutions of (3) bifurcates from the semi-trivial solution  $(0, \varphi_\omega)$  at  $\gamma = 1$ .

(2) When  $\kappa > 0$ ,  $(\alpha_-, \beta_+) \rightarrow (0, 1)$  as  $\gamma \rightarrow 1 - 0$ . That is, the branch  $\{(\alpha_- \varphi_\omega, \beta_+ \varphi_\omega) : \gamma_m < \gamma < 1\}$  of positive solutions of (3) bifurcates from the semi-trivial solution  $(0, \varphi_\omega)$  at  $\gamma = 1$ , where  $\gamma_m = \inf\{\gamma : (\kappa, \gamma) \in \mathcal{S}_{\kappa, \gamma}\}$ , and it is given by  $\gamma_m = 0$  if  $\kappa > 1/\sqrt{2}$ , and  $\gamma_m = \gamma_+$  if  $0 < \kappa \leq 1/\sqrt{2}$ .

Figures: Cases  $\kappa = 1.1$  (upper left),  $\kappa = 0.8$  (upper right),  $\kappa = 0.7$  (lower left),  $\kappa = -0.5$  (lower right).

Blue:  $\gamma \mapsto \alpha_+^2 + \beta_-^2$  (lower curves), Purple:  $\gamma \mapsto \alpha_-^2 + \beta_+^2$  (upper curves).



Recall that  $\varphi_\omega$  is the positive radial solution of (4).

**Theorem 2.** Let  $N \leq 3$  and  $(\kappa, \gamma) \in \mathcal{J}_1 \cup \mathcal{J}_2$ . For any  $\omega > 0$ , the standing wave solution  $G(\omega t)(\alpha_+ \varphi_\omega, \beta_- \varphi_\omega)$  of (1) is stable.

**Theorem 3.** Let  $N \leq 3$  and  $(\kappa, \gamma) \in \mathcal{J}_2$ . For any  $\omega > 0$ , the standing wave solution  $G(\omega t)(\alpha_- \varphi_\omega, \beta_+ \varphi_\omega)$  of (1) is unstable.

**Remark 2.** In this talk, we do not study the case  $(\kappa, \gamma) \in \mathcal{J}_3$ .

**Remark 3.** The proof of Theorem 2 is based on the abstract stability theorem of Grillakis, Shatah and Strauss [7]. While, the proof of Theorem 3 relies on the abstract instability theorem of [11] (see also Maeda [10]), which is a generalization of the classical result of [13, 7].

We also obtain the stability and instability results of semi-trivial standing wave at the bifurcation point  $\gamma = 1$ . The results depend on the sign of  $\kappa$ .

**Theorem 4.** Let  $N \leq 3$ ,  $\kappa > 0$  and  $\gamma = 1$ . For any  $\omega > 0$ , the standing wave solution  $(0, e^{2i\omega t} \varphi_\omega)$  of (1) is unstable.

**Theorem 5.** Let  $N \leq 3$ ,  $\kappa \leq 0$  and  $\gamma = 1$ . For any  $\omega > 0$ , the standing wave solution  $(0, e^{2i\omega t} \varphi_\omega)$  of (1) is stable.

**Remark 4.** The linearized operator  $S''_\omega(0, \varphi_\omega)$  around the semi-trivial standing wave is independent of  $\kappa$ . Therefore, Theorems 4 and 5 are never obtained from the linearized analysis only. The proof of Theorem 4 is based on [11]. While, the proof of Theorem 5 relies on the variational method of Shatah [12] and on the characterization of the ground states in Theorem 6 below.

**Remark 5.** For the case  $\gamma = 1$ , the kernel of  $S''_\omega(0, \varphi_\omega)$  contains a nontrivial element  $(\varphi_\omega, 0)$  other than the elements  $\nabla(0, \varphi_\omega)$  and  $J(0, \varphi_\omega)$  naturally coming from the symmetries of  $S_\omega$ .

Next, we consider the ground state problem for (3). The set  $\mathcal{G}_\omega$  of the ground states for (3) is defined as follows.

$$\begin{aligned}\mathcal{A}_\omega &= \{\vec{v} \in X : S'_\omega(\vec{v}) = 0, \vec{v} \neq 0\}, \\ d(\omega) &= \inf\{S_\omega(\vec{v}) : \vec{v} \in \mathcal{A}_\omega\}, \\ \mathcal{G}_\omega &= \{\vec{u} \in \mathcal{A}_\omega : S_\omega(\vec{u}) = d(\omega)\}.\end{aligned}$$

We define

$$\kappa_c(\gamma) = \frac{1}{2}(\gamma + 2)\sqrt{1 - \gamma}, \quad 0 < \gamma < 1. \quad (6)$$

Then,  $\kappa_c$  is strictly decreasing on the open interval  $]0, 1[$ ,  $\kappa_c(0) = 1$  and  $\kappa_c(1) = 0$ . We define a function  $\gamma_c$  on  $]0, 1[$  by the inverse function of  $\kappa_c$ . We

divide the parameter domain  $\mathcal{D} = \{(\kappa, \gamma) : \kappa \in \mathbb{R}, \gamma > 0\}$  into the following sets.

$$\begin{aligned}\mathcal{K}_1 &= \{(\kappa, \gamma) : \kappa \leq 0, \gamma > 1\} \cup \{(\kappa, \gamma) : \kappa \geq 1, \gamma > 0\} \\ &\quad \cup \{(\kappa, \gamma) : 0 < \kappa < 1, \gamma > \gamma_c(\kappa)\}, \\ \mathcal{K}_2 &= \{(\kappa, \gamma) : \kappa \leq 0, 0 < \gamma \leq 1\} \cup \{(\kappa, \gamma) : 0 < \kappa < 1, 0 < \gamma < \gamma_c(\kappa)\}, \\ \mathcal{K}_3 &= \{(\kappa, \gamma) : 0 < \kappa < 1, \gamma = \gamma_c(\kappa)\}.\end{aligned}$$

Remark that since  $\sqrt{2\gamma(1-\gamma)} < \kappa_c(\gamma)$  for  $0 < \gamma < 1$ , we have  $\mathcal{J}_0 \subset \mathcal{K}_2$ .

Moreover, we define

$$\begin{aligned}\mathcal{G}_\omega^0 &= \{G(\theta)\tau_y(0, \varphi_\omega) : \theta \in \mathbb{R}, y \in \mathbb{R}^N\}, \\ \mathcal{G}_\omega^1 &= \{G(\theta)\tau_y(\alpha_+\varphi_\omega, \beta_-\varphi_\omega) : \theta \in \mathbb{R}, y \in \mathbb{R}^N\}.\end{aligned}$$

Then, the set  $\mathcal{G}_\omega$  of the ground states for (3) is determined as follows.

**Theorem 6.** *Let  $N \leq 3$  and  $\omega > 0$ .*

- (1) *If  $(\kappa, \gamma) \in \mathcal{K}_1$ , then  $\mathcal{G}_\omega = \mathcal{G}_\omega^1$ .*
- (2) *If  $(\kappa, \gamma) \in \mathcal{K}_2$ , then  $\mathcal{G}_\omega = \mathcal{G}_\omega^0$ .*
- (3) *If  $(\kappa, \gamma) \in \mathcal{K}_3$ , then  $\mathcal{G}_\omega = \mathcal{G}_\omega^0 \cup \mathcal{G}_\omega^1$ .*

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