Proceedings of the 38th Sapporo Symposium on Partial Differential Equations

Edited by
Y. Giga, S. Jimbo, T. Ozawa, K. Tsutaya, Y. Tonegawa
H. Kubo, T. Sakajo, and H. Takaoka

Series #159. August, 2013

HOKKAIDO UNIVERSITY
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Proceedings of the 38th Sapporo Symposium on Partial Differential Equations

Edited by
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Sapporo, 2013

Partially supported by Grant-in-Aid for Scientific Research, the Japan Society for the Promotion of Science.
日本学術振興会科学研究費補助金　（基盤研究 S 課題番号 21224001）
日本学術振興会科学研究費補助金　（基盤研究 B 課題番号 24340024）
日本学術振興会科学研究費補助金　（基盤研究 B 課題番号 25287022）
日本学術振興会科学研究費補助金　（挑戦的萌芽 課題番号 23654057）

PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 21 through August 23 in 2013 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Professor Taira Shirota started the symposium more than 35 years ago. Professor Kôji Kubota and late Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

Y. Giga, S. Jimbo, T. Ozawa, K. Tsutaya, Y. Tonegawa
H. Kubo, T. Sakajo, and H. Takaoka

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The 38th Sapporo Symposium on Partial Differential Equations
（第３８回偏微分方程式論札幌シンポジウム）

組織委員: 久保英夫, 高岡秀夫

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Program Committee: Y. Giga, S. Jimbo, T. Ozawa, K. Tsutaya, Y. Tonegawa
H. Kubo, T. Sakajo, and H. Takaoka

Period (期間) August 21 , 2013 - August 23 , 2013

Venue (場所) Room 203, Faculty of Science Building #5, Hokkaido University
北海道大学 理学部 5 号館大講義室 (203 号室)

URL http://www.math.sci.hokudai.ac.jp/sympo/sapporo/program130821.html

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Application of phase space analysis to the scattering theory for continuous and discrete Schrödinger equations[1]

Shu Nakamura[2]

Abstract

We discuss several applications of phase space analysis (or microlocal analysis) to Schrödinger equations. When we study scattering theory using microlocal analytic methods, it is often useful to utilize the Hörmander pseudodifferential operator theory in the Fourier space. This calculus is sometimes called *the scattering calculus* (following Melrose), but it goes back at least to classical works by Kitada and others.

We review application to the long-range scattering theory for Schrödinger equations on $\mathbb{R}^d$, and discuss its generalizations to discrete Schrödinger equations. "Miclolocal analysis on $\mathbb{Z}^d$" sounds rather odd, but actually we can consider the problems microlocally on $T^*\mathbb{T}^d$ in a natural manner, where $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ denotes the d-dimensional torus, which is the Fourier space for $\mathbb{Z}^d$. T^*M denotes the cotangent bundle of M.

We also discuss the microlocal properties of the scattering matrices. For Schrödinger equations with short-range smooth potentials, we can show the wave operators, the scattering operator, and the scattering matrices are pseudodifferential operators in the Fourier space. Moreover, the asymptotic expansion of the symbol of the scattering matrices are shown to correspond to the classical Born expansion. We then discuss generalization of these results to discrete Schrödinger equations.

These results are still mostly in progress, and some results are preliminary.

1 Schrödinger operators on $\mathbb{R}^d$

We consider Schrödinger operator:

$$H = -\frac{1}{2}\triangle + V(x) \quad \text{on } \mathcal{H} = L^2(\mathbb{R}^d)$$

with the space dimension $d \geq 1$. During the talk, we always suppose V is a real-valued smooth function, and for any multi-index $\alpha \in \mathbb{Z}_+^d$, it satisfies

$$\left|\partial_x^\alpha V(x)\right| \leq C_\alpha \langle x\rangle^{-\mu-|\alpha|}, \quad x \in \mathbb{R}^d, \tag{1}$$

where $\mu > 0$ is a given decay rate of V, $C_\alpha > 0$, and $\langle x\rangle = (1+|x|^2)^{1/2}$. Then it is well-known that H is self-adjoint with $D(H) = H^2(\mathbb{R}^d)$, the Sobolev space of order 2. The spectrum is bounded below, the essential spectrum is $\mathbb{R}_+$, and it is absolutely continuous possibly except for negative discrete eigenvalues. The solution to the time-dependent Schrödinger equation:

$$\frac{\partial}{\partial t}u(t) = -iHu(t), \quad u(0) = u_0 \in L^2(\mathbb{R}^d),$$

is given by $u(t) = e^{-itH}u_0$, where $u \in C(\mathbb{R}, L^2(\mathbb{R}^d))$.

[1]Abstract for the 38th Sapporo Symposium on Partial Differential Equations, August 21–23, 2013. (Prepared on July 26, 2013)

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The Hamilton function of the corresponding classical mechanics is given by

$$p(x,\xi) = \frac{1}{2}|\xi|^2 + V(x), \quad (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d,$$

and the classical flow is defined by the Hamilton vector field:

$$H_p = \sum_{j=1}^{d} \left(\frac{\partial p}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j} \frac{\partial}{\partial \xi_j} \right) = \sum_{j=1}^{d} \left(\xi_j \frac{\partial}{\partial x_j} - \frac{\partial V}{\partial x_j} \frac{\partial}{\partial \xi_j} \right) \quad \text{on } \mathbb{R}^d \times \mathbb{R}^d.$$

Note that this flow is equivalent to the Newton equation:

$$\dot{x}(t) = \xi(t), \quad \dot{\xi}(t) = -\nabla V(x(t)).$$

Various properties of the Schrödinger evolution group e^{-itH} follow from the properties of the classical flow, as we will see in the context of the scattering theory later. In particular, we note the Hamilton operator H is considered as a *quatization* of $p(x,\xi)$ in the following sense. For a smooth function $a(x,\xi)$ on $\mathbb{R}^d \times \mathbb{R}^d$, the *Weyl quatization* $a(x,D_x)$ is defined by

$$a(x,D_x)u(x) = (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} a(\tfrac{x+y}{2},\xi) u(y) dy d\xi, \quad u \in \mathcal{S}(\mathbb{R}^d).$$

Then $H = p(x,D_x)$, and we may also consider the time evolution e^{-itH} as a quantization of the Hamilton flow, though the justification of this observation is not obvious in general.

2 Schrödinger operators on $\mathbb{Z}^d$

In solid state physics, the discrete Schrödinger equation, sometimes called the Anderson tight binding model, is widely used. The discrete Schrödinger operator is defined by

$$\tilde{H}u[n] = -\frac{1}{2}\triangle u[n] + \tilde{V}[n]u[n], \quad n \in \mathbb{Z}^d, \ u = u[\cdot] \in \tilde{\mathcal{H}} = \ell^2(\mathbb{Z}^d),$$

where the discrete Laplacian is defined (for example) by

$$\triangle u[n] = \sum_{|n-m|=1} u[m],$$

and $\tilde{V}[\cdot] : \mathbb{Z}^d \to \mathbb{R}$ is a potential function. If $\tilde{V}$ is bounded, then $\tilde{H}$ is a bounded self-adjoint operator on $\ell^2(\mathbb{Z}^d)$. Thus the solution to the time-dependent Schrödinger equation is also given by $e^{-it\tilde{H}}u_0$, $u_0 \in \ell^2(\mathbb{Z}^d)$.

Since the configuration space of the quantum particle is the lattice $\mathbb{Z}^d$, it is not obvious (at first) what the corresponding classical mechanics is. We suppose $\tilde{V}[\cdot]$ satisfies the following conditions: Let $\tilde{\partial}_j$ be the difference operator:

$$\tilde{\partial}_j u[n] = u[n] - u[n-e_j], \quad u \in \ell^2(\mathbb{Z}^d), \ j = 1,\dots,d,$$

where $\{e_j\}_{j=1}^d$ is the standard basis of $\mathbb{R}^d$. Then we assume for any $\alpha \in \mathbb{Z}_d^d$,

$$\left|\tilde{\partial}^\alpha \tilde{V}[n]\right| \le C_\alpha \langle n \rangle^{-\mu-|\alpha|}, \quad n \in \mathbb{Z}^d. \tag{2}$$

Then $\tilde{V}[n]$ is extended to a smooth potential function $V(x)$ which satisfies the conditions (1) in Section 1. We set

$$\tilde{p}(x,\xi) = -\sum_{j=1}^{d} \cos(\xi_j) + V(x), \quad (x,\xi) \in \mathbb{R}^d \times \mathbb{T}^d \cong T^*\mathbb{T}^d,$$

where $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$. Then we may consider $\tilde{p}(x,\xi)$ as the classical Hamilton function corresponding to $\tilde{H}$. $\tilde{p}$ generates a Hamilton flow on $T^*\mathbb{T}^d$:

$$H_{\tilde{p}} = \sum_{j=1}^{d}\left(\frac{\partial \tilde{p}}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial \tilde{p}}{\partial x_j}\frac{\partial}{\partial \xi_j}\right) = \sum_{j=1}^{d}\left(\sin(\xi_j)\frac{\partial}{\partial x_j} - \frac{\partial V}{\partial x_j}\frac{\partial}{\partial \xi_j}\right) \quad \text{on } \mathbb{R}^d \times \mathbb{T}^d.$$

We note, by the Fourier inversion formula,

$$\tilde{H}u[n] = (2\pi)^{-d}\int_{\mathbb{T}^d}\sum_m e^{i(n-m)\cdot\xi}\tilde{p}(n,\xi)u[m]d\xi,$$

and this expression is different from the standard Weyl quantization. However, we can show

$$F\tilde{H}F^* \equiv \tilde{p}(-D_\xi,\xi)$$

modulo smoothing operators on $\mathbb{T}^d$, where $F:\ \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d)$ is the discrete Fourier transform:

$$Fu(\xi) = (2\pi)^{-d/2}\sum_{n\in\mathbb{Z}^d} e^{-in\cdot\xi}u[n], \quad u \in \ell^2(\mathbb{Z}^d),\ \xi \in \mathbb{T}^d,$$

and the right hand side is defined as a pseudodifferential operaor on $\mathbb{T}^d$ with the symbol $\tilde{p}$. We will consider $\tilde{H}$ as a quantization of $\tilde{p}$ in this sense.

3 Short-range scattering theory on $\mathbb{R}^d$

Here we review the well-known formulation of the short-range scattering theory for Schrödinger equations on $\mathbb{R}^d$ (see, e.g., [5] Volume 3, [6]). We suppose V is short-range, i.e., the assumption (1) with $\mu > 1$. Then the wave operators:

$$W_\pm u = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}u, \quad u \in L^2(\mathbb{R}^d),$$

exist, where $H_0 = -\frac{1}{2}\triangle$. Moreover, it is proved that $W_\pm$ are asymptotically complete, i.e., $\operatorname{Ran} W_\pm = \mathcal{H}_c(H)$, the continuous subspace of H.

If we write $u = W_\pm u_\pm \in \mathcal{H}_c(H)$, then by the definition of $W_\pm$, we learn

$$u(t) = e^{-itH}u \sim e^{-itH_0}u_\pm \quad \text{as } t \to \pm\infty.$$

Namely, $u(t)$ asymptotically converges to the free evolutions $e^{-itH_0}u_\pm$ as $t \to \pm\infty$. $u_\pm$ is called the *scattering data*.

This result corresponds to the fact that if a classical particle $(x(t),\xi(t))$ is not trapped, then there are (classical mechanical) scattering data $(x_\pm,\xi_\pm) \in \mathbb{R}^d \times \mathbb{R}^d$ such that

$$x(t) \sim x_\pm + t\xi_\pm, \quad \xi(t) \sim \xi_\pm \quad \text{as } t \to \pm\infty.$$

This correspondence plays crucial role in the following discussions.

4 Short-range scattering theory on $\mathbb{Z}^d$

The scattering theory is constructed for the Schrödinger operators on $\mathbb{Z}^d$ similarly. (See, e.g., Isozaki-Korotyaev [4], Boutet de Monvel-Sahbani [2]). Here we also suppose $\tilde{V}$ is short-range, i.e., the assumption (2) with $\mu > 1$. Then the wave operators:

$$\tilde{W}_{\pm} u = \lim_{t \to \pm\infty} e^{it\tilde{H}} e^{-it\tilde{H}_0} u, \quad u \in \ell^2(\mathbb{Z}^d),$$

exist, where $\tilde{H}_0 = -\frac{1}{2}\triangle$. Moreover, it is also proved that $\tilde{W}_{\pm}$ are asymptotically complete, i.e., $\operatorname{Ran} \tilde{W}_{\pm} = \tilde{\mathcal{H}}_c(\tilde{H})$, the continuous subspace of $\tilde{H}$. Again, if we write $u = \tilde{W}_{\pm} u_{\pm} \in \tilde{\mathcal{H}}_c(\tilde{H})$, we learn

$$u(t) = e^{-it\tilde{H}} u \sim e^{-it\tilde{H}_0} u_{\pm} \quad \text{as } t \to \pm\infty.$$

The corresponding classical mechanical scattering is not as straightforward. We denote

$$\tilde{p}_0(\xi) = -\sum_{j=1}^{d} \cos(\xi_j) \quad \text{on } \mathbb{T}^d,$$

and we set the velocity function by

$$v_j(\xi) = \frac{\partial \tilde{p}_0}{\partial \xi_j}(\xi) = \sin(\xi_j), \quad j = 1, \ldots, d, \ \xi \in \mathbb{T}^d.$$

Then the classical free evolution is give by :

$$\exp tH_{\tilde{p}_0}(x, \xi) = x + tv(\xi), \quad t \in \mathbb{R}, \ (x, \xi) \in \mathbb{R}^d \times \mathbb{T}^d.$$

Thus, if a classical particle i.e., a trajectory along $H_{\tilde{p}}$: $(x(t), \xi(t)) \in \mathbb{R}^d \times \mathbb{T}^d$ is not trapped, then there exist scattering data $(x_{\pm}, \xi_{\pm})$ such that

$$x(t) \sim x_{\pm} + tv(\xi_{\pm}), \quad \xi(t) \sim \xi_{\pm} \quad \text{as } t \to \pm\infty.$$

We note that the "particle" moves in a *virtual configuration space* $\mathbb{R}^d$, not in the original configuration space $\mathbb{Z}^d$.

5 Long-range scattering theory on $\mathbb{R}^d$

If the potential is long-range type, i.e., if V satisfies the assumption (1) with $0 < \mu \leq 1$, then the above argument does not hold. In this case we need to modify the free evolution to approximate the solution as $t \to \pm\infty$ (see, e.g., [5] Section X.9).

If μ is modestly long-range, i.e., if $1/2 < \mu \leq 1$, then we can employ the Dollard modifier:

$$\Phi_D(t, \xi) = \int_0^t p(s\xi, \xi) ds = \frac{t}{2}|\xi|^2 + \int_0^t V(s\xi) ds, \quad t \in \mathbb{R}, \ \xi \in \mathbb{R}^d.$$

We set the Fourier multiplier

$$U_D(t) = \exp(-i\Phi_D(t, D_x)), \quad t \in \mathbb{R},$$

as the *modified free evolution.* Then the modified wave operators:

$$W_{\pm}^{D}u = \lim_{t\to\pm\infty} e^{itH}U_D(t)u, \quad u \in L^2(\mathbb{R}^d)$$

exist, and are asymptotically complete: $\operatorname{Ran} W_{\pm}^{D} = \mathcal{H}_c(H)$.

More generally, we construct solutions to the momentum space Hamilton-Jacobi equation

$$\frac{\partial\Phi}{\partial t}(t,\xi) = p(\partial_\xi\Phi(t,\xi),\xi), \quad \pm t \geq 0,\ \xi \in \mathbb{R}^d,$$

with the initial condition: $\Phi(0,\xi) = \pm R|\xi|$, where $R > 0$ is a sufficiently large constant. Then the modified free evolution is defined similarly: $U(t) = \exp(-i\Phi(t,D_x))$; the modified wave operators exist, and are complete.

We note that the construction of the modified free evolution relies on the classical mechanical argument crucially. The Dollard modifier is a first order approximate solution to the Hamilton-Jacobi equation.

6 Long-range scattering theory on $\mathbb{Z}^d$

The long-range scattering theory for the discrete Schrödinger equations can be constructed similarly using the corresponding classical mechanics on $T^*\mathbb{T}^d$.

If $1/2 < \mu \leq 1$, then we set

$$\tilde{\Phi}_D(t,\xi) = \int_0^t \tilde{p}(sv(\xi),\xi)ds = t\tilde{p}_0(\xi) + \int_0^t V(sv(\xi))ds, \quad t \in \mathbb{R},\ \xi \in \mathbb{T}^d,$$

and

$$\tilde{U}_D(t) = F^* \exp(-i\tilde{\Phi}_D(t,\xi))F,$$

Then the modified wave operators

$$\tilde{W}_{\pm}^{D}u = \lim_{t\to\pm\infty} e^{it\tilde{H}}\tilde{U}_D(t)u, \quad u \in \ell^2(\mathbb{Z}^d),$$

exist and are complete. More generally, we can construct solutions to the Hamilton-Jacobi equation

$$\frac{\partial\tilde{\Phi}}{\partial t}(t,\xi) = \tilde{p}(\partial_\xi\tilde{\Phi}(t,\xi),\xi), \quad \pm t \geq 0,\ \xi \in \mathbb{T}^d,$$

with the initial condition $\tilde{\Phi}(0,\xi) = \pm R\tilde{p}_0(\xi)$. We note the construction of $\tilde{\Phi}(t,\xi)$ is local in the energy : $\lambda = p(x,\xi)$, and we need to avoid the threshold energies $\mathcal{T} = \{-d+2j \mid j = 0,\dots,d\}$.

7 Scattering matrix for Schrödinger operators on $\mathbb{R}^d$

Here we consider scattering theory from the microlocal point of view. Here we suppose the potential V satisfies the short range condition: $\mu > 1$ in (1). The scattering operator is defined by

$$S = W_{+}^{*}W_{-} \ :\ L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d),$$

and it is unitary. Moreover, it commutes with the free energy: $SH_0 = H_0 S$, and hence

$$\tfrac{1}{2}|\xi|^2(\mathcal{F}S\mathcal{F}^*)u(\xi) = (\mathcal{F}S\mathcal{F}^*)\tfrac{1}{2}|\xi|^2 u(\xi) \quad \text{for } u(\xi) \in L^2(\mathbb{R}^d).$$

Thus $\mathcal{F}S\mathcal{F}^*$ is reduced to a family of operators on $L^2(\Sigma_\lambda)$, $\lambda > 0$, where

$$\Sigma_\lambda = \big\{\xi \in \mathbb{R}^d \;\big|\; \tfrac{1}{2}|\xi|^2 = \lambda\big\}, \quad \lambda > 0,$$

is the energy surface. We denote

$$\mathcal{F}S\mathcal{F}^* = \int^{\oplus} S(\lambda)d\lambda \quad \text{on } L^2(\mathbb{R}^d) \cong \int^{\oplus} L^2(\Sigma_\lambda)d\lambda,$$

and $S(\lambda) : L^2(\Sigma_\lambda) \to L^2(\Sigma_\lambda)$ is called the scattering matrix. The scattering matrix is one of the most fundamental observables in the quantum mechanics.

We set

$$W(t) = e^{itH_0}e^{-itH}, \quad t \in \mathbb{R}.$$

Then it is easy to see

$$\frac{d}{dt}W(t) = -i\big(e^{itH_0}He^{-itH_0} - H_0\big)W(t) = -i\big(e^{itH_0}Ve^{-itH_0}\big)W(t).$$

Using the Weyl quantization, we learn

$$\big(e^{itH_0}Ve^{-itH_0}\big) = V(x + tD_x),$$

and hence we may consider $\{W(t)\}$ as an evolution operator with a time-dependent generator $V(x+tD_x)$. Since $V(x+tD_x)$ is bounded, the solution is expressed using the Dyson expansion:

$$\begin{aligned} W(t) &= 1 + \sum_{n=1}^{\infty}(-i)^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} V(x+t_1D_x)\cdots V(x+t_nD_x)dt_n\cdots dt_1 \\ &= 1 - i\int_0^t V(x+t_1D_x)dt_1 - \int_0^t\int_0^{t_1} V(x+t_1D_x)V(x+t_2D_x)dt_2dt_1 + \cdots . \end{aligned}$$

We may consider this expansion as an asymptotic expansion as pseudodifferential operators in the Fourier space. On the other hand, $W(t)$ converges to $W_\pm^*$ weakly as $t \to \pm\infty$. Each terms in the Dyson expansion does not converge as $t \to \pm\infty$ in the standard symbol class, but we can show the wave operators are in fact pseudodifferential operators. Moreover the scattering operator is also a pseudodifferential operator and

$$S = 1 - i\int_{-\infty}^{\infty} V(x+tD_x)dt + (\text{lower order terms}),$$

in some sense. This then implies the scattering matrix is a pseudodifferential operator on Σ_λ with the symbol

$$S(\lambda; x, \xi) = 1 - i\int_{-\infty}^{\infty} V(x+t\xi)dt + (\text{lower order terms}),$$

where $(x,\xi) \in \big\{(x,\xi) \in \mathbb{R}^d \times \Sigma_\lambda \;\big|\; x \perp \xi\big\} \cong T^*\Sigma_\lambda$, i.e., (x,ξ) is an element of the cotangent bundle of Σ_λ. Using this expression, we can compute the asymptotic distribution of eigenvalues of the scattering matrix, at least formally (see, e.g., Birman-Yafaev [1], Bulgar-Pushnitski [3]).

8 Scattering matrix for Schrödinger operators on $\mathbb{Z}^d$

We can also consider the scattering matrix for discrete Schrödinger operators. For $\lambda \in [-d, d]$, the energy surface is defined by

$$\Lambda_\lambda = \{\xi \in \mathbb{T}^d \mid \tilde{p}_0(\xi) = \lambda\},$$

and it is a regular submanifold of $\mathbb{T}^d$ unless $\lambda \in \mathcal{T}$. The scattering matrix is defined as a unitary operator $\tilde{S}(\lambda)$ on $L^2(\Lambda_\lambda)$ as well as the continuous case. For the wave operators, we compute:

$$\frac{d}{dt}\big(e^{it\tilde{H}}e^{-it\tilde{H}_0}\big) = -iV_1(t)\big(e^{it\tilde{H}}e^{-it\tilde{H}_0}\big)$$

where $FV_1(t)F^*$ is a pseudodifferential operator on $\mathbb{T}^d$ with the symbol:

$$V_1(t; x, \xi) = V(x + tv(\xi)) + (\text{lower order terms}), \quad (x, \xi) \in \mathbb{R}^d \times \mathbb{T}^d \cong T^*\mathbb{T}^d.$$

Then we obtain the Dyson expansion of the evolution:

$$e^{it\tilde{H}}e^{-it\tilde{H}_0} = 1 - i\int_0^t V_1(t_1)dt_1 - \int_0^t \int_0^{t_1} V_1(t_1)V_1(t_2)dt_2dt_1 + \cdots .$$

Analogously to the previous section, we can show that the scattering matrix $S(\lambda)$ is a pseudo-differential operator on Λ_λ, and the symbol is given by

$$\tilde{S}(\lambda; x, \xi) = 1 - i\int_{-\infty}^{\infty} V(x + tv(\xi))dt + (\text{lower order terms}),$$

where $(x, \xi) \in \big\{(x, \xi) \in \mathbb{R}^d \times \Lambda_\lambda \mid x \perp v(\xi)\big\} \cong T^*\Lambda_\lambda$. We can apply this argument to obtain the asymptotic distribution of the eigenvalues of $\tilde{S}(\lambda)$.

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Phase transition solutions to an Allen-Cahn model equation

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We consider the following equation

$$-\Delta u + A(x)G'(u) = 0 \qquad \text{in} \quad \mathbb{R}^n \tag{0.1}$$

where $A(x) = A(x+i)$ for $x \in \mathbb{R}^n, i \in \mathbb{Z}^n$, $G(u+j) = G(u)$ for $u \in \mathbb{R}, j \in \mathbb{Z}$, $G(u) > 0$ for $u \in \mathbb{R} \setminus \mathbb{Z}$, $G(j) = 0, G''(j) > 0$ for $j \in \mathbb{Z}$ and $G(t) = G(-t)$ for $t \in \mathbb{R}$.

Several authors have studied Allen-Cahn phase transition models in which the spatial phase transition manifests itself as a heteroclinic or homoclinic solution of the corresponding partial differential equation. See e.g. [1]-[4], [16]-[17] and [20]. Similar type of solutions in more general settings arise in extension's of Moser's work [14] on developing an Aubry-Mather theory for PDE's: [5], [6]-[7], [9]-[11], [18]-[19], [21], and [23]. In all of these sources the transition type solutions are *unidirectional* in the sense that they change in a particular direction.

In this talk, I would like to introduce some recent works with Paul Rabinowitz about some *multidirectional* solutions. For autonomous problem, that is, $A =$ constant, there have been many works [22], [12],[13] and references therein which show that the bounded solutions of (0.1) are closely related to the minimal surfaces in $\mathbb{R}^n$.

We believe that there would be a generic condition for existence of *multi directional* solutions like gap conditions for construction of multi-transition (*unidirectional*) solutions(see [21] and references therein). We do not know how to do this yet for (0.1), but will present a certain type of non-autonomous term A for which such solutions can be found.

To describe our results, let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $A \in C^1(\mathbb{R}^n)$ be a nonnegative function that is 1-periodic in $x_i, 1 \leq i \leq n$. Set $\Omega \equiv$

$\{x \in (0,1)^n \mid A(x) > 0\}$. We assume $2d^* = |\partial\Omega - \partial[0,1]^n| > 0$ and $\partial\Omega$ is a smooth manifold. Set $\Omega_d \equiv \{x \in \Omega \mid |x - \partial\Omega| > d\}$. Then for sufficiently small $d \in (0, d^*), \partial\Omega_d$ is diffeomorphic to $\partial\Omega$. Fixing such a small d, let $\varepsilon > 0$ and $A_\varepsilon = 1 + \frac{1}{\varepsilon}A$. Our model equation is

$$-\Delta u + A_\varepsilon G'(u) = 0, \quad x \in \mathbb{R}^n. \tag{0.2}$$

First, we are interested in solutions of (0.2) satisfying $0 < U < 1$ and that are near 1 on A^T and near 0 on B^T, where

$$T \subset \mathbb{Z}^n, \;\; A^T = \cup_{i \in T}(i + \Omega), \;\; B^T = \cup_{i \in \mathbb{Z}^n \setminus T}(i + \Omega).$$

To describe our results more precisely, for $d \in (0, d^*)$ and Ω_d as above, we define

$$A_T \equiv \cup\{i + \Omega_d \mid i \in T\}$$

and

$$B_T \equiv \cup\{i + \Omega_d \mid i \in \mathbb{Z}^N \setminus T\}.$$

Set

$$L_\varepsilon(u) = \frac{1}{2}|\nabla u|^2 + A_\varepsilon G(u) \quad \text{and} \quad J_\varepsilon(u) = \int_{\mathbb{R}^n} L_\varepsilon(u)\, dx.$$

Choose $0 < b < \frac{1}{2} < a < 1$ and define

$$\tilde{\Gamma}(T) = \{u \in C^2(\mathbb{R}^n, [0,1]) \mid u \geq a > 1/2 \text{ on } A_T \text{ and } u \leq b < 1/2 \text{ on } B_T\}.$$

Whenever a solution, $u \in \tilde{\Gamma}(T)$, of (0.2) satisfies

$$J_\varepsilon(u + \varphi) - J_\varepsilon(u) = \int_{supp\, \varphi} (L_\varepsilon(u + \varphi) - L_\varepsilon(u))\, dx \geq 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $u + \varphi \in \tilde{\Gamma}(T)$, u will be said to be *minimal* in $\tilde{\Gamma}(T)$.

The solutions obtained here will have certain decay properties relative to T. To help describe them, for $x \in \mathbb{R}^n$, define

$$d(x, T) \equiv \text{dist}(x, \partial(T + [0,1]^n))$$

and for $S \subset \mathbb{R}^n$, let χ_S denote the characteristic function of S. Our first main result [8] is:

Theorem 0.3. *Under the above hypotheses on A and G, there is an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any $T \subset \mathbb{Z}^n$,*

1° *there exists a solution $U = U_{\varepsilon,T} \in \tilde{\Gamma}(T)$ of (0.2);*

2° *U is minimal in $\tilde{\Gamma}(T)$;*

3° $0 < U_{\varepsilon,T} < 1$ *;*

4° *$U_{\varepsilon,T}$ converges uniformly to 1 on A^T and to 0 on B^T as $\varepsilon \to 0$;*

5° *there exist constants $C, c > 0$, independent of $T \subset \mathbb{Z}^n$ and of $\varepsilon \in (0, \varepsilon_0)$, satisfying*

$$|U_{\varepsilon,T}(x) - \chi_{T+[0,1]^n}(x)| \leq C \exp(-cd(x,T)), \quad x \in \mathbb{R}^n.$$

Second, we are interested in unbounded solutions of (0.2). As an Aubry-Mather theory for PDE's, Moser [14] obtained among other things the following existence result.

Theorem 0.4. *For each $\alpha \in \mathbb{R}^n$, there exists a solution u_α of (0.1) such that for some $C > 0$, $\quad |u_\alpha(x) - \alpha \cdot x| \leq C,\ x \in \mathbb{R}^n$.*

Let $P : \mathbb{Z}^n + \Omega \to \mathbb{Z}$ be a function such that P is constant on $i + \Omega$ for each $i \in \mathbb{Z}^n$. We say P is Lipschitz continuous if there exists $C > 0$ such that $|P(x) - P(y)| \leq C|x - y|$ for any $x, y \in \mathbb{Z}^n + \Omega$.

As for *multidirectional* (unbounded) solutions generalizing the *unidirectional* solutions obtained by Moser in Theorem 0.4, our second main result is:

Theorem 0.5. *Let A_ε be as above. Suppose P be Lipschitz continuous on S. Then there is an ε_0 depending on the Lipschitz constant of P such that for each $\varepsilon \in (0, \varepsilon_0)$, there is a solution, $U = U_{\varepsilon,P}$ of (0.2) with $U \to P$ uniformly on $\mathbb{Z}^n + \Omega$ as $\varepsilon \to 0$.*

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Structure-preserving finite difference scheme for some sphere-valued partial differential equations *

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Keywords: (*Finite difference method, structure-preserving scheme, vector-valued PDE, energy structure, sphere-valued solution*)

1 Target PDE and aim

In this talk, we treat some typical sphere-valued partial differential equations, for examples, Heisenberg equation and Landau-Lifshitz equation. These equations describe the evolution of spin fields in continuum ferromagnetism and have the following properties:

1. length preserving,
2. energy conservation or dissipation property.

We propose a finite difference scheme for these equations which inherits the above properties and show some theoretical results on the scheme. And we also demonstrate numerical examples in order to show the effectiveness of our scheme.

2 Landau-Lifshitz equation

Landau-Lifshitz equation is the following vector-valued partial differential equation:

$$\frac{\partial u}{\partial t} = \mu u \times \Delta u - \lambda u \times (u \times \Delta u). \tag{1}$$

Here, $\lambda \geq 0$, $\mu \in \mathbb{R}$ are constants, $\Omega \subseteq \mathbb{R}^N$ is the material and $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) : \Omega \times (0,\infty) \to \mathbb{R}^3$ describes a spin field. In case $\lambda = 0$, the equation (1) is often called the Heisenberg equation.

By taking an inner product of (1) with $u(x,t)$, we obtain $|u(x,t)| = |u_0(x)|$ for any $t > 0$. Thus we usually consider the case $|u_0| = |u(x,t)| = 1$ as the spin model.

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We also obtain the following energy equality:

$$E(u(t)) = E(u_0) - 2\lambda \int_0^t \|u(\cdot, s) \times \Delta u(\cdot, s)\|^2_{L^2(\Omega)} ds. \tag{2}$$

Here, $E(u(t)) := \|\nabla u(\cdot, t)\|^2_{L^2(\Omega)}$.

In this talk, for simplicity, we only treat one dimensional case with periodic boundary condition. We remark that we can easily extend our results for higher dimensional case and other suitable boundary conditions.

3 Proposed scheme

Let $\Delta t > 0, \Delta x := L/N$ $(N \in \mathbb{N})$ be time and spatial mesh sizes, where $L > 0$ is the spatial period. We denote by $U_n^j = (U_{1,n}^j, U_{2,n}^j, U_{3,n}^j)$ an approximate vector of $u(x,t)$ at $x = x_n := n\Delta x, t = t_j := j\Delta t$. Our proposed scheme is the following:

$$\begin{aligned}
\frac{U_n^{j+1} - U_n^j}{\Delta t} &= \mu U_n^{(j+1,j)} \times \tilde{\Delta} U_n^{(j+1,j)} - \lambda U_n^{(j+1,j)} \times \left(U_n^{(j+1,j)} \times \tilde{\Delta} U_n^{(j+1,j)} \right), \\
U_n^0 &= u_0(x_n), \\
U_{N-1}^j &= U_{-1}^j, \qquad U_N^j = U_0^j.
\end{aligned}$$

Here, $U_n^{(j+1,j)} = (U_n^{j+1} + U_n^j)/2$, $\tilde{\Delta} := D^+ D^-$ is a standard discrete Laplacian, where D^+ (resp.D^-) is a forward (resp. backward) difference operator on space, that is,

$$D^+ X_n = \frac{X_{n+1} - X_n}{\Delta x}, \quad D^- X_n = \frac{X_n - X_{n-1}}{\Delta x}.$$

For a finite difference solution of the above scheme, we have the following properties on a length of the solution vector and an energy structure.

Theorem 1. *Any finite difference solution of the proposed scheme satisfies the following:*
(1) The solution keeps its length, that is, $|U_n^{j+1}| = |U_n^j|$ for all j and n.
(2) The following energy equality is satisfied:

$$E_h(U^j) = E_h(U^0) - 2\lambda \sum_{i=0}^{j-1} ||U^{(i+1,i)} \times \tilde{\Delta}_h U^{(i+1,i)}||_2^2 \Delta t.$$

Here $E_h(U^j) := ||D^+ U^j||_2^2$, where $||v||_2 = \left(\sum_{n=0}^{N-1} |v_n|^2 \Delta x \right)^{1/2}$.

Note that the above energy equality is a discrete version of that of original problem. That is, the proposed scheme inherits length-preserving property and the energy structure from the original problem.

We also have the following error estimate.

Theorem 2. *Let $T > 0$. Assume that $u_0(x), u(x,t)$ are sufficiently smooth for $x \in [0, L]$ and $t \in (0, T)$. Then, there exist constants $C_c := C_c(\lambda, \mu, u_0, L, T)$ and $C_\gamma := C_\gamma(\lambda, \mu, L, K)$. For any $\Delta t < C_c$ and $\Delta x > 0$ with $\frac{\Delta t}{\Delta x^2} < C_\gamma$, we have*

$$||U_n^j - u(x_n, t_j)||_{h^1} = O(\Delta t^2 + \Delta x^2).$$

Here, $\|X\|_{h^1} = \left(\|X\|_2^2 + \|D^+X\|_2^2\right)^{1/2}$.

4 Numerical results

In this section we show two numerical examples for the cases $\lambda = 0$(Heisenberg case) and $\lambda > 0$(Landau-Lifshitz case) to show the effectiveness of the proposed scheme. Here we use exact solutions and compare them.

First we see the Heisenberg case ($\lambda = 0$). We have $\max\left||U_n^j| - 1\right| \sim 10^{-15}$, that is, we verify the length-preserving property numerically. Figure 1 shows the behavior of the energy. We see that the numerical energy keeps constant value. Thus, we conclude that the numerical energy is conserved. Figure 2 shows the behaviors of (a) U_1 and (b) u_1 of the exact solution. The both behaviors coincide very much.

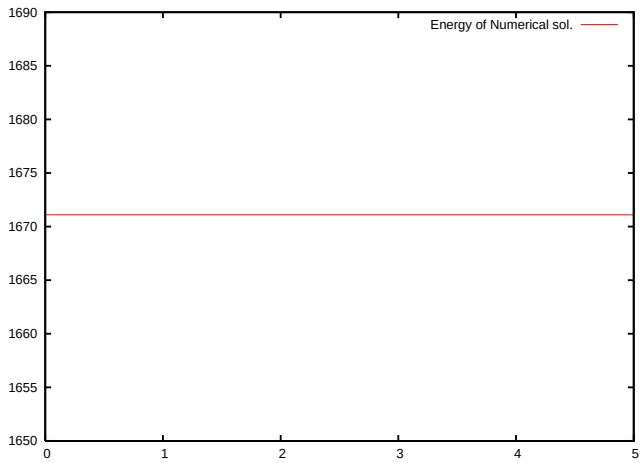
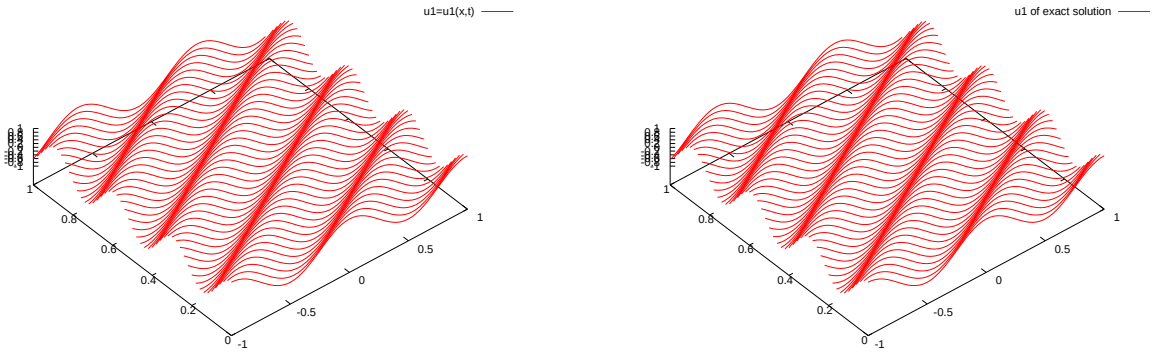


Fig. 1: Time evolution of the energy ($\lambda = 0$).



(a) (b)

Fig. 2: Time evolutions of u_1 of (a) numerical solution and (b) exact solution in the case $\lambda = 0$.

Next wee see the Landau-Lifshitz case($\lambda = 0.1$). We also have $\max\left||U_n^j| - 1\right| \sim 10^{-15}$, that is, we verify the length-preserving property numerically. Figure 3 shows the behaviors of the energy: The curved line describes the energy of numerical solution and "+" marks shows the energy of the exact solution. We see that the behavior of numerical energy is almost same as that of the exact solution. Figure 4 shows the behaviors of (a) U_1 and (b) u_1 of the exact solution. We also conclude that behaviors of numerical solution and the exact solution coincide very much.

From these numerical results via a test problem, we conclude that the proposed scheme is effective.

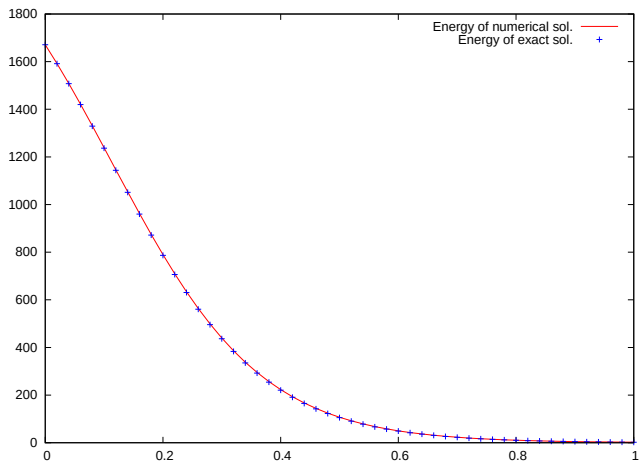
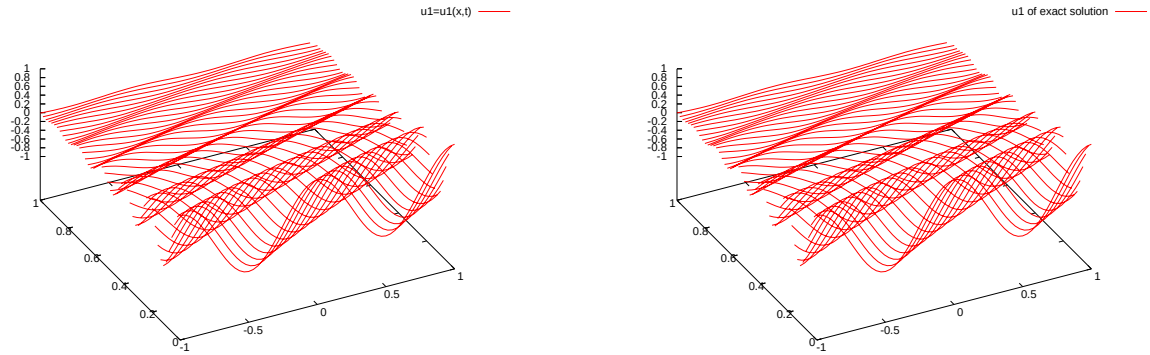


Fig. 3: Time evolution of energy for $\lambda = 0.1$: numerical solution (curved line) and exact solution ("+" mark).



(a) (b)

Fig. 4: Time evolutions of u_1 of (a) numerical solution and (b) exact solution in the case $\lambda = 0.1$.

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Boundedness in degenerate Keller-Segel systems of parabolic-parabolic type

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1. Introduction

We consider the following fully-parabolic Keller-Segel system on the whole space:

$$(\mathrm{KS})_0 \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (\nabla u^m - u^{q-1}\nabla v), & x \in \mathbb{R}^N,\ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v - v + u, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $N \in \mathbb{N}$, $m \geq 1$, $q \geq 2$ and the initial data u_0, v_0 are non-negative functions which has some regularity. The system with $m = 1$ and $q = 2$ is introduced by Keller and Segel in 1970 ([**5**]). These types of systems describe the chemitactic process of slime molds. Here $u = u(x,t)$ represents the cell density, $v = v(x,t)$ denotes the chemotaxis concentration at place $x \in \mathbb{R}^N$, time $t > 0$.

In the recent study, Sugiyama-Kunii [**6**] and Ishida-Yokota [**1, 2, 3**] consider $(\mathrm{KS})_0$ and reach the results of global solvability. However, they did not have the boundedness of the solution. Indeed, in [**6**], [**3**] the upper estimate for $\|u\|_{L^\infty(0,T;L^\infty(\mathbb{R}^N))}$ grows up as $T \to \infty$ (i.e., $\|u\|_{L^\infty(0,T;L^\infty(\mathbb{R}^N))} \leq C_1 T$). The reason of this growing is the following:

(I) When $q < m + \frac{2}{N}$, it follows that $\|u\|_{L^\infty(0,T;L^\infty(\mathbb{R}^N))} \leq C(\|u\|_{L^\infty(0,T;L^{r_*}(\mathbb{R}^N))})$ for large $r_* > 1$ and the upper estimate of $\|u\|_{L^\infty(0,T;L^{r_*}(\mathbb{R}^N))}$ is depending on T.

(II) When $q \geq m + \frac{2}{N}$, since they used the maximal Sobolev regularity of the second equation, it follows that $\|u\|_{L^\infty(0,T;L^\infty(\mathbb{R}^N))} \leq C_1 T$ even if u has the L^p-boundedness ($p \in [1,\infty)$).

Note that the approaches (I) and (II) slightly differ from one another because they use the condition $q < m + \frac{2}{N}$ or $q \geq m + \frac{2}{N}$.

On the other hand, in the bounded domain case, Tao-Winkler [**7**] give a successful result, that is, the global existence and the global-in-time boundedness of solutions to $(\mathrm{KS})_0$, i.e., $\|u\|_{L^\infty(0,\infty;L^\infty(\Omega))} \leq C_2$ where C_2 is not depending on time variable. They show the L^∞-boundedness from the L^{r_*}-boundedness of u for large $r_* \geq 1$ without using the condition between m and q. Unfortunately, they assume u solves $(\mathrm{KS})_0$ classically, and moreover, this boundedness is depending on $|\Omega|$.

Our goal is the global-in-time L^∞-boundedness of the solution to $(\mathrm{KS})_0$ on the whole space without using the condition between m and q. Note that it is difficult to apply Tao-Winkler's results for the following reasons; (1st reason) $(\mathrm{KS})_0$ does not have a classical

solution because of the degenerate diffusion; (2nd reason) their boundedness is depending on $|\Omega|$, while our system is on the whole space ($|\mathbb{R}^N| = \infty$).

For the above first reason, it might well have started with the following non-degenerate problem:

$$
(\mathrm{KS})_\delta \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (\nabla (u+\delta)^m - (u+\delta^{\frac{m-1}{q-2}})^{q-2} u \nabla v), & x \in \mathbb{R}^N,\ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v - v + u, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases}
$$

where $N \in \mathbb{N}$, $m \geq 1$, $q \geq 2$ and $\delta \in (0,1)$. The non-negative initial data u_0, v_0 belong to $C_0^\infty(\mathbb{R}^N)$. This type of systems is considered in [**6, 1, 2, 3**] as the approximated problem.

The main purpose in this work is to give a proof of the global-in-time L^∞-boundedness of the solution to $(\mathrm{KS})_\delta$ on the whole space without using the condition between m and q under assuming the L^p-boundedness of u.

Section 2 gives the main theorem and its proof which are related to L^∞-boundedness on $(\mathrm{KS})_\delta$. In Section 3 we state L^∞-boundedness on $(\mathrm{KS})_0$ when $q \geq m + \frac{2}{N}$.

2. Boundedness in $(\mathrm{KS})_\delta$

We discuss the boundedness of the solution to $(\mathrm{KS})_\delta$. Let us begin with giving the definition of solutions to $(\mathrm{KS})_\delta$.

Definition 2.1. Let $T > 0$. A pair (u, v) of non-negative functions defined on $\mathbb{R}^N \times (0,T)$ is called a *solution* to $(\mathrm{KS})_\delta$ on $[0,T)$ if for $r > N+1$

(a) $u \in W^{1,p}(0,T; L^p(\mathbb{R}^N)) \cap L^p(0,T; W^{2,p}(\mathbb{R}^N))$ for $p = \begin{cases} 2, r & N = 1, \\ \frac{N}{N-1}, r & N \geq 2, \end{cases}$

(b) $v \in W^{1,r}(0,T; L^r(\mathbb{R}^N)) \cap L^r(0,T; W^{2,r}(\mathbb{R}^N))$,

(c) (u, v) satisfies $(\mathrm{KS})_\delta$ almost everywhere on $\mathbb{R}^N \times [0,T)$.

In particular, if we can take $T = \infty$ and

$$
\|u\|_{L^\infty(0,\infty;L^\infty(\mathbb{R}^N))} < \infty, \quad \|v\|_{L^\infty(0,\infty;W^{1,\infty}(\mathbb{R}^N))} < \infty,
$$

then (u, v) is called a *global bounded solution* to $(\mathrm{KS})_\delta$.

Now we state the main theorem.

Theorem 2.1. *Let $m \geq 1$, $q \geq 2$, $n \in \mathbb{N}$. Assume that the initial data $u_0, v_0 \in C_0^\infty(\mathbb{R}^N)$ are non-negative and that the solution (u, v) satisfies $\|u\|_{L^\infty(0,\infty;L^{p_0}(\mathbb{R}^N))} < M_0$ for some large p_0, where $M > 0$ is a constant which does not depend on δ. Then the solution is globally bounded:*

$$
\|u\|_{L^\infty(0,\infty;L^\infty(\mathbb{R}^N))} < M.
$$

Remark 2.1. It seems that the L^{p_0}-boundedness is a big assumption. Fortunately, in the case where $q \geq m + \frac{2}{N}$ we have already proven it under the small initial data (see [**3**]).

Proof of Theorem 2.1. We divide the proof into the following five steps:

Step 1: Standard energy inequality derived from first equation.

Step 2: Energy inequality via the L^p-boundedness of ∇v.

Step 3: Energy inequality without ∇u.

Step 4: Creation of the ODI of $Z_r(t) := \|u(t)\|_{L^r}^r + \|u_0\|_{L^1}^r + 1$.

Step 5: Iteration for $Z_r(t)$ and conclusion of boundedness.

【Step 1: Standard energy inequality】 Multiplying the first equation in $(\mathrm{KS})_\delta$ by u^{r-1} and integrating it over $\mathbb{R}^N$, we see from $a^k+b^k \le (a+b)^k \le 2^k(a^k+b^k)$ for $a,b,k>0$ and the Young inequality that

$$
\begin{aligned}
(2.1)\qquad & \frac{1}{r}\frac{d}{dt}\|u(t)\|_{L^r(\mathbb{R}^N)}^r \\
&= -\int_{\mathbb{R}^N} \nabla(u+\delta)^m \cdot \nabla u^{r-1}\,dx + \int_{\mathbb{R}^N} (u+\delta^{\frac{m-1}{q-2}})^{q-2} u\nabla v \cdot \nabla u^{r-1}\,dx \\
&\le -\frac{m(r-1)}{2}\Big\{\int_{\mathbb{R}^N} u^{r+m-3}|\nabla u|^2\,dx + \delta^{m-1}\int_{\mathbb{R}^N} u^{r-2}|\nabla u|^2\,dx\Big\} \\
&\quad + 2^{2(q-2)}\frac{(r-1)}{m}\Big\{\int_{\mathbb{R}^N} u^{r+2q-m-3}|\nabla v|^2\,dx + \delta^{m-1}\int_{\mathbb{R}^N} u^r|\nabla v|^2\,dx\Big\}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
(2.2)\qquad & \frac{d}{dt}\|u(t)\|_{L^r(\mathbb{R}^N)}^r \\
&\quad + \frac{2mr(r-1)}{(r+m-1)^2}\big\|\nabla u^{\frac{r+m-1}{2}}(t)\big\|_{L^2(\mathbb{R}^N)}^2 + \delta^{m-1}\frac{2m(r-1)}{r}\big\|\nabla u^{\frac{r}{2}}(t)\big\|_{L^2(\mathbb{R}^N)}^2 \\
&\le 2^{2(q-2)}\frac{r(r-1)}{m}\Big\{\int_{\mathbb{R}^N} u^{r+2q-m-3}|\nabla v|^2\,dx + \delta^{m-1}\int_{\mathbb{R}^N} u^r|\nabla v|^2\,dx\Big\}.
\end{aligned}
$$

【Step 2: Energy inequality via the L^p-boundedness of ∇v】 We estimate the right-hand side of the above estimate, in particular, the first term. First we adjust the parameter γ, θ and s as like in [**7**, Appendix]. Let take $\gamma \in (2,\infty)$ (if $N=1,2$), $\gamma \in (2, \frac{2N}{N-2})$ (if $N \ge 3$) and put $\theta = \frac{\gamma}{2}\cdot\frac{r+m-1}{r-m-1}$, $\theta' = (1-\frac{1}{\theta})^{-1}$. Take $p_1 \ge 1$ such that

$$
(2.3)\qquad 2(q-1)p_1 \le p_0
$$

and pick $s \in (1,2)$. Let start deriving estimates. First, because of $\frac{1}{\theta}+\frac{1}{p_1}+\frac{p_1-\theta'}{p_1\theta'} = 1$, we have from the Young inequality that

$$
\begin{aligned}
(2.4)\qquad & \int_{\mathbb{R}^N} u^{r+2q-m-3}|\nabla v|^2\,dx \\
&\le \Big(\int_{\mathbb{R}^N} u^{(r-m-1)\theta}\,dx\Big)^{\frac{1}{\theta}}\Big(\int_{\mathbb{R}^N} u^{2(q-1)p_1}\,dx\Big)^{\frac{1}{p_1}}\Big(\int_{\mathbb{R}^N} |\nabla v|^{2\cdot\frac{p_1\theta'}{p_1-\theta'}}\,dx\Big)^{\frac{p_1-\theta'}{p_1\theta'}}.
\end{aligned}
$$

Now, from the fact that u has the mass conservation law $\|u(t)\|_{L^1(\mathbb{R}^N)} = \|u_0\|_{L^1(\mathbb{R}^N)}$ ($\forall t > 0$) and the assumption of L^p-boundedness we know

$$\|u\|_{L^\infty(0,\infty;L^p(\mathbb{R}^N))} \le K_1 \quad (1 \le p \le p_0),$$

so it follows from (2.3) that

$$\Big(\int_{\mathbb{R}^N} u^{2(q-1)p_1}(t)\,dx\Big)^{\frac{1}{p_1}} \le K_1^{2(q-1)}. \tag{2.5}$$

Let put $A := \frac{p_1\theta'}{p_1-\theta'}$. Noting that $\frac{p_1-\theta'}{\theta' p_1} \in (0,1)$ entails $2 < \frac{2p_1\theta'}{p_1-2\theta'} < \infty$, we have

$$\Big(\int_{\mathbb{R}^N} |\nabla v|^{2A}\,dx\Big)^{\frac{1}{A}} \le \|\nabla v(t)\|_{L^\infty(\mathbb{R}^N)}^{\frac{2(A-1)}{A}} \|\nabla v(t)\|_{L^2(\mathbb{R}^N)}^{\frac{2}{A}}.$$

Since $0 < \frac{2(A-1)}{A}, \frac{2}{A} < 2$, it follows that

$$\Big(\int_{\mathbb{R}^N} |\nabla v|^{2\cdot\frac{p_1\theta'}{p_1-\theta'}}\,dx\Big)^{\frac{p_1-\theta'}{p_1\theta'}} \le \max\{1, \|\nabla v(t)\|_{L^\infty}^2\}\max\{1, \|\nabla v(t)\|_{L^2}^2\}.$$

Here, the L^p-L^q estimate for the heat semigroup and the condition $p_0 > N$ derive $\|\nabla v(t)\|_{L^r} \le K_2$ ($r \in [1,\infty]$) (see [**7**, Lemma 1.2], [**4**, Lemma 2.3], for instance), and so we see that

$$\Big(\int_{\mathbb{R}^N} |\nabla v|^{2\cdot\frac{p_1\theta'}{p_1-\theta'}}\,dx\Big)^{\frac{p_1-\theta'}{p_1\theta'}} \le \big(\max\{1, K_2^2\}\big)^2. \tag{2.6}$$

Next, by the choice of γ, we can use the Gagliardo-Nirenberg inequality and find $c_1 > 0$ such that

$$\begin{aligned}\Big(\int_{\mathbb{R}^N} u^{(r-m-1)\theta}\,dx\Big)^{\frac{1}{\theta}} &= \big\|u^{\frac{r+m-1}{2}}(t)\big\|_{L^\gamma(\mathbb{R}^N)}^{\frac{\gamma}{\theta}} \\ &\le c_1\big\|\nabla u^{\frac{r+m-1}{2}}(t)\big\|_{L^2(\mathbb{R}^N)}^{\frac{\alpha\gamma}{\theta}}\big\|u^{\frac{r+m-1}{2}}(t)\big\|_{L^s(\mathbb{R}^N)}^{\frac{(1-\alpha)\gamma}{\theta}},\end{aligned} \tag{2.7}$$

where

$$\alpha = \Big(\frac{1}{s} - \frac{1}{\gamma}\Big)\Big(\frac{1}{N} + \frac{1}{s} - \frac{1}{2}\Big)^{-1} \in (0,1).$$

Hence connecting (2.4)–(2.7) gives

$$\int_{\mathbb{R}^N} u^{r+2q-m-3}|\nabla v|^2\,dx \le c_2\big\|\nabla u^{\frac{r+m-1}{2}}(t)\big\|_{L^2(\mathbb{R}^N)}^{\frac{\alpha\gamma}{\theta}}\big\|u^{\frac{r+m-1}{2}}(t)\big\|_{L^s(\mathbb{R}^N)}^{\frac{(1-\alpha)\gamma}{\theta}}, \tag{2.8}$$

where $c_2 = c_1 K_1^{2(q-1)}\max\{1, K_2^4\}$. Thanks to

$$\frac{\alpha\gamma}{\theta} = \alpha\gamma\cdot\frac{2(r-m-1)}{\gamma(r+m-1)} < 2\alpha < 2$$

because of $m \geq 1$, the Young inequality gives

$$
\begin{aligned}
(2.9)\qquad & 2^{2(q-2)}\frac{r(r-1)}{m}\int_{\mathbb{R}^N} u^{r+2q-m-3}|\nabla v|^2\,dx \\
&\leq \tfrac{mr(r-1)}{(r+m-1)^2}\big\|\nabla u^{\frac{r+m-1}{2}}(t)\big\|^2_{L^2(\mathbb{R}^N)} \\
&\qquad + r(r-1)\Big(\tfrac{2^{2(q-2)}c_2}{m}\big(\tfrac{m}{(r+m-1)^2}\big)^{\frac{-\alpha\gamma}{2\theta}}\Big)^{\frac{2\theta}{2\theta-\alpha\gamma}}\Big(\int_{\mathbb{R}^N} u^{\frac{s(r+m-1)}{2}}\,dx\Big)^{\frac{2(1-\alpha)\gamma}{s(2\theta-\alpha\gamma)}}.
\end{aligned}
$$

From the same argument as above we infer that

$$
\begin{aligned}
(2.10)\qquad & \frac{r(r-1)}{m}\int_{\mathbb{R}^N} u^r|\nabla v|^2\,dx \\
&\leq \tfrac{m(r-1)}{r}\delta^{m-1}\|\nabla u^{\frac{r}{2}}\|^2_{L^2} \\
&\qquad + r(r-1)\delta^{m-1}\Big(\tfrac{2^{2q-3}c_3}{m}\big(\tfrac{m}{r^2}\big)^{-\frac{\alpha\gamma}{2\eta}}\Big)^{\frac{2\eta}{2\eta-\alpha\gamma}}\Big(\int_{\mathbb{R}^N} u^{\frac{sr}{2}}\,dx\Big)^{\frac{2\gamma(1-\alpha)}{s(2\eta-\alpha\gamma)}},
\end{aligned}
$$

where $\eta = \frac{\gamma r}{2(r-2)}$, $c_3 = c_3(K_1, K_2, N) > 0$ is a constant. For large r we find $c_4, c_5 > 0$ which are independent of r such that

$$
\Big(\tfrac{2^{2(q-2)}c_2}{m}\big(\tfrac{m}{(r+m-1)^2}\big)^{\frac{-\alpha\gamma}{2\theta}}\Big)^{\frac{2\theta}{2\theta-\alpha\gamma}} \leq c_4, \qquad \Big(\tfrac{2^{2q-3}c_3}{m}\big(\tfrac{m}{r^2}\big)^{-\frac{\alpha\gamma}{2\eta}}\Big)^{\frac{2\eta}{2\eta-\alpha\gamma}} \leq c_5
$$

because $\theta \to \frac{\gamma}{2}$ and then from (2.2), (2.9), (2.10) we have

$$
\begin{aligned}
(2.11)\qquad & \frac{d}{dt}\|u(t)\|^r_{L^r(\mathbb{R}^N)} + \frac{mr(r-1)}{(r+m-1)^2}\big\|\nabla u^{\frac{r+m-1}{2}}\big\|^2_{L^2(\mathbb{R}^N)} \\
&\leq r(r-1)c_4\Big(\int_{\mathbb{R}^N} u^{\frac{s(r+m-1)}{2}}\,dx\Big)^{\frac{2(1-\alpha)\gamma}{s(2\theta-\alpha\gamma)}} + r(r-1)\delta^{m-1}c_5\Big(\int_{\mathbb{R}^N} u^{\frac{sr}{2}}\Big)^{\frac{2\gamma(1-\alpha)}{s(2\eta-\alpha\gamma)}}.
\end{aligned}
$$

【Step 3: Energy inequality without ∇u】 Next we consider the second and third terms on the left-hand side of (2.11). From the Gagliardo-Nirenberg inequality and the Young inequality we have

$$
\int_{\mathbb{R}^N} u^r\,dx \leq \Big(\big\|\nabla u^{\frac{r+m-1}{2}}(t)\big\|_{L^2(\mathbb{R}^N)} + \big\|u^{\frac{r+m-1}{2}}(t)\big\|_{L^s(\mathbb{R}^N)}\Big)^{\frac{2r}{r+m-1}}
$$

hence we see that

$$
(2.12)\qquad \frac{1}{2}\Big(\int_{\mathbb{R}^N} u^r\,dx\Big)^{\frac{r+m-1}{r}} - \Big(\int_{\mathbb{R}^N} u^{\frac{s(r+m-1)}{2}}\,dx\Big)^{\frac{2}{s}} \leq \big\|\nabla u^{\frac{r+m-1}{2}}(t)\big\|^2_{L^2(\mathbb{R}^N)}.
$$

Plugging (2.12) into (2.11) yields

$$
\begin{aligned}
& \frac{d}{dt}\|u(t)\|^r_{L^r(\mathbb{R}^N)} + \frac{mr(r-1)}{(r+m-1)^2}\Big(\frac{1}{2}\Big(\int_{\mathbb{R}^N} u^r\,dx\Big)^{\frac{r+m-1}{r}} - \Big(\int_{\mathbb{R}^N} u^{\frac{s(r+m-1)}{2}}\,dx\Big)^{\frac{2}{s}}\Big) \\
&\leq r(r-1)c_4\Big(\int_{\mathbb{R}^N} u^{\frac{s(r+m-1)}{2}}\,dx\Big)^{\frac{2(1-\alpha)\gamma}{s(2\theta-\alpha\gamma)}} + r(r-1)\delta^{m-1}c_5\Big(\int_{\mathbb{R}^N} u^{\frac{sr}{2}}\Big)^{\frac{2\gamma(1-\alpha)}{s(2\eta-\alpha\gamma)}},
\end{aligned}
$$

then we have

$$\begin{aligned}
&\frac{d}{dt}\|u(t)\|_{L^r(\mathbb{R}^N)}^r + \frac{mr(r-1)}{2(r+m-1)^2}\Big(\int_{\mathbb{R}^N} u^r\,dx\Big)^{\frac{r+m-1}{r}} \\
&\le r(r-1)c_4\Big(\int_{\mathbb{R}^N} u^{\frac{s(r+m-1)}{2}}\,dx\Big)^{\frac{2(1-\alpha)\gamma}{s(2\theta-\alpha\gamma)}} + \frac{mr(r-1)}{(r+m-1)^2}\Big(\int_{\mathbb{R}^N} u^{\frac{s(r+m-1)}{2}}\,dx\Big)^{\frac{2}{s}} \\
&\quad + r(r-1)\delta^{m-1}c_5\Big(\int_{\mathbb{R}^N} u^{\frac{sr}{2}}\Big)^{\frac{2\gamma(1-\alpha)}{s(2\eta-\alpha\gamma)}}.
\end{aligned}$$

To simplify this estimate let take $r > r_0$ with r_0 satisfies $\frac{1}{2} \le \frac{r_0-1}{r_0} \le 1$, $\frac{1}{2} \le \frac{r_0(r_0-1)}{(r_0+m-1)^2} \le 1$. Since

$$\frac{2(1-\alpha)\gamma}{s(2\theta-\alpha\gamma)} \le \frac{2}{s}, \quad \frac{2(1-\beta)\gamma}{s(2\eta-\beta\gamma)} \le \frac{2}{s}$$

from $\gamma > 2$ and $\theta, \eta \ge \frac{\gamma}{2}$ and $\delta^{m-1} < 1$, it follows from the Young inequality that

$$\begin{aligned}
&\frac{d}{dt}\|u(t)\|_{L^r(\mathbb{R}^N)}^r + \frac{m}{4}\Big(\int_{\mathbb{R}^N} u^r\,dx\Big)^{\frac{r+m-1}{r}} \\
&\le r^2c_6\Big\{1 + \Big(\int_{\mathbb{R}^N} u^{\frac{s(r+m-1)}{2}}\,dx\Big)^{\frac{2}{s}} + \Big(\int_{\mathbb{R}^N} u^{\frac{sr}{2}}\,dx\Big)^{\frac{2}{s}}\Big\},
\end{aligned} \tag{2.13}$$

where $c_6 = \max\{c_4 + c_5, c_4 + m, c_5\} > 0$ is a constant.

【Step 4: Creation of the ODI of $Z_r(t)$】 Here we define

$$Y_r(t) := \int_{\mathbb{R}^N} u^r\,dx.$$

Then (2.13) is rewritten as

$$\frac{d}{dt}Y_r(t) + \frac{m}{4}\big(Y_r(t)\big)^{\frac{r+m-1}{r}} \le c_6r^2\Big\{1 + \big(Y_{\frac{s(r+m-1)}{2}}(t)\big)^{\frac{2}{s}} + \big(Y_{\frac{sr}{2}}(t)\big)^{\frac{2}{s}}\Big\}.$$

Let us take r such that $1 < \frac{sr}{2}$. Since

$$\begin{aligned}
\big(Y_{\frac{sr}{2}}(t)\big)^{\frac{2}{s}} = \|u\|_{L^{\frac{sr}{2}}}^r &\le \|u_0\|_{L^1}^r + \|u(t)\|_{L^{\frac{s(r+m-1)}{2}}}^r \\
&= \|u_0\|_{L^1}^r + \Big(Y_{\frac{s(r+m-1)}{2}}\Big)^{\frac{2}{s}\cdot\frac{r}{(r+m-1)}} \\
&\le \|u_0\|_{L^1}^r + 1 + \Big(Y_{\frac{s(r+m-1)}{2}}\Big)^{\frac{2}{s}},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\frac{d}{dt}Y_r(t) + \frac{m}{4}\big(Y_r(t)\big)^{\frac{r+m-1}{r}} \\
&\le c_6r^2\Big\{1 + \big(Y_{\frac{s(r+m-1)}{2}}(t)\big)^{\frac{2}{s}} + \Big(\|u_0\|_{L^1}^r + 1 + \Big(Y_{\frac{s(r+m-1)}{2}}\Big)^{\frac{2}{s}}\Big)\Big\} \\
&\le c_6r^2\Big(2 + \|u_0\|_{L^1}^r + 2\big(Y_{\frac{s(r+m-1)}{2}}\big)^{\frac{2}{s}}\Big)\Big).
\end{aligned} \tag{2.14}$$

Next, (2.14) added $\|u_0\|_{L^1(\mathbb{R}^N)}^{r+m-1}+1$ gives

$$\begin{aligned}(2.15)\qquad &\frac{d}{dt}Y_r(t)+\frac{m}{4}\big(Y_r(t)\big)^{\frac{r+m-1}{r}}+\|u_0\|_{L^1(\mathbb{R}^N)}^{r+m-1}+1\\ &\le c_6r^2\Big(2+\|u_0\|_{L^1}^r+2\big(Y_{\frac{s(r+m-1)}{2}}\big)^{\frac{2}{s}}\big)\Big)+\|u_0\|_{L^1(\mathbb{R}^N)}^{r+m-1}+1.\end{aligned}$$

Now we create the unit term of iteration. First we clearly have

$$\frac{d}{dt}Y_r(t)=\frac{d}{dt}\Big\{Y_r(t)+\|u_0\|_{L^1(\mathbb{R}^N)}^r+1\Big\}. \tag{2.16}$$

Next we have for $r>m+1$,

$$\frac{m}{4}\big(Y_r(t)\big)^{\frac{r+m-1}{r}}+\|u_0\|_{L^1(\mathbb{R}^N)}^{r+m-1}+1\ge c_7\Big\{Y_r(t)+\|u_0\|_{L^1(\mathbb{R}^N)}^r+1\Big\}^{\frac{r+m-1}{r}}, \tag{2.17}$$

where $c_7=\frac{1}{2}\min\{\frac{m}{4},1\}$ because

$$\begin{aligned}c_7\Big\{Y_r(t)+\|u_0\|_{L^1(\mathbb{R}^N)}^r+1\Big\}^{\frac{r+m-1}{r}}&\le c_7 2^{\frac{r+m-1}{r}-1}\Big\{Y_r(t)^{\frac{r+m-1}{r}}+\|u_0\|_{L^1(\mathbb{R}^N)}^{r\cdot\frac{r+m-1}{r}}+1^{\frac{r+m-1}{r}}\Big\}\\ &\le\frac{1}{2}\cdot 2^{\frac{m-1}{r}}\Big\{\frac{m}{4}\big(Y_r(t)\big)^{\frac{r+m-1}{r}}+\|u_0\|_{L^1(\mathbb{R}^N)}^{r+m-1}+1\Big\}\\ &\le\frac{m}{4}\big(Y_r(t)\big)^{\frac{r+m-1}{r}}+\|u_0\|_{L^1(\mathbb{R}^N)}^{r+m-1}+1.\end{aligned}$$

Finally it follows that

$$\begin{aligned}(2.18)\qquad &c_6r^2\Big(2+\|u_0\|_{L^1}^r+2\big(Y_{\frac{s(r+m-1)}{2}}\big)^{\frac{2}{s}}\big)\Big)+\|u_0\|_{L^1(\mathbb{R}^N)}^{r+m-1}+1\\ &\le c_6r^2\Big(2+1+\|u_0\|_{L^1}^{r+m-1}+2\big(Y_{\frac{s(r+m-1)}{2}}\big)^{\frac{2}{s}}\big)\Big)+\|u_0\|_{L^1(\mathbb{R}^N)}^{r+m-1}+1\\ &\le 4c_6r^2\Big(1+\|u_0\|_{L^1}^{r+m-1}+\big(Y_{\frac{s(r+m-1)}{2}}\big)^{\frac{2}{s}}\big)\Big)\\ &\le 4c_6r^2\Big(1+\|u_0\|_{L^1}^{r+m-1\cdot\frac{s}{2}}+\big(Y_{\frac{s(r+m-1)}{2}}\big)\Big)^{\frac{2}{s}}.\end{aligned}$$

Hence (2.15) combined with (2.16)–(2.18) gives that

$$\begin{aligned}(2.19)\qquad &\frac{d}{dt}\Big\{Y_r(t)+\|u_0\|_{L^1(\mathbb{R}^N)}^r+1\Big\}+c_7\Big\{Y_r(t)+\|u_0\|_{L^1(\mathbb{R}^N)}^r+1\Big\}^{\frac{r+m-1}{r}}\\ &\le 4c_6r^2\Big(1+\|u_0\|_{L^1}^{r+m-1\cdot\frac{s}{2}}+\big(Y_{\frac{s(r+m-1)}{2}}\big)\Big)^{\frac{2}{s}}.\end{aligned}$$

Let $r=p_k$ where p_k is given by

$$p_k:=\frac{2}{s}p_{k-1}-m+1 \text{ with } p_0=\max\{N,r_0,r_1,\tfrac{2}{s},m+1\}.$$

Note that $p_{k-1}=\frac{s(p_k+m-1)}{2}$, $p_{k-1}<p_k$ when $p_0>\frac{s(m-1)}{2-s}$) and that

$$p_k=\Big(\frac{2}{s}\Big)^k p_0+\Big(\Big(\frac{2}{s}\Big)^{k-1}+\cdots+\frac{2}{s}+1\Big)(-m+1)$$

implies

$$p_0\Big(\frac{2}{s}\Big)^k < p_k < \big(p_0 + \tfrac{s}{2-s}\big)\Big(\frac{2}{s}\Big)^k.$$

Let us define

$$Z_r(t) = Y_r(t) + \|u_0\|^r_{L^1(\mathbb{R}^N)} + 1.$$

Then we have

$$\frac{d}{dt}Z_{p_k}(t) + c_7\big(Z_{p_k}(t)\big)^{\frac{p_k+m-1}{p_k}} \le c_8\big(\tfrac{2}{s}\big)^k \sup_{t>0}\big(Z_{p_{k-1}}(t)\big)^{\frac{2}{s}},$$

where $c_8 = 4c_6(p_0 + \frac{s}{2-s})$.

【Step 5: Iteration and conclusion】 From the Gronwall type inequality, the above ODI gives

$$Z_{p_k}(t) \le \max\Big\{Z_{p_k}(0), \Big[\tfrac{c_{10}}{c_9}\big(\tfrac{2}{s}\big)^k \sup_{t>0}\big(Z_{p_{k-1}}(t)\big)^{\frac{2}{s}}\Big]^{\frac{p_k}{p_k+m-1}}\Big\}. \tag{2.20}$$

Noting that

$$Z_{p_k}(0)^{\frac{1}{p_k}} = \Big(\int_{\mathbb{R}^N} u_0^{p_k}\,dx + \|u_0\|^{p_k}_{L^1(\mathbb{R}^N)} + 1\Big)^{\frac{1}{p_k}} \to \|u_0\|_{L^\infty(\mathbb{R}^N)} + \|u_0\|_{L^1(\mathbb{R}^N)} + 1 =: \mathcal{R}$$

as $k \to \infty$ we find that it is enough to consider the following two cases by taking a sub-series of $\{k\}$ if necessary:

(i): $\mathcal{R} < \Big[\frac{c_{10}}{c_9}\big(\frac{2}{s}\big)^k \sup_{t>0}\big(Z_{p_{k-1}}(t)\big)^{\frac{2}{s}}\Big]^{\frac{1}{p_k+m-1}}$ for any $k \gg 1$.

In this case (2.20) gives

$$\sup_{t>0} Z_{p_k}(t) \le \Big[C\big(\tfrac{2}{s}\big)^k \sup_{t>0}\big(Z_{p_{k-1}}(t)\big)^{\frac{2}{s}}\Big]^{\frac{p_k}{p_k+m-1}},$$

where $C = \max\{1, \frac{c_{10}}{c_9}\}$. Because $C, \frac{2}{s} > 1$ and $Z_k > 1$, $\frac{p_k}{p_k+m-1} < 1$ for any $k \ge 1$ it follows that

$$\sup_{t>0} Z_{p_k}(t) \le C\big(\tfrac{2}{s}\big)^k \sup_{t>0}\big(Z_{p_{k-1}}(t)\big)^{\frac{2}{s}}.$$

By Moser's iteration technique, we have

$$\sup_{t>0} Z_{p_k}(t) \le C^{1+\frac{2}{s}+(\frac{2}{s})^2+\cdots+(\frac{2}{s})^{k-2}}\big(\tfrac{2}{s}\big)^{1\cdot k+\frac{2}{s}\cdot(k-1)+(\frac{2}{s})^2\cdot(k-2)+\cdots+(\frac{2}{s})^{k-1}\cdot 1} \sup_{t>0}\big(Z_{p_1}(t)\big)^{(\frac{2}{s})^k}.$$

Noting that $(p_0 + \frac{s}{2-s})(\frac{s}{2})^k \le \frac{1}{p_k} \le p_0^{-1}(\frac{s}{2})^k$, we have

$$\lim_{k\to\infty}\frac{1}{p_k}\sum_{n=1}^{k-1}\Big(\frac{2}{s}\Big)^{n-1} \le \frac{1}{p_0}\lim_{k\to\infty}\sum_{n=1}^{k-1}\Big(\frac{s}{2}\Big)^{n+1} = \frac{s}{2(2-s)p_0},$$

$$\lim_{k\to\infty}\frac{1}{p_k}\Big\{1\cdot k + \frac{2}{s}\cdot(k-1) + \Big(\frac{2}{s}\Big)^2\cdot(k-2) + \cdots + \Big(\frac{2}{s}\Big)^{k-1}\cdot 1\Big\} = \frac{2s}{(2-s)^2 p_0},$$

and

$$\lim_{k\to\infty}\frac{1}{p_k}\Big(\frac{2}{s}\Big)^{k-1}\le\frac{s}{2p_0},$$

then we obtain

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)}=\lim_{k\to\infty}\sup_{t>0}Z_{p_k}(t)^{\frac{1}{p_k}}-\|u_0\|_{L^1(\mathbb{R}^N)}-1\le M,$$

where $M>0$ is a constant which depends on $\sup_{t>0}\|u(t)\|_{L^{p_0}(\mathbb{R}^N)}$, $\|u_0\|_{L^1(\mathbb{R}^N)}$, m, N but not on time variable.

(ii): $\Big[\frac{c_{10}}{c_9}\big(\frac{2}{s}\big)^k\sup_{t>0}\big(Z_{p_{k-1}}(t)\big)^{\frac{2}{s}}\Big]^{\frac{1}{p_k+m-1}}<\mathcal{R}$ for any $k\gg 1$.

This case implies

$$\sup_{t>0}Z_{p_k}^{\frac{1}{p_k}}(t)\le Z_{p_k}^{\frac{1}{p_k}}(0)$$

and we obtain

$$\sup_{t>0}\|u(t)\|_{L^\infty(\mathbb{R}^N)}\le\|u_0\|_{L^\infty(\mathbb{R}^N)}.$$

The L^∞-boundedness of u can be obtained from the above two cases, i.e., there exist a positive constant M which is not depending on time such that

$$\|u(t)\|_{L^\infty}\le M.$$

Moreover, the $W^{1,\infty}$-boundedness of v is given by L^p-L^q estimate of heat semigroup. □

3. Boundedness in $(\mathrm{KS})_0$ in the super-critical case

In this section we state the L^∞-boundedness in $(\mathrm{KS})_0$. As mentioned in Remark 2.1, we already had the L^p-boundedness on $(\mathrm{KS})_\delta$ as the approximate problem in super-critical case (i.e., $q\ge m+\frac{2}{N}$). Hence, this boundedness and Theorem 2.1 drive the global-in-time boundedness to the degenerate system $(\mathrm{KS})_0$:

Theorem 3.1. *Assume that m and q satisfy*

$$q\ge m+\frac{2}{N}.$$

Assume that the initial data satisfies

$$u_0,v_0\ge 0,\ \ u_0,v_0\in L^1\cap L^\infty(\mathbb{R}^N),$$
$$v_0\in B^{2-\frac{2}{r+q-1}}_{r+q-1,r+q-1}\cap B^{2-\frac{2}{r+1}}_{r+1,r+1}(\mathbb{R}^N),\quad \Delta v_0\in L^{q_c+q-1}\cap L^{q_0+q-1}\cap L^\infty(\mathbb{R}^N)$$

with some $r>1$ and $q_c:=\frac{N}{2}(q-m)$, $q_0:=\frac{N}{2}$ and satisfies following smallness:

$$\|u_0\|_{L^{q_c}}\le\delta_u,\ \ \|v_0\|_{B^{2-2/(q_c+q-1)}_{q_c+q-1,q_c+q-1}}\le\delta_v\ \textit{when}\ q\ge m+1\ (N\ge 3)\ \textit{or}\ N=2,$$

$$\|u_0\|_{L^{q_0}}\le\delta_u,\ \ \|v_0\|_{B^{2-2/(q_0+q-1)}_{q_0+q-1,q_0+q-1}}\le\delta_v\ \textit{when}\ q<m+1\ (N\ge 3),$$

where $\delta_u = \delta_u(m,q,N)$, $\delta_v = \delta_v(m,q,N)$ are positive constants. Then $(\mathrm{KS})_0$ *has a non-negative global-in-time solution (u,v) such that*

$$u \in L^\infty(0,\infty;L^p(\mathbb{R}^N)) \ (\forall\, p \in [1,\infty]), \quad u^m \in L^2(0,T;H^1(\mathbb{R}^N)) \ (\forall\, T > 0),$$
$$v \in L^\infty(0,\infty;H^1(\mathbb{R}^N)),$$
$$(u,v) \text{ satisfies } (\mathrm{KS})_0 \text{ in the sense of distributions}$$

with

$$\|u\|_{L^\infty(0,\infty;L^p(\mathbb{R}^N))} \le M,$$

where $M > 0$ is independent of time variable.

Future work: In the present work we derived the globally L^∞-boundedness on $(\mathrm{KS})_\delta$ without using the condition of m and q (Theorem 2.1), and moreover, can obtain it on $(\mathrm{KS})_0$ only in the case where $q \ge m + \frac{2}{N}$ because [**3**] allows the L^p-boundedness of u in this case (Theorem 3.1). So, our next interest is suitable for finding L^p-boundedness of u with the condition $q < m + \frac{2}{N}$.

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An error estimate of conservative finite difference scheme for the Boussinesq type equations

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1 Introduction

We consider the Boussinesq type equation

$$\partial_t^2 u + \gamma \partial_x^4 u - \alpha_1 \partial_x^2 u = \partial_x^2 f(u), \qquad (x,t) \in (0,L) \times (0,T), \tag{1.1}$$

with the periodic boundary condition $u(x,t) = u(x+L,t)$. We assume that constants γ and α_1 are positive and that f is a polynomial and a *definite energy type* such that for some constant C and a primitive F of f

$$F(u) \geq -C \qquad (u \in \mathbb{R}). \tag{1.2}$$

Examples of f are given by

$$f(u) = \alpha_\rho u^\rho \qquad (\rho \text{ is a odd number}, \alpha_\rho > 0), \tag{1.3}$$

$$f(u) = -\alpha_3 u^3 + \alpha_5 u^5 \qquad (\alpha_3,\ \alpha_5 > 0) \tag{1.4}$$

etc. The equation (1.1) with (1.4) was derived by Falk et al. as the model representing the phase transition on shape memory alloys (we refer to [2]). Originally, the equation with the opposite sign in the fourth derivative $\gamma < 0$ was derived by Boussinesq [1] as the model of water waves. In [4] Furihata proposed the discrete variational derivative method (DVDM) which is the method to derive the structure preserving finite difference scheme for evolution equations. More concretely, the finite difference scheme derived by DVDM preserves the property such as energy conservation law or dissipative law. In [4] the conservative finite difference scheme for the Cahn-Hilliard equation was derived by DVDM and the error estimate between an exact solution and a solution of the scheme was given. For more precise informations and recent progress of DVDM, we refer to the monograph by Furihata and Matsuo [5]. In [7] Matsuo proposed several numerical schemes for the good Boussinesq equation by DVDM, on the other hand, an error estimate has not been gave in [7].

In this study we give an error estimate between the solution for (1.1) and the solution for one of numerical schemes derived in [7]. Moreover our motivation of this study is to give the error estimate as general as possible in order to apply to other equations. This study is a joint work with Kyosuke Ichikawa based on [6]

2 Preliminary

We denote by ∂_t and ∂_x partial differential operators with respect to variables t and x, respectively. We split space interval $[0,L]$ into K-th parts and time interval $[0,T]$ into N-th parts, and hence we see that $L = K\Delta x$ and $T = N\Delta t$. For $k = 0,1,\ldots,K$ and $n = 0,1,\ldots,N$ we write $u_k^{(n)} = u(k\Delta x, n\Delta t)$ for the solution u of the original equation. Let $U_k^{(n)}$ an approximate solution for difference scheme corresponding $u_k^{(n)}$.

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We introduce several definitions of difference operators which are the almost same as the notation in Furihata-Matsuo [5]. We define several difference operators δ^+, δ^-, $\delta^{\langle 1\rangle}$, $\delta^{\langle 2\rangle}$ as an approximation to the differential operators by

$$\delta_k^+ u_k^{(n)} := \frac{u_{k+1}^{(n)} - u_k^{(n)}}{\Delta x}, \qquad \delta_k^- u_k^{(n)} := \frac{u_k^{(n)} - u_{k-1}^{(n)}}{\Delta x},$$
$$\delta_k^{\langle 1\rangle} u_k^{(n)} := \frac{u_{k+1}^{(n)} - u_{k-1}^{(n)}}{2\Delta x}, \qquad \delta_k^{\langle 2\rangle} u_k^{(n)} := \frac{u_{k+1}^{(n)} - 2u_k^{(n)} + u_{k-1}^{(n)}}{\Delta x^2}.$$

We approximate an integral by the trapezoid formula

$$\sum_{k=0}^{K}{}'' u_k \Delta x := \left(\frac{1}{2}u_0 + \sum_{k=1}^{K-1} u_k + \frac{1}{2}u_K\right)\Delta x.$$

Shift operators s and mean value operators μ are defined by

$$s_k^+ u_k^{(n)} := u_{k+1}^{(n)}, \qquad s_k^- u_k^{(n)} := u_{k-1}^{(n)}, \qquad s_k^{\langle 1\rangle} := \frac{s_k^+ + s_k^-}{2},$$
$$\mu_k^+ := \frac{s_k^+ + 1}{2}, \qquad \mu_k^- := \frac{1 + s_k^-}{2}, \qquad \mu_k^{\langle 1\rangle} := \frac{\mu_k^+ + \mu_k^-}{2} = \frac{s_k^+ + 2 + s_k^-}{4}.$$

In the same way as the above definitions, we also define difference, shift and mean value operators with respect to time variable by $\delta_n^\pm$, $\delta_n^{\langle m\rangle}$, $s_n^\pm$, $s_n^{\langle 1\rangle}$, $\mu_n^\pm$ and $\mu_n^{\langle 1\rangle}$.

From the direct calculation, the formulae corresponding to the fundamental theorem of differentiation-integral hold

$$\sum_{k=0}^{K}{}'' \delta_k^{\langle 1\rangle} u_k^{(n)} \Delta x = \left[\mu_k^{\langle 1\rangle} u_k^{(n)}\right]_{k=0}^{K}, \qquad \sum_{k=0}^{K}{}'' \delta_k^{\langle 2\rangle} u_k^{(n)} \Delta x = \left[\delta_k^{\langle 1\rangle} u_k^{(n)}\right]_{k=0}^{K}. \tag{2.1}$$

Moreover, we also obtain

$$\begin{aligned}\sum_{k=0}^{K}{}'' \left(\delta_k^{\langle 2\rangle} u_k^{(n)}\right) \widetilde{u}_k^{(n)} \Delta x &+ \sum_{k=0}^{K}{}'' \frac{\left(\delta_k^+ u_k^{(n)}\right)\left(\delta_k^+ \widetilde{u}_k^{(n)}\right) + \left(\delta_k^- u_k^{(n)}\right)\left(\delta_k^- \widetilde{u}_k^{(n)}\right)}{2}\Delta x \\ &= \left[\frac{\left(\delta_k^+ u_k^{(n)}\right)\left(\mu_k^+ \widetilde{u}_k^{(n)}\right) + \left(\delta_k^- u_k^{(n)}\right)\left(\mu_k^- \widetilde{u}_k^{(n)}\right)}{2}\right]_{k=0}^{K},\end{aligned} \tag{2.2}$$

which corresponds to the integration by part

$$\int_0^L u\partial_x^2 v \,\mathrm{d}x + \int_0^L \partial_x u \partial_x v \,\mathrm{d}x = \left[u\partial_x v\right]_{x=0}^{L}.$$

For simplicity, we denote by $\phi^{\langle\rho\rangle}$ the following ρ-th homogeneous polynomial:

$$\phi^{\langle\rho\rangle}(u,v) := \sum_{j=0}^{\rho} u^{\rho-j} v^j \quad (\rho \in \mathbb{N} \cup \{0\}), \tag{2.3}$$

which appears naturally through the procedure of the calculation for the discrete variational derivative.

3 Difference Scheme

In this section we introduce the derivation of the difference scheme for the Boussinesq type equation (1.1) applying DVDM to the equation. For simplicity we only treat the equation with simple nonlinearity (1.3). The Hamiltonian form of the equation (1.1) with (1.3) is rewritten as

$$\begin{cases} \partial_t u = \partial_x^2 v, \\ \partial_t v = -\gamma \partial_x^2 u + \alpha_1 u + \alpha_\rho u^\rho. \end{cases} \tag{3.1}$$

The energy of this system is given as

$$G(u,v) = \frac{1}{2}\left(\partial_x v\right)^2 + \frac{1}{2}\gamma\left(\partial_x u\right)^2 + \frac{1}{2}\alpha_1 u^2 + \frac{1}{\rho+1}\alpha_\rho u^{\rho+1}. \tag{3.2}$$

By using the energy G we may rewrite the equation (3.1) to

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta G}{\delta u} \\ \\ \frac{\delta G}{\delta v} \end{pmatrix}. \tag{3.3}$$

To introduce the discrete variational derivative, we give more precise explanation of the above. The total energy I is defined by

$$I(u,v) = \int_0^L G(u,v)dx. \tag{3.4}$$

Since from the direct calculation

$$I(u+\delta u, v) - I(u,v) = \int_0^L \left(-\gamma\partial_x^2 u + \alpha_1 u + \alpha_\rho u^\rho\right)\delta u dx,$$
$$I(u, v+\delta v) - I(u,v) = -\int_0^L \partial_x^2 v \delta v dx,$$

the variational derivatives $\dfrac{\delta G}{\delta u}$ and $\dfrac{\delta G}{\delta v}$ with respect to u and v are given as

$$\frac{\delta G}{\delta u} = -\gamma\partial_x^2 u + \alpha_1 u + \alpha_\rho u^\rho, \qquad \frac{\delta G}{\delta v} = -\partial_x^2 v.$$

From now on, we start to derive the difference scheme approximating the problem (3.3). Corresponding to the definition (3.2), we define the discrete energy G_d by

$$G_d(U_k, V_k) := \frac{1}{2}\frac{\left(\delta_k^+ V_k\right)^2 + \left(\delta_k^- V_k\right)^2}{2} + \frac{1}{2}\gamma\frac{\left(\delta_k^+ U_k\right)^2 + \left(\delta_k^- U_k\right)^2}{2} + \frac{1}{2}\alpha_1 U_k^2 + \frac{1}{\rho+1}\alpha_\rho U_k^{\rho+1}. \tag{3.5}$$

We remark that for the discretizing the energy we need to be able to apply the summation by part (2.2). Corresponding to the periodic boundary condition we assume that

$$U_k^{(n)} = U_{k \bmod K}^{(n)}, \qquad V_k^{(n)} = V_{k \bmod K}^{(n)}. \tag{3.6}$$

We define the discrete total energy I_d corresponding (3.4) by

$$I_d(\boldsymbol{U}, \boldsymbol{V}) := \sum_{k=0}^{K}{}'' G_d(U_k, V_k)\Delta x.$$

An similar calculation to (3.2) yields

$$I_d(\boldsymbol{U}, \widetilde{\boldsymbol{V}}) - I_d(\boldsymbol{U}, \boldsymbol{V}) = -\frac{1}{2} \sum_{k=0}^{K}{}'' \left(\widetilde{V}_k - V_k \right) \left\{ \delta_k^{\langle 2 \rangle} \phi^{\langle 1 \rangle}(\widetilde{V}_k, V_k) \right\} \Delta x,$$

due to (2.2). From this we define *discrete variational derivative* with respect to $\boldsymbol{V}$ by

$$\left(\frac{\delta G_d}{\delta(\widetilde{\boldsymbol{V}}; \boldsymbol{V})} \right)_k := -\frac{1}{2} \delta_k^{\langle 2 \rangle} \phi^{\langle 1 \rangle}(\widetilde{V}_k, V_k). \tag{3.7}$$

Similarly we define the *discrete variational derivative* with respect to $\boldsymbol{U}$ by

$$\left(\frac{\delta G_d}{\delta(\widetilde{\boldsymbol{U}}; \boldsymbol{U})} \right)_k := -\frac{1}{2} \gamma \delta_k^{\langle 2 \rangle} \phi^{\langle 1 \rangle}(\widetilde{U}_k, U_k) + \frac{1}{2} \alpha_1 \phi^{\langle 1 \rangle}(\widetilde{U}_k, U_k) + \frac{1}{\rho + 1} \alpha_\rho \phi^{\langle \rho \rangle}(\widetilde{U}_k, U_k). \tag{3.8}$$

Corresponding to (3.3), the difference scheme of DVDM for (1.1) is given by

$$\delta_n^+ \begin{pmatrix} U_k^{(n)} \\ V_k^{(n)} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} \left(\dfrac{\delta G_d}{\delta(\boldsymbol{U}^{(n+1)}; \boldsymbol{U}^{(n)})} \right)_k \\ \left(\dfrac{\delta G_d}{\delta(\boldsymbol{V}^{(n+1)}; \boldsymbol{V}^{(n)})} \right)_k \end{pmatrix} \right), \qquad k = 1, 2, \ldots, K. \tag{3.9}$$

Our scheme is (3.8)-(3.9) with the boundary condition (3.6).

Since the difference scheme (3.9) is nonlinear and implicit, it is not trivial whether the solution for (3.9) exists or not. We can show the existence of the solution for (3.9) by using the fixed point principle under some assumptions for Δx and Δt. If we take

$$\max \left\{ \frac{\Delta t}{\Delta x^2}, \frac{\Delta t}{\Delta x^{\frac{\rho - 1}{2}}} \right\} \tag{3.10}$$

sufficiently small, then these assumptions are satisfied.

4 Stability and Error Estimate

The equation (3.3) has the following conservation laws:

$$\frac{d}{dt} I(u(t), v(t)) = 0, \qquad \frac{d}{dt} M(u(t)) = 0,$$

where the momentum $M(u(t))$ is defined by

$$M(u(t)) := \int_0^L u(x, t) dx.$$

Correspondingly, we can easily show

$$I_d(\boldsymbol{U}^{(n)}, \boldsymbol{V}^{(n)}) = I_d(\boldsymbol{U}^{(0)}, \boldsymbol{V}^{(0)}),$$

$$M_d(\boldsymbol{U}^{(n)}) = M_d(\boldsymbol{U}^{(0)}) \quad \text{for} \quad M_d(\boldsymbol{U}^{(n)}) := \sum_{k=0}^{K}{}'' U_k^{(n)} \Delta x.$$

By using these conservation laws we easily obtain the stability result.

Theorem 4.1 (Stability). *The solution for the difference equation* (3.9) *satisfies*

$$\max_{0\le k\le K}|U_k^{(n)}| \le \left|M_d(\boldsymbol{U}^{(0)})\right| + \sqrt{\frac{2L}{\gamma}I_d(\boldsymbol{U}^{(0)},\boldsymbol{V}^{(0)})}.$$

We define error terms $e_k^{(n)}$, $\widetilde{e}_k^{(n)}$ between exact solutions $u_k^{(n)}$, $v_k^{(n)}$ and approximate solutions $U_k^{(n)}$, $V_k^{(n)}$ by

$$e_k^{(n)} := U_k^{(n)} - u_k^{(n)}, \qquad \widetilde{e}_k^{(n)} := V_k^{(n)} - v_k^{(n)}.$$

We shall estimate the error terms for the difference scheme (3.9).

Theorem 4.2 (Error Estimate). *We set*

$$C_1 := \max_{0\le \ell\le N}\left\{\max_{0\le k\le K}\left|U_k^{(\ell)}\right|, \max_{0\le k\le K}\left|u_k^{(\ell)}\right|\right\} \tag{4.1}$$

Assume that the smooth solution for (3.3) *satisfying* $u\in\mathcal{C}^4([0,L]\times[0,T])$ *exists for the initial data* (u_0,v_0) *and that* $e_k^{(0)}=0$ *and* $\widetilde{e}_k^{(0)}=0$. *If the time-step* Δt *is small satisfying*

$$\Delta t < \frac{4\sqrt{\gamma}}{\alpha_1+\alpha_\rho\rho C_1^{\rho-1}}, \tag{4.2}$$

then the solution $(U_k^{(n)},V_k^{(n)})$ *for* (3.9) *converges to* (u,v) *in the sense that*

$$\|e\|_{L^2_{\mathrm{d}}} + \|\widetilde{e}\|_{L^2_{\mathrm{d}}} = O(\Delta x^2+\Delta t^2), \qquad \text{as } \Delta x,\ \Delta t\to 0,$$

where

$$\|e\|_{L^2_{\mathrm{d}}} := \left(\sum_{k=0}^{K}{}''|e_k|^2\Delta x\right)^{1/2},$$

Remarks 4.3. (i) The above results can be extended for more general polynomial nonlinearity satisfying (1.2) by a slightly modification.
(ii) In the same fashion the above results holds true in the Neumann boundary conditions for (u,v) (see [4]).

5 Computation Example

By using the proposed scheme (3.9) we can simulate a variety of dynamics of solutions for the Boussinesq type equations through a numerical calculation. Particularly, in this section we consider the isothermal Falk model:

$$\partial_t^2 u + \gamma\partial_x^4 u - \alpha_1\partial_x^2 u = \partial_x^2\{\alpha_5 u^5 - \alpha_3 u^3\}, \qquad (x,t)\in\mathbb{R}\times\mathbb{R}^+, \tag{5.1}$$

which corresponds to the Boussinesq type equation (1.1) with nonlinearity (1.4). The equation is the isothermal model of Falk's thermoelastic system (see [2, Chapter 5]). In [3], this equation has a traveling wave solution in the austenite (high-temperature, namely large α_1) phase.

We demonstrate the travelling wave solution through a numerical computation. We first take the physical constant as the same as in [3]: $\gamma=2$, $\alpha_5=6$, $\alpha_3=4$, $\alpha_1=1/4$, $L=100$, $T=50$. Moreover, we assume that the velocity of the travelling wave c is given by $c=-\sqrt{3}/2$ which satisfies the condition to observe the travelling wave solution given in [3]. According to the paper [3],

$$u(z) = u\left(x-ct-x_0\right) = \frac{1}{\sqrt{4+2\sqrt{2}+4\sqrt{2}\sinh^2\left(\frac{x-ct-x_0}{2\sqrt{2}}\right)}} \tag{5.2}$$

is a solution for (5.1), where x_0 is a initial position of the wave. Corresponding to (5.2), we choose the initial values. For the error estimate we need to replace the assumption (4.2) to

$$\Delta t < \frac{4\sqrt{\gamma}}{\alpha_1 + 3\alpha_3 C_1^2 + 5\alpha_5 C_1^4}, \tag{5.3}$$

In our case, we easily check the following $I_d(U_k^{(0)}, V_k^{(0)}) \leq 0.73$ and $|M_d(\boldsymbol{U}^{(0)})| \leq 3.57$. Then we see that $\max_{0\leq k\leq K} |U_k^{(n)}| \leq 8.58$. Since the exact solution obviously satisfies $\|u(t)\|_{L^\infty} < 1$, we see that $C_1 \leq 8.58$. To satisfy the assumption (5.3) it is sufficient to take, for example, $\Delta t = \frac{1}{30000}$. If we take $\Delta x = 1/2$, then assumptions related to (3.10) which assure the existence of solution for the scheme are also satisfied for some M. In order to obtain new time-step solutions using our nonlinear scheme, we use Newton's method. Figure 1 shows the profiles of the numerical solutions U.

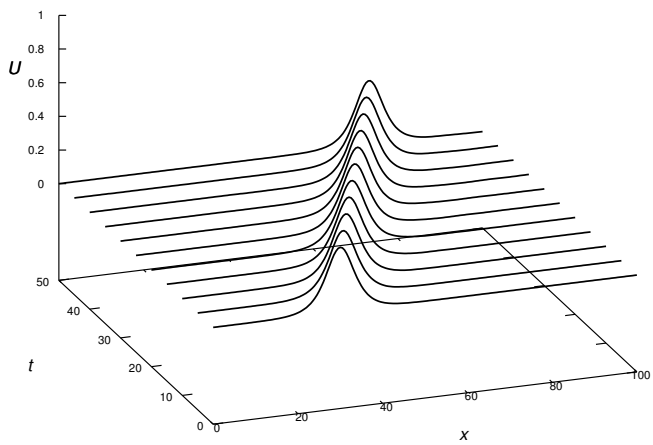

Figure 1: Numerical solution U

The energy fluctuation obtained by using our scheme is shown in Figure 2. From this we can confirm that the discrete energy values are conserved well.

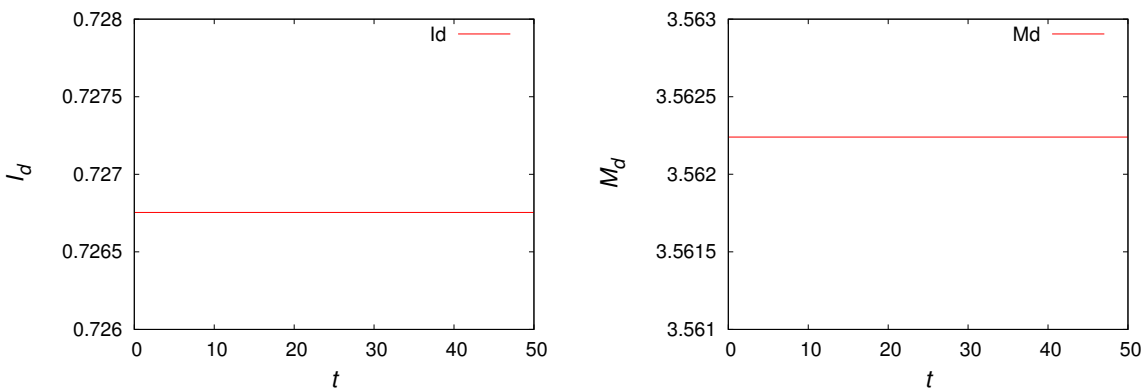

Figure 2: Discrete conserved quantities I_d and M_d

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Existence of Solutions for Isotropic Elastodynamics

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The talk will present an overview of global existence results for the initial value problem associated to the motion of isotropic elastic and viscoelastic materials. In the absence of boundaries, the motion of a body is described by a smooth one-parameter family of orientation preserving diffeomorphisms or *deformations* $x(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, $t \geq 0$, $n = 2, 3$. Under such a family of deformations, each material point $y \in \mathbb{R}^n$ follows the trajectory $t \mapsto x(t, y)$, $t \geq 0$. A homogeneous elastic material can be characterized by a smooth strain energy function

$$W : GL_+(n, \mathbb{R}) \to \mathbb{R}_+,$$

where $GL_+(n, \mathbb{R})$ is the set of $n \times n$ real matrices with positive determinant. Formally, the equations of motion are obtained by applying the principle of stationary action to the Lagrangian

$$\mathcal{L}[x] = \iint \left[\frac{1}{2}|D_t x|^2 - W(D_y x)\right] dydt. \tag{1}$$

This yields the system

$$D_t^2 x_i - \sum_{j=1}^{n} D_j[S_{ij}(D_y x)] = 0, \quad S_{ij}(F) = \frac{\partial W(F)}{\partial F_{ij}}. \tag{2}$$

It is physically reasonable to assume that this system is Galilean invariant and that the equilibrium deformation $x(t, y) = y$ has vanishing energy and stress. We also assume that the material is isotropic. This implies that $W(I) = 0$, $S(I) = 0$, and that $W(F)$ is a function only of the principal invariants of $F^\top F$.

The construction of solutions relies on a combination of energy estimates and decay estimates. The results are perturbative in that at every time t the displacement from equilibrium, $u(t, y) = x(t, y) - y$, is assumed to be "small" in an appropriate energy norm, defined below. Therefore, a key role is played by the linearized problem

$$D_t^2 x^i - \sum_{j,k,\ell=1}^{n} A_{ijk\ell} D_j D_\ell x^k = 0, \quad A_{ijk\ell} = \frac{\partial^2 W}{\partial F_{ij} \partial F_{k\ell}}(I).$$

We impose the standard Legendre-Hadamard ellipticity condition

$$(3) \qquad \sum_{i,j,k,\ell=1}^{n} A_{ijk\ell} x^i \xi_j x^k \xi_\ell = D_\varepsilon^2 W(I + \varepsilon x \otimes \xi)\big|_{\varepsilon=0} > 0, \qquad \text{for all} \quad 0 \neq x, \xi \in \mathbb{R}^n.$$

This can also be justified by physical considerations. In the isotropic case, the linearized equation takes the form

$$(4) \quad Lx = D_t^2 x - c_2^2 \Delta x - (c_1^2 - c_2^2) D(D \cdot x) = 0, \qquad \text{with} \quad c_1 > c_2 > 0, \quad D = (D_1, \ldots, D_n).$$

The energy

$$\mathcal{E}[x](t) = \frac{1}{2} \int_{\mathbb{R}^r} \left[|D_t x(t)|^2 + c_2^2 |Dx(t)|^2 + (c_1^2 - c_2^2)(D \cdot x(t))^2 \right] dy$$

is conserved for equation (4).

Equation (4) is invariant under the Galilean (but not Lorentz) group, the generators of which contain the the collection of vector fields

$$(5) \ \Gamma = \{D_j, \ S = tD_t + y \cdot D, \ \Omega_{ij} = I(y_j D_i - y_i D_j) + e_j \otimes e_i - e_i \otimes e_j\},$$

where $\{e_i\}_{i=1}^n$ is the standard basis for $\mathbb{R}^n$. This implies the commutation properties

$$LS = S(L+2) \quad \text{and} \quad L\Omega_{ij} = \Omega_{ij} L,$$

for the linear operator L. It follows that the higher order energies $\mathcal{E}_m[x](t) = \sum_{|\alpha| \leq m-1} \mathcal{E}[\Gamma^\alpha x](t)$ are also conserved for solutions of (4).

Solutions of (4) are a superposition of pressure and shear waves, propagating with speeds c_1 and c_2 respectively. This decomposition is approximated by

$$x(t,y) = \mathbb{P}(y) x(t,y) + [I - \mathbb{P}(y)] x(t,y), \quad \text{with} \quad \mathbb{P}(y) = \frac{y}{|y|} \otimes \frac{y}{|y|}.$$

Using scaling and rotational invariance of L, we obtain the uniform bound (see [6])

$$\mathcal{X}[D_j D_\ell x](t) \lesssim \mathcal{E}_2[x](t) + |t| \|Lx(t)\|_{L^2(\mathbb{R}^n)}, \quad j, \ell = 1, \ldots, n$$

in which

$$\mathcal{X}[u](t) = \|\langle c_1 t - |y| \rangle \mathbb{P}(y) u(t,y)\|_{L^2(\mathbb{R}^n)} + \|\langle c_2 t - |y| \rangle [I - \mathbb{P}(y)] u(t,y)\|_{L^2(\mathbb{R}^n)}.$$

In the full nonlinear problem, the self-interaction of the two wave families influences the existence of global solutions. For the pressure

waves, this interaction must be limited by imposing an additional *null condition*:

$$D_\varepsilon^3 W(I + \varepsilon\xi \otimes \xi)|_{\varepsilon=0} = 0, \quad \text{for all} \quad \xi \in \mathbb{R}^n. \tag{6}$$

In the isotropic case, the nonlinear interactions of shear waves is inherently null.

Theorem 1 ([4], see also [1])**.** *Let $n = 3$. Suppose that equation* (2) *arises from a strain energy function W which satisfies $W(I) = 0$, $S(I) = 0$, the Legendre-Hadamard condition* (3)*, and the null condition* (6)*.*

Choose initial data for (2) *such that the displacement from equilibrium $u(t, y) = x(t, y) - y$ satisfies $\mathcal{E}_m[u](0) < \infty$, $m \geq 9$.*

Let $\varepsilon > 0$ be sufficiently small. There is a constant $C > 0$, depending on W, such that if

$$\mathcal{E}_{m-2}[u](0) \exp\left[C\mathcal{E}_m^{1/2}[u](0)\right] < \varepsilon,$$

then (2) *has a unique global solution $x(t, y) = y + u(t, y)$ with*

$$u(t, y) \in \cap_{k=0}^m C^k(\mathbb{R}_+, H^{m-k}(\mathbb{R}^3))$$

and $\mathcal{E}_{m-2}[u](t) < \varepsilon$.

Incompressible isotropic materials can be viewed as a limiting case where the speed of the pressure waves is infinite, see [5]. In this case, the linearized problem is the wave equation, and the nonlinear interactions are null. However, with the incompressibility constraint the problem becomes nonlocal, requiring special estimates for the pressure term.

The equations of motion can be obtained by extremizing the Lagrangian (1) constrained to the class of volume preserving deformations. For simplicity, we consider only the Hookean case

$$W(F) = \frac{1}{2} \operatorname{tr} F^\top F,$$

whence the equations are

$$D_t^2 x_i - \Delta_y x_i + \sum_{j=1}^n (D_y x^{-1})_{ji} D_j \lambda = 0, \quad \det D_y x = 1. \tag{7}$$

Here, λ is a Lagrange multiplier. Using the constraint, it follows that

$$\Delta_x \lambda + \sum_{i,j=1}^n \partial_i \partial_j [D_t x_i D_t x_j - \sum_{k=1}^n D_k x_i D_k x_j] = 0,$$

where the derivatives Δ_x and ∂_i are taken in the spatial coordinates x. This shows that λ can be regarded as a nonlocal null form.

Theorem 2 ([7], [3]). *Choose initial data for* (7) *which is compatible with the incompressibility constraint*

$$\det D_y x(0) = 1, \quad \mathrm{tr}[D_y x(0)^{-1} D_t D_y x(0)] = 0.$$

When $n = 3$*, suppose that the displacement from equilibrium* $u(t,y) = x(t,y) - y$ *satisfies* $\mathcal{E}_m[u](0) < \infty$*, with* $m \geq 8$*. Let* $\varepsilon > 0$ *be sufficiently small. There is a constant* $C > 0$ *such that if*

$$\mathcal{E}_{m-2}[u](0) \exp\left[C\mathcal{E}_m^{1/2}[u](0)\right] < \varepsilon,$$

then (7) *has a unique global solution* $x(t,y) = y + u(t,y)$ *with*

$$u(t,y) \in \cap_{k=0}^m C^k(\mathbb{R}_+, H^{m-k}(\mathbb{R}^3))$$

and $\mathcal{E}_{m-2}[u](t) < \varepsilon$*.*

When $n = 2$*, suppose that* $\mathcal{E}_m[u](0) < \varepsilon$*,* $m \geq 5$*, for* ε *sufficiently small. Then* (7) *has a unique local solution* $x(t,y) = y + u(t,y)$ *with*

$$u(t,y) \in \cap_{k=0}^m C^k([0,T_\varepsilon), H^{m-k}(\mathbb{R}^2)),$$

$T_\varepsilon > \exp(C/\varepsilon)$*, and* $\mathcal{E}_m[u](t) < 2\varepsilon$*.*

Viscoelastic materials arise through the addition of weak viscous dissipation to the classical elastic case, resulting in a hyperbolic-parabolic system. Obtaining global existence results which are uniform in the magnitude of the dissipation parameter ν requires the adaptation of the hyperbolic estimates to a system without scaling invariance, see [2]. As a model, consider the nonlinear wave equation in 3-D

$$D_t^2 u - \Delta u + \nu \Delta D_t u = \sum_{i,j,k=1}^{3} B_{ijk} D_i (D_j u D_k u). \tag{8}$$

Theorem 3. *Choose initial data for* (8) *with* $\mathcal{E}_m[u](0) < \infty$*,* $m \geq 6$*. Define* $\delta = \max\{|\sum_{i,j,k=1}^3 B_{ijk}\omega_i\omega_j\omega_k| : \omega \in \mathbb{R}^3,\ |\omega| = 1\}$*. Let* ε *be sufficiently small. Put* $m^* = \left[\frac{m+5}{2}\right]$*. There is a constant* $C > 0$ *such that if*

$$\mathcal{E}_{m^*}[u](0)\left\{\exp\left[C\mathcal{E}_m^{1/2}[u](0)\right] + \exp\left[C\mathcal{E}_m[u](0)\delta^2/\nu\right]\right\} < \varepsilon,$$

then (8) *has a unique global solution with* $u(t,y) \in \cap_{k=0}^m C^k(\mathbb{R}_+, H^{m-k}(\mathbb{R}^3))$ *and* $\mathcal{E}_{m^*}[u](t) < \varepsilon$*.*

The number of occurrences of the scaling operator S is restricted to $m^* + 1$ in $\mathcal{E}_m[u](t)$. Also, since (8) is a scalar equation, we use $\Omega_{ij} = y_i D_j - y_j D_i$ instead of the vectorial version in (5). This result says, in particular, if the null condition is satisfied, $\delta = 0$, then (8) has global solutions for small initial data, *independent of the size of* ν*.*

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Mathematical modeling and numerical treatment of adhesion, exfoliation and collision

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1 Scalar problem

In this talk, we would like to treat mathematical modeling and numerical treatment of adhesion, exfoliation and collision. Phenomena of adhesion and exfoliation come from droplet motion on a plane. We describe the surface of droplet by using a graph of scalar function. For this problem, key words are 'volume constraint', 'free boundary' and 'potential energy'. The constraint produces a non-local term in the partial differential equations. Hence, we need to introduce variational method to these problems. Lagrangian will be

$$L(u) = \int_\Omega \big(- \chi_{u>0} u_t^2 + |\nabla u|^2 + Q\chi_{u>0} \big)\, dx, \tag{1}$$

where $\chi_{u>0}$ is the characteristic function and Q is a force of adhesion. If the droplet keeps its volume, we can get the following equation:

$$\chi_{u>0} u_{tt} = \Delta u - Q\chi'_{u>0} + \big(\int_\Omega [u_{tt} u + |\nabla u|^2]\, dx \big) \chi_{u>0}. \tag{2}$$

2 Shell bouncing problem

The model equation is derived by calculating the energy stored in the shell. The considered types of energy are stretching energy, bending energy, energy related to the compression of the enclosed gas, potential energy and kinetic energy. We assume that the strip of initial radius r_0 is bent to radius r and stretched by the ratio $\tilde{\mu}$. We adopt the following form of elastic energy

$$E_e(\boldsymbol{p}) = \frac{1}{24} kh^3 \int_0^{2\pi} (\kappa - \kappa_0)^2 |\boldsymbol{p}_\theta|\, d\theta + \frac{1}{2} kh \int_0^{2\pi} \Big(\frac{|\boldsymbol{p}_\theta|}{|\boldsymbol{q}_\theta|} - 1 \Big)^2 |\boldsymbol{q}_\theta|\, d\theta.$$

Denoting the mass density of the shell in equilibrium by σ, the local mass density of the shell $\boldsymbol{p}$ becomes $\sigma|\boldsymbol{q}_\theta|/|\boldsymbol{p}_\theta|$ and thus the kinetic energy is given by

$$E_k(\boldsymbol{p}) = \frac{1}{2} h \int_0^{2\pi} \sigma |\boldsymbol{q}_\theta|\, |\boldsymbol{p}_t|^2 \chi_{p^2>0}\, d\theta.$$

Physically, this corresponds to the assumption of zero reflection and infinite friction between the shell and the obstacle.

When the shell is closed, it is necessary to take into account also the energy related to the compression of the gas present inside the shell. The energy stored due to compression of the enclosed volume V of gas can now be calculated as minus the work done by pressure

forces:

$$E_g(\boldsymbol{p}) = -\int_{V_0}^{V} P\,dV = -\int_{V_0}^{V} \Big\{P_0 + c_g\big(\frac{1}{V} - \frac{1}{V_0}\big)\Big\}\,dV = -P_0(V - V_0) - c_g\Big(1 - \frac{V}{V_0} + \ln\frac{V}{V_0}\Big). \tag{3}$$

The constant c_g is the product of the total mass of the gas and the square of the sound of speed. The equilibrium volume V_0 is known and the volume of $\boldsymbol{p}(t)$ is given by

$$V = \frac{1}{2}\int_0^{2\pi} (\boldsymbol{p} \cdot A\boldsymbol{p}_\theta)\chi_{p^2>0}\,d\theta.$$

In real situations, the impact of a shell is influenced by the action of gravity. Therefore, we introduce also the gravity potential of the shell

$$E_p(\boldsymbol{p}) = gh\int_0^{2\pi} \sigma|\boldsymbol{q}_\theta|p^2\chi_{p^2>0}\,d\theta.$$

Employing the obtained elastic, kinetic and gas energies, the action integral is written as

$$I(\boldsymbol{p}) = \int_0^T \big(E_e(\boldsymbol{p}) + E_g(\boldsymbol{p}) + E_p(\boldsymbol{p}) - E_k(\boldsymbol{p})\big)\,dt. \tag{4}$$

Applying Hamilton's principle, the governing equation for the free part of the shell is obtained from

$$\frac{d}{d\epsilon}I(\boldsymbol{p} + \epsilon\boldsymbol{\phi})\big|_{\epsilon=0} = 0 \qquad \forall\boldsymbol{\phi} \in \big[C_0^\infty\big((0,T)\times(0,2\pi)\cap\{p^2>0\}\big)\big]^2, \tag{5}$$

where I is the action functional defined in (4).

In the sequel, we shall use the notation $\rho(\theta) = |\boldsymbol{p}_\theta(\theta)|/|\boldsymbol{q}_\theta(\theta)|$ for the local relative density with respect to the equilibrium state.

Taking into account the influence of the obstacle on the computed energy, one may expect that the following equation expresses, in a rough approximation, the deformation of the whole shell:

$$\begin{aligned}\sigma\chi_{p^2>0}\boldsymbol{p}_{tt} \;=\; & \big\{-\tfrac{1}{12}kh^2\rho(\kappa_{ss} + \tfrac{1}{2}\kappa^3)\chi_{p^2>0} + \tfrac{1}{24}kh^2\kappa_0^2\rho\kappa - k\rho(\rho-1)\kappa\chi_{p^2>0}\big\}\boldsymbol{\nu} \\ & + \frac{\rho}{h}\big(P_0 + c_g(\tfrac{1}{V} - \tfrac{1}{V_0})\big)\chi_{p^2>0}\boldsymbol{\nu} + k\rho\rho_s\chi_{p^2>0}\boldsymbol{\tau} + g\sigma\chi_{p^2>0}\boldsymbol{e}_2.\end{aligned} \tag{6}$$

In our talk, we will explain how to treat the above equation numerically and will show the numerical result.

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Phase field systems of grain boundaries with solidification effects

Ken Shirakawa*

Introduction

This study is based on a recent collaborations with Prof. Salvador Moll†, Dr. Hiroshi Watanabe‡ and Prof. N. Yamazaki†† (cf. [10, 11, 12, 15]), which are communicated and supported by Prof. J. M. Mazón‡‡.

Let $0 < T < \infty$ be a fixed constant, let $1 < N \in \mathbb{N}$ be a fixed number, and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$. Besides, we denote by $\nu_{\partial\Omega}$ the unit outer normal vector on $\partial\Omega$, and we set $Q := (0,T) \times \Omega$ and $\Sigma := (0,T) \times \partial\Omega$.

Let $\nu > 0$ be a fixed small constant. In this paper, a coupled system of parabolic initial-boundary value problems is considered. This system is denoted by $(\mathrm{S})_\nu$, and formally described as follows.

$(\mathrm{S})_\nu$:

$$\begin{cases} w_t - \Delta w + p(w,\eta) + \nu\beta'(w)|\nabla\theta|^2 = 0 \ \text{ in } Q, \\ \nabla w \cdot \nu_{\partial\Omega} = 0 \ \text{ on } \Sigma, \\ w(0,x) = w_0(x), \quad x \in \Omega; \end{cases}$$

$$\begin{cases} \eta_t - \Delta\eta + q(w,\eta) + \alpha'(\eta)|\nabla\theta| = 0 \ \text{ in } Q, \\ \nabla\eta \cdot \nu_{\partial\Omega} = 0 \ \text{ on } \Sigma, \\ \eta(0,x) = \eta_0(x), \quad x \in \Omega; \end{cases}$$

$$\begin{cases} \alpha_0(w,\eta)\,\theta_t - \mathrm{div}\left(\alpha(\eta)\dfrac{D\theta}{|D\theta|} + \nu\beta(w)\nabla\theta\right) = 0 \ \text{ in } Q, \\ \left(\alpha(\eta)\dfrac{D\theta}{|D\theta|} + \nu\beta(w)\nabla\theta\right) \cdot \nu_{\partial\Omega} = 0 \ \text{ on } \Sigma, \\ \theta(0,x) = \theta_0(x), \quad x \in \Omega. \end{cases}$$

The system $(\mathrm{S})_\nu$ is based on a mathematical model of planar grain boundary motions with solidification effects, proposed by Kobayashi [9] and Warren-Kobayashi-Lobkovski-Carter [14]. In accordance with the modelling method as in [9, 14], the system $(\mathrm{S})_\nu$ is

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This study is supported by Grant-in-Aid for Encouragement of Young Scientists (B) (No. 24740099) JSPS.

derived as a gradient system of a governing free-energy, defined as:

$$\begin{aligned}
[w,\eta,\theta] &\in [H^1(\Omega)\cap L^\infty(\Omega)]\times[H^1(\Omega)\cap L^\infty(\Omega)]\times H^1(\Omega)\\
&\mapsto \mathscr{F}_\nu(w,\eta,\theta) := \frac{1}{2}\int_\Omega |\nabla w|^2\,dx + \frac{1}{2}\int_\Omega |\nabla\eta|^2\,dx + \int_\Omega f(w,\eta)\,dx\\
&\qquad + \int_\Omega \alpha(\eta)|\nabla\theta|\,dx + \nu\int_\Omega \beta(w)|\nabla\theta|^2\,dx.
\end{aligned}$$

In the context, the unknown $w = w(t,x)$ is an order parameter to indicate the solidification order of grain. Unknowns $\eta = \eta(t,x)$ and $\theta = \theta(t,x)$ are order parameters to reproduce the crystalline orientation in Q, by using a vector field:

$$(t,x)\in Q \mapsto \eta(t,x)\left[\cos\theta(t,x), \sin\theta(t,x)\right] \in \mathbb{R}^2.$$

Then, the components η and θ are supposed to indicate the orientation order and the orientation angle of the grain, respectively. In particular, w and η are supposed to satisfy range constraint properties $0 \le w,\eta \le 1$ in Q, and the cases when $[w,\eta] = [1,1]$ and $[w,\eta] = [0,0]$ are supposed to reproduce two stable states of grains: *the solidified-oriented state*; *the liquefied-disoriented state*, respectively.

The major theme of this study is to ensure qualitative properties for the system $(\mathrm{S})_\nu$ from the theoretical viewpoint in mathematics. As part of this theme, we here focus on an approach based on the time-discretization, and we conclude the existence theorem of the system $(\mathrm{S})_\nu$. Furthermore, we will mention about the problems in future with the application possibilities of similar approaches by time-discretizations.

1 Statement of the Main Theorem

First of all, let us confirm the assumptions for the given functions $f = f(w,\eta)$, $p = p(w,\eta)$, $q = q(w,\eta)$, $\alpha_0 = \alpha_0(w,\eta)$, $\alpha = \alpha(\eta)$, $\beta = \beta(w)$, η_0 and θ_0 as in the system $(\mathrm{S})_\nu$.

(A1) $f \in C^2([0,1]^2)$ and $p,q \in C^1([0,1]^2)$ such that

$$\begin{cases}
\bullet\ f(w,\eta)\ge 0 \text{ and } \nabla f(w,\eta) = [p(w,\eta), q(w,\eta)] \text{ for all } [w,\eta]\in[0,1]^2,\\
\bullet\ p(0,\eta)\le 0 \text{ and } p(1,\eta)\ge 0 \text{ for all } \eta\in[0,1],\\
\bullet\ q(w,0)\le 0 \text{ and } q(w,1)\ge 0 \text{ for all } w\in[0,1].
\end{cases}$$

(A2) $\alpha_0 \in C^1([0,1]^2)$ and $\alpha,\beta \in C^2[0,1]$ such that:

$$\begin{cases}
\bullet\ \alpha_0(w,\eta) > 0 \text{ for all } [w,\eta]\in[0,1]^2,\\
\bullet\ \alpha'(0) = 0,\ \ \alpha(\eta) > 0 \text{ and } \alpha''(\eta)\ge 0 \text{ for all } \eta\in[0,1],\\
\bullet\ \beta'(0) = 0,\ \ \beta(w) > 0 \text{ and } \beta''(w)\ge 0 \text{ for all } w\in[0,1],
\end{cases}$$

where α' and β' are the differentials of α and β, respectively, and α'' and β'' are the second differentials. Note that α and β turn out to be convex functions on $\mathbb{R}$, and

$$\delta_* := \min_{[\tilde w,\tilde\eta]\in[0,1]^2}\{\alpha_0(\tilde w,\tilde\eta), \alpha(\tilde\eta), \beta(\tilde w)\} > 0.$$

(A3) The pair (triplet) of initial data $[w_0, \eta_0, \theta_0]$ belongs to a class $D_* \subset H^1(\Omega)^2 \times H^1(\Omega)$, defined as:

$$D_* := \left\{ \ [\tilde{w}, \tilde{\eta}, \tilde{\theta}] \in H^1(\Omega)^3 \ \middle| \ 0 \le \tilde{w}, \tilde{\eta} \le 1 \ \text{ a.e. in } \Omega, \ \ \tilde{\theta} \in L^\infty(\Omega) \ \right\}.$$

In addition, for the convenience of descriptions, we prepare the following notations.

Notation 1.1 For any $\tilde{v} = [\tilde{w}, \tilde{\eta}] \in L^\infty(\Omega)^2$, let $\Phi_\nu(\tilde{v}; \cdot\,) = \Phi_\nu(\tilde{w}, \tilde{\eta}; \cdot\,)$ be a proper l.s.c. and convex function on $L^2(\Omega)$, defined as:

$$\begin{aligned} \vartheta \in L^2(\Omega) \ \ &\mapsto \ \ \Phi_\nu(\tilde{v}; \vartheta) = \Phi_\nu(\tilde{w}, \tilde{\eta}; \vartheta) \\ &:= \ \ \begin{cases} \displaystyle\int_\Omega \alpha(\tilde{\eta}) |\nabla \vartheta| \, dx + \nu \int_\Omega \beta(\tilde{w}) |\nabla \vartheta|^2 \, dx, & \text{if } \vartheta \in H^1(\Omega), \\ \infty, \ \ \text{otherwise}, \end{cases} \end{aligned}$$

and let $\partial \Phi_\nu(\tilde{v}; \cdot\,) = \partial \Phi_\nu(\tilde{w}, \tilde{\eta}; \cdot\,)$ be the L^2-subdifferential of $\Phi_\nu(\tilde{v}; \cdot\,) = \Phi_\nu(\tilde{w}, \tilde{\eta}; \cdot\,)$.

Now, the Main Theorem is stated as follows.

Main Theorem. *(Solvability of the system $(S)_\nu$) Under the assumptions (A1)-(A3), the system $(S)_\nu$ admits at least one solution $[w, \eta, \theta]$, in the sense of the following items.*

(S1) $w \in W^{1,2}(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega))$, $0 \le w \le 1$ *a.e. in* Q;
$\eta \in W^{1,2}(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega))$, $0 \le \eta \le 1$ *a.e. in* Q;
$\theta \in W^{1,2}(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \cap L^\infty(Q)$, $|\theta|_{L^\infty(Q)} \le |\theta_0|_{L^\infty(\Omega)}$.

(S2) w *solves the following variational identity of parabolic type:*

$$\begin{gathered} \int_\Omega \left(w_t(t) + p(w(t), \eta(t)) + \nu \beta'(w(t)) |\nabla \theta(t)|^2 \right) \varphi \, dx + \int_\Omega \nabla w(t) \cdot \nabla \varphi \, dx = 0, \\ \text{for any } \varphi \in H^1(\Omega) \cap L^\infty(\Omega) \text{ and a.e. } t \in (0,T). \end{gathered} \tag{1.1}$$

(S3) η *solves the following variational identity of parabolic type:*

$$\begin{gathered} \int_\Omega \left(\eta_t(t) + q(w(t), \eta(t)) + \alpha'(\eta(t)) |\nabla \theta(t)| \right) \psi \, dx + \int_\Omega \nabla \eta(t) \cdot \nabla \psi \, dx = 0, \\ \text{for any } \psi \in H^1(\Omega) \text{ and a.e. } t \in (0,T). \end{gathered} \tag{1.2}$$

(S4) θ *solves the following variational inequality of parabolic type:*

$$\begin{gathered} \int_\Omega \alpha_0(w(t), \eta(t)) \, \theta_t(t) \, (\theta(t) - \omega) \, dx + \Phi_\nu(w(t), \eta(t); \theta(t)) \le \Phi_\nu(w(t), \eta(t); \omega), \\ \text{for any } \omega \in H^1(\Omega) \text{ and a.e. } t \in (0,T). \end{gathered} \tag{1.3}$$

(S5) $[w(0), \eta(0), \theta(0)] = [w_0, \eta_0, \theta_0]$ *in* $L^2(\Omega)^3$.

Remark 1.1 For the solution $[w, \eta, \theta]$ to the system $(\mathrm{S})_\nu$, let us put $v := [w, \eta]$. Then, we note that the variational identities (1.1)-(1.2) can be reduced to the following form of a single evolution equation:

$$v_t(t) + Fv(t) - v(t) + (\nabla f)(v(t)) + \big[\nu\beta'(w(t))|\nabla\theta(t)|^2,\ \alpha'(\eta(t))|\nabla\theta(t)|\big] = 0$$
$$\text{in } \ [H^1(\Omega) \cap L^\infty(\Omega)]^* \times H^1(\Omega)^*, \quad \text{a.e. } t \in (0, T),$$

where $F : H^1(\Omega) \longrightarrow H^1(\Omega)^*$ is the duality mapping between $H^1(\Omega)$ and $H^1(\Omega)^*$, and hence $F\tilde{v} := [F\tilde{w}, F\tilde{\eta}]$ for any $\tilde{v} = [\tilde{w}, \tilde{\eta}]$.

Also, we note that the variational inequality (1.3) is reformulated to the following form of an evolution equation:

$$\alpha_0(v(t))\,\theta_t(t) + \partial\Phi_\nu(v(t); \theta(t)) \ni 0 \ \text{ in } L^2(\Omega), \ \text{ a.e. } t \in (0, T),$$

governed by the subdifferentials $\partial\Phi_\nu(v(t);\ \cdot\ )$ of unknown-dependent convex functions $\Phi_\nu(v(t);\ \cdot\ )$ for $t \in [0, T]$.

2 Solution method for the system $(\mathrm{S})_\nu$

As mentioned in Introduction, the mathematical approach to the system $(\mathrm{S})_\nu$ is based on the time-discretization. To this end, we need to prepare a class $\{(\mathrm{RS})^\nu_\varepsilon \,|\, 0 < \varepsilon < 1\}$ of relaxation systems, denoted by $(\mathrm{RS})^\nu_\varepsilon$, for the original system $(\mathrm{S})_\nu$.

Let us fix a constant $N_* > 1 + N/2$, and let us fix any $\tilde{\theta}_0 \in H^{N_*}(\Omega)$. Also, for any $0 < \varepsilon < 1$ and any $\tilde{v} = [\tilde{w}, \tilde{\eta}] \in H^1(\Omega)^2$, let us define a relaxed convex function $\Psi^\nu_\varepsilon(\tilde{v};\ \cdot\ ) = \Psi^\nu_\varepsilon(\tilde{w}, \tilde{\eta};\ \cdot\ )$, by putting:

$$\begin{aligned} \vartheta \in L^2(\Omega) \quad &\mapsto \quad \Psi^\nu_\varepsilon(\tilde{v}; \vartheta) = \Psi^\nu_\varepsilon(\tilde{w}, \tilde{\eta}; \vartheta) \\ &:= \begin{cases} \Phi_\nu(\tilde{v}; \vartheta) + \dfrac{\varepsilon}{2}|\vartheta|^2_{H^{N_*}(\Omega)}, & \text{if } \vartheta \in H^{N_*}(\Omega), \\ \infty, \quad \text{otherwise}, \end{cases} \end{aligned}$$

and let us denote by $\partial\Psi^\nu_\varepsilon(\tilde{v};\ \cdot\ ) = \partial\Psi^\nu_\varepsilon(\tilde{w}, \tilde{\eta};\ \cdot\ )$ its subdifferential in $L^2(\Omega)$.

By using the above notations, the relaxation system $(\mathrm{RS})^\nu_\varepsilon$, for each $0 < \varepsilon < 1$, is prescribed as follows.

$(\mathrm{RS})^\nu_\varepsilon$: to find $[v, \theta] \in C([0, T]; L^2(\Omega))^3$ with $v = [w, \eta] \in C([0, T]; L^2(\Omega))^2$, such that

$$\begin{cases} v_t(t) - \Delta_N v(t) + (\nabla f)(v(t)) + [\nu\beta'(w(t))|\nabla\theta(t)|^2,\ \alpha'(\eta(t))|\nabla\theta(t)|] = 0 \\ \qquad\qquad \text{in } L^2(\Omega)^2, \ \text{ a.e. } t \in (0, T), \\ v(0) = [w(0), \eta(0)] = v_0 := [w_0, \eta_0] \text{ in } L^2(\Omega)^2; \end{cases} \tag{2.1}$$

$$\begin{cases} \alpha_0(v(t))\,\theta_t(t) + \partial\Psi^\nu_\varepsilon(v(t); \theta(t)) \ni 0 \text{ in } L^2(\Omega), \ \text{ a.e. } t \in (0, T), \\ \theta(0) = \tilde{\theta}_0 \text{ in } L^2(\Omega), \end{cases} \tag{2.2}$$

where Δ_N is the operator of the Laplacian subject to the Neumann-zero boundary condition, i.e.

$$\Delta_N : z \in \big\{\ \tilde{z} \in H^2(\Omega) \ \big|\ \nabla\tilde{z} \cdot \nu_{\partial\Omega} = 0 \text{ a.e. on } \partial\Omega\ \big\} \mapsto \Delta z \in L^2(\Omega).$$

On that basis, for any $0 < \varepsilon < 1$, we call a pair (triplet) of functions $[v, \theta] = [w, \eta, \theta]$ a solution to $(\mathrm{RS})_\varepsilon^\nu$ if and only if $v = [w, \eta] \in W^{1,2}(0, T; L^2(\Omega))^2 \cap L^\infty(0, T; H^1(\Omega))^2 \cap L^2(0, T; H^2(\Omega))^2$, $\theta \in W^{1,2}(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^{N_*}(\Omega))$, and the components $v = [w, \eta]$ and θ fulfill the Cauchy problems (2.1) and (2.2), respectively. Note that the embedding $H^{N_*}(\Omega) \subset W^{1,\infty}(\Omega)$ enables us to treat $\beta'(w(t))|\nabla\theta(t)|^2$ and $\alpha'(\eta(t))|\nabla\theta(t)|$ as in (2.1), as L^∞-perturbations.

Now, we prepare an approximation index $h \in (0, 1)$ of the time-step, and denote by $(\mathrm{AP})_h^\varepsilon$ the time-discretization system for $(\mathrm{RS})_\varepsilon^\nu$, formulated as follows.

$(\mathrm{AP})_h^\varepsilon$:

$$\frac{v_{h,i} - v_{h,i-1}}{h} - \Delta_N v_{h,i} + (\nabla f)(v_{h,i}) + \left[\nu\beta'(w_{h,i})|\nabla\theta_{h,i-1}|^2,\ \alpha'(\eta_{h,i})|\nabla\theta_{h,i-1}|\right] = 0 \ \text{ in } L^2(\Omega)^2, \tag{2.3}$$

$$\alpha_0(v_{h,i})\frac{\theta_{h,i} - \theta_{h,i-1}}{h} + \partial\Psi_\varepsilon^\nu(\alpha(v_{h,i}); \theta_{h,i}) \ni 0 \ \text{ in } L^2(\Omega), \tag{2.4}$$

for $i = 1, 2, 3, \cdots$, subject to:

$$[v_{h,0}, \theta_{h,0}] := [w_{h,0}, \eta_{h,0}, \theta_{h,0}] = [v_0, \tilde{\theta}_0] \ \text{ in } L^2(\Omega)^3. \tag{2.5}$$

Here, for any $0 < h < 1$, we call a pair (triplet) of sequences $[\{v_{h,i}\}, \{\theta_{h,i}\}] = [\{w_{h,i}\}, \{\eta_{h,i}\}, \{\theta_{h,i}\}] \subset L^2(\Omega)^3$ a solution to $(\mathrm{AP})_h^{(\nu)}$, or simply an approximating solution, if and only if $\{v_{h,i} \,|\, i \in \mathbb{N}\} \subset H^2(\Omega)^2$, $\{\theta_{h,i} \,|\, i \in \mathbb{N}\} \subset H^{N_*}(\Omega)$, and for any $i \in \mathbb{N}$, the components $v_{h,i}$ and $\theta_{h,i}$ fulfill the respective elliptic type problems (2.3) and (2.4) subject to (2.5).

In this paper, the class $\{(\mathrm{AP})_h^\varepsilon \,|\, 0 < \varepsilon, h < 1\}$ of the time-discretization systems is adopted as that of approximation problems for $(\mathrm{S})_\nu$. With regard to each approximation problem, we can prove the following theorem.

Theorem 2.1 *(Solvability of the approximation problem) There exists a small constant $0 < h_* < 1$, and for any $0 < \varepsilon < 1$ and any $0 < h \le h_*$, the approximation problem $(AP)_h^\varepsilon$ admits a unique solution $[\{v_{h,i}\}, \{\theta_{h,i}\}] = [\{w_{h,i}\}, \{\eta_{h,i}\}, \{\theta_{h,i}\}] \subset L^2(\Omega)^3$, such that:*

$$0 \le w_{h,i}, \eta_{h,i} \le 1 \ \text{ a.e. in } \Omega, \ \ |\theta_{h,i}|_{L^\infty(\Omega)} \le |\theta_{h,i-1}|_{L^\infty(\Omega)}, \ \ \text{and} \tag{2.6}$$

$$\frac{1}{2h}|v_{h,i} - v_{h,i-1}|^2_{L^2(\Omega)^2} + \frac{1}{h}\left|\sqrt{\alpha_0(v_{h,i})}(\theta_{h,i} - \theta_{h,i-1})\right|^2_{L^2(\Omega)} + \mathscr{E}_\varepsilon^\nu(v_{h,i}, \theta_{h,i}) \le \mathscr{E}_\varepsilon^\nu(v_{h,i-1}, \theta_{h,i-1}), \ i = 1, 2, 3, \cdots, \tag{2.7}$$

where $\mathscr{E}_\varepsilon^\nu$ is the relaxed free-energy, defined as:

$$[\tilde{v}, \tilde{\theta}] = [\tilde{w}, \tilde{\eta}, \tilde{\theta}] \in H^1(\Omega)^2 \times H^{N_*}(\Omega) \mapsto \mathscr{E}_\varepsilon^\nu(\tilde{v}, \tilde{\theta}) := \mathscr{F}_\nu(\tilde{w}, \tilde{\eta}, \tilde{\theta}) + \frac{\varepsilon}{2}|\tilde{\theta}|^2_{H^{N_*}(\Omega)}.$$

Outline of the proof. Note that (2.3) and (2.4) can be regarded as independent variational problems of elliptic types. Indeed, it is not so difficult to show that the problem (2.3) has a unique (vectorial) unknown $v_{h,i} = [w_{h,i}, \eta_{h,i}]$. Hence, after solving (2.3), we can restrict the unknown in (2.4) only to $\theta_{h,i}$. Consequently, for each step $i \in \mathbb{N}$, these problems can be solved in the order of (2.3) and (2.4) by means of the usual variational

method (e.g. [3]). The property (2.6) can be deduced on the basis of the theory of T-monotonicity (cf. [2, 7]). Furthermore, the inequality (2.7) is obtained by multiplying the both sides of (2.3) and (2.4) by $(v_{h,i} - v_{h,i-1})$ and $(\theta_{h,i} - \theta_{h,i-1})$, respectively, and taking the sum of the results. Incidentally, the smallness $0 < h \le h_*$ for h will be needed only for the discussions associated with $\{v_{h,i}\}$: the solvability of (2.3); the range constraint property as in (2.6); the derivation of the coefficient $\frac{1}{2h}$ at the head of (2.7). ■

3 Outline of the proof of the Main Theorem

In this paper, we show just the outline of the proof. The detailed arguments will be published in the forthcoming paper (in preparation).

Roughly summarized, the proof is proceeded through several steps, lined up below.

Step 1: preparation of notation. We begin with taking a sequence $\{\tilde{\theta}_0^\varepsilon\} \subset H^{N_*}(\Omega)$ such that:

$$\begin{cases} |\tilde{\theta}_0^\varepsilon|_{L^\infty(\Omega)} \le |\theta_0|_{L^\infty(\Omega)} \quad \text{for all } 0 < \varepsilon < 1, \\ \tilde{\theta}_0^\varepsilon \to \theta_0 \text{ in } H^1(\Omega) \quad \text{and} \quad \mathscr{E}_\varepsilon^\nu(v_0; \tilde{\theta}_0^\varepsilon) \to \mathscr{F}_\nu(w_0, \eta_0, \theta_0), \quad \text{as } \varepsilon \searrow 0. \end{cases} \tag{3.1}$$

To this end, we need to prove the following (Fact 1), in advance.

(Fact 1) The sequence $\{\Psi_\varepsilon^\nu(v_0; \cdot) \,|\, 0 < \varepsilon < 1\}$ of convex functions converges to the convex function $\Phi_\nu(v_0; \,\cdot\,)$ on $L^2(\Omega)$, in the sense of Γ-convergence, as $\varepsilon \searrow 0$.

Subsequently, let $0 < h_* < 1$ be the small constant as in Theorem 2.1, and for any $0 < h \le h_*$, let $[\{v_{h,i}^\varepsilon\}, \{\theta_{h,i}^\varepsilon\}] = [\{w_{h,i}^\varepsilon\}, \{\eta_{h,i}^\varepsilon\}, \{\theta_{h,i}^\varepsilon\}]$ be the solution to $(\mathrm{AP})_h^\varepsilon$ in the case when $\tilde{\theta}_0 = \tilde{\theta}_0^\varepsilon$. On that basis, we set:

$$\begin{cases} t_{h,i} := ih, \quad i = 0, 1, 2, 3, \cdots, \\ \Delta_{h,i} := [t_{h,i-1}, t_{h,i}), \quad i = 1, 2, 3, \cdots, \end{cases} \quad \text{for any } 0 < h \le h_*,$$

and we construct sequences:

$$\begin{cases} \{[\overline{v}_h^\varepsilon, \overline{\theta}_h^\varepsilon]\} &= \{[\overline{w}_h^\varepsilon, \overline{\eta}_h^\varepsilon, \overline{\theta}_h^\varepsilon] \,|\, 0 < \varepsilon < 1, \ 0 < h \le h_*\} \\ &\subset L^\infty(0, T; H^2(\Omega))^2 \times L^\infty(0, T; H^{N_*}(\Omega)), \\ \{[\underline{v}_h^\varepsilon, \underline{\theta}_h^\varepsilon]\} &= \{[\underline{w}_h^\varepsilon, \underline{\eta}_h^\varepsilon, \underline{\theta}_h^\varepsilon] \,|\, 0 < \varepsilon < 1, \ 0 < h \le h_*\} \\ &\subset L^\infty(0, T; H^1(\Omega))^2 \times L^\infty(0, T; H^{N_*}(\Omega)), \\ \{[\widehat{v}_h^\varepsilon, \widehat{\theta}_h^\varepsilon]\} &= \{[\widehat{w}_h^\varepsilon, \widehat{\eta}_h^\varepsilon, \widehat{\theta}_h^\varepsilon] \,|\, 0 < \varepsilon < 1, \ 0 < h \le h_*\} \\ &\subset W^{1,\infty}(0, T; H^1(\Omega))^2 \times W^{1,\infty}(0, T; H^{N_*}(\Omega)), \end{cases}$$

by using the following different kinds of time-interpolations:

$$\begin{cases} [\overline{v}_h^\varepsilon(t), \overline{\theta}_h^\varepsilon(t)] &= [\overline{w}_h^\varepsilon(t), \overline{\eta}_h^\varepsilon(t), \overline{\theta}_h^\varepsilon(t)] := [v_{h,i}, \theta_{h,i}] = [w_{h,i}, \eta_{h,i}, \theta_{h,i}] \\ & \quad \text{in } H^2(\Omega)^2 \times H^{N_*}(\Omega), \\ [\underline{v}_h^\varepsilon(t), \underline{\theta}_h^\varepsilon(t)] &= [\underline{w}_h^\varepsilon(t), \underline{\eta}_h^\varepsilon(t), \underline{\theta}_h^\varepsilon(t)] := [\eta_{h,i-1}, \theta_{h,i-1}] = [w_{h,i-1}, \eta_{h,i-1}, \theta_{h,i-1}] \\ & \quad \text{in } H^1(\Omega)^2 \times H^{N_*}(\Omega), \\ [\widehat{\eta}_h^\varepsilon(t), \widehat{\theta}_h^\varepsilon(t)] &= [\widehat{w}_h^\varepsilon(t), \widehat{\eta}_h^\varepsilon(t), \widehat{\theta}_h^\varepsilon(t)] := \tfrac{t_{h,i} - t}{h}[v_{h,i-1}^\varepsilon, \theta_{h,i-1}^\varepsilon] + \tfrac{t - t_{h,i-1}}{h}[v_{h,i}^\varepsilon, \theta_{h,i}^\varepsilon] \\ & \quad \text{in } H^1(\Omega)^2 \times H^{N_*}(\Omega), \end{cases}$$

$$\text{for all } 0 < h \le h_* \text{ and all } t \in \Delta_{h,i}, \ i = 1, 2, 3, \cdots.$$

Step 2: finding the approximation limits as $\varepsilon, h \searrow 0$. Taking into account the assumptions (A1)-(A3), (2.6)-(2.7) and (3.1), we can see that:

$$\begin{cases} \bullet\ \{\overline{v}_h^\varepsilon \,|\, 0<\varepsilon<1,\ 0<h\le h_*\} \text{ is bounded in } L^\infty(0,T;H^1(\Omega))^2, \\ \bullet\ \{\underline{v}_h^\varepsilon \,|\, 0<\varepsilon<1,\ 0<h\le h_*\} \text{ is bounded in } L^\infty(0,T;H^1(\Omega))^2, \\ \bullet\ \{\widehat{v}_h^\varepsilon \,|\, 0<\varepsilon<1,\ 0<h\le h_*\} \text{ is bounded in } W^{1,2}(0,T;L^2(\Omega))^2 \text{ and} \\ \quad \text{bounded in } L^\infty(0,T;H^1(\Omega))^2, \\ \bullet\ 0\le \overline{w}_h^\varepsilon, \overline{\eta}_h^\varepsilon \le 1,\ \ 0\le \underline{w}_h^\varepsilon, \underline{\eta}_h^\varepsilon \le 1 \ \text{ and } \ 0\le \widehat{w}_h^\varepsilon, \widehat{\eta}_h^\varepsilon \le 1, \ \text{ a.e. in } Q, \\ \quad \text{for all } 0<\varepsilon<1,\ 0<h\le h_*; \end{cases}$$

$$\begin{cases} \bullet\ \{\overline{\theta}_h^\varepsilon \,|\, 0<\varepsilon<1,\ 0<h\le h_*\} \text{ is bounded in } L^\infty(0,T;H^1(\Omega)), \\ \bullet\ \{\underline{\theta}_h^\varepsilon \,|\, 0<\varepsilon<1,\ 0<h\le h_*\} \text{ is bounded in } L^\infty(0,T;H^1(\Omega)), \\ \bullet\ \{\widehat{\theta}_h^\varepsilon \,|\, 0<\varepsilon<1,\ 0<h\le h_*\} \text{ is bounded in } W^{1,2}(0,T;L^2(\Omega)) \text{ and} \\ \quad \text{bounded in } L^\infty(0,T;H^1(\Omega)), \\ \bullet\ |\overline{\theta}_h^\varepsilon|_{L^\infty(Q)} \le |\theta_0|_{L^\infty(\Omega)},\ \ |\underline{\theta}_h^\varepsilon|_{L^\infty(Q)} \le |\theta_0|_{L^\infty(\Omega)} \ \text{ and } \ |\widehat{\theta}_h^\varepsilon|_{L^\infty(Q)} \le |\theta_0|_{L^\infty(\Omega)}, \\ \quad \text{for all } 0<\varepsilon<1 \text{ and } 0<h\le h_*. \end{cases}$$

Therefore, applying the compactness theory of Aubin's type [13], we find sequences $\{\varepsilon_n \,|\, n\in\mathbb{N}\}\subset(0,1)$, $\{h_n \,|\, n\in\mathbb{N}\}\subset(0,h_*]$, and a pair (triplet) of functions $[v,\theta]=[w,\eta,\theta]\in L^2(0,T;L^2(\Omega))^3$, such that:

$$\varepsilon_n \searrow 0 \text{ and } h_n \searrow 0 \ \text{ as } n\to\infty,$$

$$\begin{cases} v=[w,\eta]\in W^{1,2}(0,T;L^2(\Omega))^2\cap L^\infty(0,T;H^1(\Omega))^2,\ \ 0\le w,\eta\le 1 \text{ a.e. in } Q, \\ \theta\in W^{1,2}(0,T;L^2(\Omega))\cap L^\infty(0,T;H^1(\Omega)),\ \ |\theta|_{L^\infty(Q)}\le|\theta_0|_{L^\infty(\Omega)}, \end{cases} \tag{3.2}$$

$$\begin{cases} \overline{v}_n=[\overline{w}_n,\overline{\eta}_n]:=\overline{v}_{h_n}^{\varepsilon_n}\to v \ \text{ and } \ \underline{v}_n=[\underline{w}_n,\underline{\eta}_n]:=\underline{v}_{h_n}^{\varepsilon_n}\to v, \\ \qquad \text{in } L^\infty(0,T;L^2(\Omega))^2, \text{ weakly-}*\text{ in } L^\infty(0,T;H^1(\Omega))^2, \\ \qquad \text{weakly-}*\text{ in } L^\infty(Q)^2, \text{ and in the pointwise sense a.e. in } Q, \\ \widehat{v}_n=[\widehat{w}_n,\widehat{\eta}_n]:=\widehat{v}_{h_n}\to v \ \text{ in } C([0,T];L^2(\Omega))^2, \\ \qquad \text{weakly in } W^{1,2}(0,T;L^2(\Omega))^2, \text{ weakly-}*\text{ in } L^\infty(0,T;H^1(\Omega))^2, \\ \qquad \text{weakly-}*\text{ in } L^\infty(Q)^2, \text{ and in the pointwise sense a.e. in } Q, \end{cases} \quad \text{as } n\to\infty, \tag{3.3}$$

and

$$\begin{cases} \overline{\theta}_n:=\overline{\theta}_{h_n}^{\varepsilon_n}\to\theta \ \text{ and } \ \underline{\theta}_n:=\underline{\theta}_{h_n}^{\varepsilon_n}\to\theta \ \text{ in } L^\infty(0,T;L^2(\Omega)), \\ \qquad \text{weakly-}*\text{ in } L^\infty(0,T;H^1(\Omega)), \text{ weakly-}*\text{ in } L^\infty(Q), \text{ and} \\ \qquad \text{in the pointwise sense a.e. in } Q, \\ \widehat{\theta}_n:=\widehat{\theta}_{h_n}^{\varepsilon_n}\to\theta \ \text{ in } C([0,T];L^2(\Omega)), \text{ weakly in } W^{1,2}(0,T;L^2(\Omega)), \\ \qquad \text{weakly-}*\text{ in } L^\infty(0,T;H^1(\Omega)), \text{ weakly-}*\text{ in } L^\infty(Q), \text{ and} \\ \qquad \text{in the pointwise sense a.e. in } Q, \end{cases} \quad \text{as } n\to\infty. \tag{3.4}$$

Step 3: verification of (S1)-(S5). From (3.2)-(3.4), we easily check almost all conditions in (S1) and (S5), except for $\eta\in L^2(0,T;H^2(\Omega))$.

Alternatively, the conditions (S2)-(S4) will be verified on the basis of the following (Fact 2) associated with the Γ-convergence of functionals on $L^2(0,T;L^2(\Omega))$.

(Fact 2) A sequence $\{\hat{\Psi}_n \,|\, n \in \mathbb{N}\}$ of convex functions, defined as:

$$\vartheta \in L^2(0,T;L^2(\Omega)) \mapsto \hat{\Psi}_n(\vartheta) := \int_0^T \Psi^{\nu}_{\varepsilon_n}(\overline{v}_n(t);\vartheta(t))\,dt, \quad \text{for } n \in \mathbb{N},$$

converges to a convex function $\hat{\Phi}_\nu$, defined as:

$$\vartheta \in L^2(0,T;L^2(\Omega)) \mapsto \hat{\Phi}_\nu(\vartheta) := \int_0^T \Phi_\nu(v(t);\vartheta(t))\,dt,$$

on $L^2(0,T;L^2(\Omega))$, in the sense of Γ-convergence, as $n \to \infty$.

In fact, by this Γ-convergence, we obtain (S4), together with the following strong convergence:

$$\overline{\theta}_n \to \theta, \quad \underline{\theta}_n \to \theta \quad \text{and} \quad \widehat{\theta}_n \to \theta \quad \text{in } L^2(0,T;H^1(\Omega)), \quad \text{as } n \to \infty. \tag{3.5}$$

From (3.3)-(3.5), it further follows that:

$$\begin{cases} \displaystyle\int_0^T\!\!\int_\Omega \varphi\beta'(\overline{w}_n(t))|\nabla\underline{\theta}_n(t)|^2\,dxdt \to \int_0^T\!\!\int_\Omega \varphi\beta'(w)|\nabla\theta(t)|^2\,dxdt, \\ \qquad \text{for any } \varphi \in H^1(\Omega)\cap L^\infty(\Omega), \\ \displaystyle\int_0^T\!\!\int_\Omega \psi\alpha'(\overline{\eta}_n(t))|\nabla\underline{\theta}_n(t)|\,dxdt \to \int_0^T\!\!\int_\Omega \psi\alpha'(\eta)|\nabla\theta(t)|\,dxdt, \\ \qquad \text{for any } \psi \in H^1(\Omega), \end{cases} \quad \text{as } n \to \infty. \tag{3.6}$$

Taking into account (3.3)-(3.4) and (3.6), we verify (S2)-(S3).

Now, the remaining condition $\eta \in L^2(0,T;H^2(\Omega))$ is a direct consequence from the standard regularity theory of evolution equations (cf. [2]). ■

4 Problems in future

Finally, we mention about the problems in future.

(Problem 1) Further qualitative properties for (S)$_\nu$

Based on the solvability result, it will be possible to verify further qualitative properties for (S)$_\nu$, such as *smoothing effect*, *energy-dissipation*, *large-time behavior*, and so on. Then, the key-point will be how we derive the pointwise convergence of the energy:

$$\mathscr{E}^{\nu}_{\varepsilon_n}(\overline{v}_n(t),\overline{\theta}_n(t)) \to \mathscr{F}_\nu(w(t),\eta(t),\theta(t)) \quad \text{and} \quad \mathscr{E}^{\nu}_{\varepsilon_n}(\underline{v}_n(t),\underline{\theta}_n(t)) \to \mathscr{F}_\nu(w(t),\eta(t),\theta(t)),$$
$$\text{for a.e. } t \in (0,T), \quad \text{as } n \to \infty,$$

to obtain the limiting formula for the energy-inequality (2.7) in Theorem 2.1:

$$\frac{1}{2}\int_s^t |v_t(\tau)|^2_{L^2(\Omega)^2}\,d\tau + \int_s^t |\sqrt{\alpha_0(v(\tau))}\theta_t(\tau)|^2_{L^2(\Omega)}\,d\tau + \mathscr{F}_\nu(w(t),\eta(t),\theta(t))$$
$$\le \mathscr{F}_\nu(w(s),\eta(s),\theta(t)), \quad \text{for all } 0 \le s \le t \le T.$$

(Problem 2) The case when $\nu = 0$

This theme will be concerned with the study of association between the limiting situation of (S)$_\nu$ as $\nu \searrow 0$, and the following simplified system, denoted by (S)$_0$.

$(\mathrm{S})_0$:

$$\begin{cases} w_t - \Delta w + p(w,\eta) = 0 \ \text{ in } Q, \\ \eta_t - \Delta\eta + q(w,\eta) + \alpha'(\eta)|D\theta| = 0 \ \text{ in } Q, \\ \alpha_0(w,\eta)\theta_t - \mathrm{div}\left(\alpha(\eta)\dfrac{D\theta}{|D\theta|}\right) = 0 \ \text{ in } Q, \end{cases}$$

subject to suitable initial-boundary conditions. Then, analytic methods established in [1, 10] will act key-roles in mathematical treatments of the nonstandard terms $\alpha'(\eta)|D\theta|$ and $-\mathrm{div}(\alpha(\eta)\frac{D\theta}{|D\theta|})$ as in $(\mathrm{S})_0$.

(Problem 3) The anisotropic case

We may consider the anisotropic versions for the system $(\mathrm{S})_\nu$ (actually, also for $(\mathrm{S})_0$) by modifying the density of the functionals $\Phi_\nu(v;\,\cdot\,)$ for $v = [w,\eta] \in H^1(\Omega)^2 \cap L^\infty(\Omega)^2$. However, since *the Wulff shape*, i.e. the structural unit of the crystal, is forced to rotate with the variation of θ, the modified formula of the functional should be settled in the following form:

$$\vartheta \in H^1(\Omega) \mapsto \int_\Omega \alpha(\eta)\gamma(\theta,\nabla\theta)\,dx + \nu\int_\Omega \beta(w)\gamma(\theta,\nabla\theta)^2\,dx,$$
$$\text{for } v = [w,\eta] \in H^1(\Omega)^2 \cap L^\infty(\Omega)^2,$$

where $\gamma : \mathbb{R}\times\mathbb{R}^N \longrightarrow [0,\infty)$ is a given function to characterize the anisotropy.

(Problem 4) The non-isothermal case

According to the modelling method of [9, 14], we can consider full-versions for the systems $(\mathrm{S})_\nu$ and $(\mathrm{S})_0$ (and also for their anisotropic versions), by coupling with the equation for the temperature $u = u(t,x)$. Then, the equation for u should be described in the form of the heat equation:

$$(u+w)_t - \Delta u = h \ \text{ in } Q,$$

with the given forcing term $h = h(t,x)$ and the initial-boundary condition. Also, the perturbations $p = p(w,\eta)$, $q = q(w,\eta)$ and their potential $f = f(w,\eta)$ should be replaced by more interactive forms $p = p(u,w,\eta)$, $q = q(u,w,\eta)$ and $f = f(u,w,\eta)$, respectively.

Conclusive comments. In the mathematical analysis for the above problems, our strategy will be based on the time-discretization approach as presented here. In addition, we expect that similar kinds of approaches might be effective not only for the problems in future but also for mathematical models treated in the relevant previous studies, such as [4, 5, 6, 8, 10, 11, 12, 15].

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A CANCELLATION PROPERTY AND THE WELL-POSEDNESS FOR THE PERIODIC KDV EQUATIONS

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In this paper, we consider the Cauchy problem of the fifth order KdV equation in the periodic setting $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ as follows:.

$$\begin{cases} \partial_t u + a_1\partial_x^5 u + c_1\partial_x(\partial_x u)^2 + c_2\partial_x^2(u\partial_x u) + c_3\partial_x(u^3) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{T}, \\ u|_{t_0} = u_0 \in H^s(\mathbb{T}) \end{cases} \tag{1}$$

where an unknown function $u = u(t,x)$ is real valued and $a_1, c_1, c_2, c_3 \in \mathbb{R}$ with $a_1 \neq 0$. Such equation and its generalizations

$$\partial_t u + a_0\partial_x^3 u + a_1\partial_x^5 u + c_0\partial_x(u^2) + c_1\partial_x(\partial_x u)^2 + c_2\partial_x^2(u\partial_x u) + c_3\partial_x(u^3) = 0$$

arise as long-wave approximations to the water wave equations like to the KdV equation

$$\partial_t u + \partial_x^3 u + \partial_x(u^2) = 0 \tag{2}$$

(see Craig, Guyenne and Kalisch [5], Oiver [9] and so on). By the scaling $u(t,x) \mapsto |c_3/10a_0|^{1/2}\, u(t/a_0, x)$, (1) is rewritten into

$$\partial_t u + \partial_x^5 u + \alpha\partial_x(\partial_x u)^2 + \beta\partial_x^2(u\partial_x^2 u) + 10\gamma\partial_x(u^3) = 0 \tag{3}$$

where $\alpha, \beta \in \mathbb{R}$ and $\gamma = \operatorname{sgn} c_3/a_0 = -1, 0$ or 1. In the special case $(\alpha,\beta,\gamma) = (-5, 10, 1)$ or $(5, -10, 1)$, the equations in (3) are in the KdV hierarchy discovered by Lax [8]. In this case, (3) is complete integrable and has an infinite number of conservation laws, which implies that (3) has rich structure. Put

$$E_0(u(t)) := \frac{1}{2\pi}\int_{\mathbb{T}} u(t)\, dx, \quad E_1(u(t)) := \frac{1}{2\pi}\int_{\mathbb{T}} u^2(t)\, dx.$$

E_0 and E_1 are conserved if and only if the following condition holds.

$$2\alpha + \beta = 0. \tag{4}$$

Note that in the case (4), the Hamiltonian

$$H(u(t)) := \frac{1}{2\pi}\int_{\mathbb{T}} (\partial_x^2 u(t))^2 + 2\alpha u(t)(\partial_x u(t))^2 + 5\gamma u^4(t)\, dx$$

is well-defined. In this paper, we only consider the Hamiltonian case, namely

$$\partial_t u + \partial_x^5 u + \alpha \partial_x (\partial_x u)^2 - 2\alpha \partial_x^2 (u \partial_x u) + 10\gamma \partial_x (u^3) = 0. \tag{5}$$

In the periodic case, there are many results on the well-posedness of the KdV equation (2) but no results on that of the fifth order KdV equation (5) when initial data is given in $H^s(\mathbb{T})$. Therefore our aim is to establish the method reflecting the algebraic structure and show the well-posedness of the Cauchy problem for (5) in $H^s(\mathbb{T})$.

The strong singularity in the nonlinear terms having some derivatives makes the problem difficult. This is called a loss of derivatives. It is the most important point in this study how to cancel a loss of three derivatives in the quadratic terms of (5). We can divide problem into two parts. First one is how to cancel the resonant parts with a loss of derivatives (For the definition of the resonant parts, see blow). Second one is how to recover three derivatives on the non-resonant parts. To overcome this difficulty, we try to use the Fourier restriction norm method established by Bourgain [3] or the normal form method (differentiation by parts). Since there are the resonant parts with a loss of derivatives, the solution map associated to (5) fails to be C^3 in $H^s(\mathbb{T})$ for any $s \in \mathbb{R}$. As a consequence, we cannot solve the Cauchy problem of (5) by a Picard iteration method in $H^s(\mathbb{T})$ with $s \in \mathbb{R}$. This is because there are no smoothing effects associated to the linear pats in the periodic case unlike $\mathbb{R}$ case. Here we give the definition of the resonant parts and the non-resonant parts. We consider the following integral equation

$$u(t) = e^{-t\partial_x^5} u(0) + \int_0^t e^{-(t-s)\partial_x^5} P^{(N)}(u, \partial_x u, \cdots, \partial_x^l u)(s)\, ds$$

where $P^{(N)}$ is an Nth homogeneous polynomial. We apply a change of coordinates: $v(t) = e^{t\partial_x^5} u(t)$. We substitute v into the above integral equation and use the spatial Fourier series expansion to obtain

$$\hat{v}(t,k) = \hat{v}(0,k) + \int_0^t \sum_{k=k_1+k_2+\cdots+k_N} e^{s\Phi_N} m^{(N)}(k_1, k_2, \cdots, k_N) \prod_{i=1}^N \hat{v}(s, k_i)\, ds$$

where $m^{(N)}$ is the Nth multiplier and Φ_N is the oscillation term defined as

$$\Phi_N = i\Big[\big(\sum_{i=1}^N k_i\big)^5 - \sum_{i=1}^N k_i^5\Big].$$

The resonant parts are expressed by

$$\Omega_R^{(N)} := \big\{(k_1, k_2, \cdots, k_N) \in \mathbb{Z}^N \ : \ k = k_{1,2,\cdots N},\ \Phi_N = 0\big\}$$

and the non-resonant parts are defined as

$$\Omega_{NR}^{(N)} := \big\{(k_1, k_2, \cdots, k_N) \in \mathbb{Z}^N \ : \ k = k_{1,2,\cdots,N}, \ \Phi_N \neq 0\big\}$$

where $k_{1,2,\cdots,N} := k_1 + k_2 + \cdots + k_N$. Note that the oscillation terms Φ_2 and Φ_3 can be factorized exactly as follows:

$$\Phi_2 = i\frac{5}{2}k_1k_2k_{1,2}(k_1^2 + k_2^2 + k_{1,2}^2),$$
$$\Phi_3 = i\frac{5}{2}k_{1,2}k_{2,3}k_{3,1}(k_{1,2}^2 + k_{2,3}^2 + k_{3,1}^2),$$

which implies that the resonant cases are $k_1k_2k_{1,2} = 0$ and $k_{1,2}k_{2,3}k_{3,1} = 0$ for $N = 2$ and $N = 3$, respectively.

Let us briefly go over recent results on the well-posedness of the periodic KdV equation (2). Bourgain [3] proved the local well-posedness (LWP) of (2) for $s \geq 0$. His idea is to remove the resonant parts by using the conservation laws $E_0(u(t)) = E_0(u_0)$ and modifying the linear parts. Actually, with $v(t) = e^{t\partial_x^3}u(t)$, it follows from (2) that v satisfies

$$\partial_t \hat{v} = -\sum_{k=k_1+k_2} e^{-t\Phi_3} ik_{1,2} \prod_{i=1}^{2} \hat{v}(t, k_i)$$

where $\Phi_3 = i(k_{1,2}^3 - k_1^3 - k_2^3) = 3ik_1k_2k_{1,2}$. This implies that the resonant case is $k_1k_2k_{1,2} = 0$. Note that

$$\hat{v}(t, 0) = \hat{u}(t, 0) = \frac{1}{2\pi}\int_{\mathbb{T}} u(t)\ dx = E_0(u_0).$$

(2) is rewritten into

$$\partial_t u + \partial_x^3 u + 2E_0(u_0)\partial_x u = -\partial_x\big(u - \frac{1}{2\pi}\int_{\mathbb{T}} u\ dx\big)^2. \tag{6}$$

The nonlinear term of (6) is the non-resonant parts. A loss of one derivative in the non-resonant parts can be recovered by the Fourier restriction norm method Bouragin's result was improved to $s \geq -1/2$ by Kenig, Ponce and Vega [6] (see also [4]). Later they also the solution map associated to (2) fails to be uniformly continuous when $s < -1/2$. As a consequence, $s = -1/2$ is the critical regularity in this sense. Moreover Babin, Ilyin and Titi [1] proved LWP and the unconditional uniqueness in $C([0, T] : H^s(\mathbb{T}))$ for (2) when $s \geq 0$. The unconditional uniqueness means that uniqueness holds in the whole of $C([0, T] : H^s(\mathbb{T}))$ without using any auxiliary function space. On the other hand, the nonlinear term $\partial_x(u^2)$ cannot be defined in the distribution sense for $s < 0$. Therefore this result is optimal in the

above sense. Their idea is to recover one derivative in (6) by the normal form method (differentiation by parts). In fact, it follows form (6) that $v(t) = e^{t\partial_x^3}u(t)$ satisfies

$$\partial_t \hat{v}(t,k) = -\sum_{\Phi_2 \neq 0} e^{-t\Phi_2} i k_{1,2} \prod_{i=1}^{2} \hat{v}(t,k_i). \tag{7}$$

We can differentiate the non-resonant parts (7) by parts to have

$$\partial_t \hat{v}(t,k) = \frac{\partial}{\partial t}\Big[\sum_{\Phi_2 \neq 0} e^{-t\Phi_2} \frac{i k_{1,2}}{\Phi_2} \prod_{i=1}^{2} \hat{v}(t,k_i)\Big] \tag{8}$$

$$- \sum_{\Phi_2 \neq 0} e^{-t\Phi_2} \frac{i k_{1,2}}{\Phi_2} (\partial_t \hat{v}(t,k_1)\hat{v}(t,k_2) + \hat{v}(t,k_1)\partial_t \hat{v}(t,k_2)). \tag{9}$$

Both (8) and (9) have $\Phi_2 (\neq 0)$ in the denominators, and this provides smoothing. As a consequence, we can obtain the nonlinear terms with no loss by the above procedure i.e. the normal form method. Note that at most one derivative can be recovered by using the normal form method once.

(5) has the nonlinear terms with a loss of three derivatives. But at most two derivatives can be recovered by the Fourier restriction norm method. We apply the normal form method three times to recover three derivatives and obtain the following results.

Theorem 1. *Let $s \geq 1$. Then (5) is unconditionally locally well-posed in $H^s(\mathbb{T})$. Namely, for any $u_0 \in H^s(\mathbb{T})$, there exist $T = T(\|u_0\|_{H^s}) > 0$ and the unique solution $u \in C([0,T] : H^s(\mathbb{T}))$ to (5). Moreover the solution map, $H^s(\mathbb{T}) \ni u_0 \mapsto u \in C([0,T] : H^s(\mathbb{T}))$, is continuous.*

Theorem 2. *Let $s \geq 1/2$. (5) is locally well-posed in $X_w^s([0,T]) \cap C([0,T] : H^s(\mathbb{T}))$ in the sense of Theorem 1. X_w^s is the modified $X^{s,b}$ space defined below.*

Remark 1. When $s < 1$, the nonlinear term $\partial_x(\partial_x u)^2$ cannot be defined in the distribution sense. Therefore Theorem 1 is optimal in this sense.

Remark 2. In the case $s \geq 2$, the Hamiltonian of (5) is well-defined. The conserved quantities E_1 and H provide a control of the H^2 norm and allow to prove (5) is unconditionally global well-posed in H^2.

Remark 3. The results of Theorems 1 and 2 are corresponding to those of Babin, Ilyin and Titi [1] and Kenig, Ponce and Vega [6].

Remark 4. The results of Theorems 1 and 2 are also valid for the following equation and their proofs are similar.

$$\partial_t u + a_0\partial_x^3 u + \partial_x^5 u + c_0\partial_x(u^2) + \alpha\partial_x(\partial_x u)^2 - 2\alpha\partial_x^2(u\partial_x u) = 0.$$

The key idea in the proof of the main results is to cancel the resonant parts with a loss of derivatives by using the conserved quantities E_0 and E_1 and modifying the linear parts of (5). In fact, if E_0 and E_1 are conserved, then (5) is rewritten into

$$\begin{aligned}
&\partial_t u - \partial_x^5 u - 2\alpha E_0(u_0)\partial_x^3 u + \big(30\gamma - \frac{4}{5}\alpha^2\big)E_1(u_0)\partial_x u + \frac{4}{5}\alpha^2 E_0^2(u_0)\partial_x u \\
&\quad = -30\gamma\Big(u^2 - \frac{1}{2\pi}\int_{\mathbb{T}} u^2\,dx\Big)\partial_x u \qquad (10)\\
&\quad - \alpha\partial_x(\partial_x u)^2 + 2\alpha\partial_x^2\Big\{\Big(u - \frac{1}{2\pi}\int_{\mathbb{T}} u\,dx\Big)\partial_x u\Big\} \qquad (11)\\
&\quad - \frac{4}{5}\alpha^2\Big\{\frac{1}{2\pi}\int_{\mathbb{T}} u^2\,dx - \Big(\frac{1}{2\pi}\int_{\mathbb{T}} u\,dx\Big)^2\Big\}\partial_x u. \qquad (12)
\end{aligned}$$

Recall the resonant parts in the case $N = 2, 3$. There are no resonant parts with a loss of derivatives in the nonlinear terms except the resonant parts (12). Therefore we can apply the normal form method to (10) and (11) once. However, to cancel a loss of three derivatives in (11), we need to apply the normal form method three times. So it is necessary that the resonant parts with a loss of derivatives coming from the normal form reduction of (11) are canceled with the resonant parts (12). This is the most interesting point in our study. All nonlinear terms with no loss coming from the normal form reduction can be estimated by only Sobolev's embedding theorem when $s \geq 1$. As a consequence, we obtain the claim of Theorem 1 by a Picard iteration method.

The claim of Theorem 1 can be extended to $s \geq 1/2$ by introducing the function space X_w^s. This is the modified $X^{s,b}$ space such as Bejenaru and Tao [2] and equipped with the norm as follows:

$$\|u\|_{X_w^s} := \|w_{u_0}^s(\tau, k)\hat{u}\|_{l_k^2 L_\tau^2}$$

Here the weight function $w_{u_0}^s$ is defined as follows;

$$w_{u_0}^s(\tau, k) = \begin{cases} \langle k\rangle^s\langle\tau - p_{u_0}(k)\rangle^{1/4}, & \text{when } |k|^4 \leq |\tau - p_{u_0}(k)| \leq |k|^5 \\ \langle k\rangle^s\langle\tau - p_{u_0}(k)\rangle^{1/2}, & \text{otherwise} \end{cases}.$$

where

$$p_{u_0}(k) = k^5 - 2\alpha E_0(u_0)k^3 + (30\gamma - \frac{4}{5}\alpha^2)E_1(u_0)k + \frac{4}{5}\alpha^2 E_0^2(u_0))k.$$

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Resolvent estimates for the Stokes equations in spaces of bounded functions

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Abstract

The Stokes equation is well understood in the L^p-setting for a large class of domains including bounded and exterior domains with smooth boundaries provided $1 < p < \infty$. The situation is different for the case $p = \infty$ since in this case the Helmholtz projection does not act as a bounded operator anymore. Nevertheless, it was recently shown by a contradiction argument that the Stokes operator generates an analytic semigroup on L^∞-type spaces for a large class of domains. In this talk, we present a new approach as well as new a priori estimate to the resolvent Stokes equation. They in particular implies that the Stokes operator generates an analytic semigroup of angle $\pi/2$ on L^∞-type spaces for a large class of domains. This talk is based on a joint work with Y. Giga and M. Hieber.

1 Introduction

It is well known that the Stokes semigroup is an analytic semigroup on L^p-solenoidal spaces, $p \in (1, \infty)$, for various kind of domains, e.g., bounded and exterior domains with smooth boundaries [8], [5]. However, it was unknown whether or not the Stokes semigroup is an analytic semigroup on L^∞-type spaces except a half space where explicit solution formulas are available [4], [9], [6]. Recently, an affirmative answer to this problem was given for a large class of domains including bounded and exterior domains based on an a priori L^∞-estimate for solutions to the non-stationary Stokes equations [1], [2].

In this talk, we present a new approach as well as new a priori estimate for the resolvent Stokes equations. We consider the resolvent Stokes equations in the domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$:

$$\lambda v - \Delta v + \nabla q = f \quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div} v = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$v = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

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and establish a bound for

$$M_p(v,q)(x,\lambda) = |\lambda||v(x)| + |\lambda|^{1/2}|\nabla v(x)| + |\lambda|^{n/2p}\|\nabla^2 v\|_{L^p(\Omega_{x,|\lambda|^{-1/2}})} + |\lambda|^{n/2p}\|\nabla q\|_{L^p(\Omega_{x,|\lambda|^{-1/2}})},$$

and $p > n$ of the form,

$$\sup_{\lambda\in\Sigma_{\vartheta,\delta}} \left\|M_p(v,q)\right\|_{L^\infty(\Omega)}(\lambda) \le C\,\|f\|_{L^\infty(\Omega)} \tag{1.4}$$

for some constant $C > 0$ independent of f. Here, $\Omega_{x,r}$ denotes the intersection of Ω with an open ball $B_x(r)$ centered at $x \in \Omega$ with radius $r > 0$, and $\Sigma_{\vartheta,\delta}$ denotes the sectorial region in the complex plane given by $\Sigma_{\vartheta,\delta} = \{\lambda \in \mathbf{C}\backslash\{0\} \mid |\arg\lambda| < \vartheta,\ |\lambda| > \delta\}$ for $\vartheta \in (\pi/2,\pi)$ and $\delta > 0$. The estimate (1.4) in particular implies that the Stokes operator generates an analytic semigroup on L^∞-type spaces.

Our approach is inspired by the corresponding approach for general elliptic operators. K. Masuda was the first to prove the analyticity of the semigroup associated to general elliptic operators in the whole space [7]. This result was then extended by H. B. Stewart to the case for the Dirichlet problem [10], [11]. By now, this Masuda-Stewart method applies to many other situations. However, it seems that their localization argument does not directly apply to the resolvent Stokes equations (1.1)–(1.3) because of the presence of pressure.

In order to prove the estimate (1.4), we apply *the harmonic-pressure gradient estimate* introduced in [1], [2], i.e.,

$$\sup_{x\in\Omega} d_\Omega(x)|\nabla q(x)| \le C_\Omega\|W(v)\|_{L^\infty(\partial\Omega)}, \tag{1.5}$$

for $W(v) = -(\nabla v - \nabla^T v)n_\Omega$. When $n = 3$, $W(v)$ is nothing but the tangential component of vorticity, i.e., $-\mathrm{curl}\ \mathrm{v} \times \mathrm{n}_\Omega$. The estimate (1.5) plays a key role in transferring results from the elliptic situation to the situation of the Stokes system. Here, n_Ω denotes the unit outward normal vector field on $\partial\Omega$ and d_Ω denotes the distance from $x \in \Omega$ to the boundary $\partial\Omega$. A key observation is that the Neumann data of the pressure q is transformed into the surface divergence, i.e., $\Delta v \cdot n_\Omega = \mathrm{div}_{\partial\Omega}\, W(v)$ as $\mathrm{div}\ v = 0$ in Ω. So the estimate (1.5) is reduced to investigating an a priori estimate for solutions of the homogeneous Neumann problem:

$$\Delta q = 0 \text{ in } \Omega, \quad \frac{\partial q}{\partial n_\Omega} = \mathrm{div}_{\partial\Omega} W \text{ on } \partial\Omega.$$

Of course, the estimate (1.5) holds for a half space. It is proved in [1], [2] by a blow-up argument that the estimate (1.5) holds for bounded and exterior domains with C^3-boundaries.

To state a result, let $C_{0,\sigma}(\Omega)$ be the L^∞-closure of all smooth solenoidal vector fields with compact support in Ω. Let $L^\infty_d(\Omega)$ be the space of all locally integrable functions f such that $d_\Omega f$ is bounded in Ω. Our typical main result is:

Theorem 1.1. *Let Ω be a bounded or an exterior domain in $\mathbf{R}^n$, $n \ge 2$, with C^3-boundary. Let $p > n$. For $\vartheta \in (\pi/2,\pi)$ there exists constants δ and C such that the a priori estimate (1.4) holds for all solutions $(v,\nabla q) \in W^{2,p}_{\mathrm{loc}}(\Omega)\times(L^p_{\mathrm{loc}}\cap L^\infty_d)(\Omega)$ for $f \in C_{0,\sigma}(\Omega)$ and $\lambda \in \Sigma_{\vartheta,\delta}$.*

The a priori estimates (1.4) and (1.5) imply that the solution operator $R(\lambda) : f \mapsto v_\lambda$ is invertible on $C_{0,\sigma}$, i.e., there exists a closed operator A such that $R(\lambda) = (\lambda - A)^{-1}$. We call A *the Stokes operator* in $C_{0,\sigma}$. The estimate (1.4) says that the Stokes operator A is a sectorial operator in $C_{0,\sigma}$.

Theorem 1.2. *Let Ω be a bounded or an exterior domain with C^3-boundary. Then, the Stokes operator A generates a C_0-analytic semigroup of angle $\pi/2$ on $C_{0,\sigma}(\Omega)$.*

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Global existence for semilinear wave equations with the blow-up term in high dimensions *

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1 General theory for nonlinear wave equations

First we shall outline the general theory on the initial value problem for fully nonlinear wave equations,

$$\begin{cases} u_{tt} - \Delta u = H(u, Du, D_x Du) \quad \text{in} \quad \mathbf{R}^n \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \ u_t(x, 0) = \varepsilon g(x), \end{cases} \tag{1}$$

where $u = u(x, t)$ is a scalar unknown function of space-time variables,

$$\begin{aligned} &Du = (u_{x_0}, u_{x_1}, \cdots, u_{x_n}), \ x_0 = t, \\ &D_x Du = (u_{x_i x_j}, \ i, j = 0, 1, \cdots, n, \ i + j \geq 1), \end{aligned}$$

$f, g \in C_0^\infty(\mathbf{R}^n)$ and $\varepsilon > 0$ is a "small" parameter. We note that it is impossible to construct a general theory for "large" ε due to blow-up results. For example, see Glassey [4], Levine [7], or Sideris [17]. Let

$$\widehat{\lambda} = (\lambda; \ (\lambda_i), i = 0, 1, \cdots, n; \ (\lambda_{ij}), i, j = 0, 1, \cdots, n, \ i + j \geq 1).$$

Suppose that the nonlinear term $H = H(\widehat{\lambda})$ is a sufficiently smooth function with

$$H(\widehat{\lambda}) = O(|\widehat{\lambda}|^{1+\alpha})$$

in a neighborhood of $\widehat{\lambda} = 0$, where $\alpha \geq 1$ is an integer. Let us define the lifespan $\widetilde{T}(\varepsilon)$ of classical solutions of (1) by

$$\begin{aligned} \widetilde{T}(\varepsilon) = \sup\{t > 0 \ : \ \exists \ &\text{classical solution } u(x, t) \text{ of (1)} \\ &\text{for arbitrarily fixed data, } (f, g).\}. \end{aligned}$$

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When $\widetilde{T}(\varepsilon)=\infty$, the problem (1) admits a global in time solution, while we only have a local in time solution on $t\in[0,\widetilde{T}(\varepsilon))$ when $\widetilde{T}(\varepsilon)<\infty$. For local in time solutions, one can measure the long time stability of a zero solution by orders of ε. Because the uniqueness of the solution of (1) may yield that $\lim_{\varepsilon\to+0}\widetilde{T}(\varepsilon)=\infty$. Such an uniqueness theorem can be found in Appendix of John [12] for example.

In Chapter 2 of Li and Chen [9], we have long histories on the estimate for $\widetilde{T}(\varepsilon)$. The lower bounds of $\widetilde{T}(\varepsilon)$ are summarized in the following table. Let $a=a(\varepsilon)$ satisfy

$$a^2\varepsilon^2\log(a+1)=1 \tag{2}$$

and c stands for a positive constant independent of ε. Then, due to the fact that it is impossible to obtain an L^2 estimate for u itself by standard energy methods, we have

$\widetilde{T}(\varepsilon)\geq$	$\alpha=1$	$\alpha=2$	$\alpha\geq 3$
$n=2$	$ca(\varepsilon)$ in general case, $c\varepsilon^{-1}$ if $\int_{\mathbf{R}^2}g(x)dx=0$, $c\varepsilon^{-2}$ if $\partial_u^2H(0)=0$	$c\varepsilon^{-6}$ in general case, $c\varepsilon^{-18}$ if $\partial_u^3H(0)=0$, $\exp(c\varepsilon^{-2})$ if $\partial_u^3H(0)=\partial_u^4H(0)=0$	∞
$n=3$	$c\varepsilon^{-2}$ in general case, $\exp(c\varepsilon^{-1})$ if $\partial_u^2H(0)=0$	∞	∞
$n=4$	$\exp(c\varepsilon^{-2})$ in general case, ∞ if $\partial_u^2H(0)=0$	∞	∞
$n\geq 5$	∞	∞	∞

The result for $n=1$ is that

$$\widetilde{T}(\varepsilon)\geq\begin{cases}\varepsilon^{-\alpha/2} & \text{in general case,}\\ c\varepsilon^{-\alpha(1+\alpha)/(2+\alpha)} & \text{if } \int_{\mathbf{R}}g(x)dx=0,\\ c\varepsilon^{-\alpha} & \text{if } \partial_u^\beta H(0)=0 \text{ for } 1+\alpha\leq\forall\beta\leq 2\alpha.\end{cases} \tag{3}$$

For references on these results, see Li and Chen [9]. We shall skip to refer them here. But we note that two parts in this table are different from the one in Li and Chen [9]. One is the general case in $(n,\alpha)=(4,1)$. In this part, the lower bound of $\widetilde{T}(\varepsilon)$ is $\exp(c\varepsilon^{-1})$ in Li and Chen [9]. But later, it has been improved by Li and Zhou [10]. Another is the case for $\partial_u^3H(0)=0$ in $(n,\alpha)=(2,2)$. This part is due to Katayama [14]. But it is missing in Li and Chen [9]. Its reason is closely related to the sharpness of results in the general theory. The sharpness is achieved by the fact that there is no possibility to improve the lower bound of $\widetilde{T}(\varepsilon)$ in sense of order of ε by blow-up results for special equations and special data. It is expressed in the upper bound of $\widetilde{T}(\varepsilon)$ with the same order of ε as

the lower bound. On this matter, Li and Chen [9] says that all these lower bounds are known to be sharp except for $(n, \alpha) = (4, 1)$. But before this article, Li [8] says that $(n, \alpha) = (2, 2)$ has also open sharpness while the case for $\partial_u^3 H(0) = 0$ is still missing. Li and Chen [9] might have dropped the open sharpness in $(n, \alpha) = (2, 2)$ by conjecture that $\partial_u^4 H(0) = 0$ is a technical condition. No one disagrees with this observation because the model case of $H = u^4$ has a global solution in two space dimensions, $n = 2$. However, surprisingly, Zhou and Han [23] have obtained this final sharpness in $(n, \alpha) = (2, 2)$ by studying $H = u_t^2 u + u^4$. This puts Katayama's result into the table after 20 years from Li and Chen [9]. We note that Zhou and Han [22] have also obtained the sharpness of the case for $\partial_u^3 H(0) = \partial_u^4 H(0) = 0$ in $(n, \alpha) = (2, 2)$ by studying $H = u_t^3$. This part had been verified by $H = |u_t|^3$ only.

We now turn back to another open sharpness of the general case in $(n, \alpha) = (4, 1)$. It has been obtained by our previous work, Takamura and Wakasa [19], by studying model case of $H = u^2$. This part had been open more than 20 years in the analysis on the critical case for model equations, $u_{tt} - \Delta u = |u|^p$ $(p > 1)$. In this way, the general theory and its optimality have been completed.

2 The final problem and related results

After the completion of the general theory, we are interested in "almost" global existence, namely, the case where $\widetilde{T}(\varepsilon)$ has an lower bound of the exponential function of ε with a negative power. Such a case appears in $(n, \alpha) = (2, 2), (3, 1), (4, 1)$ in the table of the general theory. It is remarkable that Klainerman [15] and Christodoulou [3] have independently found a special structure on $H = H(Du, D_x Du)$ in $(n, \alpha) = (3, 1)$ which guarantees the global existence. This algebraic condition on nonlinear terms of derivatives of the unknown function is so-called "null condition". It has been also established by Godin [5] for $H = H(Du)$ and Katayama [13] for $H = H(Du, D_x Du)$ in $(n, \alpha) = (2, 2)$. The null condition has been supposed to be not sufficient for the global existence in $(n, \alpha) = (2, 2)$. Finally Hoshiga [6] and Kubo [16] have independently succeeded to establish "non-positive" condition in this case for $H = H(Du)$. It might be necessary and sufficient condition to the global existence. On the other hand, the situation in $(n, \alpha) = (4, 1)$ is completely different from $(n, \alpha) = (2, 2), (3, 1)$ because H has to include u^2.

In the sense of the first section, one of the final open problem on the optimality of the general theory for fully nonlinear wave equations can be established by model problem;

$$\begin{cases} u_{tt} - \Delta u = u^2 \quad \text{in} \quad \mathbf{R}^4 \times [0, \infty), \\ u(x, 0) = \varepsilon f(x), \ u_t(x, 0) = \varepsilon g(x). \end{cases} \tag{4}$$

We note that this is an extended problem of John [11] to high dimensional case which has the "critical" exponent of Strauss' conjecture [18]. The lifespan $T(\varepsilon)$ of the solution of (4) should have an estimate of the form;

$$\exp(c\varepsilon^{-2}) \leq T(\varepsilon) \leq \exp(C\varepsilon^{-2}). \tag{5}$$

This final problem on the upper bound has been solved by our previous work, Takamura and Wakasa [19]. In its proof, the analysis on $\|u(\cdot,t)\|^2_{L^2(\mathbf{R}^4)}$ is a key because we cannot use any pointwise esimate of the solution due to so-called derivative loss in fundamental solutions in high dimensions. Therefore one may have questions;

- Do we have any possibility to get $T(\varepsilon)=\infty$ if the nonlinear term is not single while it includes u^2 ?
- Do we have any possibility to get a pointwise positivity of the solution for some special nonlinear term ?

For these questions, we get the following partial answers.

Theorem 1 (Takamura and Wakasa [20]) *Even if the right-hand side of the equation in* (4) *additionally has integral terms of the form;*

$$-\frac{1}{\pi^2}\int_0^t d\tau\int_{|\xi|\le 1}\frac{(u_t u)(x+(t-\tau)\xi,\tau)}{\sqrt{1-|\xi|^2}}d\xi-\frac{\varepsilon^2}{2\pi^2}\int_{|\xi|\le 1}\frac{f(x+t\xi)^2}{\sqrt{1-|\xi|^2}}d\xi, \tag{6}$$

there is no change on the estimate the lifespan (5).

(6) comes from a removal of the derivative loss factor in the nonlinear term of the equivalent integral equation to (4). This observation already appeared in Agemi, Kubota and Takamura [1] in which a global solution is obtained for the "super-critical" case. The proof of this theorem is established by iteration argument in weighted L^∞ space.

In contrast with (6), we have the following.

Theorem 2 (Takamura and Wakasa [21]) *If the right-hand side of the equation in* (4) *additionally has integral terms of the form;*

$$-\frac{1}{2\pi^2}\int_0^t d\tau\int_{|\omega|=1}(u_t u)(x+(t-\tau)\omega,\tau)dS_\omega-\frac{\varepsilon}{4\pi^2}\int_{|\omega|=1}(\varepsilon f^2+\Delta f+2\omega\cdot\nabla g)(x+t\omega)dS_\omega, \tag{7}$$

then, $T(\varepsilon)=\infty$ holds.

(7) follows from the following fact due to Agemi and Takamura [2]. When $n\ge 3$, a classical solution u of (1) satisfies

$$\begin{aligned}(n-2)\omega_n u(x,t)=\ & \varepsilon\int_{|\omega|=1}\{t\omega\cdot\nabla f+(n-2)f+tg\}(x+t\omega)dS_\omega\\ &+(n-3)\int_0^t d\tau\int_{|\omega|=1}u_t\left(x+(t-\tau)\omega,\tau\right)dS_\omega\\ &+\int_0^t(t-\tau)d\tau\int_{|\omega|=1}H\left(x+(t-\tau)\omega,\tau\right)dS_\omega,\end{aligned} \tag{8}$$

where ω_n is an area of the unit sphere in $\mathbf{R}^n$. If we neglect the second term in the right-hand side of (8), we get (7) by replacing g by $2g$ when $n=4$ and $H=u^2$. The proof of this theorem is also established by iteration argument in weighted L^∞ space. But the key estimate is

$$\left|\int_{|\omega|=1}(t\omega\cdot\nabla f+2f+2tg)(x+t\omega)dS_\omega\right|\le\frac{C_{f,g}}{(1+t)^2},$$

where $C_{f,g}$ is a positive constant. This is faster than $(1+t)^{-3/2}$ which is a decay of a solution of the free equation.

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Timelike minimal surfaces in Minkowski space

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We discuss here some properties of timelike minimal surfaces in flat Minkowski space-time, reviewing in particular some results proved in [3, 2, 4, 10].
Recall a for a smooth lorentzian time-like minimal submanifold, its time-slices satisfy

$$a = (1 - |v|_e^2)\kappa, \tag{0.1}$$

where a, v and κ are respectively the normal acceleration, the normal velocity and the euclidean mean curvature of the time-slice of the submanifold, $|v|_e$ is the euclidean length of v, and where we normalize units so that the speed of light equals one. Equation (0.1) can be considered as a geometric evolution equation, and the lorentzian structure of the ambient space $\mathbb{R}^{1+N}$ reflects in the hyperbolic nature of the equation. Smooth lorentzian timelike minimal submanifolds are thus characterized by the fact that their (spacetime) lorentzian mean curvature is identically zero. A particular case, relevant for its applications in physics, is the one of two-dimensional surfaces of codimension one or two, namely when $N = 2$ or $N = 3$. Under these assumptions, minimal surfaces are called (classical) relativistic strings, of codimension one and two respectively. In case the minimal submanifold is a two-dimensional graph, the evolution equation (0.1) is also known as the Born-Infeld equation, a nonlinear model for electromagnetism (see [7, 6]).
Solutions to (0.1) develop generically singularities of various type in finite time [15, 8].
A general short-time existence result for smooth solutions to (0.1) has been obtained in [12]. If the manifold is supposed sufficiently close to a linear subspace, then global existence results has been proved in [5], [11]. If one is interested in non regular or weak solutions to the lorentzian minimal surface equation, globally defined in time, one may for instance use non smooth parametrizations, or introduce generalized lorentzian surfaces in the spirit of Almgren's varifold theory and Di Perna-Majda generalized Young measures solutions the Euelr and Navier-Stokes equation [4], but in this case one is faced also to the fact that, differently from the euclidean case, the space of such solutions is not compact under uniform convergence, and a satisfactory characterization of its closure is still missing for weak or generalized minimal submanifolds of dimension larger than two.
Remark also that some suggestions in the direction of a suitable definition of weak solution arise from the analysis of the asymptotic limits, as $\varepsilon \to 0^+$, of hyperbolic Ginzburg-Landau equation, a semilinear wave equation that allows concentration of energy on timelike minimal surfaces, as observed by Neu [13]. In the more simple one-codimensional case, this is the scaled semilinear hyperbolic PDE

$$u_{tt} - \Delta u + \frac{1}{\varepsilon^2} W'(u) = 0, \tag{0.2}$$

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where $W(x) = \frac{1}{4}(1-x^2)^2$ is a standard double-well potential. As showed in [13] by a formal asymptotic expansion argument, equation (0.1) can be approximated by solutions to (0.2). Next, it has been rigorously shown in [9] that solutions to (0.2) with a well-prepared initial datum converge, as $\varepsilon \to 0^+$, to a smooth lorentzian minimal hypersurface, when the latter exists, this result being generalized by the author also in codimension two. Again, due to the presence of singularities, the validity of this convergence result is restricted to short times. On the other hand, a preliminary analysis of the limit behaviour of the hyperbolic Ginzburg-Landau equations in the varifolds framework has been pursued in [3], without restricting to short times, but under rather strong assumptions on the limiting varifold solution, in the spirit of [1] for the parabolic case.

It is worth noticing that for surfaces of dimension two, i.e. the case of relativistic strings, the problem of characterizing the closure of the space of solutions of (0.1) has a reasonable positive answer. Indeed, in this case one may parametrize the string in the so called orthogonal gauge, so that the parametrization $X(s,t)$ solves the linear wave system, and hence the solution is explicitely given by D'Alembert formula $X(s,t) = (t, \gamma(t,s))$, with $\gamma(t,s) = \frac{1}{2}(a(s-t)+b(s+t))$, where a and b are arc-lenght parametrized closed curves (i.e. $|a'| = |b'| = 1$).

The closure of two-dimensional minimal surfaces has been completely charaterized, at least in a parametric setting, leading to the concept of subrelativistic string (see [15, 13, 6, 2] for a detailed discussion), i.e. a lipschitz parametrization $\gamma(t,s) = \frac{1}{2}(a(s-t)+b(s+t))$ with the relaxed constraint $|a'| \leq 1$, $|b'| \leq 1$ a.e..

In terms of lorentzian varifolds, as introduced in [4], the notion of subrelativistic string corresponds to the notion of a stationary weakly rectifiable varifold, which can be in turn thought as a limit of rectifiable varifolds, and enjoys some useful properties reflected by equation (0.1) such as energy and momentum conservation during the evolution.

In the case of evolution of strings in three dimensional Minkowski space-time we analyze also convexity preserving properties of (0.1) , in the spirit of Gage-Hamilton result for curvature flow of planar curves, and Kong-Kefeng-Zhang in the case of the (nonrelativistic) hyperbolic curvature flow, proving such a result for centrally symmetric uniformly convex planar curves initially at rest in [2], together with an analysis of the asymptotic profile near the collapsing time, which corresponds to a round circle.

Differently from the parabolic case, this collapsing singularity is non generic, while the generic singularity expected is a cusp, as proven in [15, 8].

The extinction singularity concerns also the case of non uniformly convex boundary curves (e.g. a square), but in this case the asymptotic profile will not be a circle (for the square, the profile remains a square). The D'alembert parametrization provides a global weak periodic solution (pulsating kink), which strongly lacks of uniqueness (one can for instance restart the evolution of a pulsating circle after collapsing with a suitable expanding centrally symmetric convex boundary, see [10]).

The special case of minimal immersed cylinders in Minkowski space-time has also been investigated, in particular by Nguyen-Tian in [14], where it is proved that any immersed timelike minimal cylinder in three-dimensional space-time necessarily develops singularities, and it si conjectured that this doesn't hold in dimension higher than four. This conjecture is proved to be true in [10], where also an estimate of the dimension of the singular set of parametrizations of minimal C^1 cylinders in a specific form (not necessarily immersions).

More precisely, we prove in [10] that, generically, in space dimension higher than three, given

a closed immersed curve Γ and a velocity field v with $|v| < 1$ and orthogonal to Γ, there exists a smooth globally immersed timelike minimal surface containing Γ and tangent to $(1, v)$. In the three-dimensional case, roughly speaking, both globally smooth immersed solutions and solutions that develop singularities occur for large sets of initial data (i.e. sets with nonempty interior).

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L^1 MAXIMAL REGULARITY AND LOCAL EXISTENCE OF A SOLUTION TO THE COMPRESSIBLE NAVIER-STOKES-POISSON SYSTEM IN A CRITICAL BESOV SPACE

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1. Introduction

We consider the Cauchy problem of the compressible Navier-Stokes-Poisson system in $\mathbb{R}^n$ with $n \geq 2$.

$$\begin{cases} \partial_t \rho + \mathrm{div}\ (\rho u) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ \partial_t(\rho u) + \mathrm{div}\ (\rho u \otimes u) + \nabla P(\rho) & \\ \qquad = \mu \Delta u + (\mu+\lambda)\nabla \mathrm{div}\ u + \kappa \rho \nabla \psi, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ -\Delta \psi = \rho - \bar{\rho}, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ \qquad u(0,x) = u_0(x),\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\rho = \rho(t,x)$, $u = u(t,x)$ and $\psi = \psi(t,x)$ are the unknown fluid density, the velocity vector and the potential force, respectively. $P = P(\rho)$ is the pressure given by ρ, and $u \otimes u$ denotes the tensor product of velocity vector u. μ, λ are Lamé constants satisfying $\mu + 2\lambda > 0$, $\kappa = \pm$ is a coupling constant and $\bar{\rho}$ is a given background density. Without losing generality, we assume that $\bar{\rho} = 1$. The system is strongly relevant to the simplified system of degenerate drift-diffusion equations and the Smoluchowski-Poisson system appeared in a semiconductor divise models (cf. [16]).

Introducing the perturbed density by $a(t,x) \equiv \rho(t,x) - 1$ with $a_0(x) \equiv \rho_0(x) - 1$, the problem (1.1) is reduced into the following problem of (a,u):

$$\begin{cases} \partial_t a + u \cdot \nabla a = -(1+a)\mathrm{div}\ u, \\ \partial_t u - \mathcal{L}u + \nabla(-\Delta)^{-1} a = -\dfrac{a}{1+a}\mathcal{L}u - u \cdot \nabla u - \nabla(Q(a)), \\ u(0,x) = u_0,\ a(0,x) = a_0. \end{cases} \tag{1.2}$$

Here, we denote the elliptic operator $\mathcal{L}$ by $\mathcal{L} = \mu\Delta + (\lambda+\mu)\nabla \mathrm{div}$ and Q is a smooth function determined by P by

$$Q(a) := -\int_0^t \frac{P'((1+z)^{-1})}{(1+z)^2} dz.$$

Nash [20] considered the local well-posedness of the compressible Navier-Stokes system for smooth data away from a vacuum. Itaya [15] also obtained the existence and uniqueness of the system assuming sufficient smoothness to the data. Matsumura-Nishida [19] proved the existence of global classical solution provided the initial data with high regularity is close to the equilibrium state.

We recall that the compressible Navier-Stokes system (1.1) has a scaling invariance: For $\nu > 0$,

$$\begin{cases} \rho_\nu(t,x) = \rho(\nu^2 t, \nu x), \\ u_\nu(t,x) = \nu u(\nu^2 t, \nu x) \end{cases} \tag{1.3}$$

provided the pressure term has been changed accordingly. Extending the classical idea initiated by Fujita-Kato [11] applied to the incompressible Navier-Stokes system, Danchin [5], [8] considered the local existence and the uniqueness of the solution for the problem in the *"scaling-critical"* homogeneous Besov space. Haspot [14] improved Danchin's result [8] to the general Besov space and a larger space for the density by introducing an effective velocity. In order to consider the critical solvability, we necessarily introduce the homogeneous Besov spaces. Since the system (1.1) involves the hyperbolic equation for the density equation, it is required to consider the equation in the suitable space, where the supremum of the density has to be controlled. To this end, Danchin introduced the homogeneous Besov space that embedded into $L^\infty(\mathbb{R}^n)$.

Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity satisfying that

$$\sum_{j\in\mathbb{Z}} \hat{\phi}_j(\xi) = 1$$

for all $\xi \neq 0$ and supp $\hat{\phi}_j \subset \{\xi \in \mathbb{R}^n |\ 2^{j-1} < |\xi| < 2^{j+1}\}$. For $s \in \mathbb{R}$ and $1 \le p, \sigma \le \infty$, we define the homogeneous Besov space $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ by

$$\dot{B}^s_{p,\sigma}(\mathbb{R}^n) = \{f \in \mathcal{S}^*/\mathcal{P}; \|f\|_{\dot{B}^s_{p,\sigma}} < \infty\}$$

with the norm

$$\|f\|_{\dot{B}^s_{p,\sigma}} \equiv \begin{cases} \Big(\sum_{j\in\mathbb{Z}} 2^{js\sigma}\|\phi_j * f\|_p^\sigma\Big)^{1/\sigma}, 1 \le \sigma < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\phi_j * f\|_p,\ \sigma = \infty, \end{cases} \tag{1.4}$$

where $\mathcal{P}$ denotes polynomials (see Triebel [23] for details).

Since our system involves the Poisson term and it brings our problem disturbing the low frequency in the Fourier spaces. Namely, the inverse operator of the Laplacian gives a stronger restriction for the low frequency part of the solution (cf. Yukawa potential case [2]). To handle with low frequency part, we also introduce another homogeneous Besov space of hybrid type; $\dot{B}^{s_*}_{p,\sigma} \oplus \dot{B}^{s^*}_{p,\sigma}$ by

$$\|f\|_{\dot{B}^{s_*}_{p,\sigma}\oplus\dot{B}^{s^*}_{p,\sigma}} \equiv \left(\sum_{j\le 0} 2^{\sigma s_* j}\|\phi_j * f\|_p^\sigma + \sum_{j>0} 2^{\sigma s^* j}\|\phi_j * f\|_p^\sigma\right)^{1/\sigma}$$

for all $1 \le p, \sigma \le \infty$ and s_*, $s^* \in \mathbb{R}$. We note that if $s_* < s^*$, then it holds that

$$\dot{B}^{s_*}_{p,\sigma} \oplus \dot{B}^{s^*}_{p,\sigma} = \dot{B}^{s_*}_{p,\sigma} \cap \dot{B}^{s^*}_{p,\sigma}$$

and hereafter we only use this setting. We define the *critical* inhomogeneous space as follows:

$$a \in L^\infty(0,T;\dot{B}^{\frac{N}{p}}_{p,1}),\ u \in L^\infty(0,T;\dot{B}^{\frac{N}{p}-1}_{p,1}),\ f \in L^1_{loc}(\mathbb{R}^+;\dot{B}^{\frac{N}{p}-1}_{p,1}).$$

Zheng [24] used the linearized formulation to (1.1) and solve the system by the way of integral equations. The key idea is to consider the Poisson term as the linear term and he

introduced the semi-group

$$\frac{d}{dt}\begin{pmatrix} a \\ u \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{div} \\ -|\nabla| - \nabla(-\Delta)^{-1} & \mathcal{L} \end{pmatrix}\begin{pmatrix} a \\ u \end{pmatrix}. \tag{1.5}$$

Establishing the L^p-L^q type estimate of the semi-group generated by the above operator he constructed a global solution for small data to (1.1) in the critical Besov space $\rho_0 - 1 \in \dot{B}^{\frac{n}{2}-2}_{2,1} \oplus \dot{B}^{\frac{n}{p}}_{p,1}$, $u_0 \in \dot{B}^{\frac{n}{2}-2}_{2,1} \oplus \dot{B}^{\frac{n}{p}-1}_{p,1}$. However the critical case $p = n$ was not treated, since the product formula necessarily required in the case $p = n$ such as

$$\|fg\|_{\dot{B}^{n/p-1}_{p,1}} \le C\|f\|_{\dot{B}^{n/p-1}_{p,1}}\|g\|_{\dot{B}^{n/p+1}_{p,1}}$$

failes in general.

We now recover the local existence result in the critical hybrid Besov spaces.

Theorem 1.1 ([3]). *Let $n = 3$, $1 < p \le 3$. $\mu > 0$ with $\mu + 2\lambda > 0$. For any $\rho_0 - 1 \in \dot{B}^{\frac{n}{p}-2}_{p,1}(\mathbb{R}^3) \oplus \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^3)$, $u_0 \in \dot{B}^{\frac{n}{p}-2}_{p,1}(\mathbb{R}^3) \oplus \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^3)$. Then there exists a weak solution to (1.1) such that for some $T > 0$ with $I = [0, T)$, (ρ, u, ψ): a solution of (1.1) satisfying*

$$\begin{aligned} \rho - 1 &\in C\big(I; \dot{B}^{\frac{n}{p}-1}_{p,1} \oplus \dot{B}^{\frac{n}{p}}_{p,1}\big), \\ u &\in \big(C(I; \dot{B}^{\frac{n}{p}-2}_{p,1} \oplus \dot{B}^{\frac{n}{p}-1}_{p,1}) \cap L^1(I; \dot{B}^{\frac{n}{p}+1}_{p,1})\big)^N, \\ \psi &\in C\big(I; \dot{B}^{\frac{n}{p}+1}_{p,1} \oplus \dot{B}^{\frac{n}{p}+2}_{p,1}\big). \end{aligned} \tag{1.6}$$

2. Key estimates

2.1. **The mass conservation equation.** To prove the case $p = n = 3$, we employ the following proposition. Let (a, u) solves the following equation.

$$\begin{cases} \partial_t a + u \cdot \nabla a = -(1 + a)\mathrm{div}\ u, & (t, x) \in I \times \mathbb{R}^n, \\ a(0, x) = a_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{2.1}$$

where $I = [0, T)$.

Proposition 2.1. *Let $a_0 \in \dot{B}^{-1}_{3,1}(\mathbb{R}^3)$, $u \in L^\infty(I; \dot{B}^0_{3,1}(\mathbb{R}^3)) \cap L^1(I; \dot{B}^2_{3,1}(\mathbb{R}^3))$ and $U(t) := \int_0^t \|\nabla u(\tau)\|_{\dot{B}^1_{3,1}} d\tau$. Suppose that $a \in L^\infty(I; \dot{B}^{-1}_{3,1}(\mathbb{R}^3) \cap \dot{B}^1_{3,1}(\mathbb{R}^3))$ solves the equation (2.1). Then there exists a constant $C > 0$ depending on n and p such that the following inequality holds.*

$$\|a\|_{L^\infty_t(I;\dot{B}^{-1}_{3,1})} \le e^{U(t)}\left[\|a_0\|_{\dot{B}^{-1}_{3,1}} + C\int_0^t e^{-U(\tau)}(1 + \|a\|_{\dot{B}^1_{3,1}})\|u\|_{\dot{B}^0_{3,1}} d\tau\right], \tag{2.2}$$

for $t \in [0, T]$.

In view of the above estimate (2.2), it is required that the velocity field has to have maximal regularity in L^1 in time variable. This is the key point to show the main theorem.

2.2. **Maximal L^1 Regularity.** Let u_0 be the initial data, and a and h be given functions. The momentum governed by the following linearized parabolic equation.

$$\begin{cases} \partial_t u - \mu\Delta u - (\lambda+\mu)\nabla(\mathrm{div}\ u)) = F, & (t,x) \in I \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.3}$$

It is known that the Cauchy problem for the heat equation (2.3) has maximal regularity in such non-reflexive function spaces. One of the general result can be seen in [21].

Proposition 2.2 ([21])**.** *Let $1 < \rho, \sigma \le \infty$ and $I = [0,T)$ be an interval with $T \le \infty$. For $f \in L^\rho(I; \dot{B}^0_{1,\rho}(\mathbb{R}^n))$ and $u_0 \in \dot{B}^{2(1-1/\rho)}_{1,\rho}(\mathbb{R}^n)$, let u be a solution of the Cauchy problem of the heat equation* (2.3)*. Then we have*

$$\|\partial_t u\|_{L^\rho(I;\dot{B}^0_{1,\rho})} + \left\|\nabla^2 u\right\|_{L^\rho(I;\dot{B}^0_{1,\rho})} \le C(\|u_0\|_{\dot{B}^{2(1-1/\rho)}_{1,\rho}} + \|F\|_{L^\rho(I;\dot{B}^0_{1,\rho})}). \tag{2.4}$$

The above result does not cover the end-point exponent $\rho = 1$. In general, the end-point case $p = 1$ is eliminated in the abstract theory and we need to develop the each cases. Danchin [6] (see also Haspot [14]) obtained maximal regularity in the homogeneous Besov space for the case $\rho = 1$.

Theorem 2.3 ([8], [22])**.** *Let $1 \le p \le \infty$. For $F \in L^1(\mathbb{R}_+; \dot{B}^0_{p,1}(\mathbb{R}^n))$ and $u_0 \in \dot{B}^0_{p,1}(\mathbb{R}^n)$ there exists a unique solution u to* (2.3) *which satisfies the estimate:*

$$\begin{aligned} &\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})} + \left\|\nabla^2 u\right\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})} \\ &\qquad\qquad \le C\left(\|u_0\|_{\dot{B}^0_{p,1}} + \|F\|_{L^1(\mathbb{R}_+;\dot{B}^0_{p,1})}\right), \end{aligned} \tag{2.5}$$

where constant C is depending only on n. Moreover the estimate is optimal for the class of initial data. Namely if the data is $u_0 \in L^p(\mathbb{R}^N)$ or $\dot{F}^0_{p,1}(\mathbb{R}^N)$ the above estimate fails.

Remark 2.1. The upper estimate of (2.5) was obtained by Danchin- Mucha [9, Proposition 5] with $1 < p < \infty$. For $p = 1$, Danchin essentialy obtained the same estimate even for the variable coefficient case. Giga-Saal considered time L^1 maximal regularity in some space [12].

If we replace $u_0 \in \dot{B}^0_{p,1}(\mathbb{R}^n)$ into $u_0 \in \dot{B}^0_{p,\sigma}(\mathbb{R}^n)$ for $1 < \sigma \le \infty$, then maximal regularity *fails* since the lower bound by the initial data and the strict inclusion result for the subsufix σ as $\dot{B}^0_{p,1}(\mathbb{R}^n) \subsetneq \dot{B}^0_{p,\sigma}(\mathbb{R}^n)$.

To avoid the difficulty on using the limiting case of the bi-linear estimate in the homogeneous Besov spaces, we employ the following bi-linear estimate to treat the nonlinear term of the equation (2.1).

Lemma 2.4. *For $u \in \dot{B}^0_{3,1} \cap \dot{B}^2_{3,1}$ and $\tilde{a} \in \dot{B}^{-1}_{3,1} \cap \dot{B}^1_{3,1}$ it holds that*

$$\|\mathrm{div}\ (\tilde{a}u)\|_{\dot{B}^{-1}_{3,1}} \le C\big(\|u\|_{\dot{B}^0_{3,1}}\|\tilde{a}\|_{\dot{B}^1_{3,1}} + \|u\|_{\dot{B}^2_{3,1}}\|\tilde{a}\|_{\dot{B}^{-1}_{3,1}}\big).$$

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