Affine geometry of space curves and homogeneous surfaces

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Affine geometry of space curves and homogeneous surfaces

by

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Abstract

We investigate the centroaffine space curves with constant centroaffine curvatures in $\mathbb{R}^3$. We classify them and give their explicit expressions. Moreover, we find out each centroaffine space curve with constant centroaffine curvatures can be written as the orbit of a certain one-parameter subgroup of $GL(3,\mathbb{R})$. Thus we can treat them as nondegenerate centroaffine homogeneous curves. Furthermore, for each centroaffine homogeneous curve, we check if there is a nondegenerate centroaffine homogeneous surface such that the corresponding group contains exactly, as a subgroup, the one-parameter subgroup with respect to the homogeneous curve. We obtain the similar results for equiaffine space curves with constant equiaffine curvatures.

At the end, we bring up a related topic of the centroaffine space curve theory, degenerate center maps. We investigate centroaffine ruled surfaces and determine such surfaces whose center map is degenerate. As a corollary, given a nondegenerate centroaffine space curve, we can construct a centroaffine ruled surface whose center map is precisely this curve.

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Chapter 1

Introduction

Affine differential geometry has a long history: As far as we know it was A. Transon who published the first result in affine differential geometry in 1841; he considered the affine normal of a curve. But it needed more than 70 years before a systematic and intensive study of affine properties of curves and surfaces began.

According to F. Klein's "Erlangen Program" (1872), the study of the properties of geometric figures (curves, surfaces, etc.) that are invariant under a given geometric transformation group G is called the geometry subordinated to G. Thus we have the corresponding Euclidean geometry when G is the Euclidean motion group. Following the ideas of F. Klein, G. Pick (1906), G. Tzitzeica (1912) and others proposed the study of curves and surfaces with respect to different groups. Affine differential geometry is the study of differential invariants with respect to the group of affine transformations, i.e. nonsingular linear transformation together with translations, denoted by $AGL(n, \mathbb{R}) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$. In 1916, a group of geometers: L. Berwald, W. Blaschke, H. Liebmann, G. Pick, J. Randon, and others started the systematic study of properties of curves and surfaces subordinated to the equiaffine (unimodular) transformation group, which consists of volume preserving affine transformations. This group shall be denoted by $ASL(n, \mathbb{R}) = SL(n, \mathbb{R}) \ltimes \mathbb{R}^n$. The progress made was so rapid that the first monograph [2] about (equi-)affine differential geometry appeared in 1923 by W. Blaschke.

The second monograph [21], 1934, contains the local centroaffine theory. The centroaffine transformation is the affine transformation without translation, denoted by $GL(n,\mathbb{R})$. The book [22] of father and son Schirokow, published in Russian in 1962, is the next mark in the development of affine differential geometry. It documents remarkable progress in the local theory and in the local classification of special classes of surfaces, the influence of E. Cartan's ideas and strong contributions of Russian geometers. The introduction of the monograph [14] gives more details and references on the development of the field up to the early 1990s, in particular on global results.

Affine homogeneous submanifolds as one of the most important submanifolds have been studied intensively and classified well in the case of codimension one. For equiaffine homogeneous submanifolds of codimension one in $\mathbb{R}^2$, we have: A nondegenerate plane curve has constant equiaffine curvature if and only if it is the orbit of a point under a certain one parameter group of equiaffine transformations, that is, an equiaffine homogeneous plane curve. Moreover, the equiaffine plane curves with constant equiaffine curvature can be classified as quadric curves.

In 1991, K. Nomizu and T. Sasaki [18] completed the classification of equiaffine homogeneous surfaces in $\mathbb{R}^3$. This classification stimulated further investigations on homogeneous submanifolds. In 1993, F. Dillen and L. Vrancken [4] gave a classification of 3-dimensional equiaffine homogenous, locally strongly convex hypersurfaces in $\mathbb{R}^4$ and a classification for homogeneous affine hypersurfaces with some conditions on the shape operator in [5]. H. L. Liu and C. P. Wang [15] classified all centroaffine homogeneous surfaces in $\mathbb{R}^3$. B. E. Abdalla, F. Dillen and L. Vrancken [1] classified all affine homogeneous surfaces in $\mathbb{R}^3$ with vanishing Pick invariant.

It is an important problem to *classify the equiaffine (or centroaffine) homogeneous submanifolds of codimension two in* $\mathbb{R}^3$*, i.e. equiaffine (or centroaffine) homogeneous space* *curves.*

The curve theory in the equiaffine space of dimension 3, has been contained as a section in the monographs [2] and [10]. The setup of such a theory is similar to the Euclidean curve theory. In 1997, R. Gardner and G. Wilkens [9] studied centroaffine curves in $\mathbb{R}^n$ using the classical Cartan approach to moving frames and gave the fundamental theorem for centroaffine curves. In chapter 4, we adopt Shengjin's theorem to investigate and classify the nondegenerate centroaffine space curves with constant centroaffine curvatures in $\mathbb{R}^3$, and we have

Theorem 4.2.3 ([11]). *Any nondegenerate centroaffine space curve φ with constant centroaffine curvatures κ_1, κ_2 and signature -1 is centroaffinely equivalent to one of the following curves:*

(1) $\varphi(s) = {}^t(se^{-s}, e^{-s}, s^2e^{-s} + e^{-s})$, *if* $A^2 + B^2 = 0$,

(2) $\varphi(s) = {}^t(e^{\varsigma_1 s}, e^{\varsigma_2 s}, se^{\varsigma_1 s})$, *if* $A^2 + B^2 \neq 0$ *and* $\Delta = 0$,

(3) $\varphi(s) = e^{\kappa_2 s/3} \cdot {}^t(e^{\rho_1 s}\sin(\rho_2 s), e^{\rho_1 s}\cos(\rho_2 s), e^{-2\rho_1 s})$, *if* $A^2 + B^2 \neq 0$ *and* $\Delta > 0$,

(4) $\varphi(s) = e^{\kappa_2 s/3} \cdot {}^t(e^{-2\sigma_1 s}, e^{(\sigma_1+\sigma_2)s}, e^{(\sigma_1-\sigma_2)s})$, *if* $A^2 + B^2 \neq 0$ *and* $\Delta < 0$,

where A, B, Δ *and* ς_i, ρ_i, σ_i $(i = 1, 2)$ *are constants determined by* κ_1, κ_2 *as* (4.2.2)~(4.2.6).

Moreover, we shall indicate a corresponding one-parameter subgroup of centroaffine transformations for each class of curves. Thus, we can regard the nondegenerate centroaffine space curves with constant centroaffine curvatures as the centroaffine homogeneous curves (§4.3). Finally, we compare the classes of the centroaffine homogeneous curves and the ones of the centroaffine homogeneous surfaces, and find out a corresponding relation, i.e. the groups of the centroaffine homogeneous curves are the subgroups of the ones of certain centroaffine homogeneous surfaces (Theorem 4.4.2).

In chapter 3, we can obtain the similar results for equiaffine space curves with constant equiaffine curvatures.

In chapter 5, we bring up a related topic of the centroaffine space curve theory, degenerate center maps. Center maps are firstly introduced for centroaffine hypersurfaces by H. Furuhata and L. Vrancken [8] as a generalization of the center of proper affine spheres. Proper affine spheres are exactly those affine hypersurfaces whose center map is constant, obviously, their center maps are degenerate. We are interested in the other degenerate case, that is, when the image of the center map for a centroaffine surface is a curve, in particular, a nondegenerate centroaffine space curve. We investigate centroaffine ruled surfaces and obtained that it is only one type of minimal centroaffine ruled surfaces with scalar curvature 1 whose center map can satisfy this condition (Theorem 5.2.1) and as a corollary, we proved

Corollary 5.2.5 ([12]). *Given a nondegenerate centroaffine space curve* $b(u)$ *with centroaffine arc-length parameter* u *and centroaffine second curvature* $\kappa_2(u)$*, we can construct a centroaffine ruled surface* $f(u,v)$ *whose center map is* $b(u)$*. In fact, the center map of* $f(u,v) = \phi(u)b'(u) + vb(u)$ *is* $b(u)$*, where* $\phi(u) = -\frac{2}{3}\mu^{-1}(u)\int \mu(u)du$ and $\mu(u) = e^{\frac{1}{3}\int \kappa_2(u)du}$.

Chapter 2

Preliminaries

2.1 Equiaffine plane curves

Let $[\cdot,\cdot]$ denote the standard area form of $\mathbb{R}^2$; $[x,y] := x^1y^2 - x^2y^1$ for $x = {}^t(x^1,x^2)$, $y = {}^t(y^1,y^2) \in \mathbb{R}^2$. This section is devoted to the study of the properties of plane curves invariant under the group of area-preserving affine transformations. The group is generated by the action of the equiaffine linear group $SL(2,\mathbb{R})$ and the translation group $\mathbb{R}^2$.

Definition 2.1.1. A C^∞ map φ from an interval I to $\mathbb{R}^2$ is called an *equiaffine plane curve* in $\mathbb{R}^2$ if $[\frac{d\varphi}{dt}(t), \frac{d^2\varphi}{dt^2}(t)] \neq 0$ for all $t \in I$.

Definition 2.1.2. An equiaffine plane curve is said to be parameterized by *equiaffine arc-length parameter* if

$$[\frac{d\varphi}{ds}(s), \frac{d^2\varphi}{ds^2}(s)] = 1 \text{ for all s} \in \text{I}.$$

Remark 2.1.3. An equiaffine plane curve can be reparameterized by equiaffine arc-length parameter.

It is easy to show that

$$1 = [\frac{d\varphi}{ds}(s), \frac{d^2\varphi}{ds^2}(s)] = [\frac{d\varphi}{dt}(t), \frac{d^2\varphi}{dt^2}(t)](\frac{dt}{ds})^3.$$

Thus

$$s(t) = \int [\frac{d\varphi}{dt}(t), \frac{d^2\varphi}{dt^2}(t)]^{\frac{1}{3}} dt.$$

From now on and throughout this thesis, prime denotes differentiation with respect to the equiaffine (resp. centroaffine) arc-length parameter s, thus $\varphi' = \dfrac{d\varphi}{ds}$ etc, whereas a dot is reserved for differentiation with respect to an arbitrary parameter t, thus $\dot{\varphi} = \dfrac{d\varphi}{dt}$ etc.

Definition 2.1.4. For an equiaffine plane curve parameterized by equiaffine arc-length parameter, we define the *equiaffine curvature* by

$$\kappa(s) := [\varphi''(s), \varphi'''(s)].$$

Remark 2.1.5. 1. The equiaffine curvature $\kappa(s)$ is equiaffine invariant.

2. From the definition of $\kappa(s)$, we get $\varphi'''(s) + \kappa(s)\varphi'(s) = 0$, that is

$$\frac{d}{ds}\begin{bmatrix} {}^t\varphi'(s) \\ {}^t\varphi''(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \kappa(s) & 0 \end{bmatrix}\begin{bmatrix} {}^t\varphi'(s) \\ {}^t\varphi''(s) \end{bmatrix} =: \Omega\begin{bmatrix} {}^t\varphi'(s) \\ {}^t\varphi''(s) \end{bmatrix}. \tag{2.1.1}$$

Theorem 2.1.6 ([18])**.** *An equiaffine plane curve φ with constant κ is equiaffinely equivalent to one of the following curve*

(1) $\kappa = 0$, $\varphi(s) = {}^t(s, \frac{1}{2}s^2)$*, that is a parabola* $y = \frac{1}{2}x^2$*,*

(2) $\kappa > 0$, $\varphi(s) = {}^t(\kappa^{-\frac{1}{2}}\sin(\kappa^{\frac{1}{2}}s), -\kappa^{-1}\cos(\kappa^{\frac{1}{2}}s))$*, that is an ellipse* $\kappa x^2 + \kappa^2 y^2 = 1$*,*

(3) $\kappa < 0$, $\varphi(s) = {}^t((-\kappa)^{-\frac{1}{2}}\sinh((-\kappa)^{\frac{1}{2}}s), (-\kappa)^{-1}\cosh((-\kappa)^{\frac{1}{2}}s))$*, that is a hyperbola* $\kappa x^2 + \kappa^2 y^2 = 1$.

As it is known, the equiaffine homogeneous curves are precisely the orbits of certain one-parameter subgroups $G(s)$ of $SL(2;\mathbb{R}) \ltimes \mathbb{R}^2$, that is,

$$\varphi(s) = G(s)\varphi_0,$$

where s is the equiaffine arc-length parameter of φ. Using the properties of subgroup, we have $G'(s) = G(s)G'(0)$, $G''(s) = G(s)G''(0)$ and $G'''(s) = G(s)G'''(0)$. Then we can get the following equiaffine curvature

$$\begin{aligned} \kappa(s) &= [\varphi''(s), \varphi'''(s)] = [G(s)G''(0)\varphi_0, G(s)G'''(0)\varphi_0] \\ &= [G''(0)\varphi_0, G'''(0)\varphi_0]. \end{aligned}$$

Thus we have

Theorem 2.1.7 ([10]). *The orbits of a one-parameter group of equiaffine transformations are quadratic curves or straight lines.*

2.2 Equiaffine curves in $\mathbb{R}^n$

Definition 2.2.1. Let $I \subset \mathbb{R}$ be an open interval and $\varphi : I \to \mathbb{R}^n$ a C^∞ map. $\varphi(t)$ is called an *equiaffine space curve* in $\mathbb{R}^n$ if $[\dot{\varphi}(t), \ddot{\varphi}(t), \cdots, \varphi^{(n)}(t)] \neq 0$ for all $t \in I$, where $[\cdot, \ldots, \cdot] : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ is the standard volume form of $\mathbb{R}^n$, which is given by the determinant as before.

Without of generality, we assume the equiaffine curve is dextrorse.

Definition 2.2.2. An equiaffine space curve is said to be parameterized by *equiaffine arc-length parameter* if

$$[\varphi'(s), \varphi''(s), \cdots, \varphi^{(n)}(s)] = 1 \text{ for all } s \in I.$$

Remark 2.2.3. An equiaffine space curve can be reparameterized by equiaffine arc-length parameter.

Using basic properties of determinants, it is easy to show that

$$[\varphi'(s), \varphi''(s), \cdots, \varphi^{(n)}(s)] = [\dot{\varphi}(t), \ddot{\varphi}(t), \cdots, \varphi^{(n)}(t)](\frac{dt}{ds})^{n(n+1)/2}. \tag{2.2.1}$$

Assuming that $[\varphi'(s), \varphi''(s), \cdots, \varphi^{(n)}(s)] = 1$ we obtain

$$s(t) = \int [\dot{\varphi}(t), \ddot{\varphi}(t), \cdots, \varphi^{(n)}(t)]^{\frac{2}{n(n+1)}} dt. \tag{2.2.2}$$

Thus for $t_1 \leq t \leq t_2$, equiaffine arc-length is given by

$$\int_{t_1}^{t_2} [\dot{\varphi}(t), \ddot{\varphi}(t), \cdots, \varphi^{(n)}(t)]^{\frac{2}{n(n+1)}} dt. \tag{2.2.3}$$

Here we define the equiaffine curvatures of an equiaffine space curve. Let $\varphi : I \to \mathbb{R}^n$ be parametrized by equiaffine arc-length, so that $[\varphi'(s), \varphi''(s), \cdots, \varphi^{(n)}(s)] = 1$ for all $s \in I$. Hence the set of vectors $\varphi'(s), \cdots, \varphi^{(n-1)}(s), \varphi^{(n)}(s)$ are linearly independent. Then differentiating with respect to s gives $[\varphi'(s), \cdots, \varphi^{(n-1)}(s), \varphi^{(n+1)}(s)] = 0$. Therefore, there exists functions $\kappa_i : I \to \mathbb{R}$ for $1 \leq i \leq n-1$ such that

$$\varphi^{(n+1)}(s) + \kappa_1(s)\varphi'(s) + \kappa_2(s)\varphi''(s) + \cdots + \kappa_{n-1}(s)\varphi^{(n-1)}(s) = 0. \tag{2.2.4}$$

The functions $\kappa_i(s)$ are called the *equiaffine curvatures* for the equiaffine space curve φ. Notice that

$$\kappa_i(s) = (-1)^{n-i+1}[\varphi'(s), \cdots, \varphi^{(i-1)}(s), \varphi^{(i+1)}(s), \cdots, \varphi^{(n+1)}(s)]. \tag{2.2.5}$$

Theorem 2.2.4 ([10]). *$\kappa_i(s)$ $(1 \leq i \leq n-1)$ are $n-1$ invariants which characterize $\varphi(s)$ up to an equiaffine transformation.*

2.3 Centroaffine plane curves

Definition 2.3.1. A C^∞ map φ from an interval I to $\mathbb{R}^2$ is called a *centroaffine plane curve* in $\mathbb{R}^2$ if $[\varphi(t), \dot{\varphi}(t)] \neq 0$ for all $t \in I$, where $[\cdot, \cdot]$ is the standard area form of $\mathbb{R}^2$.

Definition 2.3.2. A centroaffine plane curve φ is said to be *nondegenerate* if $[\dot{\varphi}(t), \ddot{\varphi}(t)] \neq 0$ for all $t \in I$.

Definition 2.3.3. A nondegenerate centroaffine plane curve is said to be parameterized by centroaffine arc-length parameter if $\varepsilon(s) := \dfrac{[\dot{\varphi}(s), \ddot{\varphi}(s)]}{[\varphi(s), \dot{\varphi}(s)]} = \pm 1$ for all $s \in I$. Then ε is called the *signature* of φ and s is called the *centroaffine arc-length parameter* of φ.

Remark 2.3.4. A nondegenerate centroaffine plane curve can be reparameterized by centroaffine arc-length parameter.

Definition 2.3.5. For a nondegenerate centroaffine plane curve parameterized by centroaffine arc-length parameter, we define the *centroaffine curvature* by

$$\kappa(s) := \frac{[\varphi(s), \varphi''(s)]}{[\varphi(s), \varphi'(s)]}.$$

From the definition of $\kappa(s)$ we get

$$\varphi''(s) = -\varepsilon(s)\varphi(s) + \kappa(s)\varphi'(s),$$

that is

$$\frac{d}{ds}\begin{bmatrix} {}^t\varphi(s) \\ {}^t\varphi'(s) \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon(s) \\ 1 & \kappa(s) \end{bmatrix}\begin{bmatrix} {}^t\varphi(s) \\ {}^t\varphi'(s) \end{bmatrix}.$$

It is well known that a centroaffine curve in $\mathbb{R}^2$ with constant centroaffine curvature can be represented by some exponential functions of the centroaffine arc-length parameter.

Theorem 2.3.6 ([7]). *A nondegenerate centroaffine plane curve with vanishing centroaffine curvature is centroaffinely equivalent to the one of the following curves:*

(i) If $\varepsilon = -1$,

$$\varphi(s) = {}^t(\cosh s, \sinh s),$$

(ii) if $\varepsilon = +1$,

$$\varphi(s) = {}^t(\cos s, \sin s).$$

Theorem 2.3.7 ([7]). *A nondegenerate centroaffine plane curve with constant centroaffine curvature κ is centroaffinely equivalent to one of the following curves:*

(i) $\varepsilon = -1$,

$$\varphi(s) = \frac{1}{\lambda + \lambda^{-1}} \cdot {}^t(\lambda e^{-\lambda^{-1}s} + \lambda^{-1} e^{\lambda s}, -e^{-\lambda^{-1}s} + e^{\lambda s}),$$

where $\lambda := \frac{1}{2}(\kappa + \sqrt{\kappa^2 + 4})$.

(ii-1) $\varepsilon = +1, \mid \kappa \mid > 2$,

$$\varphi(s) = \frac{1}{\lambda - \lambda^{-1}} \cdot {}^t(\lambda e^{\lambda^{-1}s} - \lambda^{-1} e^{\lambda s}, -e^{\lambda^{-1}s} + e^{\lambda s}),$$

where $\lambda := \frac{1}{2}(\kappa + \sqrt{\kappa^2 - 4})$.

(ii-2) $\varepsilon = +1, \kappa = +2$,

$$\varphi(s) = {}^t(e^s - se^s, se^s).$$

(ii-3) $\varepsilon = +1, \kappa = -2$,

$$\varphi(s) = {}^t(e^{-s} + se^{-s}, se^{-s}).$$

(ii-4) $\varepsilon = +1, \mid \kappa \mid < 2$,

$$\varphi(s) = {}^t(e^{\alpha s}\cos(\beta s) - \alpha\beta^{-1} e^{\alpha s}\sin(\beta s), -\beta^{-1} e^{\alpha s}\sin(\beta s)),$$

where $\alpha := \frac{\kappa}{2}, \beta := \frac{1}{2}\sqrt{4-\kappa^2}$.

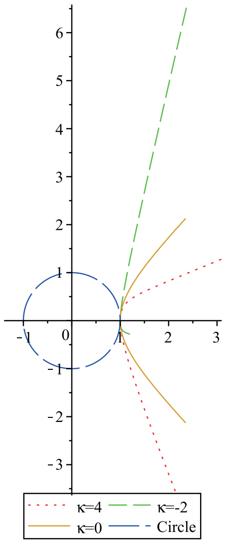

Figure 2.1: Centroaffine plane curve (i)

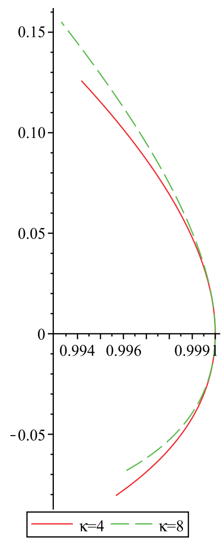

Figure 2.2: Centroaffine plane curve (ii-1)

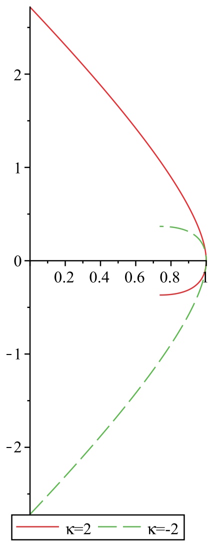

Figure 2.3: Centroaffine plane curve (ii-2,3)

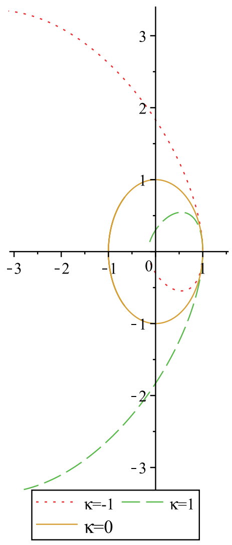

Figure 2.4: Centroaffine plane curve (ii-4)

2.4 Centroaffine curves in $\mathbb{R}^n$

Definition 2.4.1. Let $I \subset \mathbb{R}$ be an open interval and $\varphi : I \to \mathbb{R}^n$ a C^∞ map. $\varphi(t)$ is called a *centroaffine space curve* in $\mathbb{R}^n$ if $[\varphi(t), \dot\varphi(t), \cdots, \varphi^{(n-1)}(t)] \neq 0$ for all $t \in I$, where $[\cdot, \ldots, \cdot]$ is the standard volume form of $\mathbb{R}^n$, that is, the determinant.

Definition 2.4.2. A centroaffine space curve φ is said to be *nondegenerate* if for all $t \in I$, $[\dot\varphi(t), \ddot\varphi(t), \cdots, \varphi^{(n)}(t)] \neq 0$.

Definition 2.4.3. A nondegenerate centroaffine space curve is said to be parameterized by centroaffine arc-length parameter if $\varepsilon(s) := \dfrac{[\dot\varphi(s), \ddot\varphi(s), \cdots, \varphi^{(n)}(s)]}{[\varphi(s), \dot\varphi(s), \cdots, \varphi^{(n-1)}(s)]} = \pm 1$ for all $s \in I$. Then ε is called the *signature* of φ and s is called the *centroaffine arc-length parameter* of φ.

Remark 2.4.4. Any nondegenerate centroaffine space curve has a reparametrization by centroaffine arc-length parameter.

Using basic properties of determinants, we can easily show that

$$[\varphi'(s), \varphi''(s), \cdots, \varphi^{(n)}(s)] = [\dot\varphi(t), \ddot\varphi(t), \cdots, \varphi^{(n)}(t)](\frac{dt}{ds})^{n(n+1)/2}, \tag{2.4.1}$$

and

$$[\varphi(s), \varphi'(s), \cdots, \varphi^{(n-1)}(s)] = [\varphi(t), \dot\varphi(t), \cdots, \varphi^{(n-1)}(t)](\frac{dt}{ds})^{n(n-1)/2}. \tag{2.4.2}$$

Assuming that $\dfrac{[\varphi'(s), \varphi''(s), \cdots, \varphi^{(n)}(s)]}{[\varphi(s), \varphi'(s), \cdots, \varphi^{(n-1)}(s)]} = \varepsilon(s) = \pm 1$, we obtain

$$s(t) = \int \left| \frac{[\dot\varphi(t), \ddot\varphi(t), \cdots, \varphi^{(n)}(t)]}{[\varphi(t), \dot\varphi(t), \cdots, \varphi^{(n-1)}(t)]} \right|^{\frac{1}{n}} dt. \tag{2.4.3}$$

Thus the centroaffine arc-length of $\varphi([t_1, t_2])$ is given by

$$\int_{t_1}^{t_2} \left| \frac{[\dot\varphi(t), \ddot\varphi(t), \cdots, \varphi^{(n)}(t)]}{[\varphi(t), \dot\varphi(t), \cdots, \varphi^{(n-1)}(t)]} \right|^{\frac{1}{n}} dt. \tag{2.4.4}$$

Here we define the centroaffine curvatures of a nondegenerate centroaffine space curve. Let $\varphi : I \to \mathbb{R}^n$ be a nondegenerate centroaffine curve parametrized by centroaffine arc-length, so that $\varepsilon(s) := \dfrac{[\dot{\varphi}(s), \ddot{\varphi}(s), \cdots, \varphi^{(n)}(s)]}{[\varphi(s), \dot{\varphi}(s), \cdots, \varphi^{(n-1)}(s)]} = \pm 1$ for all $s \in I$. Therefore, there exists functions $\kappa_i : I \to \mathbb{R}$ for $1 \le i \le n-1$ such that

$$\varphi^{(n)}(s) = (-1)^{n-1}\varepsilon(s)\varphi(s) + \kappa_1(s)\varphi'(s) + \kappa_2(s)\varphi''(s) + \cdots + \kappa_{n-1}(s)\varphi^{(n-1)}(s). \tag{2.4.5}$$

The functions κ_i are called the *centroaffine curvatures* for the space curve φ. Notice that

$$\kappa_i(s) = (-1)^{n-i-1} \frac{[\varphi(s), \varphi'(s), \cdots, \varphi^{(i-1)}(s), \varphi^{(i+1)}(s), \cdots, \varphi^{(n)}(s)]}{[\varphi(s), \varphi'(s), \cdots, \varphi^{(n-1)}(s)]}. \tag{2.4.6}$$

$\kappa_i(s)$ $(1 \le i \le n-1)$ are $n-1$ invariants which characterize $\varphi(s)$ up to a centroaffine transformation.

Theorem 2.4.5 ([9]). *Let* $\mathrm{I} \subset \mathbb{R}$ *be an open interval,* $\varepsilon = \pm 1$, *and let* $\kappa_1(s), \cdots, \kappa_{n-1}(s)$ *be smooth functions on I. Then there is a smooth nondegenerate centroaffine immersion* $s \mapsto \varphi(s)$ *from* I *to* $\mathbb{R}^n$ *such that* s *is a centroaffine arc-length parameter,* $\kappa_1(s), \cdots, \kappa_{n-1}(s)$ *are the centroaffine curvatures, and*

$$\varphi^{(n)}(s) \equiv \varepsilon\varphi(s)\ (mod\ \varphi'(s), \cdots, \varphi^{(n-1)}(s)). \tag{2.4.7}$$

Moreover, φ *is uniquely determined up to a centroaffine transformation of* $\mathbb{R}^n$.

2.5 Affine hypersurfaces

2.5.1 Equiaffine hypersurfaces

Let $f : M \to \mathbb{R}^{n+1}$ be an immersion of an n-dimensional oriented manifold M into $\mathbb{R}^{n+1}$. We denote by $\Gamma(TM)$ the space of sections of the tangent bundle TM, by D the standard flat affine connection in $\mathbb{R}^{n+1}$, and by $[\cdot, \ldots, \cdot]$ the standard volume form of $\mathbb{R}^{n+1}$.

A vector field ξ along f is called a *transversal vector field* if it satisfies at each point x of M the tangent space $T_{f(x)}\mathbb{R}^{n+1}$ is decomposed as

$$T_{f(x)}\mathbb{R}^{n+1} = f_* T_x M \oplus \mathbb{R}\xi_x, \tag{2.5.1}$$

and that the volume form θ defined by

$$\theta(X_1, \cdots, X_n) := [f_* X_1, \cdots, f_* X_n, \xi], \tag{2.5.2}$$

for $X_1, \cdots, X_n \in \Gamma(TM)$, is compatible with the orientation of M. When we choose a transversal vector field ξ , we determine a torsion free affine connection ∇, a symmetric $(0, 2)$-tensor field h, a $(1, 1)$-tensor field S, and a 1-form τ by

$$D_X f_* Y = f_* \nabla_X Y + h(X, Y)\xi \quad \text{(the formula of Gauss)},$$

$$D_X \xi = -f_* S X + \tau(X)\xi \quad \text{(the formula of Weingarten)}$$

according to the decomposition (2.5.1). We call ∇ the *induced connection*, h the *affine metric (affine fundamental form)*, S the *affine shape operator* and τ the *transversal connection form* of (f, ξ). It is easily shown that the conformal class of h does not depend on the choice of ξ. When h is nondegenerate (resp. definite, indefinite), f is said to be nondegenerate (resp. definite, indefinite).

We begin by deriving more fundamental equations for a hypersurface immersion $f : M \to \mathbb{R}^{n+1}$. First, we consider the case where the given transversal vector field ξ is arbitrary. We have

Theorem 2.5.1 ([18]). *For a hypersurface immersion f with an arbitrary transversal vector* *field ξ the induced connection ∇, the affine metric h, the affine shape operator S, and the transversal connection form τ satisfy the following equations:*

1. Gauss:

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY;$$

2. Codazzi for h:

$$(\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z);$$

3. Codazzi for S:

$$(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)hX;$$

4. Ricci:

$$h(X,SY) - h(SX,Y) = d\tau(X,Y).$$

Lemma 2.5.2 ([18]). *We have*

$$\nabla_X \theta = \tau(X)\theta \text{ for all } X \in T_xM. \tag{2.5.3}$$

Consequently, the following two conditions are equivalent:

(1) $\nabla\theta = 0$;

(2) $\tau = 0$*, that is $D_X\xi$ is tangential for every vector field X on M.*

Theorem 2.5.3 ([18]). *If f is nondegenerate, there is a transversal vector field ξ satisfying that*

(I) $\nabla\theta = 0$;

(II) *θ coincides with the volume element ω_h of the nondegenerate affine metric h.*

Moreover, it is uniquely determined.

Definition 2.5.4. A transversal vector field satisfying (I) and (II) is called the *Blaschke normal vector field*. For each point $x \in M$ we take the line through $f(x)$ in the direction of the Blaschke normal vector ξ_x. This line, which is independent of the choice of sign for ξ, is called the *affine normal* through x. An *equiaffine hypersurface* is a hypersurface equipped with the Blaschke normal vector field. The corresponding induced connection ∇ and affine metric h are then called the *equiaffine induced connection* and *equiaffine metric*, respectively.

Theorem 2.5.5 ([18])**.** *If $f : M \to \mathbb{R}^{n+1}$ is an equiaffine immersion, then the equations of Codazzi and Ricci are as follows:*

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z); \tag{2.5.4}$$

$$(\nabla_X S)(Y) = (\nabla_Y S)(X); \tag{2.5.5}$$

$$h(X, SY) = h(SX, Y). \tag{2.5.6}$$

Definition 2.5.6. An equiaffine hypersurface $f : M \to \mathbb{R}^{n+1}$ is called an *improper affine sphere* if S is identically zero. If $S = \lambda I$, where λ is a nonzero constant, then M is called a *proper affine sphere.*

Theorem 2.5.7 ([18])**.** *If $f : M \to \mathbb{R}^{n+1}$ is an improper affine sphere, then the affine normals are parallel in $\mathbb{R}^{n+1}$. If $f : M \to \mathbb{R}^{n+1}$ is a proper affine sphere, then the affine normals meet at one point in $\mathbb{R}^{n+1}$ (called the center). The converse of each of these statements is also valid.*

From the Coddazzi equation for h we see that the *cubic form*

$$C(X, Y, Z) = (\nabla_X h)(Y, Z) \tag{2.5.7}$$

is symmetric in X, Y and Z. Now, suppose that f is nondegenerate. In addition to the equiaffine induced connection ∇ on M, we may consider the Levi-Civita connection $\widetilde{\nabla}$ for the equiaffine metric h. We consider the *difference tensor* of type $(1, 2)$

$$K(X, Y) = K_X Y = \nabla_X Y - \widetilde{\nabla}_X Y. \tag{2.5.8}$$

Since both ∇ and $\widetilde{\nabla}$ have zero torsion, we have $K(X, Y) = K(Y, X)$. For each $X \in T_x M$, K_X is a tensor of type $(1, 1)$. We can now relate the cubic form to the difference tensor as follows.

Theorem 2.5.8 ([18]). *We have*

$$C(X, Y, Z) = -2h(K_X Y, Z). \tag{2.5.9}$$

Theorem 2.5.9 ([18]). *We have the apolarity condition*

$$\operatorname{tr} K_X = 0 \quad \text{for all} \quad X \in T_x M; \tag{2.5.10}$$

in index notation, $\sum_{j=1}^{n} K_{ij}^{j} = 0$ *for each fixed i.*

Theorem 2.5.10 (Maschke-Pick-Berwald [18]). *Let* $f : M \to \mathbb{R}^{n+1}$, $n \geq 2$, *be a nondegenerate equiaffine hypersurface. If the cubic form C vanishes identically, then* $f(M)$ *is equiaffinely equivalent to a hyperquadric in* $\mathbb{R}^{n+1}$.

Proposition 2.5.11 ([18]). *The scalar curvature* ρ *of the equiaffine metric h can be expressed by*

$$\rho = H + J, \tag{2.5.11}$$

where

$$H = \frac{1}{n} \operatorname{tr} S \tag{2.5.12}$$

is called the equiaffine mean curvature *and*

$$J = \frac{1}{n(n-1)} h(K, K) \tag{2.5.13}$$

is called the Pick invariant.

2.5.2 Centroaffine hypersurfaces

Definition 2.5.12. Let f be an immersion of an n-dimensional C^∞-manifold M into the affine space $\mathbb{R}^{n+1}$ such that the hypersurface $f(M)$ does not pass through the origin and the position vector $f(x)$, $x \in M$, is transversal to $f(M)$. We call such an immersion a *centroaffine hypersurface*.

We determine a torsion free affine connection ∇ and a symmetric $(0,2)$-tensor field h by

$$D_X f_* Y = f_* \nabla_X Y + h(X,Y) f. \tag{2.5.14}$$

We call ∇ and h the *centroaffine induced connection* and *centroaffine metric* of f respectively. The hypersurface is called elliptic if the centroaffine metric is negative definite. We denote the *difference tensor* of the centroaffine induced connection ∇ and the Levi-Civita connection $\widetilde{\nabla}$ of the centroaffine metric h by

$$K = \nabla - \widetilde{\nabla} \in \Gamma(TM^{(1,2)}), \tag{2.5.15}$$

its associate cubic form C, defined by

$$C(X,Y,Z) = -2h(K(X,Y),Z), \quad X,Y,Z \in \Gamma(TM), \tag{2.5.16}$$

is totally symmetric.

Theorem 2.5.13. *Let $f : M \to \mathbb{R}^{n+1}$ be a nondegenerate centroaffine hypersurface. If the difference tensor $K = 0$ vanishes identically, then $f(M)$ is centroaffinely equivalent to an ellipsoid or a hyperboloid of two-sheets.*

We define the *centroaffine Tchebychev vector field* T and the *centroaffine Tchebychev operator* $\mathfrak{T}$ by

$$T := \operatorname{tr} K \in \Gamma(TM), \tag{2.5.17}$$

$$\mathfrak{T} := \widetilde{\nabla} T \in \Gamma(TM^{(1,1)}). \tag{2.5.18}$$

Theorem 2.5.14 ([7])**.** *Let $f : M \to \mathbb{R}^{n+1}$ be a nondegenerate centroaffine hypersurface. The centroaffine Tchebychev vector field T vanishes identically if and only if $f(M)$ is centroaffinely equivalent to proper affine sphere with center at the origin.*

In 1994, C.P.Wang studied the centroaffine hypersurfaces as extremals for the volume integral of the centroaffine metric and gave the conditions for centroaffine minimal hypersurfaces.

Theorem 2.5.15 ([25])**.** *The critical hypersurfaces for the volume integral are exactly the hypersurfaces with vanishing centroaffine mean curvature, where the* centroaffine mean curvature *is defined by*

$$H = \operatorname{tr} \mathfrak{T}. \tag{2.5.19}$$

The function J on M defined by

$$J := \frac{1}{n(n-1)} h(K, K), \tag{2.5.20}$$

is called the *Pick invariant*. The centroaffine theorema egregium is

$$\rho = J - \frac{n}{n-1} h(T, T) + 1, \tag{2.5.21}$$

where ρ is the scalar curvature of the centroaffine metric h.

Chapter 3

Equiaffine space curves

3.1 Basic notions of equiaffine space curves

Let $\varphi : I \to \mathbb{R}^3$ be an equiaffine space curve in $\mathbb{R}^3$ parameterized by equiaffine arc-length parameter s. Let $\kappa_1(s)$ and $\kappa_2(s)$ be the equiaffine first and second curvature of φ, respectively. By definition, they are given by

$$\kappa_1(s) := -[\varphi''(s), \varphi'''(s), \varphi''''(s)],$$

and

$$\kappa_2(s) := [\varphi'(s), \varphi'''(s), \varphi''''(s)].$$

From the definition of $\kappa_1(s)$ and $\kappa_2(s)$, we get

$$\varphi''''(s) + \kappa_1(s)\varphi'(s) + \kappa_2(s)\varphi''(s) = 0,$$

that is

$$\frac{d}{ds}\begin{bmatrix} {}^t\varphi'(s) \\ {}^t\varphi''(s) \\ {}^t\varphi'''(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\kappa_1(s) & -\kappa_2(s) & 0 \end{bmatrix}\begin{bmatrix} {}^t\varphi'(s) \\ {}^t\varphi''(s) \\ {}^t\varphi'''(s) \end{bmatrix} =: \Omega\begin{bmatrix} {}^t\varphi'(s) \\ {}^t\varphi''(s) \\ {}^t\varphi'''(s) \end{bmatrix}. \tag{3.1.1}$$

3.2 Equiaffine space curves with constant equiaffine curvatures

From the fundamental theorem for equiaffine space curves, the equiaffine space curves are uniquely determined by the equiaffine curvatures up to an equiaffine transformation in $\mathbb{R}^3$. In the following, we obtain the curves with constant curvatures by solving the ODE: $\varphi''''(s) + \kappa_1\varphi'(s) + \kappa_2\varphi''(s) = 0$.

Lemma 3.2.1 (Shengjin's formulas). *For a one variable cubic equation $ax^3+bx^2+cx+d = 0$, where $(a, b, c, d \in \mathbb{R}, a \neq 0)$, if we set $A := b^2 - 3ac$, $B := bc - 9ad$, $C := c^2 - 3bd$, $\Delta := B^2 - 4AC$, we have the following solutions:*

(1) *if $A = B = 0$, then*

$$x_1 = x_2 = x_3 = -\frac{b}{3a} = -\frac{c}{b} = -\frac{3d}{c},$$

(2) *if $\Delta = B^2 - 4AC = 0$, then*

$$x_1 = -\frac{b}{a} + k,\ x_2 = x_3 = -\frac{k}{2},$$

where $k = \frac{B}{A}$, $(A \neq 0)$,

(3) *if $\Delta = B^2 - 4AC > 0$, then*

$$x_1 = \frac{-b - \sqrt[3]{y_1} - \sqrt[3]{y_2}}{3a},$$

$$x_2, x_3 = \frac{-2b + \sqrt[3]{y_1} + \sqrt[3]{y_2} \pm i\sqrt{3}(\sqrt[3]{y_1} - \sqrt[3]{y_2})}{6a},$$

where $y_1, y_2 = Ab + \frac{3a}{2}(-B \pm \sqrt{B^2 - 4AC})$,

(4) *if* $\Delta = B^2 - 4AC < 0$, *then*

$$x_1 = \frac{-b - 2\sqrt{A}\cos\frac{\theta}{3}}{3a},$$

$$x_2, x_3 = \frac{-b + \sqrt{A}(\cos\frac{\theta}{3} \pm \sqrt{3}\sin\frac{\theta}{3})}{3a},$$

where $\theta = \arccos t$, $t = \frac{2Ab-3aB}{2\sqrt{A^3}}$, $(A > 0,\ -1 < t < 1)$.

First, using Shengjin's formula we get the eigenvalues of the coefficient matrix Ω in (3.1.1) which is one variable cubic equation:

$$\left|\lambda I - \Omega\right| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ \kappa_1 & \kappa_2 & \lambda \end{vmatrix} = \lambda^3 + \kappa_2\lambda + \kappa_1 = 0, \tag{3.2.1}$$

whose discriminants of multiple root are:

$$\begin{cases} A := -3\kappa_2, \\ B := -9\kappa_1, \\ C := \kappa_2^2, \end{cases} \tag{3.2.2}$$

and principal discriminant is:

$$\Delta := B^2 - 4AC = 3(27\kappa_1^2 + 4\kappa_2^3). \tag{3.2.3}$$

Then using the eigenvalues to get the fundamental system of solution of this ODE.

We obtain the following classification theorem:

Theorem 3.2.2. *Any nondegenerate equiaffine space curve φ with constant equiaffine curva-*

tures κ_1, κ_2 is equiaffinely equivalent to one of the following curves:

(1) $A^2 + B^2 = 0$,

$$\varphi(s) = {}^t(s, \frac{1}{2}s^2, \frac{1}{6}s^3),$$

(2) $A^2 + B^2 \neq 0$ *and* $\Delta = 0$,

$$\varphi(s) = {}^t(e^{\sigma s}, se^{\sigma s}, -\frac{1}{18\sigma^5}e^{-2\sigma s}),$$

(3) $A^2 + B^2 \neq 0$ *and* $\Delta > 0$,

$$\varphi(s) = \begin{cases} \kappa_2^{-1} \cdot {}^t(-\kappa_2^{\frac{1}{2}} s, \sin(\kappa_2^{\frac{1}{2}} s), \cos(\kappa_2^{\frac{1}{2}} s)), & if\ \kappa_1 = 0, \\ {}^t(\frac{1}{2\sigma_1\sigma_2(9\sigma_1^2+\sigma_2^2)(\sigma_1^2+\sigma_2^2)}e^{-2\sigma_1 s}, e^{\sigma_1 s}\sin(\sigma_2 s), e^{\sigma_1 s}\cos(\sigma_2 s)), & if\ \kappa_1 \neq 0, \end{cases}$$

(4) $A^2 + B^2 \neq 0$ *and* $\Delta < 0$,

$$\varphi(s) = \begin{cases} -\kappa_2^{-1} \cdot {}^t(-(-\kappa_2)^{\frac{1}{2}} s, \sinh((-\kappa_2)^{\frac{1}{2}} s), \cosh((-\kappa_2)^{\frac{1}{2}} s)), & if\ \kappa_1 = 0, \\ {}^t(\frac{1}{4\sigma_1\sigma_2(9\sigma_1^2-\sigma_2^2)(\sigma_1^2-\sigma_2^2)}e^{-2\sigma_1 s}, e^{(\sigma_1+\sigma_2)s}, e^{(\sigma_1-\sigma_2)s}), & if\ \kappa_1 \neq 0, \end{cases}$$

where σ, σ_i $(i = 1, 2)$ *are constants determined by equiaffine curvatures* κ_1, κ_2 *as in* (3.2.4)~(3.2.6), *respectively.*

Proof. Using Shengjin's formulas, we separate out four cases:

(1) When $A = B = 0$, that is $\kappa_1 = \kappa_2 = 0$, one variable cubic equation (3.2.1) has a triple root:

$$\lambda_1 = \lambda_2 = \lambda_3 = 0.$$

Then 1, s and s^2 are the fundamental solutions. Thus we get

$$\varphi'(s) = c_1 + c_2 s + c_3 s^2,$$

where $c_i(i = 1, 2, 3)$ are the constant vectors in $\mathbb{R}^3$. In order to make s as the equiaffine arc-length parameter, we should make sure the determinant $[\varphi'(s), \varphi''(s), \varphi'''(s)] = 1$, that is, $[c_1, c_2, c_3] = \frac{1}{2}$. Without loss of generality, if we set $c_1 = {}^t(1, 0, 0)$, $c_2 = {}^t(0, 1, 0)$, $c_3 = {}^t(0, 0, \frac{1}{2})$, we get

$$\varphi(s) = {}^t(s, \frac{1}{2}s^2, \frac{1}{6}s^3) + c_0$$

for some constant vector c_0 in $\mathbb{R}^3$. If we set $c_0 = {}^t(0, 0, 0)$, the space curve is equiaffinely equivalent to the following curve:

$$\varphi(s) = {}^t(s, \frac{1}{2}s^2, \frac{1}{6}s^3).$$

(2) When $A^2 + B^2 \neq 0$, $\Delta = 0$, then $\kappa_2 = -3(\frac{\kappa_1}{2})^{\frac{2}{3}} < 0$, one variable cubic equation (3.2.1) has a double root:

$$\lambda_1 = \lambda_2 = \sigma, \lambda_3 = -2\sigma,$$

where

$$\sigma = (\frac{\kappa_1}{2})^{\frac{1}{3}}. \tag{3.2.4}$$

Then $e^{\sigma s}$, $se^{\sigma s}$ and $e^{-2\sigma s}$ are the fundamental solutions. Thus we get

$$\varphi'(s) = c_1 e^{\sigma s} + c_2 s e^{\sigma s} + c_3 e^{-2\sigma s},$$

where $c_i (i = 1, 2, 3)$ are the constant vectors in $\mathbb{R}^3$. In order to make s as the equiaffine arc-length parameter, we should make sure the determinant $[\varphi'(s), \varphi''(s), \varphi'''(s)] = 1$, that is, $[c_1, c_2, c_3] = \frac{1}{9\sigma^2}$. Without loss of generality, if we set $c_1 = {}^t(\sigma, 1, 0)$, $c_2 = {}^t(0, \sigma, 0)$, $c_3 = {}^t(0, 0, \frac{1}{9\sigma^4})$, we get

$$\varphi(s) = {}^t(e^{\sigma s}, se^{\sigma s}, -\frac{1}{18\sigma^5}e^{-2\sigma s}) + c_0$$

for some constant vector c_0 in $\mathbb{R}^3$. If we set $c_0 = {}^t(0, 0, 0)$, the space curve is equiaffinely equivalent to the following curve:

$$\varphi(s) = {}^t(e^{\sigma s}, se^{\sigma s}, -\frac{1}{18\sigma^5}e^{-2\sigma s}).$$

(3) When $A^2 + B^2 \neq 0$, $\Delta > 0$, one variable cubic equation (3.2.1) has a pair of conjugate imaginary roots:

$$\lambda_1 = -2\sigma_1, \lambda_2 = \sigma_1 + i\sigma_2, \lambda_3 = \sigma_1 - i\sigma_2,$$

where

$$\begin{cases} \sigma_1 = \frac{1}{6}(\sqrt[3]{\frac{3(9\kappa_1+\sqrt{12\kappa_2^3+81\kappa_1^2})}{2}} + \sqrt[3]{\frac{3(9\kappa_1-\sqrt{12\kappa_2^3+81\kappa_1^2})}{2}}), \\ \sigma_2 = \frac{\sqrt{3}}{6}(\sqrt[3]{\frac{3(9\kappa_1+\sqrt{12\kappa_2^3+81\kappa_1^2})}{2}} - \sqrt[3]{\frac{3(9\kappa_1-\sqrt{12\kappa_2^3+81\kappa_1^2})}{2}}). \end{cases} \tag{3.2.5}$$

Then $e^{-2\sigma_1 s}$, $e^{\sigma_1 s}\cos(\sigma_2 s)$ and $e^{\sigma_1 s}\sin(\sigma_2 s)$ are the fundamental solutions. Thus we get

$$\varphi'(s) = c_1 e^{-2\sigma_1 s} + c_2 e^{\sigma_1 s}\cos(\sigma_2 s) + c_3 e^{\sigma_1 s}\sin(\sigma_2 s),$$

where $c_i (i = 1, 2, 3)$ are the constant vectors in $\mathbb{R}^3$. In order to make s as the equiaffine arc-length parameter, we should make sure the determinant $[\varphi'(s), \varphi''(s), \varphi'''(s)] = 1$, that is, $[c_1, c_2, c_3] = \frac{1}{(9\sigma_1^2+\sigma_2^2)\sigma_2}$.

(a) $\kappa_1 = 0$,

we have $\sigma_1 = 0$, and $[c_1, c_2, c_3] = \kappa_2^{-\frac{3}{2}}$. Without loss of generality, if we set $c_1 = {}^t(-\kappa_2^{-\frac{1}{2}}, 0, 0)$, $c_2 = {}^t(0, \kappa_2^{-\frac{1}{2}}, 0)$, $c_3 = {}^t(0, 0, -\kappa_2^{-\frac{1}{2}})$, we get

$$\varphi(s) = \kappa_2^{-1} \cdot {}^t(-\kappa_2^{\frac{1}{2}} s, \sin(\kappa_2^{\frac{1}{2}} s), \cos(\kappa_2^{\frac{1}{2}} s)) + c_0$$

for some constant vector c_0 in $\mathbb{R}^3$. If we set $c_0 =^t (0, 0, 0)$, the space curve is equiaffinely equivalent to the following curve:

$$\varphi(s) = \kappa_2^{-1} \cdot {}^t(-\kappa_2^{\frac{1}{2}} s, \sin(\kappa_2^{\frac{1}{2}} s), \cos(\kappa_2^{\frac{1}{2}} s)).$$

(b) $\kappa_1 \neq 0$,

we have $\sigma_1 \neq 0$, and $[c_1, c_2, c_3] = \frac{1}{(9\sigma_1^2+\sigma_2^2)\sigma_2}$. Without loss of generality, if we set $c_1 = {}^t(-\frac{1}{\sigma_2(9\sigma_1^2+\sigma_2^2)(\sigma_1^2+\sigma_2^2)}, 0, 0)$, $c_2 = {}^t(0, \sigma_2, \sigma_1)$, $c_3 = {}^t(0, \sigma_1, -\sigma_2)$, we get

$$\varphi(s) = {}^t(\frac{1}{2\sigma_1\sigma_2(9\sigma_1^2 + \sigma_2^2)(\sigma_1^2 + \sigma_2^2)} e^{-2\sigma_1 s}, e^{\sigma_1 s}\sin(\sigma_2 s), e^{\sigma_1 s}\cos(\sigma_2 s)) + c_0$$

for some constant vector c_0 in $\mathbb{R}^3$. If we set $c_0 = {}^t(0, 0, 0)$, the space curve is equiaffinely equivalent to the following curve:

$$\varphi(s) = {}^t(\frac{1}{2\sigma_1\sigma_2(9\sigma_1^2 + \sigma_2^2)(\sigma_1^2 + \sigma_2^2)} e^{-2\sigma_1 s}, e^{\sigma_1 s}\sin(\sigma_2 s), e^{\sigma_1 s}\cos(\sigma_2 s)).$$

(4) When $A^2 + B^2 \neq 0$, $\Delta < 0$, one variable cubic equation (3.2.1) has three different real roots:

$$\lambda_1 = -2\sigma_1, \lambda_2 = \sigma_1 + \sigma_2, \lambda_3 = \sigma_1 - \sigma_2,$$

where

$$\begin{cases} \sigma_1 = \frac{1}{3}\sqrt{-3\kappa_2}\cos(\frac{1}{3}\arccos(\frac{27}{2}\kappa_1(-3\kappa_2)^{-\frac{3}{2}})), \\ \sigma_2 = \sqrt{-\kappa_2}\sin(\frac{1}{3}\arccos(\frac{27}{2}\kappa_1(-3\kappa_2)^{-\frac{3}{2}})), \\ \kappa_2 < 0, \ \frac{27}{2}\kappa_1(-3\kappa_2)^{-\frac{3}{2}} \in (-1, 1). \end{cases} \tag{3.2.6}$$

Then $e^{-2\sigma_1 s}$, $e^{(\sigma_1+\sigma_2)s}$ and $e^{(\sigma_1-\sigma_2)s}$ are the fundamental solutions. Thus we get

$$\varphi'(s) = c_1 e^{-2\sigma_1 s} + c_2 e^{(\sigma_1+\sigma_2)s} + c_3 e^{(\sigma_1-\sigma_2)s},$$

where $c_i (i = 1, 2, 3)$ are the constant vectors in $\mathbb{R}^3$. In order to make s as the equiaffine arc-length parameter, we should make sure the determinant $[\varphi'(s), \varphi''(s), \varphi'''(s)] = 1$, that is, $[c_1, c_2, c_3] = \frac{1}{2\sigma_2(\sigma_2^2 - 9\sigma_1^2)}$.

(a) $\kappa_1 = 0$,

we can set $\sigma_1 = 0$, $\sigma_2 = (\kappa_2)^{-\frac{1}{2}}$, and $[c_1, c_2, c_3] = \frac{1}{2}(-\kappa_2)^{-\frac{3}{2}}$. Without loss of generality, if we set $c_1 = {}^t(-(-\kappa_2)^{-\frac{1}{2}}, 0, 0)$, $c_2 = {}^t(0, \frac{1}{2}(-\kappa_2)^{-\frac{1}{2}}, \frac{1}{2}(-\kappa_2)^{-\frac{1}{2}})$, $c_3 = {}^t(0, \frac{1}{2}(-\kappa_2)^{-\frac{1}{2}}, -\frac{1}{2}(-\kappa_2)^{-\frac{1}{2}})$, we get the space curve is equiaffinely equivalent to the following curve:

$$\varphi(s) = -\kappa_2^{-1} \cdot {}^t(-(-\kappa_2)^{\frac{1}{2}} s, \sinh((-\kappa_2)^{\frac{1}{2}} s), \cosh(-\kappa_2^{\frac{1}{2}} s)) + c_0$$

for some constant vector c_0 in $\mathbb{R}^3$. If we set $c_0 = {}^t(0, 0, 0)$, the space curve is equiaffinely equivalent to the following curve:

$$\varphi(s) = -\kappa_2^{-1} \cdot {}^t(-(-\kappa_2)^{\frac{1}{2}} s, \sinh((-\kappa_2)^{\frac{1}{2}} s), \cosh((-\kappa_2)^{\frac{1}{2}} s)).$$

(b) $\kappa_1 \neq 0$,

we have $\sigma_1 + \sigma_2 \neq 0$, $\sigma_1 - \sigma_2 \neq 0$, and $[c_1, c_2, c_3] = \frac{1}{2\sigma_2(\sigma_2^2 - 9\sigma_1^2)}$. Without loss of generality, if we set $c_1 = {}^t(0, \sigma_1 + \sigma_2, 0)$, $c_2 = {}^t(0, 0, \sigma_1 - \sigma_2)$, $c_3 = {}^t(\frac{1}{2\sigma_2(\sigma_2^2 - 9\sigma_1^2)(\sigma_1^2 - \sigma_2^2)}, 0, 0)$, we get the space curve is equiaffinely equivalent to the following curve:

$$\varphi(s) = {}^t(\frac{1}{4\sigma_1\sigma_2(9\sigma_1^2 - \sigma_2^2)(\sigma_1^2 - \sigma_2^2)} e^{-2\sigma_1 s}, e^{(\sigma_1+\sigma_2)s}, e^{(\sigma_1-\sigma_2)s}) + c_0$$

for some constant vector c_0 in $\mathbb{R}^3$. If we set $c_0 = {}^t(0, 0, 0)$, the space curve is equiaffinely equivalent to the following curve:

$$\varphi(s) = {}^t(\frac{1}{4\sigma_1\sigma_2(9\sigma_1^2 - \sigma_2^2)(\sigma_1^2 - \sigma_2^2)} e^{-2\sigma_1 s}, e^{(\sigma_1+\sigma_2)s}, e^{(\sigma_1-\sigma_2)s}). \qquad \square$$

3.3 Groups of equiaffine space curves with constant curvatures

As it is known, the equiaffine homogeneous curves are precisely the orbits of certain one-parameter subgroups $G(s)$ of $SL(3;\mathbb{R}) \ltimes \mathbb{R}^3$, that is,

$$\varphi(s) = G(s)\varphi_0,$$

where s is the equiaffine arc-length parameter of φ.

Using the properties of subgroup, we have $G'(s) = G(s)G'(0)$, $G''(s) = G(s)G''(0)$, $G'''(s) = G(s)G'''(0)$ and $G''''(s) = G(s)G''''(0)$. Then we can get the following equiaffine first and second curvatures

$$\begin{aligned} \kappa_1(s) &= -[\varphi''(s), \varphi'''(s), \varphi''''(s)] = -[G(s)G''(0)\varphi_0, G(s)G'''(0)\varphi_0, G(s)G''''(0)\varphi_0] \\ &= -[G''(0)\varphi_0, G'''(0)\varphi_0, G''''(0)\varphi_0], \end{aligned}$$

$$\begin{aligned} \kappa_2(s) &= [\varphi'(s), \varphi'''(s), \varphi''''(s)] = [G(s)G'(0)\varphi_0, G(s)G'''(0)\varphi_0, G(s)G''''(0)\varphi_0] \\ &= [G'(0)\varphi_0, G'''(0)\varphi_0, G''''(0)\varphi_0]. \end{aligned}$$

Thus we have

Proposition 3.3.1. *The equiaffine curvatures of a nondegenerate equiaffine homogeneous curve are constant.*

Conversely, we give the one-parameter subgroup of $SL(3,\mathbb{R}) \ltimes \mathbb{R}^3$ for each class of the non-degenerate equiaffine space curves with constant equiaffine curvatures in $\mathbb{R}^3$ as follows

(1) We take

$$G_1 := \left\{ \begin{bmatrix} 1 & s & \frac{1}{2}s^2 & \frac{1}{6}s^3 \\ 0 & 1 & s & \frac{1}{2}s^2 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}. \tag{3.3.1}$$

We can check that G_1 is a subgroup of $SL(3, \mathbb{R}) \ltimes \mathbb{R}^3$ as a group acting on ${}^t(z, y, x, 1) \in \mathbb{R}^4$. Let φ be a nondegenerate equiaffine space curve with constant equiaffine curvatures obtained in (1) of Theorem 3.2.2; that is, $\varphi(s) = {}^t(s, \frac{1}{2}s^2, \frac{1}{6}s^3)$. Then $\varphi(\mathbb{R})$ is the G_1-orbit of the point $x_0 := {}^t(0, 0, 0, 1)$.

(2) We take

$$G_2 := \left\{ \begin{bmatrix} e^{\sigma s} & 0 & 0 & 0 \\ 0 & e^{-2\sigma s} & 0 & 0 \\ se^{\sigma s} & 0 & e^{\sigma s} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}. \tag{3.3.2}$$

We can check that G_2 is a subgroup of $SL(3, \mathbb{R}) \ltimes \mathbb{R}^3$ as a group acting on ${}^t(x, z, y, 1) \in \mathbb{R}^4$. Let φ be a nondegenerate equiaffine space curve with constant equiaffine curvatures obtained in (2) of Theorem 3.2.2; that is, $\varphi(s) = {}^t(e^{\sigma s}, se^{\sigma s}, -\frac{1}{18\sigma^5}e^{-2\sigma s})$. Then $\varphi(\mathbb{R})$ is the G_2-orbit of the point $x_0 := {}^t(1, -\frac{1}{18\sigma^5}, 0, 1)$.

(3) (a) If $\kappa_1 = 0$, we take

$$G_3 := \left\{ \begin{bmatrix} \cos \kappa_2^{\frac{1}{2}} s & \sin \kappa_2^{\frac{1}{2}} s & 0 & 0 \\ -\sin \kappa_2^{\frac{1}{2}} s & \cos \kappa_2^{\frac{1}{2}} s & 0 & 0 \\ 0 & 0 & 1 & -\kappa_2^{-\frac{1}{2}} s \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}. \tag{3.3.3}$$

We can check that G_3 is a subgroup of $SL(3, \mathbb{R}) \ltimes \mathbb{R}^3$ as a group acting on ${}^t(y, z, x, 1) \in \mathbb{R}^4$. Let φ be a nondegenerate equiaffine space curve with constant equiaffine curvatures obtained in (3) of Theorem 3.2.2; that is, $\varphi(s) = \kappa_2^{-1} \cdot {}^t(-\kappa_2^{\frac{1}{2}} s, \sin(\kappa_2^{\frac{1}{2}} s), \cos(\kappa_2^{\frac{1}{2}} s))$. Then $\varphi(\mathbb{R})$ is the G_3-orbit of the point $x_0 := {}^t(0, \kappa_2^{-1}, 0, 1)$.

(b) If $\kappa_1 \neq 0$, we take

$$G_3 := \left\{ \left[\begin{array}{cccc} e^{\sigma_1 s}\cos\sigma_2 s & e^{\sigma_1 s}\sin\sigma_2 s & 0 & 0 \\ -e^{\sigma_1 s}\sin\sigma_2 s & e^{\sigma_1 s}\cos\sigma_2 s & 0 & 0 \\ 0 & 0 & e^{-2\sigma_1 s} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \middle| \, s \in \mathbb{R} \right\}. \tag{3.3.4}$$

We can check that G_3 is a subgroup of $SL(3,\mathbb{R}) \ltimes \mathbb{R}^3$ as a group acting on ${}^t(y, z, x, 1) \in \mathbb{R}^4$. Let φ be a nondegenerate equiaffine space curve with constant equiaffine curvatures obtained in (3) of Theorem 3.2.2; that is, $\varphi(s) = {}^t(\frac{1}{2\sigma_1\sigma_2(9\sigma_1^2+\sigma_2^2)(\sigma_1^2+\sigma_2^2)} e^{-2\sigma_1 s}$, $e^{\sigma_1 s}\sin(\sigma_2 s), e^{\sigma_1 s}\cos(\sigma_2 s))$. Then $\varphi(\mathbb{R})$ is the G_3-orbit of the point $x_0 := {}^t(0, 1, \frac{1}{2\sigma_1\sigma_2(9\sigma_1^2+\sigma_2^2)(\sigma_1^2+\sigma_2^2)}, 1)$.

(4) (a) If $\kappa_1 = 0$, we take

$$G_4 := \left\{ \left[\begin{array}{cccc} \cosh(-\kappa_2)^{\frac{1}{2}} s & \sinh(-\kappa_2)^{\frac{1}{2}} s & 0 & 0 \\ \sinh(-\kappa_2)^{\frac{1}{2}} s & \cosh(-\kappa_2)^{\frac{1}{2}} s & 0 & 0 \\ 0 & 0 & 1 & -(-\kappa_2)^{-\frac{1}{2}} s \\ 0 & 0 & 0 & 1 \end{array} \right] \middle| \, s \in \mathbb{R} \right\}. \tag{3.3.5}$$

We can check that G_4 is a subgroup of $SL(3,\mathbb{R}) \ltimes \mathbb{R}^3$ as a group acting on ${}^t(y, z, x, 1) \in \mathbb{R}^4$. Let φ be a nondegenerate equiaffine space curve with constant equiaffine curvatures obtained in (4) of Theorem 3.2.2; that is, $\varphi(s) = -\kappa_2^{-1} \cdot {}^t(-(-\kappa_2)^{\frac{1}{2}} s, \sinh((-\kappa_2)^{\frac{1}{2}} s)$, $\cosh((-\kappa_2)^{\frac{1}{2}} s))$. Then $\varphi(\mathbb{R})$ is the G_4-orbit of the point $x_0 := {}^t(0, -\kappa_2^{-1}, 0, 1)$.

(b) If $\kappa_1 \neq 0$, we take

$$G_4 := \left\{ \begin{bmatrix} e^{-2\sigma_1 s} & 0 & 0 & 0 \\ 0 & e^{(\sigma_1+\sigma_2)s} & 0 & 0 \\ 0 & 0 & e^{(\sigma_1-\sigma_2)s} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}. \tag{3.3.6}$$

We can check that G_4 is a subgroup of $SL(3,\mathbb{R}) \ltimes \mathbb{R}^3$ as a group acting on ${}^t(x, y, z, 1) \in \mathbb{R}^4$. Let φ be a nondegenerate equiaffine space curve with constant equiaffine curvatures obtained in (4) of Theorem 3.2.2; that is, $\varphi(s) = {}^t(\frac{e^{-2\sigma_1 s}}{4\sigma_1\sigma_2(9\sigma_1^2-\sigma_2^2)(\sigma_1^2-\sigma_2^2)}, e^{(\sigma_1+\sigma_2)s}, e^{(\sigma_1-\sigma_2)s})$. Then $\varphi(\mathbb{R})$ is the G_4-orbit of the point $x_0 := {}^t(\frac{1}{4\sigma_1\sigma_2(9\sigma_1^2-\sigma_2^2)(\sigma_1^2-\sigma_2^2)}, 1, 1, 1)$.

3.4 Equiaffine homogeneous surfaces on which equiaffine space curves with constant curvatures lie

In this section, we study the relations between the equiaffine homogeneous curves and equiaffine homogeneous surfaces. An equiaffine surface $f : M \to \mathbb{R}^3$ is called locally homogeneous if for all points p and q of M, there exists a neighborhood U_p of p in M, and an equiaffine transformation A of $\mathbb{R}^3$, i.e. $A \in SL(3,\mathbb{R}) \ltimes \mathbb{R}^3$, such that $A(f(p)) = f(q)$ and $A(f(U_p)) \subset f(M)$. If $U_p = M$ for all p, then M is called homogeneous. It was H.Guggenheimer who first attempted to classify these surfaces in $\mathbb{R}^3$. His work was completed by K.Nomizu and T.Sasaki who obtained the following result:

Proposition 3.4.1 ([17]). *Any nondegenerate surface in $\mathbb{R}^3$ that is homogeneous under equiaffine transformations is a quadric or is affinely congruent to one of the following surfaces:*

(i) $xyz = 1$,

(ii) $(x^2 + y^2)z = 1$,

(ii) $x^2(z - y^2)^3 = 1$,

(iv) $x^2(z - y^2)^3 = -1$,

(v) $z = xy - \frac{1}{3}x^3$,

(vi) $z = xy + \log x$.

And for the degenerate case in $\mathbb{R}^3$, L.Vrancken gave the classification in 1994.

Proposition 3.4.2 ([24]). *Let $f : M \to \mathbb{R}^3$ be a degenerate equiaffine homogeneous surface in $\mathbb{R}^3$. Then, f is affine equivalent to an open part of*

(vii) *a plane*

(viii) *a cylinder on an ellipse*

(ix) *a cylinder on a hyperbola*

(x) *a cylinder on a parabola*

(xi) *the surface given by $xz - \frac{1}{2}y^2 = 0$.*

Each equiaffine space curve obtained in Theorem 3.2.2 is lying on a corresponding equiaffine homogeneous surface with the following group, respectively.

(1) We take

$$\widetilde{G}_1 := \left\{ \left. \begin{bmatrix} 1 & u & v & uv - \frac{1}{3}u^3 \\ 0 & 1 & u & v \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{bmatrix} \right| u, v \in \mathbb{R} \right\}. \tag{3.4.1}$$

We can check that $\widetilde{G}_1$ is a subgroup of $SL(3, \mathbb{R}) \ltimes \mathbb{R}^3$ of dimension two. The $\widetilde{G}_1$-orbit of the point $x_0 = {}^t(0, 0, 0, 1)$ is a nondegenerate equiaffine homogeneous surface given as (v) of Proposition 3.4.1, which is called a Cayley surface. A group G_1 in (3.3.1) is given by setting $u = s$, $v = \frac{1}{2}s^2$, from $\widetilde{G}_1$, and so it is a subgroup of $\widetilde{G}_1$. Hence, the curve of class (1) is lying on the surface (v).

(2) Unfortunately, we are not able to find certain equiaffine homogeneous surfaces for the curve of class (2).

(3) (a) If $\kappa_1 = 0$, we take

$$\widetilde{G_3} := \left\{ \begin{bmatrix} \cos u & \sin u & 0 & 0 \\ -\sin u & \cos u & 0 & 0 \\ 0 & 0 & 1 & v \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| u, v \in \mathbb{R} \right\}. \tag{3.4.2}$$

We can check that $\widetilde{G_3}$ is a subgroup of $SL(3,\mathbb{R}) \ltimes \mathbb{R}^3$ of dimension two. The $\widetilde{G_3}$-orbit of the point $x_0 = {}^t(0, \kappa_2^{-1}, 0, 1)$ is a degenerate equiaffine homogeneous surface given as (viii) of Proposition 3.4.2, which is a cylinder on a circle. A group G_3 in (3.3.3) is given by setting $u = \kappa_2^{\frac{1}{2}} s$, $v = -\kappa_2^{-\frac{1}{2}} s$, from $\widetilde{G_3}$, and so it is a subgroup of $\widetilde{G_3}$. Hence, the curve of class (3) is lying on the surface (viii).

(b) If $\kappa_1 \neq 0$, we take

$$\widetilde{G_3} := \left\{ \begin{bmatrix} u\cos v & u\sin v & 0 & 0 \\ -u\sin v & u\cos v & 0 & 0 \\ 0 & 0 & u^{-2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| u, v \in \mathbb{R} \right\}. \tag{3.4.3}$$

We can check that $\widetilde{G_3}$ is a subgroup of $SL(3,\mathbb{R}) \ltimes \mathbb{R}^3$ of dimension two. The $\widetilde{G_3}$-orbit of the point $x_0 = {}^t(0, 1, \frac{1}{2\sigma_1\sigma_2(9\sigma_1^2+\sigma_2^2)(\sigma_1^2+\sigma_2^2)}, 1)$ is a nondegenerate equiaffine homogeneous surface given as (ii) of Proposition 3.4.1. A group G_3 in (3.3.4) is given by setting $u = e^{\sigma_1 s}$, $v = \sigma_2 s$, from $\widetilde{G_3}$, and so it is a subgroup of $\widetilde{G_3}$. Hence, the curve of class (3) is lying on the surface (ii).

(4) (a) If $\kappa_1 = 0$, we take

$$\widetilde{G}_4 := \left\{ \begin{bmatrix} \cosh u & \sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & v \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| u, v \in \mathbb{R} \right\}. \tag{3.4.4}$$

We can check that $\widetilde{G}_4$ is a subgroup of $SL(3, \mathbb{R}) \ltimes \mathbb{R}^3$ of dimension two. The $\widetilde{G}_4$-orbit of the point $x_0 = {}^t(0, -\kappa_2^{-1}, 0, 1)$ is a degenerate equiaffine homogeneous surface given as (ix) of Proposition 3.4.2, which is a cylinder on a hyperbola. A group G_4 in (3.3.5) is given by setting $u = (-\kappa_2)^{\frac{1}{2}} s$, $v = -(-\kappa_2)^{-\frac{1}{2}} s$, from $\widetilde{G}_4$, and so it is a subgroup of $\widetilde{G}_4$. Hence, the curve of class (4) is lying on the surface (ix).

(b) If $\kappa_1 \neq 0$, we take

$$\widetilde{G}_4 := \left\{ \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & \frac{1}{uv} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| u, v \in \mathbb{R} \right\}. \tag{3.4.5}$$

We can check that $\widetilde{G}_4$ is a subgroup of $SL(3, \mathbb{R}) \ltimes \mathbb{R}^3$ of dimension two. The $\widetilde{G}_4$-orbit of the point $x_0 = {}^t(\frac{1}{4\sigma_1\sigma_2(9\sigma_1^2 - \sigma_2^2)(\sigma_1^2 - \sigma_2^2)}, 1, 1, 1)$ is a nondegenerate equiaffine homogeneous surface given as (i) of Proposition 3.4.1. A group G_4 in (3.3.6) is given by setting $u = e^{-2\sigma_1 s}$, $v = e^{(\sigma_1 + \sigma_2)s}$, from $\widetilde{G}_4$, and so it is a subgroup of $\widetilde{G}_4$. Hence, the curve of class (4) is lying on the surface (i).

Theorem 3.4.3. *A nondegenerate equiaffine space curve with constant equiaffine curvatures is an equiaffine homogeneous curve, and can be written as follows:*

$$(1)\quad G_1\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\subset \widetilde{G}_1\begin{bmatrix}0\\0\\0\\1\end{bmatrix}: \{z = xy - \tfrac{1}{3}x^3\} \subset \mathbb{R}^3,$$

$$(2)\quad G_2\begin{bmatrix}1\\\alpha\\0\\1\end{bmatrix},$$

$$(3)\quad \begin{cases} G_3\begin{bmatrix}1\\\beta\\0\\1\end{bmatrix}\subset \widetilde{G}_3\begin{bmatrix}1\\\beta\\0\\1\end{bmatrix}: \textit{a cylinder on a circle in } \mathbb{R}^3,\ if\ \kappa_1 = 0\\ G_3\begin{bmatrix}0\\1\\\gamma\\1\end{bmatrix}\subset \widetilde{G}_3\begin{bmatrix}0\\1\\\gamma\\1\end{bmatrix}: \{z(x^2+y^2) = \gamma\} \subset \mathbb{R}^3,\ if\ \kappa_1 \neq 0\end{cases}$$

$$(4)\quad \begin{cases} G_4\begin{bmatrix}0\\\eta\\0\\1\end{bmatrix}\subset \widetilde{G}_4\begin{bmatrix}0\\\eta\\0\\1\end{bmatrix}: \textit{ a cylinder on a hyperbola in } \mathbb{R}^3,\ if\ \kappa_1 = 0\\ G_4\begin{bmatrix}\delta\\1\\1\\1\end{bmatrix}\subset \widetilde{G}_4\begin{bmatrix}\delta\\1\\1\\1\end{bmatrix}: \{xyz = \delta\} \subset \mathbb{R}^3,\ if\ \kappa_1 \neq 0\end{cases}$$

where $\alpha = -\frac{1}{18\sigma^5}$, $\beta = \frac{1}{\kappa_2}$, $\gamma = \frac{1}{2\sigma_1\sigma_2(9\sigma_1^2+\sigma_2^2)(\sigma_1^2+\sigma_2^2)}$, $\eta = -\frac{1}{\kappa_2}$, $\delta = \frac{1}{4\sigma_1\sigma_2(9\sigma_1^2-\sigma_2^2)(\sigma_1^2-\sigma_2^2)}$, $G_i(i =$

$1, 2, 3, 4)$ *are one-parameter subgroups of* $SL(3,\mathbb{R}) \ltimes \mathbb{R}^3$ *given by* (3.3.1) ~(3.3.6)*,* $\widetilde{G}_i$ $(i = 1, 3, 4)$ *are two-parameter subgroups of* $SL(3,\mathbb{R}) \ltimes \mathbb{R}^3$ *given by* (3.4.1),(3.4.2),(3.4.3),(3.4.4) *and* (3.4.5)*,* $\sigma, \sigma_1, \sigma_2$ *are constants with respect to the equiaffine curvatures given by* (3.2.4)~ (3.2.6)*, respectively.*

Chapter 4

Centroaffine space curves

4.1 Basic notions of centroaffine space curves

Let $\varphi : I \to \mathbb{R}^3$ be a centroaffine space curve in $\mathbb{R}^3$ parameterized by centroaffine arc-length parameter s and of signature ε. Let κ_1 and κ_2 be the centroaffine first and second curvature of φ, respectively. By definition, they are given by

$$\kappa_1(s) := -\frac{[\varphi(s), \varphi''(s), \varphi'''(s)]}{[\varphi(s), \varphi'(s), \varphi''(s)]},$$

and

$$\kappa_2(s) := \frac{[\varphi(s), \varphi'(s), \varphi'''(s)]}{[\varphi(s), \varphi'(s), \varphi''(s)]}.$$

From the definition of $\kappa_1(s)$ and $\kappa_2(s)$, we get

$$\varphi'''(s) = \varepsilon(s)\varphi(s) + \kappa_1(s)\varphi'(s) + \kappa_2(s)\varphi''(s),$$

that is

$$\frac{d}{ds}\begin{bmatrix} {}^t\varphi(s) \\ {}^t\varphi'(s) \\ {}^t\varphi''(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon(s) & \kappa_1(s) & \kappa_2(s) \end{bmatrix}\begin{bmatrix} {}^t\varphi(s) \\ {}^t\varphi'(s) \\ {}^t\varphi''(s) \end{bmatrix} =: \Omega\begin{bmatrix} {}^t\varphi(s) \\ {}^t\varphi'(s) \\ {}^t\varphi''(s) \end{bmatrix}. \tag{4.1.1}$$

For a nondegenerate centroaffine space curve φ, the *osculating plane* at $\varphi(s)$ is defined as the one spanned by $\{\varphi'(s), \varphi''(s)\}$, the *normal plane* by $\{\varphi''(s), \varphi'''(s)\}$ and the *rectifying plane* by $\{\varphi'''(s), \varphi'(s)\}$, respectively. φ is called a *rectifying curve* if the position vector lies on the rectifying plane for each point. Since $\pm\varphi = \varphi''' - \kappa_1\varphi' - \kappa_2\varphi''$ holds, we have the following.

Proposition 4.1.1 ([11])**.** *The centroaffine first curvature of a centroaffine curve vanishes if and only if the position vector field of this curve lies in its normal plane. The centroaffine second curvature of a centroaffine curve vanishes if and only if it is a rectifying curve.*

Consider the relations between the curvatures of a curve in the centroaffine geometry and those in the Euclidean geometry. Denoting the Euclidean arc-length parameter by $\tilde{s}$, the Euclidean curvature by $\tilde{\kappa}_1$ and the Euclidean torsion by $\tilde{\kappa}_2$, we will get the Frenet formulas of a curve:

$$\begin{cases} \dfrac{d\alpha(\tilde{s})}{d\tilde{s}} = \tilde{\kappa}_1(\tilde{s})\beta(\tilde{s}), \\ \dfrac{d\beta(\tilde{s})}{d\tilde{s}} = -\tilde{\kappa}_1(\tilde{s})\alpha(\tilde{s}) + \tilde{\kappa}_2(\tilde{s})\gamma(\tilde{s}), \\ \dfrac{d\gamma(\tilde{s})}{d\tilde{s}} = -\tilde{\kappa}_2(\tilde{s})\beta(\tilde{s}), \end{cases} \tag{4.1.2}$$

where $\alpha(\tilde{s}), \beta(\tilde{s}), \gamma(\tilde{s})$ are the Euclidean unit tangent vector, normal vector and binormal vector of the space curve, respectively. For the functions a, b, c defined by

$$\varphi(\tilde{s}) = a(\tilde{s})\alpha(\tilde{s}) + b(\tilde{s})\beta(\tilde{s}) + c(\tilde{s})\gamma(\tilde{s}),$$

we will get

$$[\varphi(\tilde{s}), \frac{d\varphi(\tilde{s})}{d\tilde{s}}, \frac{d\varphi^2(\tilde{s})}{d^2\tilde{s}}] = c(\tilde{s})\tilde{\kappa}_1(s),$$

and

$$[\frac{d\varphi(\tilde{s})}{d\tilde{s}}, \frac{d\varphi^2(\tilde{s})}{d^2\tilde{s}}, \frac{d\varphi^3(\tilde{s})}{d^3\tilde{s}}] = \tilde{\kappa}_1(\tilde{s})^2\tilde{\kappa}_2(\tilde{s}).$$

Accordingly, a curve with nonvanishing c and $\tilde{\kappa}_1$ is a centroaffine curve, and a centroaffine curve with nonvanishing $\tilde{\kappa}_2$ is nondegenerate. As $\varepsilon(\tilde{s}) = \frac{\tilde{\kappa}_1(\tilde{s})\tilde{\kappa}_2(\tilde{s})}{c(\tilde{s})}$, we have the following.

Proposition 4.1.2 ([11]). *The Euclidean arc-length parameter is precisely the centroaffine arc-length parameter if and only if*

$$c = \pm\tilde{\kappa}_1\tilde{\kappa}_2.$$

4.2 Centroaffine space curve with constant centroaffine curvatures

We assume the signature $\varepsilon = -1$ for a while. From the fundamental theorem for centroaffine space curves, the centroaffine space curves are uniquely determined by the centroaffine curvatures up to a centroaffine transformation in $\mathbb{R}^3$. In the following, we obtain the curves with constant curvatures by solving the ODE: $\varphi'''(s) = -\varphi(s) + \kappa_1\varphi'(s) + \kappa_2\varphi''(s)$ with constant coefficients. First, using Shengjin's formula we get the eigenvalues of the coefficient matrix Ω in (4.1.1) which is one variable cubic equation:

$$\left|\lambda I - \Omega\right| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 1 & -\kappa_1 & \lambda - \kappa_2 \end{vmatrix} = \lambda^3 - \kappa_2\lambda^2 - \kappa_1\lambda + 1 = 0, \qquad (4.2.1)$$

whose discriminants of multiple root are:

$$\begin{cases} A := \kappa_2^2 + 3\kappa_1, \\ B := \kappa_1\kappa_2 - 9, \\ C := \kappa_1^2 + 3\kappa_2, \end{cases} \qquad (4.2.2)$$

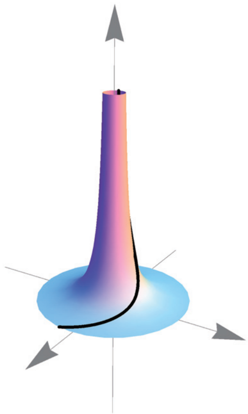

Figure 4.1: The curve of type (3) ($\kappa_1 = \kappa_2 = 0$) and the surface $(x^2 + y^2)z = 1$.

and principal discriminant is:

$$\Delta := B^2 - 4AC = 3(-4\kappa_1^3 - 4\kappa_2^3 - 18\kappa_1\kappa_2 - \kappa_1^2\kappa_2^2 + 27). \tag{4.2.3}$$

Then using the eigenvalues to get the fundamental system of solution of this ODE.

Proposition 4.2.1 ([11])**.** *The nondegenerate centroaffine space curve with vanishing curvatures and signature −1 is centroaffinely equivalent to the following curve:*

$$\varphi(s) = {}^t(\sin(\sqrt{3}s/2)e^{s/2}, \cos(\sqrt{3}s/2)e^{s/2}, e^{-s}).$$

Remark 4.2.2. It is belonging to the type (3) in Theorem 4.2.3 below and it is easy to see this curve lies on the surface $(x^2 + y^2)z = 1$, which is known as a proper affine sphere with center at the origin.

Theorem 4.2.3 ([11])**.** *Any nondegenerate centroaffine space curve with constant centroaffine curvatures κ_1, κ_2 and signature −1 is centroaffinely equivalent to one of the following curves:*

(1) $\varphi(s) = {}^t(se^{-s}, e^{-s}, s^2e^{-s} + e^{-s})$, *if* $A^2 + B^2 = 0$,

(2) $\varphi(s) = {}^t(e^{\varsigma_1 s}, e^{\varsigma_2 s}, se^{\varsigma_1 s})$, *if* $A^2 + B^2 \neq 0$ *and* $\Delta = 0$,

(3) $\varphi(s) = e^{\kappa_2 s/3} \cdot {}^t(e^{\rho_1 s}\sin(\rho_2 s), e^{\rho_1 s}\cos(\rho_2 s), e^{-2\rho_1 s})$, *if* $A^2 + B^2 \neq 0$ *and* $\Delta > 0$,

(4) $\varphi(s) = e^{\kappa_2 s/3} \cdot {}^t(e^{-2\sigma_1 s}, e^{(\sigma_1+\sigma_2)s}, e^{(\sigma_1-\sigma_2)s})$, *if* $A^2 + B^2 \neq 0$ *and* $\Delta < 0$,

where A, B, Δ *and* ς_i, ρ_i, σ_i $(i = 1, 2)$ *are constants defined in* (4.2.2)~(4.2.6) *by using centroaffine curvatures.*

Proof. Using Shengjin's formulas, we separate out four cases:

(1) When $A = B = 0$, that is $\kappa_1 = \kappa_2 = -3$, one variable cubic equation (4.2.1) has a triple root:

$$\lambda_1 = \lambda_2 = \lambda_3 = -1.$$

Then e^{-s}, se^{-s} and s^2e^{-s} are the fundamental solutions. The space curve is centroaffinely equivalent to the following curve:

$$\varphi(s) = c_1e^{-s} + c_2se^{-s} + c_3s^2e^{-s},$$

where $c_i (i = 1, 2, 3)$ are the constant vectors in $\mathbb{R}^3$. If we set $c_1 = {}^t(0, 1, 1)$, $c_2 = {}^t(1, 0, 0)$, $c_3 = {}^t(0, 0, 1)$, we get

$$\varphi(s) = {}^t(se^{-s}, e^{-s}, s^2e^{-s} + e^{-s}).$$

(2) When $A^2 + B^2 \neq 0$, $\Delta = 0$, one variable cubic equation (4.2.1) has a double root:

$$\lambda_1 = \lambda_2 = \varsigma_1, \lambda_3 = \varsigma_2,$$

where

$$\begin{cases} \varsigma_1 = -\frac{B}{2A}, \\ \varsigma_2 = \kappa_2 + \frac{B}{A}. \end{cases} \tag{4.2.4}$$

Then $e^{\varsigma_1 s}$, $se^{\varsigma_1 s}$ and $e^{\varsigma_2 s}$ are the fundamental solutions. The space curve is centroaffinely equivalent to the following curve:

$$\varphi(s) = c_1 e^{\varsigma_1 s} + c_2 s e^{\varsigma_1 s} + c_3 e^{\varsigma_2 s},$$

where $c_i (i = 1, 2, 3)$ are the constant vectors in $\mathbb{R}^3$. If we set $c_1 = {}^t(1, 0, 0)$, $c_2 = {}^t(0, 0, 1)$, $c_3 = {}^t(0, 1, 0)$, we get

$$\varphi(s) = {}^t(e^{\varsigma_1 s}, e^{\varsigma_2 s}, s e^{\varsigma_1 s}).$$

(3) When $A^2 + B^2 \neq 0$, $\Delta > 0$, one variable cubic equation (4.2.1) has a pair of conjugate imaginary roots:

$$\lambda_1 = \frac{\kappa_2}{3} - 2\rho_1, \lambda_2 = \frac{\kappa_2}{3} + \rho_1 + i\rho_2, \lambda_3 = \frac{\kappa_2}{3} + \rho_1 - i\rho_2,$$

where

$$\begin{cases} \rho_1 = \dfrac{1}{6}[(-\kappa_2 A + \dfrac{3}{2}(-B + \Delta^{\frac{1}{2}}))^{\frac{1}{3}} + (-\kappa_2 A + \dfrac{3}{2}(-B - \Delta^{\frac{1}{2}}))^{\frac{1}{3}}], \\ \rho_2 = \dfrac{1}{6}[(-\kappa_2 A + \dfrac{3}{2}(-B + \Delta^{\frac{1}{2}}))^{\frac{1}{3}} - (-\kappa_2 A + \dfrac{3}{2}(-B - \Delta^{\frac{1}{2}}))^{\frac{1}{3}}]. \end{cases} \tag{4.2.5}$$

Then $e^{(\kappa_2/3 - 2\rho_1)s}$, $e^{(\kappa_2/3+\rho_1)s}\cos(\rho_2 s)$ and $e^{(\kappa_2/3+\rho_1)s}\sin(\rho_2 s)$ are the fundamental solutions. The space curve is centroaffinely equivalent to the following curve:

$$\varphi(s) = c_1 e^{(\kappa_2/3-2\rho_1)s} + c_2 e^{(\kappa_2/3+\rho_1)s}\cos(\rho_2 s) + c_3 e^{(\kappa_2/3+\rho_1)s}\sin(\rho_2 s),$$

where $c_i (i = 1, 2, 3)$ are the constant vectors in $\mathbb{R}^3$. If we set $c_1 = {}^t(0, 0, 1)$, $c_2 = {}^t(0, 1, 0)$, $c_3 = {}^t(1, 0, 0)$, we get

$$\varphi(s) = e^{\kappa_2 s/3} \cdot {}^t(e^{\rho_1 s}\sin(\rho_2 s), e^{\rho_1 s}\cos(\rho_2 s), e^{-2\rho_1 s}).$$

(4) When $A^2 + B^2 \neq 0$, $\Delta < 0$, one variable cubic equation (4.2.1) has three different real

roots:

$$\lambda_1 = \frac{\kappa_2}{3} - 2\sigma_1, \lambda_2 = \frac{\kappa_2}{3} + \sigma_1 + \sigma_2, \lambda_3 = \frac{\kappa_2}{3} + \sigma_1 - \sigma_2,$$

where

$$\begin{cases} \sigma_1 = \frac{1}{3}A^{\frac{1}{2}} \cos(\arccos(\frac{-\kappa_2 A - 3B}{2A^{3/2}})/3), \\ \sigma_2 = \frac{1}{3}(3A)^{\frac{1}{2}} \sin(\arccos(\frac{-\kappa_2 A - 3B}{2A^{3/2}})/3), \ A > 0, \ \frac{-\kappa_2 A - 3B}{2A^{3/2}} \in (-1, 1). \end{cases} \tag{4.2.6}$$

Then $e^{(\kappa_2/3-2\sigma_1)s}$, $e^{(\kappa_2/3+\sigma_1+\sigma_2)s}$ and $e^{(\kappa_2/3+\sigma_1-\sigma_2)s}$ are the fundamental solutions. The space curve is centroaffinely equivalent to the following curve:

$$\varphi(s) = c_1 e^{(\kappa_2/3-2\sigma_1)s} + c_2 e^{(\kappa_2/3+\sigma_1+\sigma_2)s} + c_3 e^{(\kappa_2/3+\sigma_1-\sigma_2)s},$$

where $c_i (i = 1, 2, 3)$ are the constant vectors in $\mathbb{R}^3$. If we set $c_1 = {}^t(1, 0, 0)$, $c_2 = {}^t(0, 1, 0)$, $c_3 = {}^t(0, 0, 1)$, we get

$$\varphi(s) = e^{\kappa_2 s/3} \cdot {}^t(e^{-2\sigma_1 s}, e^{(\sigma_1+\sigma_2)s}, e^{(\sigma_1-\sigma_2)s}). \qquad \square$$

4.3 Groups of centroaffine space curves with constant curvatures

As it is known, the centroaffine homogeneous curves are precisely the orbits of certain one-parameter subgroups $G(s)$ of $GL(3, \mathbb{R})$, that is,

$$\varphi(s) = G(s)\varphi_0,$$

where s is the centroaffine arc-length parameter of φ.

Using the properties of subgroup, we have $G'(s) = G(s)G'(0)$, $G''(s) = G(s)G''(0)$ and

$G'''(s) = G(s)G'''(0)$. Then we can get the following centroaffine first and second curvatures

$$\begin{aligned}
\kappa_1(s) &= -\frac{[\varphi(s), \varphi''(s), \varphi'''(s)]}{[\varphi(s), \varphi'(s), \varphi''(s)]} = -\frac{[G(s)\varphi_0, G(s)G''(0)\varphi_0, G(s)G'''(0)\varphi_0]}{[G(s)\varphi_0, G(s)G'(0)\varphi_0, G(s)G''(0)\varphi_0]} \\
&= -\frac{[\varphi_0, G''(0)\varphi_0, G'''(0)\varphi_0]}{[\varphi_0, G'(0)\varphi_0, G''(0)\varphi_0]}, \\
\kappa_2(s) &= \frac{[\varphi(s), \varphi'(s), \varphi'''(s)]}{[\varphi(s), \varphi'(s), \varphi''(s)]} = \frac{[G(s)\varphi_0, G(s)G'(0)\varphi_0, G(s)G'''(0)\varphi_0]}{[G(s)\varphi_0, G(s)G'(0)\varphi_0, G(s)G''(0)\varphi_0]} \\
&= \frac{[\varphi_0, G'(0)\varphi_0, G'''(0)\varphi_0]}{[\varphi_0, G'(0)\varphi_0, G''(0)\varphi_0]}.
\end{aligned}$$

Thus we have

Proposition 4.3.1. *The centroaffine curvatures of a nondegenerate centroaffine homogeneous curve are constant.*

Conversely, we give the one-parameter subgroup of $GL(3, \mathbb{R})$ for each class of the nondegenerate centroaffine space curves with constant centroaffine curvatures in $\mathbb{R}^3$ as follows

(1) We set

$$G_1 := \left\{ \left[\begin{array}{ccc} e^{-s} & se^{-s} & 0 \\ 0 & e^{-s} & 0 \\ 2se^{-s} & s^2e^{-s} & e^{-s} \end{array} \right] \middle| s \in \mathbb{R} \right\}, \tag{4.3.1}$$

and show that G_1 is a subgroup of $GL(3, \mathbb{R})$. Let φ be a nondegenerate centroaffine space curve with constant centroaffine curvatures obtained in (1) of Theorem 4.2.3. Then $\varphi(\mathbb{R})$ is the G_1-orbit of the point ${}^t(0, 1, 1)$.

(2) We set

$$G_2 := \left\{ \left[\begin{array}{ccc} e^{\varsigma_1 s} & 0 & 0 \\ 0 & e^{\varsigma_2 s} & 0 \\ se^{\varsigma_1 s} & 0 & e^{\varsigma_1 s} \end{array} \right] \middle| s \in \mathbb{R} \right\}, \tag{4.3.2}$$

and show that G_2 is a subgroup of $GL(3,\mathbb{R})$. Let φ be a nondegenerate centroaffine space curve with constant centroaffine curvatures obtained in (2) of Theorem 4.2.3. Then $\varphi(\mathbb{R})$ is the G_2-orbit of the point ${}^t(1,1,0)$.

(3) We set

$$G_3 := \left\{ e^{\kappa_2 s/3} \begin{bmatrix} e^{\rho_1 s}\cos(\rho_2 s) & e^{\rho_1 s}\sin(\rho_2 s) & 0 \\ -e^{\rho_1 s}\sin(\rho_2 s) & e^{\rho_1 s}\cos(\rho_2 s) & 0 \\ 0 & 0 & e^{-2\rho_1 s} \end{bmatrix} \middle| \, s \in \mathbb{R} \right\}, \tag{4.3.3}$$

and show that G_3 is a subgroup of $GL(3,\mathbb{R})$. Let φ be a nondegenerate centroaffine space curve with constant centroaffine curvatures obtained in (3) of Theorem 4.2.3. Then $\varphi(\mathbb{R})$ is the G_3-orbit of the point ${}^t(0,1,1)$.

(4) We set

$$G_4 := \left\{ e^{\kappa_2 s/3} \begin{bmatrix} e^{-2\sigma_1 s} & 0 & 0 \\ 0 & e^{(\sigma_1+\sigma_2)s} & 0 \\ 0 & 0 & e^{(\sigma_1-\sigma_2)s} \end{bmatrix} \middle| \, s \in \mathbb{R} \right\}, \tag{4.3.4}$$

and show that G_4 is a subgroup of $GL(3,\mathbb{R})$. Let φ be a nondegenerate centroaffine space curve with constant centroaffine curvatures obtained in (4) of Theorem 4.2.3. Then $\varphi(\mathbb{R})$ is the G_4-orbit of the point ${}^t(1,1,1)$.

4.4 Centroaffine homogeneous surfaces on which centroaffine space curves with constant curvatures lie

In this section, we want to consider the relations of centroaffine space curves with constant centroaffine curvatures and homogeneous surfaces. An centroaffine surface $f : M \to \mathbb{R}^3$ is called locally centroaffine homogeneous if for all points p and q of M, there exist a neighborhood U_p of p in M, and an centroaffine transformation A of $\mathbb{R}^3$, i.e. $A \in GL(3,\mathbb{R})$, such that

$A(f(p)) = f(q)$ and $A(f(U_p)) \subset f(M)$. If $U_p = M$ for all p, then f is called centroaffinely homogeneous. First, we review the following fact,

Theorem 4.4.1 ([15])**.** *Let $f : M \to \mathbb{R}^3$ be a nondegenerate centroaffinely homogeneous surface. Then f is centroaffinely equivalent to one of the following surfaces in $\mathbb{R}^3$:*

(i) $x^2 + y^2 + z^2 = 1$;

(ii) $z = -x(\alpha \log x + \beta \log y), \alpha + \beta \neq 0$;

(iii) $\exp(-\alpha \arctan \frac{x}{y})(x^2 + y^2)^\beta z^\gamma = 1, \alpha^2 + \beta^2 \neq 0, 2\beta + \gamma \neq 0$;

(iv) $x^\alpha y^\beta z^\gamma = 1, \alpha \neq 0, \alpha + \beta + \gamma \neq 0$;

(v) $yz = \pm x^2 + y^\alpha, \alpha \neq 1, 2$;

where α, β, γ are constants.

The centroaffine space curves are lying on the nondegenerate centroaffine homogeneous surfaces with the following groups, respectively.

(1) Unfortunately, we are not able to find a nondegenerate centroaffine homogeneous whose group is the overgroup of the group for the curve (i) in Theorem 4.2.3.

(2) (a) If $\kappa_2 = 0$, we set

$$\widetilde{G_2} := \left\{ \left[\begin{array}{ccc} e^u & 0 & 0 \\ 0 & e^v & 0 \\ -2^{\frac{1}{3}} e^u(u+v) & 0 & e^u \end{array} \right] \middle| u, v \in \mathbb{R} \right\}, \tag{4.4.1}$$

and show that $\widetilde{G_2}$ is a subgroup of $GL(3, \mathbb{R})$ of dimension two. The $\widetilde{G_2}$-orbit of the point $x_0 = {}^t(1, 1, 0)$ is a nondegenerate centroaffine homogeneous surface given as (ii) of Theorem 4.4.1 with

$$\alpha = \beta = 2^{\frac{1}{3}}. \tag{4.4.2}$$

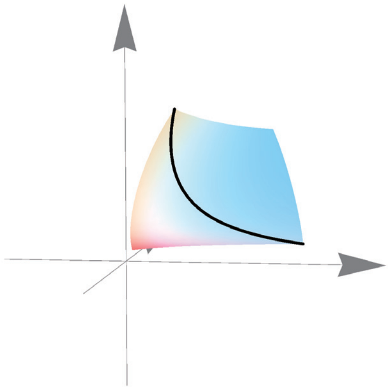

Figure 4.2: The curve of type (2) ($\kappa_1 = \kappa_2 = 1$) and the surface $z = x(2\log x + \log y)$.

A group G_2 in (4.3.2) is given by setting $u = \varsigma_1 s$, $v = \varsigma_2 s$, from $\widetilde{G}_2$, and so it is a subgroup of $\widetilde{G}_2$. Hence, the curve of class (2) is lying on the surface (ii).

(b) If $\kappa_2 \neq 0$, we set

$$\widetilde{G}_2 := \left\{ \left[\begin{bmatrix} e^u & 0 & 0 \\ 0 & e^v & 0 \\ \frac{1}{\kappa_2} e^u (2u + v) & 0 & e^u \end{bmatrix} \right] u, v \in \mathbb{R} \right\}, \tag{4.4.3}$$

and show that $\widetilde{G}_2$ is a subgroup of $GL(3, \mathbb{R})$ of dimension two. The $\widetilde{G}_2$-orbit of the point $x_0 = {}^t(1, 1, 0)$ is a nondegenerate centroaffine homogeneous surface given as (ii) of Theorem 4.4.1 with

$$\alpha = -2/\kappa_2, \ \beta = -1/\kappa_2. \tag{4.4.4}$$

A group G_2 in (4.3.2) is given by setting $u = \varsigma_1 s$, $v = \varsigma_2 s$, from $\widetilde{G}_2$, and so it is a subgroup of $\widetilde{G}_2$. Hence, the curve of class (2) is lying on the surface (ii).

(3) We set

$$\widetilde{G}_3 := \left\{ \left[\begin{array}{ccc} e^u \cos(v) & e^u \sin(v) & 0 \\ -e^u \sin(v) & e^u \cos(v) & 0 \\ 0 & 0 & e^{-2u+\frac{\kappa_2}{\rho_2}v} \end{array} \right] \middle| u, v \in \mathbb{R} \right\}, \qquad (4.4.5)$$

and show that $\widetilde{G}_3$ is a subgroup of $GL(3, \mathbb{R})$ of dimension two. The $\widetilde{G}_3$-orbit of the point $x_0 = {}^t(0, 1, 1)$ is a nondegenerate centroaffine homogeneous surface given as (iii) of Theorem 4.4.1 with

$$\alpha = \kappa_2/\rho_2, \ \beta = \gamma = 1. \qquad (4.4.6)$$

A group G_3 in (4.3.3) is given by setting $u = (\rho_1 + \frac{\kappa_2}{3})s$, $v = \rho_2 s$, from $\widetilde{G}_3$, and so it is a subgroup of $\widetilde{G}_3$. Hence, the curve of class (3) is lying on the surface (iii).

(4) (a) If $\kappa_2 = 0$, we set

$$\widetilde{G}_4 := \left\{ \left[\begin{array}{ccc} e^u & 0 & 0 \\ 0 & e^v & 0 \\ 0 & 0 & e^{-u-v} \end{array} \right] \middle| u, v \in \mathbb{R} \right\}, \qquad (4.4.7)$$

and show that $\widetilde{G}_4$ is a subgroup of $GL(3, \mathbb{R})$ of dimension two. The $\widetilde{G}_4$-orbit of the point $x_0 = {}^t(1, 1, 1)$ is a nondegenerate centroaffine homogeneous surface given as (iv) of Theorem 4.4.1 with

$$\alpha = \beta = \gamma = 1. \qquad (4.4.8)$$

A group G_4 in (4.3.4) is given by setting $u = -2\sigma_1 s$, $v = (\sigma_1 + \sigma_2)s$, from $\widetilde{G}_4$, and so it is a subgroup of $\widetilde{G}_4$. Hence, the curve of class (4) is lying on the surface (iv).

(b) If $\kappa_2 \neq 0$, we set

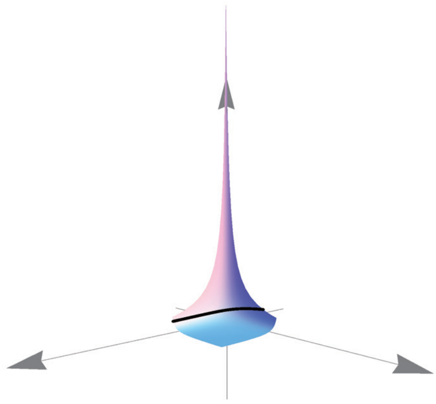

Figure 4.3: The curve of type (4) ($\kappa_1 = 3, \kappa_2 = 0$) and the surface $xyz = 1$.

$$\widetilde{G}_4 := \left\{ \begin{bmatrix} e^{(\sigma_1-\sigma_2-\frac{2A}{3\kappa_2})u} & 0 & 0 \\ 0 & e^{(\sigma_1-\sigma_2-\frac{2A}{3\kappa_2})v} & 0 \\ 0 & 0 & e^{(2\sigma_1+\frac{2A}{3\kappa_2})u-(\sigma_1+\sigma_2-\frac{2A}{3\kappa_2})v} \end{bmatrix} \middle| u, v \in \mathbb{R} \right\}. \tag{4.4.9}$$

and show that $\widetilde{G}_4$ is a subgroup of $GL(3, \mathbb{R})$ of dimension two. The $\widetilde{G}_4$-orbit of the point $x_0 = {}^t(1, 1, 1)$ is a nondegenerate centroaffine homogeneous surface given as (iv) of Theorem 4.4.1 with

$$\alpha = -2\sigma_1 - \frac{2A}{3\kappa_2},\ \beta = \sigma_1 + \sigma_2 - \frac{2A}{3\kappa_2},\ \gamma = \sigma_1 - \sigma_2 - \frac{2A}{3\kappa_2}. \tag{4.4.10}$$

A group G_4 in (4.3.4) is given by setting $u = \frac{\kappa_2(\kappa_2-6\sigma_1)}{3\kappa_2(\sigma_1-\sigma_2)-2A}s$, $v = \frac{\kappa_2(\kappa_2+3\sigma_1+3\sigma_2)}{3\kappa_2(\sigma_1-\sigma_2)-2A}s$, from $\widetilde{G}_4$, and so it is a subgroup of $\widetilde{G}_4$. Hence, the curve of class (4) is lying on the surface (iv). Then we have the following theorem.

Theorem 4.4.2 ([11]). *Any nondegenerate centroaffine space curve with constant centroaffine curvatures and signature -1 is a centroaffine homogeneous curve, and can be written as follows:*

(1) $G_1\begin{bmatrix}0\\1\\1\end{bmatrix}\subset\mathbb{R}^3,$

(2) $G_2\begin{bmatrix}1\\1\\0\end{bmatrix}\subset\widetilde{G}_2\begin{bmatrix}1\\1\\0\end{bmatrix}=\{z=-x(\alpha\log x+\beta\log y)\}\subset\mathbb{R}^3,$

(3) $G_3\begin{bmatrix}0\\1\\1\end{bmatrix}\subset\widetilde{G}_3\begin{bmatrix}0\\1\\1\end{bmatrix}=\{\exp(-\alpha\arctan\frac{x}{y})(x^2+y^2)^\beta z^\gamma=1\}\subset\mathbb{R}^3,$

(4) $G_4\begin{bmatrix}1\\1\\1\end{bmatrix}\subset\widetilde{G}_4\begin{bmatrix}1\\1\\1\end{bmatrix}=\{x^\alpha y^\beta z^\gamma=1\}\subset\mathbb{R}^3,$

where $G_i (i = 1, 2, 3, 4)$ *are one-parameter subgroups of* $GL(3,\mathbb{R})$ *given by* (4.3.1) ~(4.3.4), $\widetilde{G}_i$ $(i = 2, 3, 4)$ *are two-parameter subgroups of* $GL(3,\mathbb{R})$ *given by* (4.4),(4.4.3),(4.4.5),(4.4.7) *and* (4.4.9), α, β, γ *are constants with respect to the centroaffine curvatures given by* (4.4.2), (4.4.4), (4.4.6), (4.4.8),(4.4.10), *respectively.*

It is well known that centroaffine minimal surfaces were originally defined for centroaffine surfaces by C.P.Wang [25] as extremals for the area integral of the centroaffine metric. In particular, proper affine spheres centered at the origin are important examples of such surfaces. Then we get

Theorem 4.4.3 ([11]). *Any nondegenerate centroaffine homogenerous space curve with vanishing centroaffine second curvature lies on a centroaffine minimal surface which is determined by the centroaffine first curvature of the curve.*

Proof. A nondegenerate centroaffine homogeneous curve $\varphi(s)$ with vanishing centroaffine second curvature lies on a flat proper affine sphere except $\Delta = 0$, that is, $\kappa_1 = 3 \cdot 2^{-2/3}$. From (4.4.2), through a straightforward calculation it is easy to see that $z = -2^{\frac{1}{3}} x(\log x + \log y)$ is a nondegenerate centroaffine minimal surface. □

For nondegenerate centroaffine space curves with constant centroaffine curvatures and signature 1, we can get similar results as follows.

Theorem 4.2.3'([11]) *Any nondegenerate centroaffine space curve with constant centroaffine curvatures* κ_1, κ_2 *and signature* 1 *is centroaffinely equivalent to one of the following curves:*

(1) $\varphi(s) = {}^t(se^s, e^s, s^2e^s + e^s)$, *if* $A^2 + B^2 = 0$,

(2) $\varphi(s) = {}^t(e^{\varsigma_1 s}, e^{\varsigma_2 s}, se^{\varsigma_1 s})$, *if* $A^2 + B^2 \neq 0$ *and* $\Delta = 0$,

(3) $\varphi(s) = e^{\kappa_2 s/3} \cdot {}^t(e^{\rho_1 s}\sin(\rho_2 s), e^{\rho_1 s}\cos(\rho_2 s), e^{-2\rho_1 s})$, *if* $A^2 + B^2 \neq 0$ *and* $\Delta > 0$,

(4) $\varphi(s) = e^{\kappa_2 s/3} \cdot {}^t(e^{-2\sigma_1 s}, e^{(\sigma_1+\sigma_2)s}, e^{(\sigma_1-\sigma_2)s})$, *if* $A^2 + B^2 \neq 0$ *and* $\Delta < 0$,

where $A := \kappa_2^2 + 3\kappa_1$, $B := \kappa_1\kappa_2 + 9$, $C := \kappa_1^2 - 3\kappa_2$,

$\Delta := B^2 - 4AC = -3(4\kappa_1^3 - 4\kappa_2^3 - 18\kappa_1\kappa_2 + \kappa_1^2\kappa_2^2 - 27)$.

$\varsigma_1 = -\frac{\kappa_1\kappa_2+9}{2(\kappa_2^2+3\kappa_1)}$, $\varsigma_2 = \kappa_2 + \frac{\kappa_1\kappa_2+9}{\kappa_2^2+3\kappa_1}$,

$\rho_1 = \frac{1}{6}[(-\kappa_2 A + \frac{3}{2}(-B + \Delta^{\frac{1}{2}}))^{\frac{1}{3}} + (-\kappa_2 A + \frac{3}{2}(-B - \Delta^{\frac{1}{2}}))^{\frac{1}{3}}]$,

$\rho_2 = \frac{1}{6}[(-\kappa_2 A + \frac{3}{2}(-B + \Delta^{\frac{1}{2}}))^{\frac{1}{3}} - (-\kappa_2 A + \frac{3}{2}(-B - \Delta^{\frac{1}{2}}))^{\frac{1}{3}}]$,

$\sigma_1 = \frac{1}{3}A^{\frac{1}{2}}\cos(\arccos(\frac{-\kappa_2 A - 3B}{2A^{3/2}})/3)$,

$\sigma_2 = \frac{1}{3}(3A)^{\frac{1}{2}}\sin(\arccos(\frac{-\kappa_2 A - 3B}{2A^{3/2}})/3)$, $A > 0$, $\frac{-\kappa_2 A - 3B}{2A^{3/2}} \in (-1, 1)$.

Theorem 4.2.4'([11]) *Any nondegenerate centroaffine space curve with constant centroaffine curvatures and signature* 1 *is a centroaffine homogeneous curve, and can be written as follows:*

(1) $G_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \subset \mathbb{R}^3$,

(2) $G_2\begin{bmatrix}1\\1\\0\end{bmatrix} \subset \widetilde{G}_2\begin{bmatrix}1\\1\\0\end{bmatrix} = \{z = -x(\alpha\log x + \beta\log y)\} \subset \mathbb{R}^3,$

(3) $G_3\begin{bmatrix}0\\1\\1\end{bmatrix} \subset \widetilde{G}_3\begin{bmatrix}0\\1\\1\end{bmatrix} = \{\exp(-\alpha\arctan\frac{x}{y})(x^2+y^2)^\beta z^\gamma = 1\} \subset \mathbb{R}^3,$

(4) $G_4\begin{bmatrix}1\\1\\1\end{bmatrix} \subset \widetilde{G}_4\begin{bmatrix}1\\1\\1\end{bmatrix} = \{x^\alpha y^\beta z^\gamma = 1\} \subset \mathbb{R}^3,$

where

$$G_1 := \left\{\begin{bmatrix} e^s & se^s & 0\\ 0 & e^s & 0\\ 2se^s & s^2e^s & e^s\end{bmatrix}\middle|\, s \in \mathbb{R}\right\},$$

$G_i(i = 2,3,4)$ *are one-parameter subgroups of* $GL(3,\mathbb{R})$ *given by* (4.3.2) ~(4.3.4), $\widetilde{G}_i$ $(i = 2,3,4)$ *are two-parameter subgroups of* $GL(3,\mathbb{R})$*, for the curve* (2), if $\kappa_2 = 0$

$$\widetilde{G}_2 := \left\{\begin{bmatrix} e^u & 0 & 0\\ 0 & e^v & 0\\ 2^{\frac{1}{3}}e^u(u+v) & 0 & e^u\end{bmatrix}\middle|\, u,v \in \mathbb{R}\right\},$$

the others are given by (4.4.3),(4.4.5),(4.4.7) and (4.4.9), α, β, γ are constants with respect to the centroaffine curvatures, for the curve (2), if $\kappa_2 = 0$, $\alpha = \beta = -2^{\frac{1}{3}}$, the others are given by (4.4.4), (4.4.6), (4.4.8),(4.4.10), respectively.

Chapter 5

A related topic of centroaffine space curve theory

In this chapter, we want to investigate centroaffine surfaces with degenerate center map and consider its relationship with centroaffine space curves.

5.1 Center map

Center map was firstly introduced for centroaffine hypersurfaces by H.Furuhata and L.Vrancken [8] as a generalization of the center of proper affine spheres. They studied affine hypersurfaces whose center map is centroaffinely congruent with the original hypersurface, called to be self congruent. In particular, they showed that the center map of a definite centroaffine surface in the 3-space which is not a proper affine sphere centered at the origin is self congruent if and only if the centroaffine Tchebychev operator vanishes.

Let r be the equiaffine support function of f with respect to the origin $o \in \mathbb{R}^{n+1}$. By definition, it is a function on M written as

$$f(x) = Z_x + r(x)\xi_x, \tag{5.1.1}$$

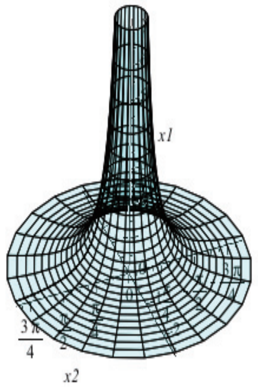

Figure 5.1: Proper affine sphere $f(u,v) = {}^t(e^{-2u}, e^u \cos v, e^u \sin v)$ with center at the origin

where Z is an $\mathbb{R}^{n+1}$-valued function tangent to f, ξ is the Blaschke normal vector field.

Definition 5.1.1. For an immersion $f : M \to \mathbb{R}^{n+1}$, we set $c : M \to \mathbb{R}^{n+1}$ by

$$c(x) := c_f(x) := f(x) - r(x)\xi_x, \text{ for } x \in M,$$

and call it the *center map* of f.

Proposition 5.1.2 ([8])**.** *An immersion $f : M \to \mathbb{R}^{n+1}$ is a proper affine sphere if and only if the center map c of f is constant.*

Proposition 5.1.3 ([8])**.** *The center map c of a centroaffine immersion $f : M \to \mathbb{R}^{n+1}$ is an immersion if and only if*

$$\mathrm{Ker}(\mathrm{id} + rS) \cap \mathrm{Ker}\, dr = \{0\},$$

where S is the equiaffine shape operator of f.

Remark 5.1.4. The center map of an improper affine sphere is an immersion.

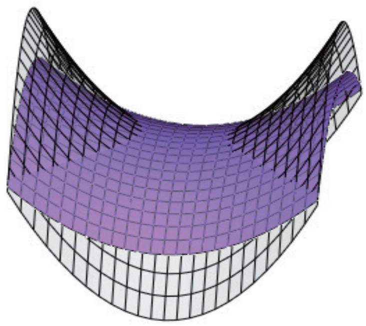

Figure 5.2: Hyperbolic paraboloid (improper affine shpere) and its center map

We assume that $f : M \to \mathbb{R}^3$ is a nondegenerate centroaffine surface as well. Let ∇, h and K be the centroaffine induced connection, the centroaffine metric, the difference tensor of ∇ and the Levi-Civita connection $\widetilde{\nabla}$ of h, respectively. We formulate the center map for centroaffine immersion.

Proposition 5.1.5 ([8])**.** *Let c be the center map of an immersion $f : M \to \mathbb{R}^{n+1}$. Then the following formula holds:*

$$c = -\frac{2}{n+2} f_* T,$$

where T is the centroaffine Tchebychev vector field of f.

5.2 Degenerate center map for centroaffine ruled surfaces and centroaffine space curves

In this section, we investigate center maps of centroaffine ruled surfaces.

Let $a(u)$, $b(u)$ be linearly independent $\mathbb{R}^3$-valued functions in one variable. Suppose that $b(u)$ is a nondegenerate centroaffine space curve and u is the centroaffine arc-length parameter of

Figure 5.3: Ruled surface $f(u, v) = b'(u) + vb(u)$ and its center map, where $b(u) = {}^t(u^2, u^3, u^4)$

$b(u)$, $\kappa_1(u)$, $\kappa_2(u)$ are the centroaffine curvatures.

Theorem 5.2.1. *Let $f : M \to \mathbb{R}^3$ be a nondegenerate centroaffine ruled surface given by $f(u, v) = a(u) + vb(u)$. The center map c of f is degenerate to a curve if and only if $a(u)$ is written as $a(u) = \phi(u)b'(u)$, where $\phi(u)$ is a nonzero function such that $3\phi'(u) + \phi(u)\kappa_2(u) \neq 0$, $\kappa_2(u)$ is the centroaffine second curvature of $b(u)$. Moreover, the center map of f is given by*

$$c(u, v) = -\frac{1}{2}(3\phi'(u) + \phi(u)\kappa_2(u))b(u).$$

Proof. Let $f : M \to \mathbb{R}^3$ be a nondegenerate centroaffine ruled surface given by $f(u, v) = a(u) + vb(u)$. Because

$$\begin{cases} f_u = a' + vb', \\ f_v = b, \end{cases}$$

we have $[f_u, f_v, f] = [a', b, a] + v[b', b, a] \neq 0$. We define functions in u as follows:

$$
\begin{aligned}
&s_1 := [b', b, a],\ \ s_2 := [a', b, a],\ \ s_3 := [a'', a', b],\\
&s_4 := [a'', b', b] + [b'', a', b],\ \ s_5 := [b'', b', b] \neq 0,\\
&s_6 := [b', a', b] \neq 0,\ \ s_7 := [b', a', a].
\end{aligned}
$$

(1) If $s_1 \neq 0$, the centroaffine metric is given by

$$
\begin{cases}
h_{11} = \dfrac{s_5 v^2 + s_4 v + s_3}{s_1 v + s_2},\\
h_{12} = h_{21} = \dfrac{s_6}{s_1 v + s_2},\\
h_{22} = 0,
\end{cases}
$$

the centroaffine induced connection as

$$
\begin{cases}
\Gamma^1_{12} = \dfrac{s_1}{s_1 v + s_2},\\
\Gamma^2_{12} = -\dfrac{s_6 v + s_7}{s_1 v + s_2},\\
\Gamma^1_{22} = 0,\\
\Gamma^2_{22} = 0,
\end{cases}
$$

the Levi-Civita connection $\widetilde{\nabla}$ with respect to the centroaffine metric as

$$
\begin{cases}
\widetilde{\Gamma}^1_{12} = 0,\\
\widetilde{\Gamma}^2_{12} = \dfrac{s_1 s_5 v^2 + 2 s_2 s_5 v + s_2 s_4 - s_1 s_3}{2 s_6 (s_1 v + s_2)},\\
\widetilde{\Gamma}^1_{22} = 0,\\
\widetilde{\Gamma}^2_{22} = -\dfrac{s_1}{s_1 v + s_2},
\end{cases}
$$

and the difference tensor K as

$$\begin{cases} K^1_{12} = \dfrac{s_1}{s_1 v + s_2}, \\ K^2_{12} = -\dfrac{-s_1 s_5 v^2 - 2(s_2 s_5 + s_6^2)v - s_2 s_4 + s_1 s_3 - 2 s_6 s_7}{2 s_6 (s_1 v + s_2)}, \\ K^1_{22} = 0, \\ K^2_{22} = \dfrac{s_1}{s_1 v + s_2}. \end{cases}$$

Then we can calculate the coefficients of the Tchebychev vector field as

$$\begin{cases} T^1 = \dfrac{2 s_1}{s_6}, \\ T^2 = -\dfrac{2 s_1 s_5 v^2 + (2 s_2 s_5 + 2 s_6^2 + s_1 s_4)v + s_2 s_4 + 2 s_6 s_7}{s_6^2}. \end{cases}$$

We can see the center map is given by

$$\begin{aligned} c &= -\frac{1}{2} f_* T = -\frac{1}{2}(T^1 f_u + T^2 f_v) \\ &= -\frac{s_1}{s_6}(a' + v b') + \frac{2 s_1 s_5 v^2 + (2 s_2 s_5 + 2 s_6^2 + s_1 s_4)v + s_2 s_4 + 2 s_6 s_7}{2 s_6^2} b. \end{aligned}$$

We then define the functions in u as follows.

$$\begin{aligned} t_1 &:= -\frac{s_1}{s_6}, \ t_2 := -\frac{s_1 s_5}{s_6^2}, \\ t_3 &:= \frac{2 s_2 s_5 + 2 s_6^2 + s_1 s_4}{2 s_6^2}, \ t_4 := \frac{s_2 s_4 + 2 s_6 s_7}{2 s_6^2}. \end{aligned}$$

We can rewrite c as

$$c = t_1 a' + t_1 v b' + (t_2 v^2 + t_3 v + t_4) b,$$

and then

$$\begin{cases} c_u = t_1'a' + t_1a'' + (t_2'v^2 + t_3'v + t_4')b + (t_2v^2 + (t_3 + t_1')v + t_4)b' + t_1vb'', \\ c_v = t_1b' + (2t_2v + t_3)b. \end{cases}$$

Then

$$\begin{aligned} [c_u, c_v, c] = [&t_1'a' + t_1a'' + (t_2'v^2 + t_3'v + t_4')b + (t_2v^2 + (t_3 + t_1')v + t_4)b' \\ &+ t_1vb'', t_1b' + (2t_2v + t_3)b, t_1a' + t_1vb' + (t_2v^2 + t_3v + t_4)b]. \end{aligned}$$

Noticing the coefficients of v^3, that is,

$$[t_2b', 2t_2b, t_1a'] + [t_1b'', t_1b', -t_2b] = \frac{3s_1^3s_5^2}{s_6^4} \neq 0,$$

we have that the center map is not degenerate.

(2) If $s_1 = 0$, it is easy to prove f is centroaffinely equivalent to $f(u, v) = \phi(u)b'(u) + vb(u)$, for some nonzero function $\phi(u)$, the centroaffine metric is given by

$$\begin{cases} h_{11} = -\phi^{-2}v^2 - (3\phi'\phi^{-2} + \kappa_2\phi^{-1})v + \kappa_1 - 2(\phi')^2\phi^{-2} + \phi''\phi^{-1} - \phi'\phi^{-1}\kappa_2, \\ h_{12} = h_{21} = \phi^{-1}, \\ h_{22} = 0. \end{cases}$$

Through a serious of calculation, the difference tensor is given by

$$\begin{cases} K_{12}^1 = K_{22}^1 = K_{22}^2 = 0, \\ K_{12}^2 = \frac{1}{2}(3\phi'\phi^{-1} + \kappa_2). \end{cases}$$

Then we can get the center map

$$c = -\frac{1}{2} f_* T = -\frac{1}{2}(3\phi' + \phi\kappa_2) b(u).$$

The center map is degenerate to be a curve if the nonzero function ϕ satisfy $3\phi' + \phi\kappa_2 \neq 0$. □

Remark 5.2.2. (1) Centroaffine ruled surfaces with $3\phi'(u) + \phi(u)\kappa_2(u) = 0$ are proper affine spheres.

(2) Centroaffine ruled surface $f(u, v) = \phi(u)b'(u) + vb(u)$ is minimal and its centroaffine scalar curvature 1.

Example. For a cubic curve $b(u) = {}^t(u, u^2, u^3)$, we can change the parameter to be the centroaffine arc-length parameter s and rewrite the curve as

$$b(s) = {}^t(e^{6^{-1/3}s}, e^{2 \cdot 6^{-1/3}s}, e^{3 \cdot 6^{-1/3}s}).$$

The second centroaffine curvature is given by

$$\kappa_2 = 6^{2/3}.$$

Then we can get the center map of $f(s, v) = b'(s) + vb(s)$ as follows,

$$c(s, v) = -\frac{1}{2} 6^{2/3} \cdot {}^t(e^{6^{-1/3}s}, e^{2 \cdot 6^{-1/3}s}, e^{3 \cdot 6^{-1/3}s}) = -\frac{1}{2}\kappa_2 b(s).$$

Corollary 5.2.3. *The center map of the centroaffine ruled surface $f(u, v) = b'(u) + vb(u)$ is centroaffinely equivalent to $b(u)$ if and only if the second centroaffine curvature of $b(u)$ is constant.*

Proof. By Theorem 5.2.1, the center map of $f(u,v) = b'(u) + vb(u)$ is given by

$$c = -\frac{1}{2}\kappa_2(u)b(u).$$

It is obvious that c is centroaffinely equivalent to $b(u)$ if κ_2 is constant. Conversely, if the center map c is centroaffinely equivalent to $b(u)$, that is, there exists a matrix $A \in GL(3;\mathbb{R})$ such that $c = Ab(u)$, then from $\det(A + \frac{1}{2}\kappa_2 I) = 0$, where I is the identity matrix, we get κ_2 is constant. □

Corollary 5.2.4. *The center map of the centroaffine ruled surface $f(u,v) = b'(u) + vb(u)$ is projectively equivalent to $b(u)$.*

Corollary 5.2.5. *Given a nondegenerate centroaffine space curve $b(u)$ with centroaffine arc-length parameter u and centroaffine second curvature $\kappa_2(u)$, we can construct a centroaffine ruled surface $f(u,v)$ whose center map is $b(u)$. In fact, the center map of $f(u,v) = \phi(u)b'(u) + vb(u)$ is $b(u)$, where $\phi(u) = -\frac{2}{3}\mu^{-1}(u)\int \mu(u)du$ and $\mu(u) = e^{\frac{1}{3}\int \kappa_2(u)du}$.*

Proof. Solving the ODE $3\phi'(u) + \phi(u)\kappa_2(u) = -2$. □

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