# Mathematical Analysis for Stability in Nonlinear Dynamics

\- in honor of Professor Vladimir Georgiev on his 60th birthday -

Organizers:
H. Kubo, T. Ozawa, H. Takamura

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# Mathematical Analysis for Stability in Nonlinear Dynamics

- in honor of Professor Vladimir Georgiev on his 60th birthday -

Organizers:
H. Kubo, T. Ozawa, H. Takamura

Faculty of Science,
Hokkaido University

August 24 – 26, 2016

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# Contents

# Preface

We welcome all the participants to the conference: Mathematical Analysis for Stability in Nonlinear Dynamics – in honor of Professor Vladimir Georgiev on his 60th birthday. This volume is intended as the proceeding of this conference, held for the period of August 24 - 26, 2016 in Sapporo.

Mathematics is supposed to have a potential power to integrate different disciplines. Since the origin of partial differential equations goes back to observations of various phenomena in our world, it would play a central role in that respect. In particular, nonlinear dynamics is of special importance from the theoretical point of view, as well as from the viewpoint of possible applications to different fields. Such stream in scientific research motivated us to organize this conference for promoting mutual communications among active participants based on the lectures given by strong leaders from the wide range of subjects of nonlinear analysis. Professor Vladimir Georgiev has been making a significant contribution to these subjects for several decades. We hope this conference would provide a nice opportunity to recognize his notable achievement in academic activity.

We wish you enjoy the conference "Mathematical Analysis for Stability in Nonlinear Dynamics" and your stay in Sapporo.

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# Program

Aug. 24 (Wed.)

| | |
|---|---|
| 13:20-13:30 | Opening |
| 13:30-14:15 | **Yoshihiro Shibata (Waseda University)**<br>On some free boundary problem for the viscous fluid flow |
| 14:30-15:15 | **Atanas Stefanov (University of Kansas)**<br>Scattering of small solutions of the cubic NLS with short range potential |
| 15:45-16:30 | **Luis Vega (University of the Basque Country UPV/EHU)**<br>Shell interactions for Dirac operators |
| 16:45-17:45 | Poster Session |

Aug. 25 (Thu.)

| | |
|---|---|
| 10:00-10:45 | **Gustavo Ponce (University of California)**<br>Unique continuation results for some evolution equations |
| 11:00-11:45 | **Jean-Claude Saut (Université Paris-Sud)**<br>Full dispersion water wave models |
| 13:30-14:15 | **Yoshio Tsutsumi (Kyoto University)**<br>Existence of global solutions and global attractor for the third order Lugiato-Lefever equation on **T** |
| 14:30-15:15 | **Maria J. Esteban (University of Paris-Dauphine)**<br>Nonlinear flows and optimality for functional inequalities |
| 15:45-16:15 | **Sandra Lucente (Università Degli Studi di Bari)**<br>Breaking symmetry in focusing NLKG equation |
| 16:15-16:45 | **Nicola Visciglia (Università Degli Studi di Pisa)**<br>On the growth of Sobolev norms for NLS on 2D and 3D compact manifolds |
| 18:00- | Banquet |

Aug. 26 (Fri.)

| | |
|---|---|
| 10:00-10:45 | **Masahito Ohta (Tokyo University of Science)**<br>Strong instability of standing waves for nonlinear Schrödinger equations with a harmonic potential |
| 11:00-11:45 | **Andrew Comech (Texas A&M University and IITP, Moscow)**<br>On spectral stability of the nonlinear Dirac equation |
| 13:30-14:15 | **Tokio Matsuyama (Chuo University)**<br>Decay estimates for wave equation with a potential on exterior domains |
| 14:30-15:15 | **Piero D'Ancona (Università "La Sapienza" di Roma)**<br>Global existence of small equivariant wave maps on rotationally symmetric manifolds |
| 15:15-15:25 | Closing |

# On some free boundary problem for the viscous fluid flow

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## 1 Free Boundary Problem for the Navier-Stokes equations

In this talk, I will consider a free boundary problem for the Navier-Stokes equtions. Let $\Omega$ be a domain in $\mathbb{R}^N$ with boundary $\Gamma$, which assumed to be suitably smooth. Let $x = \varphi(\xi, t)$ be a smooth functions defined on $\overline{\Omega}$ which gives one to one correspondence from $\overline{\Omega}$ onto $\overline{\Omega_t}$, where $\Omega_t = \{x = \varphi(\xi, t) \mid \xi \in \Omega\}$. The equation for the mass conservation is

$$\rho_t + \mathrm{div}\,(\rho\mathbf{u}) = 0 \quad \text{in } \Omega_t, \tag{1}$$

where $\rho = \rho(x, t)$ is a mass density and $\mathbf{u} = \mathbf{u}(x, t) = {}^\top(u_1, \ldots, u_N)$, where ${}^\top M$ stands for the transposed $M$, is a velocity field. Let $J = J(\xi, t)$ is the Jacobian of the transformation $x = \varphi(\xi, t)$ for each $t \geq 0$, and then we hve the Reynolds formula: $\dfrac{\partial J}{\partial t} = (\mathrm{div}\,_x\tilde{\mathbf{u}}(x, t))J$, where $\tilde{\mathbf{u}}(x, t) = \partial_t\varphi(\xi, t)$ with $x = \varphi(\xi, t)$. By the Reynolds formula and (1) we $\dfrac{d}{dt}\displaystyle\int_{\Omega_t} \rho\, dx = \int_{\Gamma_t} \rho(\tilde{\mathbf{u}} - \mathbf{u}) \cdot \mathbf{n}_t\, d\sigma$, where $\mathbf{n}_t$ stands for the unit outer normal to $\Gamma_t$. Assuming that

$$(\tilde{\mathbf{u}} - \mathbf{u}) \cdot \mathbf{n}_t = 0 \quad \text{on } \Gamma_t \tag{2}$$

we have $\dfrac{d}{dt}\int_{\Omega_t} \rho\, dx = 0$, which implies the conservation of mass. The formula (2) is called a kinematic condition. Let $F = F(x, t)$ be a function such that $\Gamma_t$ is locally represented by $F(x, t) = 0$, and then the condition (2) is equivalent to the condition:

$$F_t + \mathbf{u} \cdot (\nabla_x F) = 0 \quad \text{on } \Gamma_t. \tag{3}$$

In this talk, we assume that $\rho$ is a positive constant, so that $\mathrm{div}\,\mathbf{u} = 0$ in $\Omega_t$. From the mass conservation, we have $|\Omega_t| = |\Omega|$. Here and hereafter, $|D|$ stands for the Lebesgue measure of Lebesgure measurable set $D$ in $\mathbb{R}^N$.

Let $T \in (0, \infty]$. **Free boundary problem** is to find unknowns $\Omega_t$, $\mathbf{u}$ and $\mathfrak{p}$, which satisfy the Navier-Stokes equation with free boundary condition given by

$$\begin{cases} \rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla\mathbf{u}) - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I}) = 0, \quad \mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega_t \text{ for } 0 < t < T, \\ (\mu\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I})\mathbf{n}_t = \sigma\mathcal{H}(\Gamma_t)\mathbf{n}_t & \text{on } \Gamma_t \text{ for } 0 < t < T, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \Omega_t|_{t=0} = \Omega \end{cases} \tag{4}$$

with kinemtic condition (3). Here, $\mu$ is a positive constant denoting the viscosity coefficient, $\sigma \geq 0$ a constant denoting the coefficient of the surface tension, $\mathbf{D}(\mathbf{u}) = \nabla\mathbf{u} + {}^\top\nabla\mathbf{u}$, $\mathbf{I}$ the $N \times N$ identity matrix, and $\mathcal{H}(\Gamma_t)$ the doubled mean curvature of $\Gamma_t$ given by $\mathcal{H}(\Gamma_t)\mathbf{n}_t = \Delta_{\Gamma_t}x$ for $x \in \Gamma_t$ with Laplace-Beltrami operator $\Delta_{\Gamma_t}$ on $\Gamma_t$. As for the remaining notations, for any $N \times N$ matrix valued function $\mathbf{K} = (K_{ij})$ $\mathrm{Div}\,\mathbf{K}$ denotes an $N$-vector with $i^{\text{th}}$ component $\sum_{j=1}^N \partial_j K_{ij}$, $\partial_j = \partial/\partial x_j$, and for any $N$-vector of functions $\mathbf{v} = {}^\top(v_1, \ldots, v_N)$ and $\mathbf{w} = {}^\top(w_1, \ldots, w_N)$, $\mathbf{v} \cdot \nabla\mathbf{w}$ denotes an $N$-vector with $i^{\text{th}}$ component $\sum_{j=1}^N v_j\partial_j w_i$. **In this talk, we only consider the case** $\mu > 0$.

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## 2 Lagrange description of problem (4)

Let $\mathbf{x} = \mathbf{x}(\xi, t)$ be a unique solution of the Cauchy problem:

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t) \quad (t > 0), \quad \mathbf{x}|_{t=0} = \xi \in \overline{\Omega}. \tag{5}$$

Let $X(\xi, t) = \xi + \mathbf{x}(\xi, t)$, and $\mathbf{v}(\xi, t) = \mathbf{u}(X(\xi, t))$. The Lagrange transformation is given by

$$x = \xi + \int_0^t \mathbf{v}(\xi, s)\, ds \equiv \mathbf{X}_{\mathbf{v}}(\xi, t). \tag{6}$$

Let

$$\Gamma_t = \{x = \xi + \int_0^t \mathbf{v}(\xi, s)\, ds \mid \xi \in \Gamma\}, \quad \Omega_t = \{x = \xi + \int_0^t \mathbf{v}(\xi, s)\, ds \mid \xi \in \Omega\}. \tag{7}$$

Let

$$\int_0^T \|\mathbf{v}(\cdot, s)\|_{W^1_\infty(\Omega)}\, ds \leq \sigma. \tag{8}$$

Choose $\sigma > 0$ so small that the map $x = \mathbf{X}_{\mathbf{v}}(\xi, t)$ is one to one for $t \in [0, T]$, so that $\mathbf{X}_{\mathbf{v}}(\cdot, t)$ gives a bijection from $\Omega$ onto $\Omega_t$ and from $\Gamma$ onto $\Gamma_t$. This expresses the fact that the free surface $\Gamma_t$ consists for all $t > 0$ of the same fluid particles, which do not leave it and are not incident on it from $\Omega_t$. Moreover, if $F(x, t)$ is a function such that $\Gamma_t$ is represented locally by $F(x, t) = 0$, then (3) is a necessary condition for (7).

Let us discuss the Lagrange description of problem (4). Let $\mathbf{A}$ be the inverse matrix of the Jacobi matrix of the transformation: $x = \mathbf{X}_{\mathbf{v}}(\xi, t)$, that is the inverse matrix of the matrix whose $(j, k)$ components are

$$\frac{\partial x_j}{\partial \xi_k} = \delta_{jk} + \int_0^t \frac{\partial v_j}{\partial \xi_k}\, ds,$$

and then $\nabla_x = \mathbf{A}\nabla_\xi$, where $\mathbf{A} = \mathbf{I} + \mathbf{V}_0(\nabla \int_0^t \mathbf{v}(\xi, s)\, ds)$ with some matrix $\mathbf{V}_0 = \mathbf{V}_0(\mathbf{w})$ of polynomials with respect to $\mathbf{w} = (w_{ij} \mid i, j = 1, \ldots, N)$, where $\mathbf{V}_0(0) = 0$.

Let $\mathfrak{p}(X_{\mathbf{v}}(\xi, t), t) = \mathfrak{q}(\xi, t)$, and then problem (4) is transformed to the following equations:

$$\begin{cases} \partial_t \mathbf{v} - \mathrm{Div}\,(\mu \mathbf{D}(\mathbf{v}) - \mathfrak{q}\mathbf{I}) = \mathbf{F}(\mathbf{u}), \quad \mathrm{div}\,\mathbf{v} = G_1(\mathbf{v}) = \mathrm{div}\,\mathbf{G}_2(\mathbf{v}) & \text{in } \Omega \times (0, T), \\ (\mu \mathbf{D}(\mathbf{v}) - \mathfrak{q}\mathbf{I}) + \mathbf{H}(\mathbf{v}))\mathbf{n}_t - \sigma \mathcal{H}(\Gamma_t)\mathbf{n}_t = 0 & \text{on } \Gamma \times (0, T), \\ \mathbf{v}|_{t=0} = \mathbf{u}_0 & \text{in } \Omega. \end{cases} \tag{9}$$

Here $\mathbf{n}_t$ is given by $\mathbf{n}_t = {}^\top\mathbf{A}^{-1}\mathbf{n}|{}^\top\mathbf{A}^{-1}\mathbf{n}|^{-1}$, $\mathbf{n}$ being the unit outer normal to $\Gamma$, and $\mathbf{F}(\mathbf{v})$, $G_1(\mathbf{v})$, $\mathbf{G}_2(\mathbf{v})$ and $\mathbf{H}(\mathbf{v})$ are nonlinear functions of the forms:

$$\begin{aligned} \mathbf{F}(\mathbf{v}) &= -\mathbf{V}_0(\int_0^t \nabla \mathbf{v}\, ds)\partial_t \mathbf{v} + \mathbf{V}_2(\int_0^t \nabla \mathbf{v}\, ds)\nabla^2 \mathbf{v} + \Big(\mathbf{V}_3(\int_0^t \nabla \mathbf{v}\, ds) \int_0^t \nabla^2 \mathbf{v}\, ds\Big)\nabla \mathbf{v}, \\ G_1(\mathbf{v}) &= \mathbf{V}_4(\int_0^t \nabla \mathbf{v}\, ds)\nabla \mathbf{v}, \quad \mathbf{G}_2(\mathbf{v}) = \mathbf{V}_5(\int_0^t \nabla \mathbf{v}\, ds)\mathbf{v}, \quad \mathbf{H}(\mathbf{v}) = \mathbf{V}_6(\int_0^t \nabla \mathbf{v}\, ds)\nabla \mathbf{v} \end{aligned} \tag{10}$$

with some matrices $\mathbf{V}_j = \mathbf{V}_j(\mathbf{w})$ of polynomials with respect to $\mathbf{w}$ such that $\mathbf{V}_j(0) = 0$ for $j = 0$, 2, 4, 5, and 6.

**Linearization principle for the boundary condition in the $\sigma > 0$ case.** First, we drive the equivalent boundary conditions to the boundary condition:

$$(\mu \mathbf{D}(\mathbf{v}) - \mathfrak{q}\mathbf{I} + \mathbf{H}(\mathbf{v}))\mathbf{n}_t - \sigma \mathcal{H}(\Gamma_t)\mathbf{n}_t = 0 \quad \text{on } \Gamma \text{ for } t > 0. \tag{11}$$

Let $\mathbf{\Pi}_t \mathbf{d} = \mathbf{d} - < \mathbf{d}, \mathbf{n}_t > \mathbf{n}_t$ and $\mathbf{\Pi}_0 \mathbf{d} = \mathbf{d} - < \mathbf{d}, \mathbf{n} > \mathbf{n}$, and then we have

**Lemma 1.** *If $\mathbf{n}_t \cdot \mathbf{n} \neq 0$, then for arbitrary vector $\mathbf{d}$, $\mathbf{d} = 0$ is equivalent to*

$$\mathbf{\Pi}_0 \mathbf{\Pi}_t \mathbf{d} = 0 \ \textit{and} \ \mathbf{n} \cdot \mathbf{d} = 0. \tag{12}$$

By Lemma 1, the boundary condition (11) is equivalent to

$$\mathbf{\Pi}_0\mathbf{\Pi}_t(\mu\mathbf{D}(\mathbf{v})+\mathbf{H}(\mathbf{v}))\mathbf{n}_t=0, \tag{13}$$

$$\mathbf{n}\cdot(\mu\mathbf{D}(\mathbf{v})-\mathfrak{q}\mathbf{I}+\mathbf{H}(\mathbf{v}))\mathbf{n}_t-\sigma\mathbf{n}\cdot\Delta_{\Gamma_t}(\xi+\int_0^t\mathbf{v}\,ds)=0, \tag{14}$$

where we have used the fact that $\mathcal{H}(\Gamma_t)\mathbf{n}_t=\Delta_{\Gamma_t}\mathbf{X}_\mathbf{v}(\xi,t)=\Delta_{\Gamma_t}(\xi+\int_0^t\mathbf{v}\,ds)$. The condition (13) is equivalent to

$$\mathbf{\Pi}_0\mu\mathbf{D}(\mathbf{v})\mathbf{n}=\mathbf{\Pi}_0(\mathbf{\Pi}_0-\mathbf{\Pi}_t)\mu\mathbf{D}(\mathbf{v})\mathbf{n}_t+\mathbf{\Pi}_0\mu\mathbf{D}(\mathbf{v})(\mathbf{n}-\mathbf{n}_t)-\mathbf{\Pi}_0\mathbf{\Pi}_t\mathbf{H}(\mathbf{v})\mathbf{n}_t. \tag{15}$$

On the other hand, the formula (14) is written as follows:

$$\begin{aligned}
&\mathbf{n}\cdot(\mu\mathbf{D}(\mathbf{v})-\mathfrak{q}\mathbf{I})\mathbf{n}+\sigma(m-\Delta_\Gamma)\int_0^t\mathbf{n}\cdot\mathbf{v}\,ds\\
&\quad+\sigma\Big[(\Delta_\Gamma\mathbf{n})\cdot\int_0^t\mathbf{v}\,ds+\mathbf{n}\cdot(\Delta_\Gamma-\Delta_{\Gamma_t})\int_0^t\mathbf{v}\,ds+\mathbf{n}\cdot(\Delta_\Gamma-\Delta_{\Gamma_t})\xi\Big]\\
=&\sigma m\int_0^t\mathbf{n}\cdot\mathbf{v}\,ds+\mathbf{n}\cdot(\mu\mathbf{D}(\mathbf{v})-\mathfrak{q}\mathbf{I})(\mathbf{n}-\mathbf{n}_t)-\mathbf{n}\cdot\mathbf{H}(\mathbf{v})\mathbf{n}_t+\sigma\mathcal{H}(\Gamma)-\sigma<<\nabla_\Gamma\mathbf{n},\int_0^t\nabla_\Gamma\mathbf{v}\,ds>>,
\end{aligned} \tag{16}$$

where $m$ has been chosen so large positive number that $(m-\Delta_\Gamma)^{-1}$ exists. Let

$$\begin{aligned}
K(\mathbf{v})&=(\Delta_\Gamma\mathbf{n})\cdot\int_0^t\mathbf{v}\,ds+\mathbf{n}\cdot(\Delta_\Gamma-\Delta_{\Gamma_t})\int_0^t\mathbf{v}\,ds+\mathbf{n}\cdot(\Delta_\Gamma-\Delta_{\Gamma_t})\xi,\\
\eta&=\int_0^t\mathbf{n}\cdot\mathbf{v}\,ds+(m-\Delta_\Gamma)^{-1}K(\mathbf{v}).
\end{aligned} \tag{17}$$

Then, setting $\mathfrak{q}=(1-\mathbf{n}\cdot(\mathbf{n}-\mathbf{n}_t))^{-1}\mathfrak{p}$, by (15), (16) and (17) we have

$$\begin{cases}
\partial_t\mathbf{v}-\mathrm{Div}\,(\mu\mathbf{D}(\mathbf{v})-\mathfrak{p}\mathbf{I})=\mathbf{F}(\mathbf{v})+\nabla\Big(\frac{\mathbf{n}\cdot(\mathbf{n}-\mathbf{n}_t)\mathfrak{p}}{1-\mathbf{n}\cdot(\mathbf{n}-\mathbf{n}_t)}\Big) & \text{in } \Omega\times(0,T),\\
\mathrm{div}\,\mathbf{v}=G_1(\mathbf{v})=\mathrm{div}\,\mathbf{G}_2(\mathbf{v}) & \text{in } \Omega\times(0,T),\\
\partial_t\eta-\mathbf{n}\cdot\mathbf{v}=(m-\Delta_\Gamma)^{-1}\partial_tK(\mathbf{v}) & \text{on } \Gamma\times(0,T),\\
(\mu\mathbf{D}(\mathbf{v})-\mathfrak{p}\mathbf{I})\mathbf{n}+(\sigma(m-\Delta_\Gamma)\eta)\mathbf{n}=\sigma\mathcal{H}(\Gamma)\mathbf{n}+\mathbf{I}(\mathbf{v}) & \text{on } \Gamma\times(0,T),\\
(\mathbf{v},\eta)|_{t=0}=(\mathbf{u}_0,\eta_0) & \text{in } \Omega\times\Gamma,
\end{cases} \tag{18}$$

which are equivalent to (9). Here, $\mathcal{H}(\Gamma)$ is the doubled mean curvature of $\Gamma$.

# 3 Linearized equations

The linearized equations for problem (18) are formulated by

$$\begin{cases}
\partial_t\mathbf{u}-\mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u})-\mathfrak{q}\mathbf{I})=\mathbf{f},\quad \mathrm{div}\,\mathbf{u}=f_d=\mathrm{div}\,\mathbf{f}_d & \text{in } \Omega\times(0,T),\\
\partial_t\eta-\mathbf{n}\cdot\mathbf{u}=g & \text{in } \Gamma\times(0,T),\\
(\mu\mathbf{D}(\mathbf{u})-\mathfrak{q}\mathbf{I})\mathbf{n}+\sigma(m-\Delta_\Gamma)\eta)\mathbf{n}=\mathbf{h} & \text{in } \Gamma\times(0,T),\\
(\mathbf{u},\eta)|_{t=0}=(\mathbf{u}_0,\eta_0) & \text{on } \Omega\times\Gamma.
\end{cases} \tag{19}$$

The assumption of the domain $\Omega$ is

- $\Omega$ is a $C^3$ uniform domain.
- The weak Dirichlet problem: $(\nabla u,\nabla\varphi)_\Omega=(\mathbf{f},\nabla\varphi)$ for $\varphi\in\dot H^1_{r',0}(\Omega)$, admits a unique solution $u\in\dot H^1_{r,0}(\Omega)$ for any $\mathbf{f}\in L_r(\Omega)^N$ with $r=q$ and $r=q'=q/(q-1)$, where

$$\dot H^1_{r,0}(\Omega)=\{u\in L_{r,\mathrm{loc}}(\Omega)\mid\nabla u\in L_q(\Omega)^N,\ \ u|_\Gamma=0\}.$$

The example of $\Omega$ is a bounded domain, half space, perturbed half-space, layer, perturbed layer, straight tube and an exterior domain. By Shibata [2, 3], we know the following theorem.

**Theorem 2** (Maximal $L_p$-$L_q$ regularity theorem)**.** *Let $T>0$, $1<p,q<\infty$. Let $\mathbf{u}_0\in B^{2-2/p}_{q,p}(\Omega)$, $\eta_0\in W^{3-1/p-1/q}$ and*

$$\begin{aligned}&\mathbf{f}\in L_p((0,T),L_q(\Omega)^N),\quad f_d\in L_p((0,T),H^1_q(\Omega))\cap H^1_p((0,T),\mathbf{W}^{-1}_q(\Omega)),\\&\mathbf{f}_d\in H^1_p((0,T),L_q(\Omega)^N),\quad g\in L_p((0,T),W^{2-1/q}_q(\Gamma)),\\&\mathbf{h}\in L_p((0,T),H^1_q(\Omega)^N)\cap H^1_p((0,T),\mathbf{W}^{-1}_q(\Omega)^N),\end{aligned}$$

*where $\mathbf{W}^{-1}_q(\Omega)$ is the set of all $u\in L_{1,\mathrm{loc}}(\Omega)$ such that $\iota u\in W^{-1}_q(\mathbb{R}^N)$ with some suitable extension operator $\iota$ from $\Omega$ into $\mathbb{R}^N$. Assume that the compatibility conditions:*

$$\operatorname{div}\mathbf{u}_0=\mathbf{f}_d|_{t=0}\ \text{ in }\Omega,\quad \mathbf{\Pi}_0\mu\mathbf{D}(\mathbf{u}_0)\mathbf{n}=\mathbf{\Pi}_0\mathbf{h}|_{t=0}\ \text{ on }\Gamma\text{ provided that }2/p+N/q<1 \tag{20}$$

*are satisfied. Then, problem (19) admits unque solutions $\mathbf{u}$, $\mathfrak{q}$ and $\eta$ with*

$$\begin{aligned}&\mathbf{u}\in L_p((0,T),H^2_q(\Omega)^N)\cap H^1_p((0,T),L_q(\Omega)^N),\quad \mathfrak{q}\in L_p((0,T),H^1_q(\Omega)+\dot H^1_{q,0}(\Omega)),\\&\eta\in L_p((0,T),W^{3-1/q}_q(\Gamma))\cap H^1_p((0,T),W^{2-1/q}_q(\Gamma)).\end{aligned}$$

# 4 Local Well-Posedness

**Theorem 3.** *Let $2<p<\infty$ and $N<q<\infty$. Assume that $2/p+N/q<1$. Let $\mathbf{u}_0\in B^{2-2/p}_{q,p}(\Omega)$ which satisfies the compatibility condition:*

$$\operatorname{div}\mathbf{u}_0=G_1(\mathbf{u}_0)\ \text{ in }\Omega,\quad \mathbf{\Pi}(\mu\mathbf{D}(\mathbf{u}_0)-\mathbf{I}(\mathbf{u}_0))=0\ \text{ on }\Gamma, \tag{21}$$

*and let $\eta_0\in W^{3-1/p-1/q}_{q,p}(\Gamma)$. Assume that $\|\mathcal{H}(\Gamma)\|_{W^{2-/q}_q(\Gamma)}<\infty$. Then, there exists a time $T>0$ such that problem (18) admits unique solutions $\mathbf{v}$, $\mathfrak{q}$ and $\eta$ with*

$$\begin{aligned}&\mathbf{v}\in L_p((0,T),H^2_q(\Omega)^N)\cap H^1_p((0,T),L_q(\Omega)^N),\quad \mathfrak{q}\in L_p((0,T),H^1_q(\Omega)+\dot H^1_{q,0}(\Omega)),\\&\eta\in L_p((0,T),W^{3-1/q}_q(\Gamma))\cap H^1_p((0,T),W^{2-1/q}_q(\Gamma)).\end{aligned}$$

# 5 Global Well-Posedness

To obtain the global well-posedness, so far we can not use the Lagrange coodinate. We have to start with the formulation of problem. In what follows, $\Omega$ is assumed to be a bounded domain with $C^3$ compact boundary $\Gamma$. Let $B_R=\{x\in\mathbb{R}^N\mid |x|<R\}$ and $S_R=\{x\in\mathbb{R}^N\mid |x|=R\}$. We assume that

(A.1) the initial surface $\Gamma$ is given by $\Gamma=\{x=y+h_0(y)(y/|y|)\mid y\in S_R\}$ with some given function $h_0$.

Let $\Gamma_t=\{x=y+h(y,t)(y/|y|)+\xi(t)\mid y\in S_R\}$, where $h$ is a unknown function with $h|_{t=0}=h_0$ and $\xi(t)=\frac{1}{|\Omega|}\int_{\Omega_t}x\,dx$, which is also unknown. Let $\tilde H(y,t)$ be a solution to the Dirichlet problem: $(1-\Delta)\tilde H(\cdot,t)=0$ in $B_R$ and $\tilde H(\cdot,t)|_{S_R}=R^{-1}h(\cdot,t)$. Let

$$\Omega_t=\{x=y+H(\xi,t)y+\xi(t)\mid y\in B_R\}.$$

Let $x=e_h(y,t):=y+H(y,t)y+\xi(t)$ and let $\mathbf{u}(\xi,t)=\mathbf{v}\circ e_h$ and $\mathfrak{q}(\xi,t)=\mathfrak{p}\circ e_h-\frac{(N-1)\sigma}{r}$. Then, problem (4) is transformed to the following problem:

$$\begin{cases}\partial_t\mathbf{u}-\operatorname{Div}(\mu\mathbf{D}(\mathbf{u})-\mathfrak{q}\mathbf{I})=F(\mathbf{u},\mathfrak{q},H) & \text{in } B_R\times(0,T),\\ \operatorname{div}\mathbf{u}=F_d(\mathbf{u},H)=\operatorname{div}\mathbf{F}_d(\mathbf{u},H) & \text{in } B_R\times(0,T),\\ \mathbf{\Pi}_0(\mu\mathbf{D}(\mathbf{u})\mathbf{n})=\mathbf{G}'(\mathbf{u},h) & \text{in } S_R\times(0,T),\\ \mathbf{n}\cdot(\mu\mathbf{D}(\mathbf{u})\mathbf{n})-\mathfrak{q}-\sigma\mathcal{B}h=g_n(\mathbf{u},h) & \text{in } S_R\times(0,T),\\ \partial_t h-\mathbf{n}\cdot P\mathbf{u}=G_{kin}(\mathbf{u},h) & \text{on } S_R\times(0,T),\\ (\mathbf{u},h)|_{t=0}=(\mathbf{u}_0,h_0) & \text{on } B_R\times S_R.\end{cases} \tag{22}$$

Here, $P\mathbf{u} = \mathbf{u} - |B_R|^{-1}\int_{B_R} \mathbf{u}\,d\xi$, $\mathbf{n} = y/|y| \in S_1$, $\mathcal{B} = R^{-2}(N-1+\Delta_0)$, and $\Delta_0$ is the Laplace-Beltrami operator on $S_1$. We assume that

(A2) $|\Omega| = |B_R|$.

(A3) $\displaystyle\int_{\Omega_0} x\,dx = 0$, that is $\xi(0) = 0$.

The assumptions (A2) and (A3) lead the following compatibility conditions for $h_0$:

(A4) $\displaystyle\sum_{k=1}^{N} {}_N C_k \int_{|\omega|=R} h_0(\omega)^k\,d\omega = 0.$

(A5) $\displaystyle\sum_{k=1}^{N+1} {}_{N+1} C_k \int_{|\omega|=R} \omega_i h_0(\omega)^k\,d\omega = 0 \quad (i = 1, \ldots, N).$

**Theorem 4** ([4])**.** *Let $p$ and $q$ be real numbers such that $2 < p < \infty$, $N < q < \infty$ and $1/p + (N+1)/q < 1$. Then, there exists a small number $\epsilon \in (0,1)$ such that for any initial data $\mathbf{u}_0 \in B^{2-2/p}_{q,p}(B_R)$ and $h_0 \in B^{3-1/p-1/q}_{q,p}(S_R)$ satisfying the smallness condition:*

$$\|\mathbf{u}_0\|_{B^{2-2/p}_{q,p}(B_R)} + \|h_0\|_{B^{3-1/p-1/q}_{q,p}(S_R)} \leq \epsilon, \tag{23}$$

*the compatibility conditions:*

$$\operatorname{div}\mathbf{u}_0 = f_d(\mathbf{u}_0, h_0) = \operatorname{div}\mathbf{f}_d(\mathbf{u}_0, h_0) \quad \text{in } B_R, \quad \Pi_0(\mu\mathbf{D}(\mathbf{u}_0)\omega) = \mathbf{g}'(\mathbf{u}_0, h_0) \quad \text{on } S_R, \tag{24}$$

*and orthogonal condition: $(\mathbf{v}_0, \mathbf{p}_\ell)_\Omega = 0$ ($\ell = 1, \ldots M$), where $\{\mathbf{p}_\ell\}_{\ell=1}^M$ is the orthogonal base of the rigid space: $\mathcal{R}_d = \{\mathbf{u} \mid \mathbf{D}(\mathbf{u}) = 0\} = \{Ax + \mathbf{b} \mid A + A^\top = 0\}$, as well as* (A4) *and* (A5)*, problem* (30) *with $T = \infty$ admits unique solutions $\mathbf{u}$, $\mathfrak{q}$ and $h$ with*

$$\begin{aligned} &\mathbf{u} \in H^1_p((0,\infty), L_q(B_R)) \cap L_p((0,\infty), H^2_q(B_R)), \quad \mathfrak{q} \in L_p((0,\infty), H^1_q(B_R)), \\ &h \in H^1_p((0,\infty), W^{2-1/q}_q(S_R)) \cap L_p((0,\infty), W^{3-1/q}_q(S_R)), \end{aligned}$$

*which exponentially decay at $t = \infty$.*

**Remark 5.** The global well-posedness in the $\sigma = 0$ case can be treated in the Lagrange formulation (cf. Shibata [1]).

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# SCATTERING OF SMALL SOLUTIONS OF THE CUBIC NLS WITH SHORT RANGE POTENTIAL

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Abstract. We consider 1-D Hamiltonian Schrödinger equation with cubic nonlinearity $|u|^2u$ and even short range potential. Under the (crucial) assumption for absence of eigenvalue and resonances at zero, we show that small odd data gives rise to global solutions , which scatter at infinity at the rate of the free solution. This solves an outstanding open problem in the area and improves upon an earlier result in [2], where the authors have obtained scattering for the problem with nonlinearity $|u|^{p-1}u, p > 3$.

## 1. Introduction

We consider the nonlinear Schrödinger (NLS) equation with gauge invariant nonlinearity

$$i\partial_t\psi - \mathcal{H}\psi = F(|\psi|^2)\psi, \tag{1.1}$$

where the Hamiltonian $\mathcal{H} = -\partial_x^2 + V(x)$ can be considered as a real – valued potential perturbation of the free hamiltonian $\mathcal{H}_0 = -\partial_x^2$ on the real line $x \in \mathbb{R}$. This model is standard in quantum theory.

It is very well - known (see [9], [6], [4], [1]) that in the case of free Hamiltonian on the real line the cubic nonlinearity

$$i\partial_t\psi - \mathcal{H}_0\psi = \pm\psi|\psi|^2 \tag{1.2}$$

scatters, if the initial data $\psi(0, x) = \psi_0(x)$ is small in certain weighted norms. In any case, we assume, at the very least, that

$$\psi(0, x) = \psi_0(x) \in H^1(\mathbb{R}). \tag{1.3}$$

We consider $F : \mathbb{R} \to \mathbb{R}$ such that

$$F \in C^1(\mathbb{R}), \quad F(u) = Cu + O(u^q) \quad 1 < q < 2, \text{ for } 0 < u < 1 \tag{1.4}$$

so that we can consider the prototypical example

$$F(|\psi|^2) = C|\psi|^2 + C_1|\psi|^{2q}, \ 1 < q < 2,$$

when the nonlinearity is gauge invariant, but the scale invariance of (1.2) is broken. In order to simplify our presentation, we will henceforth take $F(u) = u$, but our results will hold, if $F$ is a reasonably smooth function with the property (1.4).

### 1.1. Decay and spectral assumptions on $V$.

The presence of the potential $V$ also breaks the translation invariance of the NLS. In this work we consider only potentials decaying sufficiently rapidly at infinity, namely we require

$$\int_{\mathbf{R}} \langle x\rangle^\gamma |V(x)| dx < \infty, \quad \gamma \in (3/2, 2). \tag{1.5}$$

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Note that under this assumption, the operator $\mathcal{H} := -\partial_x^2 + V$ is self-adjoint and $\sigma_{ess}(\mathcal{H}) = \sigma_{ess}(-\partial_x^2) = [0,\infty)$ and there are no embedded eigenvalues in $(0,\infty)$. We require

$$V : \mathbb{R} \to \mathbb{R}, V(x) = V(-x). \tag{1.6}$$

This allows the solution to preserve at least the reflection symmetry. That is, if the initial data in (1.3) are odd function

$$\psi(0,x) = \psi_0(x) \in H^s_{odd}(\mathbb{R}) = \{f \in H^s(\mathbb{R}); f(-x) = -f(x)\},$$

then the solution flow preserves this symmetry, i.e. $\psi(t,x) = -\psi(t,-x)$ for any time interval, where the solution flow of (1.1) is well - defined.

We assume that there is no spectrum outside of $(0,\infty)$. That is, there are no eigenvalues in $(-\infty,0)$ and also, zero is neither eigenvalue nor resonance for $\mathcal{H}$. Characterizing such potentials is rather tricky. Note that due to the assumption (1.5), this is equivalent to $\mathcal{H} \geq 0$, in addition to the requirement that zero is neither eigenvalue nor resonance for $\mathcal{H}$.

For the purpose we use a suitable characterization of the fact that $V$ generates a non-negative Schrödinger operator appeared only recently in [5]. In it, it was shown that for potentials with reasonable decay at $\pm\infty$, such as (1.5), we have

$$\mathcal{H} = -\partial_x^2 + V \geq 0 \Longleftrightarrow V(x) = w'(x) + w^2(x) = Miura(w). \tag{1.7}$$

1.2. **Main result.** The following is the main result of this work.

**Theorem 1.** *Assume the potential satisfies the assumptions* (1.5), (1.6) *and for some* $\delta > 0$, $(1+\delta)V = Miura(w)$. *Then, there exists constants* $C > 0$ *and* $\varepsilon > 0$ *so that whenever the odd initial data* $\psi_0 \in H^1_{odd}, x\psi_0 \in L^2(\mathbf{R})$ *with*

$$\|\psi_0\|_{H^1_{odd}(\mathbf{R})} + \|x\psi_0\|_{L^2(\mathbf{R})} \leq \varepsilon,$$

*the unique global solution* $\psi \in C([0,\infty), H^1_{odd}(\mathbf{R})) \cap L^\infty_{t,x}(\mathbf{R}\times\mathbf{R})$ *to the Cauchy problem* (1.1) *satisfies*

$$\sup_{t>0}(1+t)^{1/2}\|\psi(t,\cdot)\|_{L^\infty} \leq C\varepsilon. \tag{1.8}$$

## 2. Some reductions

Our next result establishes the equivalence of the Sobolev spaces $H^1_V(\mathbf{R}) \cap L^2_{odd}$ and $H^1_{odd}(\mathbf{R})$, under the assumptions put forward in Theorem 1.

**Lemma 1.** *Assume that* $V$ *satisfies* (1.5) *and* $(1+\epsilon)V = Miura(w)$. *Then, there exists a constant* $C$*, so that for all* **odd** *Schwartz functions* $f$,

$$\frac{1}{C}\|f'\|_{L^2} \leq \|\sqrt{\mathcal{H}}f\|_{L^2} \leq C\|f'\|_{L^2}. \tag{2.1}$$

*In other words,* $\|f\|_{H^1_V(\mathbf{R})\cap L^2_{odd}} \sim \|f\|_{H^1(\mathbf{R})}$.

2.1. **Some reductions and plan of the proof.** We give a rough the idea of the proof. For simplicity, we consider only the case $F(u) = u$, but all the arguments go through for higher order terms.

One can make a time translation and assuming the initial data is given at $t = 1$, define $\psi(t)$ as a solution to the integral equation

$$\psi(t) = e^{-i(t-1)\mathcal{H}}\psi_0 - i\int_1^t e^{-i(t-s)\mathcal{H}}\psi(s)|\psi(s)|^2 ds, \ t > 1. \tag{2.2}$$

The existence of global solutions when the initial data are in $H^1(\mathbb{R})$ and then in $L^2(\mathbb{R})$ are clear. Our main goal is control the decay of the $L^\infty$ norm of the solution provided we have small initial data. Making the transformation

$$(t,\psi) \implies (T,\Psi),$$

where

$$t = \frac{1}{T}, \quad \Psi(T,x) = \overline{\psi\left(\frac{1}{T},x\right)}. \tag{2.3}$$

We can rewrite (2.4) as follows

$$\Psi(T) = e^{i(\mathcal{H}/T - \mathcal{H}/1)}\overline{\psi_0} \pm i\int_T^1 e^{i(\mathcal{H}/T-\mathcal{H}/S)}\Psi(S)|\Psi(S)|^2\frac{dS}{S^2}. \tag{2.4}$$

The key point now is the construction of appropriate isometry

$$B(T) : L^2(\mathbb{R}) \ \to \ L^2(\mathbb{R}),$$

so that

$$\Phi(T) = B(T)\Psi(T),$$

satisfies the integral equation

$$\Phi(T) = U(T,1)B(1)\left(\overline{\psi_0}\right) \pm i\int_T^1 U(T,S)\Phi(S)|\Phi(S)|^2\frac{dS}{S}. \tag{2.5}$$

To be more precise, we choose

$$U(T,S) = B(T)e^{i\mathcal{H}/T}e^{-i\mathcal{H}/S}B^*(S), \tag{2.6}$$

where $B(T)$ is defined by

$$B(T) = M(T)\sigma_T, \tag{2.7}$$

with

$$M(T)f(x) = e^{ix^2/(4T)}f(x), \quad \sigma_T(f)(x) = T^{-1/2}f(T^{-1}x). \tag{2.8}$$

The proof of Theorem 1 is reduced to the proof of the following estimate.

**Theorem 2.1.** *Suppose the conditions* (1.6), (1.5) *are fulfilled, the operator* $\mathcal{H}$ *has no point spectrum,* 0 *is not a resonance for* $\mathcal{H}$ *and* $\Phi(T)$ *is the solution to the integral equation* (2.5) *with small initial data* $\psi_0$. *Then we have the estimate*

$$\|\Phi(T,\cdot)\|_{L^\infty(\mathbb{R})} \le C\varepsilon.$$

## 3. Some ideas for the proof

The main task is to define appropriate leading term and the modified profile for the solution $\Phi(T)$ to the integral equation

$$\Phi(T) = U(T,1)\left(\psi_1\right) \pm i\int_T^1 U(T,S)\Phi(S)|\Phi(S)|^2\frac{dS}{S}, \tag{3.1}$$

where

$$\psi_1(x) = M(1)\overline{\psi_0}(x) = e^{ix^2/4}\overline{\psi_0}(x). \tag{3.2}$$

3.1. **Modified profile for the perturbed Hamiltonian: idea for remainder estimates.** The requirement $T_2 < T_1$ implies that we need appropriate modification of the approach followed in the unperturbed case. If we can justify the existence of the strong limits

$$U(0,1)f = \lim_{\varepsilon\to 0} U(\varepsilon,1)f \tag{3.3}$$

and

$$\int_0^1 [U(0,S)-I]\Phi(S)|\Phi(S)|^2\frac{dS}{S} = \lim_{\varepsilon\to 0}\int_\varepsilon^1 [U(\varepsilon,S)-I]\Phi(S)|\Phi(S)|^2\frac{dS}{S} \tag{3.4}$$

exist, then we can set

$$\Phi_0 = U(0,1)\left(B(1)\overline{\psi_0}\right) \pm i\int_0^1 [U(0,S)-I]\Phi(S)|\Phi(S)|^2\frac{dS}{S}, \quad \psi_1 = B(1)\overline{\psi_0},$$

we can go further and we can define the leading term $\Phi_{lead}(T)$ of the solution $\Phi(T)$ as in (**??**), i.e.

$$\Phi_{lead}(T) = \Phi_0 \pm i\int_T^1 \Phi_{lead}(S)|\Phi(S)|^2\frac{dS}{S}. \tag{3.5}$$

One can set

$$\Theta(T) = \int_T^1 |\Phi(S)|^2\frac{dS}{S} \tag{3.6}$$

and find

$$\Phi_{lead}(T) = \Phi_0 e^{\pm i\Theta(T)}. \tag{3.7}$$

Our goal is to prove some a priori bounds for the remainder

$$\Phi_{rem}(T) = \Phi(T) - \Phi_{lead}(T).$$

Taking the difference between the equation (3.1) and the equation (3.5) for the leading term, we find

$$\Phi_{rem}(T) = U(T,1)\psi_1 - U(0,1)\psi_1 \pm i[U(T,0)-I]\int_T^1 U(0,S)\Phi(S)|\Phi(S)|^2\frac{dS}{S} \mp$$
$$\mp i\int_0^T [U(0,S)-I]\Phi(S)|\Phi(S)|^2\frac{dS}{S} \pm i\int_T^1 \Phi_{rem}(S)|\Phi(S)|^2\frac{dS}{S}.$$

3.2. **Decay of the Sobolev norms of the remainder.** The plan is to obtain our main a-priori estimate (for some small $\delta$)

$$\|\Phi(T) - \Phi_{lead}(T)\|_{H^\alpha_{odd}(\mathbb{R})} \le C\varepsilon T^{\delta/N}. \tag{3.8}$$

Our approach is based on the following key Lemmas.

**Lemma 3.1.** *The perturbed group $U(T,S)$ satisfies the estimates*

$$\|[U(T_1,T_2)-I]\,g\|_{H^\alpha(\mathbb{R})} \le C|T_1-T_2|^{\theta/4}\,\|g\|_{H^{\alpha+\theta}(\mathbb{R})} \tag{3.9}$$

*provided*

$$\alpha\in[0,3/4),\ \theta\in[0,1],\quad \frac{4\alpha}{3}+\theta<1,\quad 0\le T_2,T_1\le 1. \tag{3.10}$$

**Lemma 3.2.** *If $\Theta(x)$ is a real valued function and*

$$\Theta\in H^s(\mathbb{R}),\quad f\in H^s_{odd}(\mathbb{R})\cap L^\infty_{odd}(\mathbb{R}),$$

*for some $s\in[0,1)$, then*

$$\left\|e^{i\Theta}f\right\|_{H^s(\mathbb{R})} \le C\|f\|_{H^s_{odd}(\mathbb{R})} + C\|\Theta\|_{H^s(\mathbb{R})}\|f\|_{L^\infty_{odd}(\mathbb{R})}. \tag{3.11}$$

**Note:** By Sobolev embedding ( for $s > \frac{1}{2}$, $\|f\|_{L^\infty} \le C_s \|f\|_{H^s}$),

$$\left\| e^{i\Theta} f \right\|_{H^s(\mathbb{R})} \le C(1 + \|\Theta\|_{H^s(\mathbb{R})}) \|f\|_{H^s_{odd}(\mathbb{R})} \tag{3.12}$$

Finally,

**Lemma 3.3.** *If $\Phi(T)$ is a solution to the integral equation* (3.1)*, i.e.*

$$\Phi(T) = U(T,1)\,(\psi_1) \pm i \int_T^1 U(T,S)\Phi(S)|\Phi(S)|^2 \frac{dS}{S}, \tag{3.13}$$

*then for any $s \in (1/2, 3/4)$ we have*

$$\|\Phi(T)\|_{H^s(\mathbb{R})} \le C\|\psi_1\|_{H^s(\mathbb{R})} + C \int_T^1 \|\Phi(S)\|_{H^s(\mathbb{R})} \|\Phi(S)\|^2_{L^\infty(\mathbb{R})} \frac{dS}{S}. \tag{3.14}$$

3.3. **Final energy estimate.** One realizes that we need an estimate for a quantity in the form

$$|||\Phi|||_{\alpha,\theta} = \sup_{T \in (0,1]} \left( T^{\theta/32} \|\Phi(T)\|_{H^{\alpha+\theta}_{odd}(\mathbb{R})} \right) + \|\Phi\|_{L^\infty((0,1]\times\mathbb{R})}, \tag{3.15}$$

and we show

$$|||\Phi|||_{\alpha,\theta} \le C\|\psi_1\|_{H^{\alpha+\theta}_{odd}(\mathbb{R})} + C|||\Phi|||^2_{\alpha,\theta} \left( 1 + |||\Phi|||^3_{\alpha,\theta} \right) \tag{3.16}$$

which for small initial data, implies the requires bounds.

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# SHELL INTERACTIONS FOR DIRAC OPERATORS

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ABSTRACT. The self-adjointness of $H + V$ is studied, where $H = -i\alpha \cdot \nabla + m\beta$ is the free Dirac operator in $\mathbb{R}^3$ and $V$ is a measure-valued potential. The potentials $V$ under consideration are given by singular measures with respect to the Lebesgue measure, with special attention to surface measures of bounded regular domains.

We are interested in the free Dirac Operator in $\mathbb{R}^3$: $H = -i\alpha\cdot\nabla+m\beta$, with $m = \text{mass} > 0$, where

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) = \left( \begin{pmatrix} 0 & \widehat{\sigma}_1 \\ \widehat{\sigma}_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \widehat{\sigma}_2 \\ \widehat{\sigma}_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \widehat{\sigma}_3 \\ \widehat{\sigma}_3 & 0 \end{pmatrix} \right),$$

is the vector of the $4 \times 4$ complex Dirac matrices and

$$\widehat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \widehat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \widehat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the Pauli matrices. They satisfy the well known properties:

$$\beta = \begin{pmatrix} I_2 & 0 \\ -I_2 & 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{cases} \alpha_i^2 = \beta^2 = I_d & i = 1, 2, 3 \\ \{\alpha_i, \beta\} = \{\alpha_i, \alpha_j\} = 0 & i \neq j, \end{cases}$$

so that $H^2 = \left(-\Delta + m^2\right) I_d$, and $H$ is the first order symmetric differential operator introduced by Dirac in 1928 as a local version of $\sqrt{-\Delta + m^2}$.

We will considered the so called electrostatic shell interactions associated to $\Omega \subset \mathbb{R}^3$ a bounded smooth domain with $\sigma$ the corresponding surface measure on $\partial\Omega$ and $N$ the outward unit normal vector field on $\partial\Omega$.

We define the electrostatic shell potential $V_\lambda = \lambda\delta_{\partial\Omega}$:

$$\lambda \in \mathbb{R}, \qquad V_\lambda(\varphi) = \frac{\lambda}{2}\left(\varphi_+ + \varphi_-\right)$$

with

$$\begin{aligned} \varphi_\pm \quad = \quad & \text{non–tangential boundary values of } \varphi : \mathbb{R}^3 \longrightarrow \mathbb{C}^4 \\ & \text{when approaching from } \Omega \text{ or } \mathbb{R}^3 \setminus \overline{\Omega}. \end{aligned}$$

Accordingly we define the electrostatic shell interaction of $H$ as $H+V_\lambda$, with $a \in (-m, m)$.

Let us also consider the fundamental solution of $H - a$:

$$\phi^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}\left[a + m\beta + \left(1 - \sqrt{m^2 - a^2}|x|\right) i\alpha \cdot \frac{x}{x^2}\right].$$

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Then the domain of $H + V_\lambda$ is given by [AMV1]

$$\mathcal{D}(H+V_\lambda) = \Big\{\varphi : \quad \varphi = \phi^0 * (Gdx + gd\sigma,\ G \in L^2((R)3)4,\ g \in L^2(\partial\Omega)^4, \\ \lambda\left(\phi^0 * (Gdx)\right)\big\|_{\partial\Omega} = -(1+\lambda C^0_{\partial\Omega})g, \Big\}$$

where $C^a_{\partial\Omega}(g)(x) = \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}\phi^a(x-y)g(y)d\sigma(y)$ , $x \in \partial\Omega$.

It is proved in [AMV1] and [AMV2] that if $\lambda \neq \pm 2$ then $H + V_\lambda$ is self–adjoint on $\mathcal{D}(H+V_\lambda)$. Previous results where obtained in a more general setting by [P] and [PR], and in the specific case of the ball in [DES].

We will focus in the study of the point spectrum on $(-m, m)$ for $H + V_\lambda$ following the so called **Birman–Schwinger principle:** $a \in (-m,m)$, $\lambda \in \mathbb{R}\backslash 0$,

$$\ker(H+V_\lambda - a) \neq 0 \quad \text{if and only if} \quad \ker\left(\frac{1}{\lambda} + C^a_{\partial|Omega}\right) \neq 0,$$

so that a problem in $\mathbb{R}^3$ is reduced to a problem in $\partial\Omega$.

We will need the following properties of $C^a_{\partial\Omega}$, $a \in [-m,m]$:

(a) $C^a_{\partial\Omega}$ is a bounded self–adjoint operator in $L^2(\partial\Omega)^4$;

(b) $[C^a_{\partial\Omega}(\alpha\cdot N)]^2 = -\frac{1}{4}I_d$, with $\alpha\cdot N = \sum_{j=1}^{3}\alpha_j N_j$ a multiplication operator.

It is proved in [AMV1] that

$$\ker\left(\frac{1}{\lambda} + C^a_{\partial\Omega}\right) \neq 0 \quad \begin{cases} \overset{(a)}{\Longrightarrow} & |\lambda| \geq \lambda_l(\partial\Omega) > 0 \text{ and } \lambda_l(\partial\Omega) \leq 2 \\ \overset{(b)}{\Longrightarrow} & |\lambda| \leq \lambda_u(\partial\Omega) < +\infty \text{ and } \lambda_u(\partial\Omega) \geq 2. \end{cases}$$

Therefore, $\ker(H + V_\lambda - a) \neq 0$ implies that $|\lambda| \in [\lambda_l(\partial\Omega), \lambda_u(\partial\Omega)]$.

We want to address the following questions:

1.- How small can $[\lambda_l(\partial\Omega), \lambda_u(\partial\Omega)]$ be?
2.- Is there an isoperimetric–type statement w.r.t. $\Omega$?
3.- Which are the optimizers?

The examples we have in mind are: if $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain then

- Isoperimetric inequality: $\mathrm{Vol}(\Omega)^2 \leq \frac{1}{36}\mathrm{Area}(\partial\Omega)^3$.
- Pólya–Szegö inequality: If

$$\mathrm{Cap}(\overline{\Omega}) = \left(\inf_\nu \iint \frac{d\nu(x)d\nu(y)}{4\pi|x-y|}\right)^{-1},$$

with $\nu$ a probability Borel measure with $\mathrm{supp}\,\nu \subset \overline{\Omega}$, then (Pólya, Szegö, 1951)

$$\mathrm{Cap}(\overline{\Omega}) \geq 2(6\pi^2\mathrm{Vol}(\Omega))^{1/3}.$$

In both cases, the identity holds if and only if $\Omega$ is a ball.

We have the following result [AMV3].

**Theorem 0.1.** *Take $\Omega \subset \mathbb{R}^3$ a bounded smooth domain. If*

$$m\frac{\mathit{Area}(\partial\Omega)}{\mathit{Cap}(\overline{\Omega})} > \frac{1}{4\sqrt{2}},$$

*then*

$$\lambda(a) := \sup\{|\lambda| : \ker(H + V_\lambda - a) \neq 0 \ \textit{for some} \ a \in (-m, m)\}$$

$$\geq 4\left(m\frac{\textit{Area}\,(\partial\Omega)}{\textit{Cap}\,(\overline{\Omega})} + \sqrt{m^2\frac{\textit{Area}\,(\partial\Omega)^2}{\textit{Cap}\,(\overline{\Omega})^2} + \frac{1}{4}}\right)$$

*and*

$$\inf\{|\lambda| : \ker(H + V_\lambda - a) \neq 0 \ \textit{for some} \ a \in (-m, m)\}$$

$$\geq 4\left(-m\frac{\textit{Area}\,(\partial\Omega)}{\textit{Cap}\,(\overline{\Omega})} + \sqrt{m^2\frac{\textit{Area}\,(\partial\Omega)^2}{\textit{Cap}\,(\overline{\Omega})^2} + \frac{1}{4}}\right).$$

*In both cases, the identity holds if and only if* $\Omega$ *is a ball.*

The main ingredients of the proof are:

(1) The monotonictiy of $\lambda(a)$ reduces the problem to take $a = \pm m$.
(2) The quadratic form inequality relates

$$\sup\{|\lambda| \ : \ \ker(H + V_\lambda - a) \neq 0 \ \text{for some} \ a \in (-m, m)\}$$

with the optimal constant of an inequality involving the single layer potential $K$ and a Singular Integral Operator (SIO). (It is in this step where the constant $1/4\sqrt{2}$ appears as a sufficient condition).
(3) Isoperimetric type statement for $K$ in terms of $\text{Area}\,(\partial\Omega)\backslash\text{Cap}|,(\overline{\Omega})$.

**Sketch of the proof:**

(1) If

$$\ker\left(\frac{1}{\lambda(a)} + C^a_{\partial\Omega}\right) \neq 0$$

then

$$C^a_{\partial\Omega} g_a = \frac{1}{\lambda(a)} g_a \ , \ \|g_a\| = 1$$

and

$$\frac{1}{\lambda(a)} = \frac{1}{\lambda(a)}\langle g_a, g_a\rangle = \langle C^a_{\partial\Omega} g_a, g_a\rangle.$$

Also, if we understand $C^a_{\partial\Omega}$ as $(H - a)^{-1}$ then $\frac{d}{da}C^a_{\partial\Omega}$ behaves as $(H - a)^{-2}$ and $\frac{d}{da}\left(\frac{1}{\lambda(a)}\right)$ as $\langle (H-a)^{-2} g_a, g_a\rangle = \|(H-a)^{-1} g_a\|^2 \geq 0$, assuming $g_a$ is independent of $a$.

(2)

$$\left.\begin{aligned} Kf(x) &= \frac{1}{4\pi}\int \frac{f(y)}{|x-y|}d\sigma(y), \ \text{compact positive operator} \\ Wf(x) &= \frac{1}{4\pi}\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} i\cdot\widehat{\sigma}\cdot\frac{x-y}{|x-y|^3}f(y)d\sigma(y), \ \textit{SIO} \end{aligned}\right\} C^a_{\partial\Omega} = \begin{pmatrix} 2mK & W \\ W & 0 \end{pmatrix}.$$

Recall that

$$\widehat{\sigma} = (\widehat{\sigma}_1.\widehat{\sigma}_2, \widehat{\sigma}_3) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right).$$

Then,

(1) $$[C^m_{\partial\Omega}(\alpha\cdot N)]^2=-\frac{1}{4}\quad\Longrightarrow\quad\begin{cases}\{(\widehat{\sigma}\cdot N)K,(\widehat{\sigma}\cdot N)W\}=0\\[2mm] [(\widehat{\sigma}\cdot N)W]^2=-\frac{1}{4}.\end{cases}$$

(3)

$$\ker\left(\frac{1}{\lambda}+C^m_{\partial\Omega}\right)\neq 0\;\Longrightarrow\; C^m_{\partial\Omega}g=\frac{1}{\lambda}g\qquad\text{with } g=\begin{pmatrix}\mu\\ h\end{pmatrix}$$

$$\Longrightarrow\begin{cases}2mK\mu+Wh &= \;=-\frac{1}{\lambda}\mu\\[2mm] W\mu &= -\frac{1}{\lambda}.\end{cases}$$

Multiplying by $\overline{g}$, integrating on $\partial\Omega$ and after using (1) we obtain the quadratic form

$$\left(\frac{4}{\lambda}\right)^2\int_{\partial\Omega}|Wf|^2+\frac{8m}{\lambda}\underbrace{\int_{\partial\Omega}Kf\cdot\overline{f}}_{\geq 0}=\int_{\partial\Omega}|f|^2,$$

which is decreasing for $\lambda>0$.

As a consequence we have:

$$\lambda_\Omega=\inf\left\{\lambda>0\,:\,\left(\frac{4}{\lambda}\right)^2\int_{\partial\Omega}|Wf|^2+\frac{8m}{\lambda}\int_{\partial\Omega}Kf\cdot\overline{f}\leq\int_{\partial\Omega}|f|^2\qquad\forall f\in L^2(\partial\Omega)^2\right\}.$$

Our second result is the following one [AMV3]:

**Theorem 0.2.** (a)

$$4\left(m\|K\|_{\partial\Omega}+\sqrt{m^2\|K\|^2_{\partial\Omega}+\frac{1}{4}}\right)\leq\lambda_\Omega\leq 4\left(m\|K\|_{\partial\Omega}+\sqrt{m^2\|K\|^2_{\partial\Omega}+\|W\|^2_{\partial\Omega}}\right).$$

(b) *If* $\lambda>0$ *and* $\ker\left(\frac{1}{\lambda}+C^m_{\partial\Omega}\right)\neq 0\quad\Longrightarrow\quad\lambda\leq\lambda_\Omega.$

(c) *If* $\lambda_\Omega>2\sqrt{2}\left(\hookrightarrow\frac{1}{4\sqrt{2}}\right)$, *equality is attained and minimizers are related to the fact that* $\ker\left(\frac{1}{\lambda_\Omega}+C^m_{\partial\Omega}\right)\neq 0.$

Notice that

- We are looking for an isoperimetric–type result for $\lambda_\Omega$.
- Parts (b) and (c) ensure that

$$\lambda_\Omega=\sup\left\{|\lambda|\,:\,\ker\left(1/\lambda+C^m_{\partial\Omega}\right)\neq 0\right\}.$$

- We can use the monotonicity of $\lambda(a)$ to replace "for some $a\in(-m,m)$" by $a=m$.

Let us say a few words about the isoperimetric question. If $\Omega$ is a ball then $\|W\|^2_{\partial\Omega}=\frac{1}{4}$.
The opposite implication is proved in [HM-OMP-ET]. Then $\lambda_\Omega=4\left(m\|K\|_{\partial\Omega}+\sqrt{m^2\|K\|^2_{\partial\Omega}+\|W\|^2_{\partial\Omega}}\right).$

For a general $\Omega$ we have

$$\|K\|_{\partial\Omega} = \sup_{f\neq 0} \frac{1}{\|f\|^2_{\partial\Omega}} \int_{\partial\Omega} Kf \cdot \overline{f} \underset{(f=1)}{\geq} \overset{\inf D}{\iint \frac{d\sigma(y)}{4\pi|x-y|}} \frac{d\sigma(x)}{\sigma(\partial\Omega)} \geq \frac{\mathrm{Area}\,(\partial\Omega)}{\mathrm{Cap}\,(\overline{\Omega})}.$$

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# UNIQUE CONTINUATION RESULTS FOR SOME EVOLUTION EQUATIONS

GUSTAVO PONCE

## 1. Introduction

We shall discuss several results concerning unique continuation properties of some partial differential equations. These equations arise in several different physical contexts and have been extensively studied. Among these models we include :

the Camassa-Holm (CH) equation

$$\partial_t u + 3u\partial_x u - \partial_t \partial_x^2 = 2\partial_x u \partial_x^2 u + u\partial_x^3 u, \quad t,\, x \in \mathbb{R}, \tag{1.1}$$

the generalized Korteweg-de Vries (gKdV) equation

$$\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad t,\, x \in \mathbb{R}, \;\; k \in \mathbb{Z}^+, \tag{1.2}$$

the generalized Benjamin-Ono (gBO) equation

$$\partial_t u - \mathcal{H}\partial_x^u + u^k \partial_x u = 0, \quad t,\, x \in \mathbb{R}, \;\; k \in \mathbb{Z}^+, \tag{1.3}$$

where $\mathcal{H}$ denotes the Hilbert transform

$$\mathcal{H}f(x) = \frac{1}{\pi} p.v.\frac{1}{x} * f(x) = -i(\text{sgn}(\xi)\widehat{f}(\xi))^{\vee}(x),$$

Schrödinger equations of the form

$$i\partial_t u + \Delta u = V(x,t)u, \quad t \in \mathbb{R},\, x \in \mathbb{R}^n, \tag{1.4}$$

and the heat equation of the following kind

$$\partial_t u + \Delta u + V(x,t)u = 0, \quad t \in \mathbb{R},\, x \in \mathbb{R}^n. \tag{1.5}$$

Although, the models in (1.4) and (1.5) are linear, the goal is to extend the uniqueness result for these linear model to a nonlinear ones.

In general, our aim is to deduce the uniqueness of the solution as a consequence of the decay at infinity of it. For example, let us consider the following result due to Meshkov [7] : if $u \in H^2_{loc}(\mathbb{R}^n)$ is solution of the stationary problem

$$\Delta u + V(x)u = 0, \qquad x \in \mathbb{R}^n, \tag{1.6}$$

with $V \in L^\infty(\mathbb{R})$ such that

$$\int_{\mathbb{R}^n} e^{a|x|^{4/3}} |u(x)|^2 dx < \infty, \qquad \text{for any} \quad a > 0, \tag{1.7}$$

then

$$u \equiv 0.$$

It was proved in [7] that if the potential $V$ takes complex values, then the exponent $4/3$ in (1.7) is optimal. However, it has been conjectured that for real valued potentials the optimal exponent should be 1.

In the case of time evolution equations, we are interested in having similar uniqueness results by assuming an appropriate decay at two different times. Moreover, we want to see that these decay assumptions at two different times are optimal. As we shall see in some cases a condition at one time suffices (mainly in the parabolic case) in other the condition should involved three different times.

More precisely, we shall consider the following kind of results :

**Theorem 1.1.** *([5]) Assume that for some $T > 0$ and $s > 3/2$,*

$$u \in C([0,T] : H^s(\mathbb{R})) \tag{1.8}$$

*is a strong solution of the IVP associated to the RCH equation* (1.1). *If $u_0(x) = u(x,0)$ satisfies that for some $\alpha \in (1/2, 1)$*

$$|u_0(x)| \sim o(e^{-x}), \quad \text{and} \quad |\partial_x u_0(x)| \sim O(e^{-\alpha x}) \quad \text{as} \quad x \uparrow \infty \tag{1.9}$$

*and there exists $t_1 \in (0,T]$ such that*

$$|u(x,t_1)| \sim o(e^{-x}), \qquad \text{as} \quad x \uparrow \infty, \tag{1.10}$$

*then $u \equiv 0$.*

In this particular case, we shall see how a recent result of Linares-Ponce-Sideris shows that Theorem 1.1 can be extended to a class of solutions which include the so called "peakons" solutions, which are traveling waves of the form

$$u_c(x,t) = c\,\phi(x - ct), \quad c > 0, \tag{1.11}$$

where

$$\phi(x) = e^{-|x|}.$$

Notice, that $e^{-|x|} \notin H^{3/2}(\mathbb{R})$.

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Full dispersion water waves models
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By "Full dispersion models" we mean models of water waves, in various asymptotic regimes, which keep the original dispersion of the full water waves system. The aim is to obtain "good" models on larger frequency ranges. In usual water waves models (Korteweg-de Vries, Boussinesq, Kadomtsev- Petviashvili, Davey-Stewartson, nonlinear Schrödinger,...) the original dispersion is Taylor expanded at a given frequency, yielding local equations or systems with nice dispersive properties. Full dispersion models are nonlocal and in some sense only weakly dispersive, making their mathematical analysis in general delicate, in particular by the lack of useful dispersive estimates. Many examples of those models can be found *eg* in [2, 4, 3, 6, 1] but the same idea can be applied in nonlinear optics for instance (see [5]).

A typical example is the *full dispersion Kadomtsev-Petviashvili equations* (FDKP) introduced in [9] and studied in [10]:

$$\partial_t u + c_{WW}(\sqrt{\mu}|D^\mu|)(1+\mu\frac{D_2^2}{D_1^2})^{1/2}u_x + \mu\frac{3}{2}uu_x = 0, \tag{1}$$

where $c_{WW}(\sqrt{\mu}k)$ is the phase velocity of the linearized water waves system, namely

$$c_{WW}(\sqrt{\mu}k) = \left(\frac{\tanh\sqrt{\mu}k}{\sqrt{\mu}k}\right)^{1/2}$$

and

$$|D^\mu| = \sqrt{D_1^2+\mu D_2^2}, \quad D_1 = \frac{1}{i}\partial_x, \quad D_2 = \frac{1}{i}\partial_y.$$

$h$ = a typical depth of the fluid layer, $a$ = typical amplitude of the wave, $\lambda_x$ and $\lambda_y$ = typical wave lengths in $x$ and $y$ respectively, the relevant regime here is:

$$\mu \sim \frac{a}{h} \sim \left(\frac{\lambda_x}{\lambda_y}\right)^2 \sim \left(\frac{h}{\lambda_x}\right)^2 \ll 1.$$

When adding surface tension effects, one has to replace (1) by

$$\partial_t u + \tilde{c}_{WW}(\sqrt{\mu}|D^\mu|)(1+\mu\frac{D_2^2}{D_1^2})^{1/2}u_x + \mu\frac{3}{2}uu_x = 0, \tag{2}$$

with

$$\tilde{c}_{WW}(\sqrt{\mu}k) = (1+\beta\mu k^2)^{\frac{1}{2}}\left(\frac{\tanh\sqrt{\mu}k}{\sqrt{\mu}k}\right)^{1/2},$$

where $\beta > 0$ is a dimensionless coefficient measuring the surface tension effects,

In the one dimensional case, the FDKP reduce to the *Whitham equation* that displays challenging properties (see [8, 7]):

$$u_t + \mu u u_x + T_\mu u_x = 0, \quad T_\mu = \left((1+\beta\mu D^2)\frac{\tanh\sqrt{\mu}D}{\sqrt{\mu}D}\right)^{1/2}. \tag{3}$$

Moreover, it reduces formally to the usual KP equations in the long wave limit and is reminiscent of the *Full dispersion KP equations* in the high frequency regime :

$$u_t + uu_x - D_x^\alpha u_x + \epsilon\partial_x^{-1}u_{yy} = 0, \quad \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \quad u(\cdot,0) = u_0, \quad -1 < \alpha \tag{4}$$

where $\epsilon = 1$ corresponds to the fKP II equation and $\epsilon = -1$ to the fKP I equation. Here $D_x^\alpha$ denotes the Riesz potential of order $-\alpha$ in the $x$ direction, *i.e.* $D_x^\alpha$ is defined via Fourier transform by $\left(D_x^\alpha f\right)^\wedge(\xi,\eta) = |\xi|^\alpha \widehat{f}(\xi,\eta)$.

This equation can be thought as a two-dimensional weakly transverse version of the fractionary KdV equation (fKdV)

$$u_t + uu_x \pm D_x^\alpha u_x = 0 \tag{5}$$

that has been studied in [11, 13]. In particular (4) is the very relevant KP version of the Benjamin-Ono equation when $\alpha = 1$.

When$-1 < \alpha < 1$, the fractionary KdV equation is a useful toy model to understand the effects of a weak dispersive perturbation on the hyperbolic Burgers equation.

We will present recent and on going results on the Cauchy problem and solitary wave solutions, based on [10, 12] for the nonlocal KP equations above.

If time allows we will also discuss some results and open questions ([14, 15]) related to a very relevant *full dispersion surface wave model* derived under a small steepness assumption on the wave.

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# Existence of Global Solutions and Global Attractor for the Third Order Lugiato-Lefever Equation on $\mathbf{T}$

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## 1 Introduction and Main Theorems

We consider the third order Lugiato-Lefever equation:

$$\partial_t u - \partial_x^3 u + i\alpha\partial_x^2 u + u + i|u|^2 u = f, \qquad t > 0, \quad x \in \mathbf{T}, \tag{1}$$

$$u(0,x) = u_0(x), \qquad x \in \mathbf{T}, \tag{2}$$

where $\alpha$ is a real constant such that $2\alpha/3 \notin \mathbf{Z}$. In (1), all the parameters are normalized except for $\alpha$. The case $\alpha > 0$ is called focusing and the case $\alpha < 0$ is called defocusing. In the physical context, the third order Lugiato-Lefever equation includes the detuning term $i\theta u$ ($\theta \in \mathbf{R}$) on the left hand side, but we omit the detuning term because it does not matter in this paper. Recently the generalized Lugiato-Lefever equation has been attracting a great interest especially in the field of nonlinear optics (see, e.g., [3] and [12]). An increasing attention among theoretical and experimental physicists in that field has been paid to the role of third order dispersion, i.e. the third order derivative in (1) (see [11], [12] and [16]).

In this note, we present the results on the global well-posedness in $L^2(\mathbf{T})$ of the Cauchy problem (1) and (2) and the existence of the global attractor in $L^2$ for flows generated by the third order Lugiato-Lefever equation (1), which have recently been obtained in collaboration with Miyaji Tomoyuki, Meiji Institute for Adv. Stud. Math. Sci., Meiji University. To prove the former, we use the Strichartz estimate and to prove the latter, we take full advantage of the smoothing effect of the cubic nonlinear interaction. Without damping and forcing the solution $u$ of (1) and (2) formally satisfies the following three conservations, that is, the mass, the momentum and the energy conservations for $t > 0$ (see [16, lines 7 to 10 on page 2326]).

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \tag{3}$$

$$\mathrm{Im}\,(\partial_x u(t), u(t))) = \mathrm{Im}\,(\partial_x u_0, u_0), \tag{4}$$

$$\begin{aligned} &\beta\|\partial_x u(t)\|_{L^2}^2 + \mathrm{Im}\,(\partial_x^2 u(t), \partial_x u(t)) - \tfrac{1}{2}\|u(t)\|_{L^4}^4 \\ &= \beta\|\partial_x u_0\|_{L^2}^2 + \mathrm{Im}\,(\partial_x^2 u_0, \partial_x u_0) - \tfrac{1}{2}\|u_0\|_{L^4}^4, \end{aligned} \tag{5}$$

where $(\cdot,\cdot)$ denotes the scalar product of $L^2(\mathbf{T})$. The energy functional defined as in (5) is neither positive definite nor negative definite, because it includes the $L^2$ scalar product of the second and the first derivatives of the solution. This suggests that the energy is not useful for controlling the global behavior of the solution. Therefore, we need to consider the global solution in $L^2$ and as a result, we need to construct the global attractor in $L^2$ instead of the $H^1$ global attractor. The construction of global attractor in $L^2$ causes a serious problem on the compactness of orbit.

In this paper, we prove the following two theorems concerning the well-posedness of the Cauchy problem (1) and (2) and the global attractor.

**Theorem 1.1.** *(i) (Local existence) Assume that* $u_0 \in L^2$ *and* $f \in C([0,\infty); L^2)$. *Then, there exists a positive constant* $T$ *such that the Cauchy problem* (1) *and* (2) *has a unique solution* $u$ *on* $[0,T]$ *satisfying*

$$u \in C([0,T]; L^2) \bigcap L^4([0,T] \times \mathbf{T}).$$

*(ii) (Global existence and a priori estimate) The solution given by part (i) can be extended to any positive times and satisfies the following identity*

$$\|u(t)\|_{L^2}^2 = e^{-2t}\|u_0\|_{L^2}^2 + 2\int_0^t e^{-2(t-s)}\mathrm{Re}\,\big(u(s), f(s)\big)\,ds, \quad t > 0.$$

**Theorem 1.2.** *Assume that* $2\alpha/3 \notin \mathbf{Z}$ *and that* $f$ *is a time-independent function in* $L^2(\mathbf{T})$. *The third order Lugiato-Lefever equation* (1) *has the global attractor in* $L^2(\mathbf{T})$.

**Remark 1.3.** (i) In Theorem 1.1, the external forcing term $f$ is a function of variables $t$ and $x$, while $f$ is a time-independent function in Theorem 1.2. This is because equation (1) should be autonomous as we consider the global attractor in Theorem 1.2.

(ii) Theorem 1.1 holds for all $\alpha \in \mathbf{R}$, while our proof of Theorem 1.2 requires the assumption that $2\alpha/3 \notin \mathbf{Z}$. It is an interesting problem what influence the resonance of the third and second order dispersion coupling has on the regularity and the global behavior of the solution for (1).

To show the global well-posedness in $L^2(\mathbf{T})$, we prove the space-time integrability of solution for the linear inhomogeneous third order Schrödinger equation, which is called the Strichartz estimate. We now consider the following inhomogeneous linear Schrödinger equation with third order dispersion on one dimensional torus $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$.

$$\partial_t u - \partial_x^3 u + i\alpha\partial_x^2 u = f, \qquad t \in \mathbf{R}, \quad x \in \mathbf{T}, \tag{6}$$

$$u(0,x) = u_0(x), \qquad x \in \mathbf{T}, \tag{7}$$

where $\alpha$ is a real constant. We have the following $L^4$ space-time integrability estimate of solution for (6) and (7). Theorem 1.1 is an immediate consequence of the Strichartz estimate.

**Theorem 1.4.** *Let $T > 0$ and let $1/2 > b > 1/3$. Then, we have*

$$\begin{aligned} \|u\|_{L^4((-T,T)\times\mathbf{T})} \le CT^{1/2}\mathcal{T}^{-b}\big[&\|u_0\|_{L^2(\mathbf{T})} \\ &+T^{1/2}\mathcal{T}^{-b}\|f\|_{L^{4/3}((-T,T)\times\mathbf{T})}\big], \end{aligned} \tag{8}$$

*where $\mathcal{T} = \min\{T, 1\}$ and $C$ is a positive constant dependent only on $b$.*

The proof of Theorem 1.4 follows from the argument by Kenig, Ponce and Vega [9].

**Remark 1.5.** Theorem 1.4 holds valid for all $\alpha$.

To show the existence of the global attractor, we prove a kind of the smoothing effect, which is not the same as that of the parabolic equation. Instead of the original equation (1), we consider the so-called reduced equations resulting from the removal of terms which have “bad” effects on the solution. In [4], Erdoğan and Tzirakis use the smoothing effect of the Duhamel term to construct the global attractor for the KdV. However, the whole of the Duhamel term can not become more regular than the initial datum in the case of the third order Lugiato-Lefever equation (1), which is in sharp contrast to the KdV equation.

**Remark 1.6.** . We should mention the difference between the proofs of our Theorem 1.2 and the result by Molinet [14]. In [14], Molinet shows the global attractor in $L^2$ of the cubic nonlinear Schrödinger equation with damping and forcing terms. The new ingredient of his proof is the application of the argument by Ball [1] to the weak limit equation keeping the same structure as the original equation. It would be possible to apply the proof by Molinet [14] to our problem for $2\alpha/3 \notin \mathbf{Z}$. But the smoothing property we proved for Theorem 1.2 is stronger than that in the paper [14]. Furthermore, it seems difficult to apply Molinet's proof to the case $2\alpha/3 \in \mathbf{Z}\backslash\{0\}$ for the same reason as in our proof. Indeed, when $2\alpha/3 \in \mathbf{Z}\backslash\{0\}$, the reduced equation is written as follows.

$$\begin{aligned} &\partial_t \hat{v}(t,k) + \big(i(k^3 - \alpha k^2) + 1\big)\hat{v}(t,k) \\ +i \sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3)\neq 0 \\ k_3+k_1\neq 2\alpha/3}} &\hat{v}(t,k_1)\hat{\bar{v}}(t,k_2)\hat{v}(t,k_3) - i|\hat{v}(t,k)|^2 v(t,k) \\ &+i \sum_{k_1\in\mathbf{Z}} \hat{v}(t,k_1)\hat{v}(t,2\alpha/3-k_1)\hat{\bar{v}}(t,k-2\alpha/3) \\ &= \hat{f}(k) e^{-\frac{i}{\pi}\int_0^t \|v(s)\|_{L^2}^2\,ds}, \qquad t>0. \end{aligned} \tag{9}$$

The last term on the left hand side of (9) prevents us from applying not only our proof but also the proof by Molinet [14].

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# NONLINEAR FLOWS AND OPTIMALITY FOR FUNCTIONAL INEQUALITIES

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**Work done in collaboration with Jean Dolbeault and Michael Loss**

This talk consists of two parts. Firstly, we introduce the method of linear and nonlinear flows to prove rigidity results on a sphere. Then, we will present several rigidity results for nonnegative solutions of semilinear elliptic equations on infinite cylinder-like domains or in the Euclidean espace and as a consequence, about optimal symmetry properties for the optimizers of the Caffarelli-Kohn-Nirenberg inequalities. All the results will be stated in the simpler case of spherical cylinders. Similar, but less precise, results can also be stated and proved for general cylinders generated by any compact smooth Riemannian manifold without a boundary.

Other consequences from the results below are optimal estimates for the principal eigenvalue of Schrödinger operators on infinite cylinders

## 1. Linear and nonlinear flows to prove rigidity for interpolation inequalities on spheres

On the $d$-dimensional sphere, let us consider the interpolation inequality

$$\|\nabla u\|^2_{L^2(S^d)} + \frac{d}{p-2}\,\|u\|^2_{L^2(S^d)} \geq \frac{d}{p-2}\,\|u\|^2_{L^p(S^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}, d\mu)\,, \tag{1.1}$$

where the measure $d\mu$ is the uniform probability measure on $\mathbb{S} \subset \mathbb{R}^{d+1}$ corresponding to the measure induced by the Lebesgue measure on $\mathbb{R}^{d+1}$, and the exponent $p \geq 1$, $p \neq 2$, is such that $p \leq 2^* := \frac{2\,d}{d-2}$ if $d \geq 3$.

The case $p = \frac{2\,d}{d-2}$ corresponds to the Sobolev inequality (equivalent via the use of the stereographic projection).

$$\int_{\mathbb{R}^d} |\nabla v|^2\, dx \geq S\left(\int_{\mathbb{R}^d} |v|^{\frac{2\,d}{d-2}}\, dx\right)^{\frac{d-2}{d}} \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^d)\,,$$

Bidaut-Véron and Véron proved in 1991 ([2]) the above optimal inequality and the fact that for $2 < p < 2^*$, the optimizers are the constant functions. They did it by using a PDE rigidity method. Beckner used harmonic analysis to prove the same result in 1993 (see [3]). Later, in [1], Bakry and Ledoux used the *carré du champ* method to prove the same result by using a flow method, which applied only in the case $2 < p \leq 2^\# := \frac{2\,d^2+1}{(d-1)^2} < 2^*$. Even if this result is less general, the method is very interesting and we outline it below.

Let us define $\rho = |u|^p$. Then, (1.1) can be written as

$$\int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\, d\omega \geq \frac{d}{p-2}\left[\left(\int_{\mathbb{S}^d} \rho\, d\omega\right)^{\frac{2}{p}} - \int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\, d\omega\right].$$

If we define the functionals $\mathcal{E}_p$ and $\mathcal{I}_p$ respectively by

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2 \, d\omega \;, \quad \mathcal{E}_p[\rho] := \frac{1}{p-2} \left[ \left( \int_{\mathbb{S}^d} \rho \, d\omega \right)^{\frac{2}{p}} - \int_{\mathbb{S}^d} \rho^{\frac{2}{p}} \, d\omega \right] \quad \text{if} \quad p \neq 2 \,,$$

then the above inequality amount to $\mathcal{I}_p[\rho] \geq d\, \mathcal{E}_p[\rho]$. To establish such inequalities, Bakry and Ledoux used the linear heat flow $\frac{\partial \rho}{\partial t} = \Delta \rho$, where $\Delta$ denotes the Laplace-Beltrami operator on $S^d$. We have $\frac{d}{dt}\left(\int_{\mathbb{S}^d} \rho \, d\omega\right) = 0$

$$\text{If } p \leq 2^{\#}, \qquad \frac{d}{dt}\mathcal{E}_p[\rho] = -\,\mathcal{I}_p[\rho] \quad \text{and} \quad \frac{d}{dt}\mathcal{I}_p[\rho] \leq -\,d\,\mathcal{I}_p[\rho]\,,$$

that is,

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - \,d\,\mathcal{E}_p[\rho]\Big) \leq 0\,.$$

It is not difficult to prove that $\rho$ converges to a constant as $t \to +\infty$ and

$$\lim_{t\to+\infty} \Big(\mathcal{I}_p[\rho] - \,d\,\mathcal{E}_p[\rho]\Big) = 0\,.$$

and hence the nonnegativity of $\mathcal{I}_p[\rho] - \,d\,\mathcal{E}_p[\rho]$ is proved for all $\rho$.

What happens if $2^{\#} < p < 2^*$? The result below shows that the strategy based on the linear heat equation cannot work.

**Lemma 1.1** ([7])**.** *When $2^{\#} < p < 2^*$, one can find a function $\rho_0$ such that $\rho$ solution of $\frac{\partial \rho}{\partial t} = \Delta \rho$, $\rho(t=0) = \rho_0$, and*

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - \,d\,\mathcal{E}_p[\rho]\Big)\Big|_{t=0} > 0\,.$$

But in [7] it was also proved that if we consider instead the nonlinear flow $\frac{d\rho}{dt} = \Delta \rho^m$, for a well-chosen $m \neq 1$, the same strategy can be followed. The computations are much more involved, but the idea is "more or less" the same. And note that this result covers also the case $p \in (1,2)$. In the critical case $p = 2$, similar results can be found for a log-Sobolev inequality on the sphere (see [7]).

## 2. Rigidity results

The main result presented in this talk is the following rigidity theorem, which is contained, with its proof, in [6]. Many references about previous works and related topics can be found in that article.

**Theorem 2.1.** *For $d \geq 2$ define $2^* = 2d/(d-2)$ if $d \geq 3$, $2^* = +\infty$ if $d = 2$. And consider the cylinder $\mathcal{C}_1 := \mathbb{R} \times \mathbb{S}^{d-1}$. For all $p \in (2, 2^*)$ and $0 < \Lambda \leq \Lambda_{\rm FS} := 4\,\frac{d-1}{p^2-4}$, any positive solution $\varphi \in \mathrm{H}^1(\mathcal{C}_1)$ of*

$$-\,\partial_s^2\,\varphi - \,\Delta_\omega\,\varphi + \Lambda\,\varphi = \varphi^{p-1} \quad \text{in} \quad \mathcal{C}_1 \tag{2.1}$$

*is equal to $\varphi_\Lambda$, up to a translation in the $s$-direction, where*

$$\varphi_\Lambda(s) = \left(\tfrac{p}{2}\,\Lambda\right)^{\frac{1}{p-2}} \left(\cosh\left(\tfrac{p-2}{2}\,\sqrt{\Lambda}\,s\right)\right)^{-\frac{2}{p-2}}\,. \tag{2.2}$$

By using the Endem-Fowler transformation

$$v(r,\omega) = r^{a-a_c}\,\varphi(s,\omega) \quad \text{with} \quad r = |x|\,, \quad s = -\log r \quad \text{and} \quad \omega = \frac{x}{r}\,, \tag{2.3}$$

it can be easily seen that Theorem 2.1 is equivalent to the following result

**Theorem 2.2.** *Assume that $d \geq 2$. If either $a \in [0,(d-2)/2)$ and $b > 0$, or $a < 0$ and $b \geq b_{\rm FS}(a)$, with*

$$b < b_{\rm FS}(a) := \frac{d\,(a_c - a)}{2\,\sqrt{(a_c-a)^2 + d - 1}} + a - a_c\,, \tag{2.4}$$

*then any nonnegative solution $v$ of*

$$-\,\nabla\cdot\left(|x|^{-2\,a}\,\nabla v\right) = |x|^{-b\,p}\,|v|^{p-2}\,v \quad \text{in } \mathbb{R}^d\setminus\{0\} \tag{2.5}$$

*which satisfies $\int_{\mathbb{R}^d}\frac{|v|^p}{|x|^{b\,p}}\,dx < \infty$, is equal to $v_\star$ up to a scaling, with*

$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \forall\,x\in\mathbb{R}^d\,.$$

Next pick $n$ and $\alpha$ such that $n = \frac{d-b\,p}{\alpha} = \frac{d-2\,a-2}{\alpha} + 2 = \frac{2\,p}{p-2}$. Then, defining $v(r,\omega) = w(r^\alpha,\omega)\quad\forall\,(r,\omega)\in\mathbb{R}^+\times\mathbb{S}^{d-1}$, it can again be easily seen that the two above theorems are equivalent to

**Theorem 2.3.** *Assume that $d \geq 2$. If $0 < \alpha < \alpha_{\rm FS} := \sqrt{\frac{d-1}{n-1}}$, then any nonnegative solution $w(x) = w(r,\omega)$ $(r\in\mathbb{R}_+,\ \omega\in S^{d-1})$ of*

$$-\,\alpha^2\,w'' + \alpha^2\,\frac{n-1}{r}\,w' + \frac{\Delta\,w}{r^2} = w^{p-1} \quad \text{in } \mathbb{R}^d\setminus\{0\}\,, \tag{2.6}$$

*which satisfies $\int_{\mathbb{R}^d}|x|^{n-d}\,|w|^p\,dx < \infty$, is equal to $w_\star$ up to a scaling, and multiplication by a constant, with*

$$w_\star(x) = \left(1+|x|^2\right)^{-n} \quad \forall\,x\in\mathbb{R}^d\,.$$

Notice that if $n$ is an integer, then, (2.6) is the Euler-Lagrange equation associated with the so-called critical Sobolev equation $-\alpha^2\,\Delta w = w^{\frac{n+2}{n-2}}$ in $\mathbb{R}^n$.

Notice also that the above definitions imply the equivalence of the above three conditions

$$0 < \Lambda \leq \Lambda_{\rm FS} := 4\,\frac{d-1}{p^2-4}\,;\quad 0 < \alpha < \alpha_{\rm FS} := \sqrt{\frac{d-1}{n-1}}\,;$$

$$a < (d-2)/2 \ \text{ and } \ b > 0, \ \text{ or } \ a < 0 \ \text{ and } \ b \geq b_{\rm FS}(a)$$

Finally, let us remark that the above three results are optimal, since as it is proved in [8, 5], when the above conditions are not satisfied, there are nonnegative solutions of the corresponding equations that are not radially symmetric.

## 3. Consequence: optimal symmetry result for optimizers of the critical Caffarelli-Kohn-Nirenberg inequalities

The Caffarelli-Kohn-Nirenberg inequalities

$$\left(\int_{\mathbb{R}^d}\frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \leq \mathsf{C}_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\,v\in\mathcal{D}_{a,b} \tag{3.1}$$

have been established in [4], under the conditions that $a \leq b \leq a+1$ if $d \geq 3$, $a < b \leq a+1$ if $d = 2$, $a + 1/2 < b \leq a+1$ if $d = 1$, and $a < (d-2)/2$, where the exponent

$$p = \frac{2\,d}{d-2+2\,(b-a)} \tag{3.2}$$

is determined by the invariance of the inequality under scalings. Here $\mathsf{C}_{a,b}$ denotes the optimal constant in (3.1) and the space $\mathcal{D}_{a,b}$ is defined by

$$\mathcal{D}_{a,b} := \Big\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \Big\}\,.$$

Note that, up to scaling and multiplication by a constant, any optimal solution for the above inequality is a nonegative solution of (2.5). It was proved in [8] (see also [5] for a partial result) that whenever $a < 0$ and $b < b_{\rm FS}(a)$, the optimizers of (2.5) are never radially symmetric. What Theorem 2.2 implies is that whenever $b \geq b_{\rm FS}(a)$ or $a \in [0, a_c)$, the optimizers, which can be taken as nonnegative functions, thus yielding an optimal symmetry result.

## 4. Outline of the proof

Let us now quickly present the main ideas of the proof of the above results in the case $d \geq 3$. We will explain it for in the context of Theorem 2.3. Let us introduce some notation:

$$u^{\frac12-\frac1n} = |w| \quad\Longleftrightarrow\quad u = |w|^p \quad \text{with} \quad p = \frac{2\,n}{n-2} \tag{4.1}$$

and notice that, up to a multiplicative constant, the r.h.s. in (3.1) is transformed into a generalized *Fisher information*

$$\mathcal{I}[u] := \int_{(0,\infty)\times S^{d-1}} u\,|\mathsf{D}\mathsf{p}|^2\,d\mu \quad \text{where} \quad \mathsf{p} = \frac{m}{1-m}\,u^{m-1} \quad \text{and} \quad m = 1 - \frac1n\,, \tag{4.2}$$

with $\mathsf{D}p = \left(\alpha\,\frac{\partial \mathsf{p}}{\partial r}, \frac1r\,\nabla_\omega \mathsf{p}\right)$, while the l.h.s. in (3.1) is now proportional to a *mass*, $\int_{(0,\infty)\times S^{d-1}} u\,d\mu$, where the measure $d\mu$ is defined as $r^{n-1}\,dr\,d\omega$ on $(0,\infty)\times S^{d-1}$. Here $\mathsf{p}$ is the *pressure function*, as in [9, 5.7.1 p. 98]. If we replace $m$ by $1-\frac1n$, we get that

$$\mathsf{p} = (n-1)\,u^{-\frac1n}\,. \tag{4.3}$$

Let us next introduce the fast diffusion flow

$$\frac{\partial u}{\partial t} = \mathcal{L}\,u^m\,, \quad m = 1 - \frac1n\,, \tag{4.4}$$

with

$$\mathcal{L}\,w := -\,\mathsf{D}^*\,\mathsf{D}\,w = \alpha^2\,w'' + \alpha^2\,\frac{n-1}{r}\,w' + \frac{\Delta\,w}{r^2}\,, \quad {}' = d/dr\,,$$

and assume that it is well defined for all times. It is immediate to verify that $\frac{d}{dt}\int_{(0,\infty)\times S^{d-1}} u\,d\mu = 0$. Moreover, long calculations, the study of the regularity of the solutions of (2.1) at $\pm\infty$ and the use of the Bochner-Lichnerowicz-Weitzenböck formula

$$\tfrac12\,\Delta_\omega\left(|\nabla_\omega f|^2\right) = \|\mathrm{Hess} f\|^2 + \nabla_\omega(\Delta_\omega f)\cdot\nabla_\omega f + \mathrm{Ric}(\nabla_\omega f, \nabla_\omega f)\,,$$

among others, allow us to prove the following proposition.

**Proposition 4.1.** *With the notations defined by* (4.3) *if $u$ is a smooth minimizer of $\mathcal{I}[u]$ under a mass constraint, with $\alpha \leq \alpha_{\rm FS}$, then there exists a positive constant $\zeta_\star$ such that*

$$\frac{d}{dt}\mathcal{I}[u(t,\cdot)] = -\,2\,(n-1)^{n-1}\int_{(0,\infty)\times S^{d-1}} \mathsf{k}[\mathsf{p}(t,\cdot)]\,\mathsf{p}(t,\cdot)^{1-n}\,d\mu\,,$$

*with*

$$\mathsf{k}[\mathsf{p}] = \alpha^4 \left(1 - \frac{1}{n}\right) \left[\mathsf{p}'' - \frac{\mathsf{p}'}{r} - \frac{\Delta_\omega \, \mathsf{p}}{\alpha^2 \, (n-1) \, r^2}\right]^2 + 2 \, \alpha^2 \, \frac{1}{r^2} \left|\nabla_\omega \mathsf{p}' - \frac{\nabla_\omega \mathsf{p}}{r}\right|^2 + \frac{1}{r^4} \, \mathsf{k}_{\mathfrak{M}}[\mathsf{p}]$$

*and*

$$\int_{\mathbb{S}^{d-1}} \mathsf{k}_{\mathfrak{M}}[\mathsf{p}] \, \mathsf{p}^{1-n} \, d\omega \geq (n-2) \big(\alpha_{\mathrm{FS}}^2 - \alpha^2\big) \int_{\mathbb{S}^{d-1}} |\nabla_\omega \mathsf{p}|^2 \, \mathsf{p}^{1-n} \, d\omega + \zeta_\star \, (n-d) \int_{\mathbb{S}^{d-1}} |\nabla_\omega \mathsf{p}|^4 \, \mathsf{p}^{1-n} \, d\omega \, .$$

Therefore, if $\alpha \leq \alpha_{\mathrm{FS}}$, the *Fisher information* $\mathcal{I}[u]$ is nonincreasing along the flow defined by (4.4). But actually we do not need to study the flow's properties, and we only use it as a guide for a complete rigorous results of Theorem 2.3. This can be done as follows. Let $u$ be a critical point of $\mathcal{I}[u]$ under the mass constraint. Then, by Proposition 4.1, taking $u[0] = u$, and assuming $\alpha \leq \alpha_{\mathrm{FS}}$,

$$0 = \mathcal{I}'[u] \cdot \mathcal{L} \, u^m = \frac{d}{dt} \, \mathcal{I}[u(t)]_{|t=0} \geq \zeta_\star \, (n-d) \int_{(0,\infty) \times S^{d-1}} |\nabla_\omega \mathsf{p}|^4 \, \mathsf{p}^{1-n} \, d\mu \, ,$$

and hence, if $\alpha \leq \alpha_{\mathrm{FS}}$, $\nabla_\omega \mathsf{p} \equiv 0$ and therefore, $u$ is radially symmetric, since it does not depend on the angular variables. The precise shape of $u$ is given by

$$\mathsf{p}'' - \frac{\mathsf{p}'}{r} - \frac{\Delta_\omega \, \mathsf{p}}{\alpha^2 \, (n-1) \, r^2} \equiv 0 \, .$$

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# BREAKING SYMMETRY IN FOCUSING NLKG EQUATION

VLADIMIR GEORGIEV AND SANDRA LUCENTE

Abstract. Let $u : \mathbb{R}_t \times \mathbb{R}^d \to \mathbb{C}$. We consider the nonlinear Klein Gordon equation

$$\begin{cases} u_{tt} - \Delta u + m^2 u = V(x)|u|^{p-1}u \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^d) \\ u_t(0,x) = u_1(x) \in L^2(\mathbb{R}^d). \end{cases} \tag{1}$$

where

$$m \neq 0\,, \quad V(x) > 0\,, \quad d \geq 3.$$

for *subcritical* $p > 1$. We will discuss the global existence and blow up of the solution according to the size of initial data with respect to the ground energy.

## 1. Introduction.

Let $u : \mathbb{R}_t \times \mathbb{R}^d \to \mathbb{C}$. We consider the nonlinear Klein Gordon equation

$$\begin{cases} u_{tt} - \Delta u + m^2 u = V(x)|u|^{p-1}u \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^d) \\ u_t(0,x) = u_1(x) \in L^2(\mathbb{R}^d). \end{cases} \tag{2}$$

where

$$m \neq 0\,, \quad V(x) > 0\,, p > 1\,, \quad d \geq 3.$$

For $m = 0$, $V(x) = 1$ and $p = 2^* = \frac{2d}{d-2}$, Kenig and Merle in [KM08] establish a dichotomy between scattering or blow up according with the size of the $\dot{H}^1$ of $u_0$ with respect to the $\dot{H}^1$ norm of the ground state of the equation.

In [IMN14] the authors consider the case $m \neq 0$, $V(x) = 1$ and $p = 2^* = \frac{2d}{d-2}$. They give a similar dichotomy result by using the ground state of the wave equation. For the case $d = 2$, they take a different nonlinear term.

Here we broke the symmetry by taking $m \neq 0$, $V(x) \neq 1$ and we aspect that the dichotomy is preserved by using the ground states of nonlinear Klein Gordon equation with potential when $p$ is a *subcritical* where the *critical* exponent depends on the singularity of $V$.

We deal with a radial potential $V(x) = V(|x|)$ and we impose the assumptions

$$V(r) \sim r^{-b} \quad \text{as } r \to \infty, \tag{3}$$

$$V(r) \sim r^{-a} \quad \text{as } r \to 0 \tag{4}$$

for some constants

$$0 < a < b < 2.$$

We require that the potential satisfies the properties

$$V \in C^1(0,\infty), \quad V > 0, \quad V \text{ strictly decreasing.} \tag{5}$$

Our next assumption seems rather technical, but it is crucial to deduce the uniqueness of ground state. More precisely we assume

$$\begin{array}{ll} \text{the function } \frac{rV'(r)}{V(r)} & \text{is a constant or strictly decreasing} \\ -b < \frac{rV'(r)}{V(r)} < -a & \text{for any } r > 0. \end{array} \tag{6}$$

Typical example is $d \geq 3$ and

$$V(x) = \frac{1}{|x|^a + |x|^b} \tag{7}$$

with $0 < a < b < 2$. Further we can notify that this potential belongs to the weak Lebesgue space

$$V \in L^{d/b,\infty} \cap L^{d/a,\infty}. \tag{8}$$

The same property holds in the general case when (3) and (4) are fulfilled.

The required Gagliardo-Nirenberg inequality is the following.

**Lemma 1.** *If $d \geq 3$, $q > 1$ and $V \in L^{q,\infty}$. Suppose in addiction that*

$$1 < p \leq 1 + \frac{4q - 2d}{q(d-2)}$$

*then there exists $C > 0$ such that for any $u \in H^1(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d} V|u|^{p+1} \leq C\|u\|_{L^2}^{\theta}\|\nabla u\|_{L^2}^{p+1-\theta}, \tag{9}$$

*where*

$$\theta = d\left(1 - \frac{1}{q}\right) - \frac{(d-2)(p+1)}{2} \in [0, p+1]$$

*In particular for $q = \frac{d}{a}$ with $0 < a < 2$, the relation (9) holds for*

$$1 < p \leq 1 + \frac{4-2a}{d-2} \quad \theta = d - a - \frac{(d-2)(p+1)}{2} \in [0, p+1]$$

**Definition 1.** *We call* GN exponent *the value*

$$p_{GN} = 1 + \frac{4-2a}{d-2}$$

*which appears in Lemma 1.*

## 2. Main results

First we prove a uniqueness result for the stationary equation.

**Lemma 2.** *If $d \geq 3$ and the assumptions (3), (4), (5) and (6) are satisfies, then for any $p > 1$, satisfying*

$$1 < p \leq p_{GN} \tag{10}$$

*and for any $\omega > 0$ the equation*

$$-\Delta Q + m^2 Q - \omega V(x) Q^p(x) = 0 \tag{11}$$

*has at most one positive, radial solution $Q(x)$, decaying exponentially to 0 as $r \to \infty$.*

Hence we consider the existence of ground states

**Theorem 1.** *Suppose that $d \geq 3$, the assumptions (3), (4), (5) and (6) are satisfies and*

$$1 < p < 1 + \frac{4-2a}{d-2}. \tag{12}$$

*Let $\mu > 0$, and let us consider the minimization problem*

$$\inf_{g \in N(\mu)} \left( \frac{1}{2} \|\nabla g\|^2_{L^2(\mathbb{R}^d)} + \frac{m^2}{2} \|g\|^2_{L^2(\mathbb{R}^d)} \right) = I(\mu), \tag{13}$$

*where $N(\mu)$ is the constraint determined by the nonlinear term*

$$N(\mu) = \left\{ g \in H^1(\mathbb{R}^d); \int_{\mathbb{R}^d} V(x)|g(x)|^{p+1} dx \geq \mu \right\}. \tag{14}$$

*For any $\mu > 0$ one can find unique $Q = Q_\mu \in N(\mu)$, radial, positive, decaying exponentially to 0 as $r \to \infty$, which solves the problem (13), that is*

$$I(\mu) = \frac{1}{2} \|\nabla Q\|^2_{L^2(\mathbb{R}^d)} + \frac{m^2}{2} \|Q\|^2_{L^2(\mathbb{R}^d)}. \tag{15}$$

*In addiction*

$$\int_{\mathbb{R}^d} V(x)|Q(x)|^{p+1} dx = \mu. \tag{16}$$

*Moreover the function $Q_\mu$ satisfies the equation*

$$-\Delta Q + m^2 Q - \omega(\mu) V(x) Q^p(x) = 0, \tag{17}$$

*where $w(1) := c_0$ and $\omega(\mu) = c_0 \mu^{p/(p+1)}$.*

*Moreover there exists a unique $\mu_0 > 0$ such that $\omega(\mu_0) = 1$ so that $Q = Q_{\mu_0} \in N(\mu_0)$ is the unique radial, positive, decaying exponentially to 0 as $r \to \infty$ solution of the equation*

$$-\Delta Q + m^2 Q - V(x) Q^p(x) = 0. \tag{18}$$

*Finally the energy functional*

$$\tilde{E}[g] = \frac{1}{2} \|\nabla g\|^2_{L^2(\mathbb{R}^d)} + \frac{m^2}{2} \|g\|^2_{L^2(\mathbb{R}^d)} - \frac{1}{p+1} \int_{\mathbb{R}^d} V(x)|g(x)|^{p+1} dx \tag{19}$$

*satisfies*

$$I(\mu_0) - \frac{\mu_0}{p+1} = \tilde{E}[Q_{\mu_0}]. \tag{20}$$

Another characterization of the "mountain"is the following link to the classical Payne-Sattinger energy critical level determined by

$$I_K = \inf_{\substack{g \in H^1 \\ K(g) \leq 0}} \left( \frac{1}{2} \|\nabla g\|^2_{L^2(\mathbb{R}^d)} + \frac{m^2}{2} \|g\|^2_{L^2(\mathbb{R}^d)} \right), \tag{21}$$

where $K(g)$ determines the first Pohozaev relation

$$K(g) = \|\nabla g\|^2_{L^2(\mathbb{R}^d)} + m^2 \|g\|^2_{L^2(\mathbb{R}^d)} - \int_{\mathbb{R}^d} V(x)|g(x)|^{p+1} dx. \tag{22}$$

We have the following result.

**Theorem 2.** *Suppose that $d \geq 3$, the assumptions (3), (4), (5) and (6) are satisfies and $p > 1$ obeys (12). Then one can find unique $Q \in H^1_{rad}(\mathbb{R}^d)$, radial, positive, decaying exponentially to 0 as $r \to \infty$ , that solves the problem (21), so that*

$$I_K = \tfrac{1}{2}\|\nabla Q\|^2_{L^2(\mathbb{R}^d)} + \tfrac{m^2}{2}\|Q\|^2_{L^2(\mathbb{R}^d)}, \tag{23}$$

$$K(Q) = 0. \tag{24}$$

*Moreover, we have*

$$Q = Q_{\mu_0}, \ \text{and } I_K = I(\mu_0), \tag{25}$$

*where $\mu_0 > 0$ is the parameter from point b), Theorem 1.*

Our next global existence result shows that in the energy subcritical case, for suitable controlled energy data, we have global solution.

**Theorem 3.** *Suppose that $d \geq 3$, and the assumptions (3), (4), (5) and (6) are satisfies. Let*

$$1 < p < p_{GN}\,.$$

*If $Q$ is the unique radial, positive, decaying exponentially to 0 as $r \to \infty$, solution $Q$ to the equation*

$$-\Delta Q + m^2 Q - V(x) Q^{p+1}(x) = 0$$

*given by Theorem 1 b), then for any initial data $(u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ with*

$$E[u](0) < \tilde{E}(Q)$$

*we have*

$$\int V(x)|u_0|^{p+1}\, dx > \mu_0$$

*and the existence of a unique global solution to the Cauchy problem (2), moreover*

$$u(t,x) \in C(\mathbb{R}_t; H^1), \ \text{s.t. } u_t(t,x) \in C(\mathbb{R}_t; \times L^2).$$

Let $(-T_-, T_+)$the maximal time existence interval for the solution $u \in C((-T_-, T_+); H^1 \times L^2)$ to the Cauchy problem (2) and shall concentrate on the energy critical case

$$\mathcal{E}[u](0) = \tilde{E}(Q). \tag{26}$$

In this case the treshold separating global existence ($T_- = T_+ = \infty$) and blow up (at least one of $T_-, T_+$ is finite) is determined by the functional $K$ defined in (22).

First, we consider the case

$$K(u_0) \geq 0. \tag{27}$$

**Theorem 4.** *Suppose that $d \geq 3$, and the assumptions (3), (4), (5) and (6) are satisfies. Let*

$$1 < p < p_{GN}\,.$$

*If $Q$ is the unique radial, positive, decaying exponentially to 0 as $r \to \infty$, solution $Q$ to the equation*

$$-\Delta Q + m^2 Q - V(x) Q^{p+1}(x) = 0$$

*given by Theorem 1 b), then for any initial data $(u_0, u_1) \in\in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ satisfying (26) and (27) we have unique global solution*

$$u(t,x) \in C(\mathbb{R}_t; H^1), \ \text{s.t. } u_t(t,x) \in C(\mathbb{R}_t; \times L^2).$$

*to the Cauchy problem* (2)*.*

If

$$K(u_0) < 0 \tag{28}$$

then we have the following blow up result.

**Theorem 5.** *Suppose that* $d \geq 3$*, and the assumptions* (3)*,* (4)*,* (5) *and* (6) *are satisfies. Let*

$$1 < p < p_{GN} .$$

*If* $Q$ *is the unique radial, positive, decaying exponentially to* $0$ *as* $r \to \infty$*, solution* $Q$ *to the equation*

$$-\Delta Q + m^2 Q - V(x) Q^{p+1}(x) = 0$$

*given by Theorem 1* b)*. If the initial data* $u_0, u_1$ *satisfy* (26) *and* (28)*, then we have at least one of the following possibilities*

- $T_+ < \infty$ *and*
$$\lim_{t \nearrow T_+} \|u_2(t)\|^2_{L^2(\mathbb{R}^d)} + \|\nabla u_1(t)\|^2_{L^2(\mathbb{R}^d)} + m^2 \|u_1(t)\|^2_{L^2(\mathbb{R}^d)} = \infty;$$
- $T_- < \infty$ *and*
$$\lim_{t \searrow -T_-} \|u_2(t)\|^2_{L^2(\mathbb{R}^d)} + \|\nabla u_1(t)\|^2_{L^2(\mathbb{R}^d)} + m^2 \|u_1(t)\|^2_{L^2(\mathbb{R}^d)} = \infty.$$

Finally we will discuss the dichotomy in the critical case $p = p_{GN}$.

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# ON THE GROWTH OF SOBOLEV NORMS FOR NLS ON 2D AND 3D COMPACT MANIFOLDS

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We consider NLS posed on a generic 2-d compact manifold $M^2$, with a general integer nonlinearity:

$$\begin{cases} i\partial_t u + \Delta_g u = |u|^{p-1}u, \quad p = 2n+1, \\ u(0,x) = \varphi \in H^m(M^2) \end{cases}$$

as well as cubic NLS on a 3-d compact manifold $M^3$

$$\begin{cases} i\partial_t u + \Delta_g u - u|u|^2 = 0, \\ u(0,x) = \varphi \in H^m(M^3). \end{cases}$$

The main point is the study of the growth of higher order Sobolev norms $H^m$ for $m > 1$ and for large times $t >> 1$. This topic has been pioneered by Bourgain in [1], in the case of cubic NLS posed on $\mathbf{T}^2$.

More precisely we prove that in the 2-d case we have polynomial growth, namely:

**Theorem 0.1.** *Let $(M^2, g)$ be a Riemanian manifold and $p = 2n+1$ for $n \geq 1$. Then for every $\epsilon > 0$, $m \in \mathbb{N}$ and for every $u(t,x)$ solution :*

$$\sup_{(0,T)} \|u(t,x)\|_{H^m(M^2)} \leq CT^{\frac{m-1}{1-2s_0}+\epsilon}, \ \forall \epsilon > 0$$

*where $s_0 \geq 0$ is such that*

$$\|e^{it\Delta_g}\varphi\|_{L^4((0,1)\times M^2)} \leq C\|\varphi\|_{H^{s_0}(M^2)}.$$

Notice that the bound does not depend on the order of the nonlinearity. At the best of our knowledge this is the first result where it is proved a polynomial bound on the growth of higher order Sobolev norms, that covers the super-cubic case in 2-d.

Concerning cubic NLS on 3-d compact manifold we show exponential growth, namely

**Theorem 0.2.** *Let $(M^3, g)$ be a Riemanian manifold. Then for every $m \in \mathbb{N}$ and for every $u(t,x)$ solution we have:*

$$\sup_{(0,T)} \|u(t,x)\|_{H^m(M^3)} \leq C\exp(CT).$$

It is worth mentioning that this result is an improvement compared with the previous result by Burq-Gérard-Tzvetkov (see [2]), who proved double exponential bound on the growth of the higher order Sobolev norms, namely

$$\sup_{(0,T)} \|u(t,x)\|_{H^m(M^3)} \leq C \exp(\exp(CT)).$$

## 1. Idea of the proof

The main novelty in our approach is the use of suitable modified energies, following the approach in Ozawa-Visciglia [3]. More precisely we introduce the following energy

$$\mathcal{E}_{2k}(u) = \|\partial_t^k u\|_{L^2(M)}^2 - \frac{p-1}{4} \int_M |\partial_t^{k-1} \nabla_g (|u|^2)|_g^2 |u|^{p-3} \mathrm{dvol}_g$$
$$- \int_M |\partial_t^{k-1}(|u|^{p-1}u)|^2 \mathrm{dvol}_g.$$

As first step one can show by using the equation solved bu $u(t,x)$ that the quantity $\mathcal{E}_{2k}(u)$ is equivalent to the Sobolev norm $H^{2k}$; as a second step one has to control the quantity

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{2k}(u(t,x)) = &-\frac{p-1}{4} \int_M |\partial_t^{k-1} \nabla_g (|u|^2)|_g^2 \partial_t (|u|^{p-3}) \mathrm{dvol}_g \\
&+ 2 \int_M \partial_t^k (|u|^{p-1}) \partial_t^{k-1} (|\nabla_g u|_g^2) \mathrm{dvol}_g \\
&+ \sum_{j=0}^{k-1} c_j \int_M (\partial_t^j \nabla_g (|u|^2), \partial_t^{k-1} \nabla_g (|u|^2)_g \partial_t^{k-j} (|u|^{p-3}) \mathrm{dvol}_g \\
&+ \mathrm{Re} \sum_{j=0}^{k-2} c_j \int_M \partial_t^k (|u|^{p-1}) \partial_t^j (\Delta_g \bar{u}) \partial_t^{k-1-j} u \mathrm{dvol}_g \\
&+ \mathrm{Im} \sum_{j=1}^{k-1} c_j \int_M \partial_t^j (|u|^{p-1}) \partial_t^{k-j} u \partial_t^k \bar{u} \mathrm{dvol}_g \\
&+ \mathrm{Re} \sum_{j=0}^{k-1} c_j \int_M \partial_t^j (|u|^{p-1}) \partial_t^{k-j} u \partial_t^{k-1} (|u|^{p-1} \bar{u}) \mathrm{dvol}_g.\end{aligned}$$

The right hand side can be estimated thanks to Strichartz estimates, more specifically one can use in $2-d$ the estimate (with $s_0$ derivative loss):

$$\|v\|_{L^4((0,1)\times M^2)} \lesssim \|\varphi\|_{H^{s_0}(M^2)} + T\|F\|_{L^\infty((0,T);H^{s_0}(M^2))}$$

where $v$ solves the linear Schroedinger equation with forcing term $F$ and initial datum $\varphi$; in $3-d$ the key estimate is the following one

$$\begin{aligned}\|v\|_{L^2((0,1);L^6(M^3))} \lesssim &\|v\|_{L^\infty((0,1);H^\epsilon(M^3))} \\
&+ \|v\|_{L^2((0,1);H^{1/2}(M^3))} + \|F\|_{L^2(0,T);L^{6/5}(M^3))}.\end{aligned}$$

Both linear estimates above have been obtained in [2].

The results are based on joint work with F. Planchon (Nice) and N. Tzvetkov (Cergy-Pontoise).

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# Strong instability of standing waves for nonlinear Schrödinger equations with a harmonic potential

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We study the instability of standing waves $e^{i\omega t}\phi_\omega(x)$ for the nonlinear Schrödinger equation with a harmonic potential

$$i\partial_t u = -\Delta u + |x|^2 u - |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1}$$

where $N \geq 1$ and $1 < p < 2^* - 1$. Here, $2^*$ is defined by $2^* = 2N/(N-2)$ if $N \geq 3$, and $2^* = \infty$ if $N = 1, 2$.

It is known that for any $\omega \in (-N, \infty)$, there exists a unique positive solution (ground state) $\phi_\omega(x)$ of the stationary problem

$$-\Delta\phi + |x|^2\phi + \omega\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^N \tag{2}$$

in the energy space

$$X := \{v \in H^1(\mathbb{R}^N) : |x|v \in L^2(\mathbb{R}^N)\}.$$

Note that the condition $\omega > -N$ appears naturally in the existence of positive solutions for (2), because the first eigenvalue of $-\Delta + |x|^2$ is $N$. For the uniqueness of positive solutions for (2), see [7, 8, 9, 12].

The Cauchy problem for (1) is locally well-posed in the energy space $X$ (see [3, §9.2] and [11]). That is, for any $u_0 \in X$ there exist $T_{\max} = T_{\max}(u_0) \in (0, \infty]$ and a unique solution $u \in C([0, T_{\max}), X)$ of (1) with initial condition $u(0) = u_0$ such that either $T_{\max} = \infty$ (global existence) or $T_{\max} < \infty$ and $\lim_{t \to T_{\max}} \|u(t)\|_X = \infty$ (finite time blowup). Moreover, the solution $u(t)$ satisfies the conservations of charge and energy

$$\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2, \quad E(u(t)) = E(u_0) \tag{3}$$

for all $t \in [0, T_{\max})$, where the energy $E$ is defined by

$$E(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{1}{2}\|xv\|_{L^2}^2 - \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1}. \tag{4}$$

Here we give the definitions of stability and instability of standing waves.

**Definition 1.** We say that the standing wave solution $e^{i\omega t}\phi_\omega$ of (1) is *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in X$ and $\|u_0 - \phi_\omega\|_X < \delta$, then the solution $u(t)$ of (1) with $u(0) = u_0$ exists globally and satisfies

$$\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}} \|u(t) - e^{i\theta}\phi_\omega\|_X < \varepsilon.$$

Otherwise, $e^{i\omega t}\phi_\omega$ is said to be *unstable*.

**Definition 2.** We say that $e^{i\omega t}\phi_\omega$ is *strongly unstable* if for any $\varepsilon > 0$ there exists $u_0 \in X$ such that $\|u_0 - \phi_\omega\|_X < \varepsilon$ and the solution $u(t)$ of (1) with $u(0) = u_0$ blows up in finite time.

Before we state our main result, we recall some known results on the stability and instability of standing waves $e^{i\omega t}\phi_\omega$ for (1). When $\omega$ is sufficiently close to $-N$, the standing wave solution $e^{i\omega t}\phi_\omega$ of (1) is stable for any $p \in (1, 2^* - 1)$ (see [5]). On the other hand, when $\omega$ is sufficiently large, the standing wave solution $e^{i\omega t}\phi_\omega$ of (1) is stable if $1 < p \leq 1 + 4/N$ (see [5, 4]), and unstable if $1 + 4/N < p < 2^* - 1$ (see [6]). More precisely, it is proved in [6] that $e^{i\omega t}\phi_\omega$ is unstable if $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$, where $v^\lambda(x) = \lambda^{N/2}v(\lambda x)$ is the $L^2$-invariant scaling.

However, the strong instability of $e^{i\omega t}\phi_\omega$ has been unknown for (1), although there are some results on blowup (see, e.g., [2, 14]).

Now we state our main result in this paper.

**Theorem 1.** *Let $N \geq 1$, $1 + 4/N < p < 2^* - 1$, $\omega > -N$, and let $\phi_\omega$ be the positive solution of* (2)*. Assume that $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} \leq 0$. Then, the standing wave solution $e^{i\omega t}\phi_\omega$ of* (1) *is strongly unstable.*

We remark that by the scaling $v^\lambda(x) = \lambda^{N/2}v(\lambda x)$ for $\lambda > 0$, we have $\|v^\lambda\|_{L^2}^2 = \|v\|_{L^2}^2$ and

$$E(v^\lambda) = \frac{\lambda^2}{2}\|\nabla v\|_{L^2}^2 + \frac{\lambda^{-2}}{2}\|xv\|_{L^2}^2 - \frac{\lambda^\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1}. \tag{5}$$

Here and hereafter, we put

$$\alpha := \frac{N}{2}(p-1) > 2.$$

Moreover, we define $S_\omega(v) = E(v) + \frac{\omega}{2}\|v\|_{L^2}^2$ for $v \in X$. Then, $\phi_\omega$ satisfies $S'_\omega(\phi_\omega) = 0$, and

$$\begin{aligned} 0 = \partial_\lambda S_\omega(\phi_\omega^\lambda)\big|_{\lambda=1} &= \|\nabla\phi_\omega\|_{L^2}^2 - \|x\phi_\omega\|_{L^2}^2 - \frac{\alpha}{p+1}\|\phi_\omega\|_{L^{p+1}}^{p+1}, \\ \partial_\lambda^2 E(\phi_\omega^\lambda)\big|_{\lambda=1} &= \|\nabla\phi_\omega\|_{L^2}^2 + 3\|x\phi_\omega\|_{L^2}^2 - \frac{\alpha(\alpha-1)}{p+1}\|\phi_\omega\|_{L^{p+1}}^{p+1} \\ &= 4\|x\phi_\omega\|_{L^2}^2 - \frac{\alpha(\alpha-2)}{p+1}\|\phi_\omega\|_{L^{p+1}}^{p+1}. \end{aligned}$$

Thus, the condition $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} \le 0$ is equivalent to

$$\frac{\|x\phi_\omega\|_{L^2}^2}{\|\phi_\omega\|_{L^{p+1}}^{p+1}} \le \frac{\alpha(\alpha-2)}{4(p+1)}.$$

Furthermore, it is proved in Section 2 of [6] that

$$\lim_{\omega\to\infty} \frac{\|x\phi_\omega\|_{L^2}^2}{\|\phi_\omega\|_{L^{p+1}}^{p+1}} = 0.$$

Therefore, as a corollary of Theorem 1, we have the following.

**Corollary 2.** *Let $N \ge 1$, $1 + 4/N < p < 2^* - 1$, $\omega > -N$, and let $\phi_\omega$ be the positive solution of* (2). *Then, there exists $\omega_0 \in (-N, \infty)$ depending only on $N$ and $p$ such that the standing wave solution $e^{i\omega t}\phi_\omega$ of* (1) *is strongly unstable for all $\omega \in (\omega_0, \infty)$.*

# Proof of Theorem 1

In what follows, we assume that $1 + 4/N < p < 2^* - 1$, $\omega > -N$, and $\phi_\omega$ is the positive solution of (2). We put $\alpha = N(p-1)/2 > 2$.

The proofs of blowup and strong instability of standing waves for nonlinear Schrödinger equations rely on the virial identity (see, e.g., [1, 3, 10, 13]). Let $u(t)$ be the solution of (1) with $u(0) = u_0 \in X$. Then, the function $t \mapsto \|xu(t)\|_{L^2}^2$ is in $C^2[0, T_{\max})$, and satisfies

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 16P(u(t)) \tag{6}$$

for all $t \in [0, T_{\max})$, where

$$P(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{1}{2}\|xv\|_{L^2}^2 - \frac{\alpha}{2(p+1)}\|v\|_{L^{p+1}}^{p+1}. \tag{7}$$

Moreover, we define

$$R(v) = \|\nabla v\|_{L^2}^2 + 3\|xv\|_{L^2}^2 - \frac{\alpha(\alpha-1)}{p+1}\|v\|_{L^{p+1}}^{p+1}. \tag{8}$$

Note that by (5), we have

$$P(v^\lambda) = \frac{1}{2}\lambda\partial_\lambda E(v^\lambda), \quad R(v^\lambda) = \lambda^2\partial_\lambda^2 E(v^\lambda) \tag{9}$$

for $\lambda > 0$. We also define

$$\mathcal{A}_\omega = \left\{v \in X : E(v) < E(\phi_\omega),\ \|v\|_{L^2}^2 = \|\phi_\omega\|_{L^2}^2,\ \|v\|_{L^{p+1}}^{p+1} > \|\phi_\omega\|_{L^{p+1}}^{p+1}\right\},$$
$$\mathcal{B}_\omega = \{v \in \mathcal{A}_\omega : P(v) < 0\}.$$

**Lemma 1.** *Assume that $R(\phi_\omega) \le 0$. Then, $\phi_\omega^\lambda \in \mathcal{B}_\omega$ for all $\lambda > 1$.*

Since $\phi_\omega^\lambda \to \phi_\omega$ in $X$ as $\lambda \to 1$, Theorem 1 follows from Lemma 1 and the following Theorem 3.

**Theorem 3.** *Let $N \ge 1$, $1 + 4/N < p < 2^* - 1$, $\omega > -N$, and assume that $R(\phi_\omega) \le 0$. If $u_0 \in \mathcal{B}_\omega$, then the solution $u(t)$ of (1) with $u(0) = u_0$ blows up in finite time.*

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# On spectral stability of the nonlinear Dirac equation

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## 1 Introduction

The question of stability of solitary waves is answered in many cases for the nonlinear Schrödinger, Klein–Gordon, and Korteweg–de Vries equations (see e.g. the review [Str89]). In these systems, at the points represented by solitary waves, the hamiltonian function is of finite Morse index. In simpler cases, the Morse index is equal to one, and the perturbations in the corresponding direction are prohibited by conservation law when the Vakhitov–Kolokolov condition [VK73] is satisfied. In other words, the solitary waves could be demonstrated to correspond to conditional minimizers of the energy under the charge constraint; this results not only in spectral stability but also in orbital stability [CL82, GSS87]. The nature of stability of solitary wave solutions of the nonlinear Dirac equation seems completely different from the picture. The Hamiltonian function is not bounded from below, and is of infinite Morse index; the NLS-type approach to stability fails. As a consequence, we do not know how to prove the *orbital stability* [CL82, GSS87] but via proving the asymptotic stability first. The asymptotic stability, in turn, is based on knowing the spectrum of the linearized problem; this will be our main focus.

Given a real-valued function $f \in C^1(\mathbb{R} \setminus \{0\})$, $f(0) = 0$, we consider the nonlinear Dirac equation in $\mathbb{R}^n$, $n \geq 1$, which is known as the Soler model [Sol70]:

$$\mathrm{i}\partial_t\psi = D_m\psi - f(\psi^*\beta\psi)\beta\psi, \qquad \psi(x,t) \in \mathbb{C}^N, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $D_m = -\mathrm{i}\boldsymbol{\alpha} \cdot \nabla + \beta m$ is the free Dirac operator. Here $\boldsymbol{\alpha} = (\alpha^\jmath)_{1\leq \jmath \leq n}$, with $\alpha^\jmath$ and $\beta$ the self-adjoint $N \times N$ Dirac matrices; $m > 0$ is the mass. We are interested in the stability properties of solitary wave solutions to (1.1):

$$\psi(x,t) = \phi_\omega(x)e^{-\mathrm{i}\omega t}, \qquad \phi_\omega \in H^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{C}^N). \tag{1.2}$$

Given a particular solitary wave (1.2), we consider its perturbation in the form of the Ansatz

$$\psi(x,t) = (\phi_\omega(x) + \rho(x,t))e^{-\mathrm{i}\omega t}, \tag{1.3}$$

and study the spectrum of the linearized equation on $\rho$. We will say that a particular solitary wave is *spectrally stable* if the spectrum of the equation linearized at this wave does not contain eigenvalues with positive real part. The spectrum of the linearization at solitary waves of the cubic nonlinear Dirac equation in (1+1)D was computed numerically in [BC12a], suggesting spectral stability of all solitary waves in that model.

Let us mention that the linear instability in the nonrelativistic limit $\omega \lesssim m$ in the cases $k > 2/n$ follows from [CGG14]. We notice, though, that our numerics show that this spectral instability disappears when $\omega \in (0, m)$ becomes sufficiently small [CMKS$^+$16]; this is a particular feature of the nonlinear Dirac equations which is absent in the NLS case. Moreover, we note that quintic nonlinear Schrödinger equation in (1+1)D and the cubic one in (2+1)D are "charge critical" (all solitary waves have the same charge), and as a consequence the linearization at any solitary wave has a $4 \times 4$ Jordan block at $\lambda = 0$, resulting in dynamic instability of all solitary waves. On the contrary, for the nonlinear Dirac with the critical-power nonlinearity, the charge of solitary waves is no longer the same, and small amplitude solitary waves are spectrally stable. In all these cases, our conclusion is that in the nonrelativistic limit $\omega \lesssim m$ the spectral stability of a solitary wave solution (1.2) to the nonlinear Dirac equation (1.1) is formally described by the Vakhitov–Kolokolov stability criterion $\partial_\omega Q(\phi_\omega) < 0$, where $Q(\phi_\omega) = \int_{\mathbb{R}^n} |\phi_\omega(x)|^2\,dx$ is the corresponding charge.

Below is the short summary of results from [BC12b, BC16a, BC16b]. Let $\phi_\omega(x)e^{-\mathrm{i}\omega t}$ be a solitary wave solution to (1.1). The linearization at this solitary wave is the linearized equation on $\rho$ from (1.3):

$$\mathrm{i}\partial_t \rho = \mathcal{L}(\omega)\rho, \tag{1.4}$$

where

$$\mathcal{L}(\omega) = D_m - \omega - f(\phi_\omega^*\beta\phi_\omega)\beta - 2f'(\phi_\omega^*\beta\phi_\omega)\beta\phi_\omega \operatorname{Re}(\phi_\omega^*\beta\ \cdot). \tag{1.5}$$

The operator $\mathcal{L}(\omega)$ is not $\mathbb{C}$-linear because of the term with $\operatorname{Re}(\phi_\omega^*\beta\ \cdot)$. To work with $\mathbb{C}$-linear operators, one could consider this operator as acting on $(\operatorname{Re}\rho, \operatorname{Im}\rho)$ and then to study its complexification.

In this spirit, we proceed as follows. Let $J \in \operatorname{End}(\mathbb{C}^N)$ be skew-adjoint and invertible, such that $J^2 = -I_{\mathbb{C}^N}$, $[J, D_m] = 0$. We denote

$$L(\omega) = D_m - \omega + V(x, \omega), \qquad \omega \in [-m, m],$$

where $V(\cdot, \omega) \in L^\infty\big(\mathbb{R}^n, \operatorname{End}(\mathbb{C}^N)\big)$ for each $\omega \in [-m, m]$. We will study the spectrum of $JL(\omega)$, where $J$ does not necessarily commute with $V$; this parallels the absence of $\mathbb{C}$-linearity in (1.5) (that is, $JL$ represents the operator $-\mathrm{i}\mathcal{L}$ from (1.4) acting on $(\operatorname{Re}\rho, \operatorname{Im}\rho)$).

**Theorem 1.1** (Absence of embedded eigenvalues)**.** *Let $n \ge 1$, $\omega \in [-m, m]$, and $V \in L^n_{\mathrm{loc}}(\mathbb{R}^n, \operatorname{End}(\mathbb{C}^N))$. Let $\lambda \in \mathbb{R} \setminus [-m - |\omega|, m + |\omega|]$ and assume that there are $\kappa \in (0, 1]$ and $R < \infty$ such that*

$$\|(n + 16\Lambda_+^2 r^2 + 8r\tau)^{1/2} V v\|^2 \le \kappa^2 \left(\|\nabla v\|^2 + \|(\Lambda_-^2 - m^2 + \tau^2)^{1/2} v\|^2\right), \tag{1.6}$$

*for all $\tau \ge 1$ and $v \in H_0^1(\Omega_R, \mathbb{C}^N)$, with $\Lambda_\pm = |\lambda| \pm |\omega|$. Then $\pm\mathrm{i}\lambda \notin \sigma_{\mathrm{p}}(JL(\omega))$.*

**Theorem 1.2** (Bifurcation of point eigenvalues)**.** *Let $n \ge 1$. Let $(\omega_j)_{j\in\mathbb{N}}$, $\omega_j \in [-m, m]$, be a sequence with $\lim_{j\to\infty} \omega_j = \omega_0 \in [-m, m]$, and assume that $V$ is hermitian and that there is $\varepsilon > 0$ such that*

$$\|\langle r\rangle^{1+\varepsilon} V(\omega_0)\|_{L^\infty(\mathbb{R}^n, \operatorname{End}(\mathbb{C}^N))} < \infty, \qquad \lim_{j\to\infty} \|\langle r\rangle^{1+\varepsilon} \left(V(\omega_j) - V(\omega_0)\right)\|_{L^\infty(\mathbb{R}^n, \operatorname{End}(\mathbb{C}^N))} = 0,$$

*where $\|\langle r\rangle^{1+\varepsilon} V(\omega)\| = \|\langle\cdot\rangle^{1+\varepsilon} V(\cdot, \omega)\|$. Let $\lambda_j \in \sigma_{\mathrm{p}}(JL(\omega_j))$, $j \in \mathbb{N}$ be a sequence such that $\operatorname{Re}\lambda_j \ne 0$, $\lambda_j \underset{j\to\infty}{\longrightarrow} \lambda_0 \in \mathrm{i}\mathbb{R}$, and $\lambda_0 \ne \pm\mathrm{i}(m + |\omega_0|)$. If $\omega_0 = \pm m$, additionally assume that $\lambda_0 \ne 0$. Then $\lambda_0 \in \sigma_{\mathrm{p}}(JL(\omega_0))$.*

These two theorems are based on Jensen–Kato theory [JK79] adapted for the Dirac operator and on the Carleman estimates from [BG87]. Combining Theorem 1.1 and Theorem 1.2, we conclude that for the linearizations at solitary waves the bifurcations of point eigenvalues from the continuous spectrum beyond the embedded thresholds $\pm\mathrm{i}(m + |\omega|)$ are not possible.

**Theorem 1.3** (Bifurcation of point eigenvalues from the spectrum of the free Dirac operator)**.** *Let $n \ge 1$. Let $J \in \operatorname{End}(\mathbb{C}^N)$ be skew-adjoint and invertible, $\sigma(J) = \{\pm\mathrm{i}\}$, with $[J, D_m] = 0$. Let $(\omega_j)_{j\in\mathbb{N}}$, $\omega_j \in (-m, m)$, be a Cauchy sequence, $\omega_j \to m$, and assume that there is $\delta > 0$ such that*

$$\lim_{j\to\infty} \|\langle r\rangle^{1+\delta} V(\cdot, \omega_j)\|_{L^\infty(\mathbb{R}^n, \operatorname{End}(\mathbb{C}^N))} = 0. \tag{1.7}$$

*Let $\lambda_j \in \sigma_{\mathrm{p}}(JL(\omega_j))$, and let $\lambda_0 \in \mathrm{i}\mathbb{R} \cup \{\infty\}$ be an accumulation point of the sequence $(\lambda_j)_{j\in\mathbb{N}}$. Then:*

*1.* $\lambda_0 \in \{0; \pm 2m\mathrm{i}\}$. *In particular,* $\lambda_0 \neq \infty$.

*2. If additionally* $\operatorname{Re}\lambda_j \neq 0$, $\lambda_j \to \lambda_0 = 0$, *the potential* $V(x, \omega_j)$ *is hermitian (for all* $x \in \mathbb{R}^n$ *and* $j \in \mathbb{N}$*) and satisfies*

$$\|V(\cdot, \omega_j)\|_{L^\infty(\mathbb{R}^n, \operatorname{End}(\mathbb{C}^N))} \leq C(m - \omega_j), \tag{1.8}$$

*then* $\lambda_j = O(m^2 - \omega_j^2)$.

Now we consider the nonlinear Dirac equation (1.1). We assume that $f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$, and that there are $k > 0$, $K > k$ such that

$$|f(\tau) - |\tau|^k| = o(|\tau|^K), \qquad |\tau f'(\tau) - k|\tau|^k| = o(|\tau|^K), \qquad |\tau| \leq 1. \tag{1.9}$$

If $n \geq 3$, we additionally assume that $k < 2/(n-2)$.

In the nonrelativistic limit $\omega \lesssim m$, the solitary wave solutions $\phi_\omega(x)e^{-\mathrm{i}\omega t}$ to nonlinear Dirac equation could be obtained as bifurcations of the solitary wave solutions $\varphi_\omega(x)e^{-\mathrm{i}\omega t}$ to the nonlinear Schrödinger equation. We recall that, by [Str77, BL83, BGK83], the stationary nonlinear Schrödinger equation

$$-u = -\Delta u - |u|^{2k} u, \qquad u(x) \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad n \geq 1 \tag{1.10}$$

has a strictly positive spherically symmetric exponentially decaying solution $u_k \in C^2(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$ (called the ground state) if and only if $0 < k < 2/(n-2)$ (any $k > 0$ if $n \leq 2$); in the next theorem, Dirac solitary wave profiles $\phi_\omega$ are obtained as bifurcations from $u_k$. The main difficulty of the proof is insufficient regularity of $f(\tau) = |\tau|^k + \dots$ near $\tau = 0$ when $k \leq 1$. As a result, the contraction mapping theorem can not be used; in [BC16a], we base the argument on the Schauder fixed point theorem.

**Theorem 1.4** (Solitary waves in the nonrelativistic limit)**.** *In* (1.1)*, let* $n \in \mathbb{N}$.

*1. There is* $\omega_0 \in (m/2, m)$ *such that there is a* $C^1$ *map* $\omega \mapsto \phi_\omega \in H^1(\mathbb{R}^n, \mathbb{C}^N)$*, with* $\partial_\omega \phi_\omega \in H^1(\mathbb{R}^n, \mathbb{C}^N)$*. One has* $\|e^{|x|\sqrt{m^2-\omega^2}} \phi_\omega\|_{L^\infty(\mathbb{R}^n, \mathbb{C}^N)} = O\big((m^2 - \omega^2)^{\frac{1}{2k}}\big)$*, and moreover*

$$\phi_\omega(x)^* \beta \phi_\omega(x) \geq |\phi_\omega(x)|^2/2, \qquad \forall x \in \mathbb{R}^n, \qquad \forall \omega \in (\omega_0, m). \tag{1.11}$$

*2. Additionally, assume that either* $k < 2/n$, $K > k$*, or* $k = 2/n$, $K > 4/n$*. Then there is* $\omega_1 < m$ *such that* $\partial_\omega Q(\omega) < 0$ *for all* $\omega \in (\omega_1, m)$*. If instead* $k > 2/n$*, then there is* $\omega_1 < m$ *such that* $\partial_\omega Q(\omega) > 0$ *for all* $\omega \in (\omega_1, m)$.

**Theorem 1.5** (Spectral stability of solitary waves in the nonrelativistic limit)**.** *In* (1.1)*, let* $n \leq 3$, $N \leq 4$*. Let* $f \in C^1(\mathbb{R}\setminus\{0\}) \cap C(\mathbb{R})$ *satisfy the assumption* (1.9)*, with* $k$, $K$ *such that either* $k < 2/n$, $K > k$*, or* $k = 2/n$, $K > 4/n$*. Further, assume that* $k > k_n$*, with* $k_1 = 1$, $k_2 \approx 0.621$, $k_3 \approx 0.461$ *(this is a technical assumption which guarantees that the linearization of the corresponding NLS at a solitary wave has a particularly simple spectrum, making the analysis more straightforward). Let* $\phi_\omega(x)e^{-\mathrm{i}\omega t}$, $\phi_\omega \in H^2(\mathbb{R}^n, \mathbb{C}^N)$, $\omega \lesssim m$*, be a family of solitary waves from Theorem 1.4. Then there is* $\omega_* \in (0, m)$ *such that for each* $\omega \in (\omega_*, m)$ *the corresponding solitary wave is spectrally stable.*

We note that if either $k < 2/n$, $K > k$ or $k = 2/n$, $K > 4/n$, then, by Theorem 1.4 *(2)*, for $\omega \lesssim m$ one has $\partial_\omega Q(\phi_\omega) < 0$; thus, the spectral stability in the nonrelativistic limit is formally described by the Vakhitov–Kolokolov stability criterion [VK73].

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# DECAY ESTIMATES FOR WAVE EQUATION WITH A POTENTIAL ON EXTERIOR DOMAINS

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This talk is based on the joint work with Professor Vladimir Georgiev.

Let $\Omega$ be an exterior domain in $\mathbb{R}^3$ such that the obstacle

$$\mathcal{O} := \mathbb{R}^3 \setminus \Omega$$

is compact and its boundary $\partial\Omega$ is of $C^{2,1}$. For the sake of simplicity, we assume that the origin does not belong to $\overline{\Omega}$. In this talk we consider the initial-boundary value problem for the wave equations with a potential in the exterior domain $\Omega$ and our main goals are to study the local energy decay estimates and dispersive estimates for the corresponding evolution flow.

We are concerned with the following initial-boundary value problem, for a function $u = u(t, x)$:

$$\partial_t^2 u - \Delta u + V(x)u = F(t, x), \quad t \neq 0, \quad x \in \Omega, \tag{0.1}$$

with the initial condition

$$u(0, x) = f(x), \quad \partial_t u(0, x) = g(x), \tag{0.2}$$

and the boundary condition

$$u(t, x) = 0, \quad t \in \mathbb{R}, \quad x \in \partial\Omega, \tag{0.3}$$

where $V$ is a real-valued measurable function on $\Omega$ satisfying

$$- c_0|x|^{-\delta_0} \leq V(x) \leq c_1|x|^{-\delta_0} \quad \text{for some } 0 < c_0 < \frac{1}{4},\ c_1 > 0 \text{ and } \delta_0 > 2. \tag{0.4}$$

Let us introduce some operators and function spaces. We denote by

$$\mathbb{G}_0 = -\Delta \text{ the free Hamiltonian in } \mathbb{R}^3,$$

and by

$$\mathbb{G}_V \text{ a self-adjoint realization on } L^2(\Omega) \text{ of the operator } -\Delta_{|D} + V,$$

where

$$\mathbb{G} := -\Delta_{|D} \text{ is the Dirichlet Laplacian}$$

with domain

$$D(\mathbb{G}) = D(\mathbb{G}_V) = H^2(\Omega) \cap H_0^1(\Omega).$$

Then $\mathbb{G}_V$ is non-negative on $L^2(\Omega)$ on account of (0.4), and it is shown that zero is neither an eigenvalue nor a resonance of $\mathbb{G}_V$. Also, it is known that no eigenvalues are present on $(0, \infty)$ (see Mochizuki [5], and also (0.12) below). Hence the continuous

*Key words and phrases.* Wave equation, local energy decay, dispersive estimates.

spectrum of $\mathbb{G}_V$ coincides with the interval $[0,\infty)$. The main theorem involves the perturbed Besov spaces $\dot{B}^s_{p,q}(\mathbb{G}_V)$ generated by $\mathbb{G}_V$. Following Iwabuchi, Matsuyama and Taniguchi [3], we define these spaces in the following way. Let $\{\varphi_j(\lambda)\}_{j=-\infty}^{\infty}$ be the Littlewood-Paley partition of unity: $\varphi(\lambda)$ is a non-negative function having its compact support in $\{\lambda : 1/2 \le \lambda \le 2\}$ such that

$$\sum_{j=-\infty}^{+\infty} \varphi(2^{-j}\lambda) = 1 \quad (\lambda \neq 0), \quad \varphi_j(\lambda) = \varphi(2^{-j}\lambda), \quad (j \in \mathbb{Z}).$$

For any $s \in \mathbb{R}$ and $1 \le p, q \le \infty$ we define the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{G}_V)$ by letting

$$\dot{B}^s_{p,q}(\mathbb{G}_V) := \{f \in \mathcal{Z}'_V(\Omega) \, : \, \|f\|_{\dot{B}^s_{p,q}(\mathbb{G}_V)} < \infty\}, \tag{0.5}$$

where

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{G}_V)} := \left\| \left\{ 2^{sj} \|\varphi_j(\sqrt{\mathbb{G}_V}) f\|_{L^p(\Omega)} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})}.$$

Here $\mathcal{Z}'_V(\Omega)$ is the dual space of a linear topological space $\mathcal{Z}_V(\Omega)$ which is defined by letting

$$\mathcal{Z}_V(\Omega) := \Big\{ f \in L^1(\Omega) \cap D(\mathbb{G}_V) \, : \, \mathbb{G}_V^M f \in L^1(\Omega) \cap D(\mathbb{G}_V) \quad \text{and} \\ \sup_{j\le 0} 2^{M|j|} \|\varphi_j(\sqrt{\mathbb{G}_V}) f\|_{L^1(\Omega)} < \infty \text{ for all } M \in \mathbb{N} \Big\}$$

equipped with the family of semi-norms $\{q_{V,M}(\cdot)\}_{M=1}^{\infty}$ given by

$$q_{V,M}(f) := \|f\|_{L^1(\Omega)} + \sup_{j\in\mathbb{Z}} 2^{M|j|} \|\varphi_j(\sqrt{\mathbb{G}_V}) f\|_{L^1(\Omega)}.$$

It is proved in Theorem 2.5 from [3] that the norms $\|f\|_{\dot{B}^s_{p,q}(\mathbb{G}_V)}$ are independent of the choice of $\varphi_j$. We shall also use the perturbed Sobolev spaces over $\Omega$:

$$\dot{H}^s_V(\Omega) := \dot{B}^s_{2,2}(\mathbb{G}_V).$$

In particular case $V = 0$, replacing $\varphi_j(\sqrt{\mathbb{G}_V})$ by $\varphi_j(\sqrt{\mathbb{G}})$ in the definition, we define

$$\dot{B}^s_{p,q}(\mathbb{G}) \quad \text{and} \quad \dot{H}^s(\Omega) = \dot{B}^s_{2,2}(\mathbb{G}),$$

where we recall

$$\mathbb{G} = -\Delta_{|D}$$

with domain

$$D(\mathbb{G}) = H^2(\Omega) \cap H^1_0(\Omega).$$

We shall use the inhomogeneous Sobolev spaces $H^s_V(\Omega)$ for $s > 0$. We say that $f \in H^s_V(\Omega)$ ($f \in H^s(\Omega)$ resp.) for $s > 0$ if

$$\left\|(I + \mathbb{G}_V)^{s/2} f\right\|_{L^2(\Omega)} < \infty \quad \left(\left\|(I + \mathbb{G})^{s/2} f\right\|_{L^2(\Omega)} < \infty \text{ resp.}\right)$$

Local energy for wave equations is defined by letting

$$E_R(u)(t) = \int_{\Omega\cap\{|x|\le R\}} \left\{ |\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2 \right\} dx,$$

where, here and below, $R > 0$ is chosen such that

$$\mathcal{O} = \mathbb{R}^3 \setminus \Omega \subseteq \{|x| \leq R\}.$$

The result due to Ralston [9] concerns the case that

$$\mathcal{O} \text{ is a compact and trapping obstacle,}$$

and his result asserts that, given any $\mu \in (0,1)$ and any $T > 0$, there exist $f, g \in C_0^\infty(\Omega)$ with

$$\int_\Omega \left\{ |\nabla f(x)|^2 + |g(x)|^2 \right\} dx = 1$$

such that the solution to the initial-boundary value problem

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & t \neq 0, \quad x \in \Omega, \\ u(t,x) = 0, & t \in \mathbb{R}, \quad x \in \partial\Omega, \\ u(0,x) = f(x), \quad \partial_t u(0,x) = g(x), & x \in \Omega \end{cases}$$

satisfies the inequality

$$E_R(u)(T) \geq 1 - \mu.$$

On the other hand, the scattering theory developed by Lax and Phillips gives a construction of the scattering operator by using weaker form of local energy decay

$$\liminf_{t \to \infty} E_R(u)(t) = 0. \tag{0.6}$$

Note that (0.6) follows directly from the RAGE (or simply ergodic type) theorem

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T E_R(u)(t) dt = 0 \tag{0.7}$$

and the property that zero is not eigenvalue of $\mathbb{G}$, i.e.,

$$u \in D(\mathbb{G}), \quad \mathbb{G}u = 0 \quad \Longrightarrow u = 0.$$

An important consequence of weak energy decay (0.7) is the existence of the wave operators

$$W_\mp := s - \lim_{t \to \pm\infty} e^{\mathrm{i}t\sqrt{\mathbb{G}}} J_0 e^{-\mathrm{i}t\sqrt{\mathbb{G}_0}},$$

where $J_0$ is the orthogonal projection

$$J_0 : L^2(\mathbb{R}^3) \to L^2(\Omega).$$

This observation implies that scattering theory and existence of wave operators are established without appealing to additional geometric assumption of type

$$\mathcal{O} = \mathbb{R}^3 \setminus \Omega \text{ is non-trapping obstacle.} \tag{0.8}$$

The condition (0.8) is crucial for the strong local energy decay in view of the results of Morawetz, Ralston and Strauss [8] and Ralston [9].

Our main decay estimates (0.9)–(0.11) below are obtained also without appealing to assumption (0.8).

We shall prove the following:

**Theorem 0.1.** *Assume that the measurable potential $V$ satisfies* (0.4). *Let $\sigma \geq 2$. If $f, g \in C_0^\infty(\Omega)$ and $R > 0$ is such that*

$$\mathcal{O} \subseteq \{|x| \leq R\},$$

*then the solution $u$ to the initial-boundary value problem* (0.1)–(0.3) *with $F \equiv 0$ satisfies the estimate*

$$E_R(u)(t) \leq \frac{C}{t^2}\left(\|f\|^2_{H_V^{2\sigma+1}(\Omega)} + \|g\|^2_{H_V^{2\sigma}(\Omega)}\right) \tag{0.9}$$

*for any $t \neq 0$.*

Interpolation between (0.9) and standard energy estimate

$$E_R(u)(t) \leq C\left(\|f\|^2_{H^1(\Omega)} + \|g\|^2_{L^2(\Omega)}\right)$$

gives the following:

**Corollary 0.2.** *Assume that the measurable potential $V$ satisfies* (0.4). *If $f, g \in C_0^\infty(\Omega)$ and $R > 0$ is such that*

$$\mathcal{O} \subseteq \{|x| \leq R\},$$

*then for any $k \in (0, 1]$, the solution $u$ to the initial-boundary value problem* (0.1)–(0.3) *with $F \equiv 0$ satisfies the estimate*

$$E_R(u)(t) \leq \frac{C}{t^{k/2}}\left(\|f\|^2_{H^{k+1}(\Omega)} + \|g\|^2_{H^k(\Omega)}\right) \tag{0.10}$$

*for any $t \neq 0$.*

**Remark 0.1.** If $V = 0$, then we are able to prove (0.10) for any $k > 0$. In the case of presence of potential satisfying (0.4), we use the fact that

$$D(\mathbb{G}_V^{s/2}) = D(\mathbb{G}^{s/2}), \quad \|f\|_{H_V^s(\Omega)} \sim \|f\|_{H^s(\Omega)}, \quad f \in D(\mathbb{G}^{s/2})$$

for any $s \in [0, 2]$. Therefore we need the restriction $0 < k \leq 1$ in Corollary 0.2, when there is a potential.

**Remark 0.2.** It should be mentioned that the estimate (0.10) is slightly better local energy decay estimate compared with the estimate

$$E_R(u)(t) \leq \frac{C}{\log(2+t)^{2k}}\left(\|f\|^2_{H^{k+1}(\Omega)} + \|g\|^2_{H^k(\Omega)}\right),$$

which is obtained by Burq (see [1]).

The second result is concerned with $L^p$-$L^{p'}$-estimates:

**Theorem 0.3.** *Let $1 \leq p' \leq 2 \leq p \leq \infty$ and $1/p + 1/p' = 1$. Suppose that the measurable potential $V$ satisfies* (0.4). *Then there exists a constant $C > 0$ such that*

$$\left\|\left(\sqrt{\mathbb{G}_V}\right)^{-1} e^{it\sqrt{\mathbb{G}_V}} g\right\|_{\dot{B}^{-(1/2)+(2/p)}_{p,2}(\mathbb{G}_V)} \leq C|t|^{-1+(2/p)}\|g\|_{\dot{B}^{(1/2)-(2/p)}_{p',2}(\mathbb{G}_V)} \tag{0.11}$$

*for any $g \in \dot{B}^{(1/2)-(2/p)}_{p',2}(\mathbb{G}_V)$ and any $t \neq 0$.*

The strategy of proof of Theorems 0.1 and 0.3 is based on the spectral representation of an operator $\varphi(\sqrt{\mathbb{G}_V})$. More precisely, we shall use the identity:

$$\varphi(\sqrt{\mathbb{G}_V}) = \frac{1}{\pi \mathrm{i}} \int_0^\infty \varphi(\lambda) \left[ R_V(\lambda^2 + \mathrm{i}0) - R_V(\lambda^2 - \mathrm{i}0) \right] \lambda \, d\lambda,$$

where

$$\varphi(\lambda) = \varphi_j(\lambda) \lambda^{-1} e^{\mathrm{i}\lambda t},$$

and $R_V(\lambda^2 \pm \mathrm{i}0)$ are the operators induced by the resolvent operator

$$R_V(z) = (\mathbb{G}_V - z)^{-1} \quad \text{for } z \in \mathbb{C},$$

whose existence is assured by the limiting absorption principle (see Mochizuki [5, 6, 7]): Let $\delta_0 > 1$. Then there exist the limits

$$\text{s}-\lim_{\varepsilon \searrow 0} R_V(\lambda^2 \pm \mathrm{i}\varepsilon) = R_V(\lambda^2 \pm \mathrm{i}0) \quad \text{in } \mathscr{B}(L^2_s(\Omega), H^2_{-s}(\Omega)) \tag{0.12}$$

for some $s > 1/2$ and for any $\lambda > 0$. It should be mentioned that the limiting absorption principle (0.12) is true for an arbitrary exterior domain with a compact boundary. If one considers the uniform resolvent estimates obtained in [5, 6, 7], the geometrical condition (0.8) on $\Omega$ is imposed. However, the argument in this paper does not require any geometrical condition.

Once the dispersive estimates are established, Strichartz estimates are obtained by $TT^*$ argument of [2] (see also Yajima [11]). This result will be presented in the talk.

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# GLOBAL EXISTENCE OF SMALL EQUIVARIANT WAVE MAPS ON ROTATIONALLY SYMMETRIC MANIFOLDS

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This talk is based on a joint work with Qidi Zhang (School of Science, East China University of Science and Technology, Shanghai, e-mail: `qidizhang@ecust.edu.cn`), to appear on *Int. Math. Res. Not.*

*Wave maps* are functions $u : M^{1+n} \to N^\ell$ from a Lorentzian manifold $(M^{1+n}, h)$ to a Riemannian manifold $(N^\ell, g)$, which are critical points for the functional on $M^{1+n}$ with Lagrangian density $L(u) = Tr_h(u^*g)$, the trace with respect to the metric $h$ of the pullback of the metric $g$ through the map $u$. The space $M^{1+n}$ is usually called the *base manifold* and $N^\ell$ the *target manifold*; both are assumed to be smooth, complete and without boundary. This notion extends to a Lorentzian setting the usual definition of harmonic maps between Riemannian manifolds. Wave maps arise in several different physical theories, and in particular they play an important role in general relativity.

When the base manifold is the flat Minkowski space $\mathbb{R}\times\mathbb{R}^n$, in local coordinates on the target, the Euler-Lagrange equations for $L(u)$ reduce to a system of derivative nonlinear wave equations

$$\Box u^a + \Gamma^a_{bc}(u)\partial_\alpha u^b \partial^\alpha u^c = 0, \tag{1.1}$$

where $\Gamma^a_{bc}$ are the Christoffel symbols on $N^\ell$ and we use implicit summation over repeated indices. The natural setting is then the Cauchy problem with data at $t = 0$

$$u(0, x) = u_0, \qquad u_t(0, x) = u_1. \tag{1.2}$$

The data are taken in suitable $N^\ell$-valued Sobolev spaces

$$(u_0, u_1) \in H^s(\mathbb{R}^m, N^\ell) \times H^{s-1}(\mathbb{R}^m, TN^\ell) \tag{1.3}$$

which can be defined as follows, if $N^\ell$ is isometrically embedded in a euclidean $\mathbb{R}^{\ell'}$:

$$H^s(\mathbb{R}^m; N^\ell) := \{v \in H^s(\mathbb{R}^m; \mathbb{R}^{\ell'}),\ v(\mathbb{R}^m) \subseteq N^\ell\}. \tag{1.4}$$

Solutions belong to the space $C([0, T); H^s)$, with $T \le \infty$. Starting with [7], [6] Problem (1.1), (1.2) has been studied extensively; see [16] and [5] for a review of the classical theory.

Since equation (1.1) is invariant for the scaling $u(t, x) \mapsto u(\lambda t, \lambda x)$, the critical Sobolev space for the data corresponds to $s = \frac{n}{2}$. In dimension $n = 1$ energy conservation is sufficient to prove global well posedness, thus in the following we assume $n \ge 2$. Concerning local existence, the behaviour is rather clear; Problem (1.1)–(1.3) is

- locally well posed if $s > \frac{n}{2}$ (see [8], [10]). Note that classical energy estimates only allow to prove local existence for $s > \frac{n}{2} + 1$, and the sharp result requires bilinear methods which exploit the null structure of the nonlinearity.
- ill posed if $s < \frac{n}{2}$ (see [15], [3], [4]).

The problem of global existence with small data has been completely understood through the efforts of many authors during the last 20 years (see among the others [18], [20], [21], [9], [17], [11], [22]). The end result is that if the initial data belong to $H^{\frac{n}{2}} \times H^{\frac{n}{2}-1}$, and their homogeneous

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$\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}-1}$ norm is sufficiently small, then there exists a global solution, continuous with values in $H^{\frac{n}{2}}$, for general targets. Note that the solution also belongs to a suitable Strichartz space (more on this below), and uniqueness holds only under this additional constraint.

When the initial data are large, the geometry of the target manifold comes into play, and the problem presents additional difficulties; in particular, blow up in finite time may occur. For targets with positive curvature, when the dimension of the base space is $n \geq 3$, blow up examples with self similar structure were constructed already in [15], [18]. On the other hand, when the target is negatively curved, the available blow up examples require $n \geq 7$ [1]. The case $n = 2$ is especially interesting since the critical norm $\dot{H}^{\frac{n}{2}}$ coincides with the energy norm, which is conserved. However we shall not consider it here and we refer to [23] for further information.

The more general case of a nonflat base manifold has received much less attention. If we restrict to maps defined on a product $\mathbb{R} \times M^n$, with $M^n$ a Riemannian manifold, the wave map system in local coordinates (1.1) becomes

$$u^a_{tt} - \Delta_M u^a + \Gamma^a_{bc}(u)\partial_\alpha u^b \partial^\alpha u^c = 0, \tag{1.5}$$

where $\Delta_M$ is the negative Laplace-Beltrami operator on $M^n$. To our knowledge, there are few results on (1.5). In [19] the stability of equivariant, stationary wave maps on $\mathbb{S}^2$ with values in $\mathbb{S}^2$ is proved, while [2] considers the local existence on Robertson-Walker spacetimes. More recently, in [12] global existence of small wave maps is proved in the case when $M^n = M^4$ is a four dimensional small perturbation of flat $\mathbb{R}^4$, and the stability of equivariant wave maps defined on $\mathbb{H}^2$ is studied in [13].

We plan to initiate the study of equivariant solutions of (1.5) on more general base manifolds $M^n$, $n \geq 3$. Our main result is the global existence of equivariant wave maps for small data in the critical norm, provided the base manifold belongs to a class of manifolds which we call *admissible*. The class of admissible manifolds is rather large, and includes in particular asymptotically flat manifolds and perturbations of real hyperbolic spaces; see some examples in Remark 1.3 below. The precise definition is the following:

**Definition 1.1** (Admissible manifolds)**.** Let $n \geq 3$. We say that a smooth manifold $M^n$ is *admissible* if its metric has the form $dr^2 + h(r)^2 d\omega^2_{\mathbb{S}^{n-1}}$ and $h(r)$ satisfies:

*(i)* $\exists h_\infty \geq 0$ such that $H(r) := h^{\frac{1-n}{2}}(h^{\frac{n-1}{2}})'' = h_\infty + O(r^{-2})$ for $r \gg 1$.

*(ii)* $H^{(j)}(r) = O(r^{-1})$ and $(h^{-\frac{1}{2}})^{(j)} = O(r^{-\frac{1}{2}-j})$ for $r \gg 1$ and $1 \leq j \leq [\frac{n-1}{2}]$.

*(iii)* There exist $c, \delta_0 > 0$ such that for $r > 0$ we have $h(r) \geq cr$ while the function $P(r) = rH(r) - rh_\infty + \frac{1-\delta_0}{4r}$ satisfies the condition $P(r) \geq 0 \geq P'(r)$.

Note that (i) is a form of asymptotic convexity, while (iii) is effective essentially on a bounded region. Condition (ii), on the other hand, is weaker and excludes singularities of the metric at infinity. The parameter $h_\infty$ can be understood as a measure of the curvature of the manifold at infinity; $h_\infty = 0$ means essentially that the manifold is asymptotically flat, while the case $h_\infty > 0$ includes examples with large asymptotic curvature, like the hyperbolic spaces.

Now assume both $M^n$ and $N^\ell$ are rotationally symmetric manifolds, with global metrics

$$M^n: \quad dr^2 + h(r)^2 d\omega^2_{\mathbb{S}^{n-1}}, \qquad N^\ell: \quad d\phi^2 + g(\phi)^2 d\chi^2_{\mathbb{S}^{\ell-1}} \tag{1.6}$$

where $d\omega^2_{\mathbb{S}^{n-1}}$ and $d\chi^2_{\mathbb{S}^{\ell-1}}$ are the standard metrics on the unit sphere. We recall the *equivariant ansatz* (see [16]): writing the map $u = (\phi, \chi)$ in coordinates on $N^\ell$, the radial component $\phi = \phi(t, r)$ depends only on time and $r$, the radial coordinate on $M^n$, while the angular component $\chi = \chi(\omega)$ depends only on the angular coordinate $\omega$ on $M^n$. It follows that $\chi : \mathbb{S}^{n-1} \to \mathbb{S}^{\ell-1}$ must be a harmonic polynomial map of degree $k$, whose energy density is $k(k + n - 2)$ for some integer $k \geq 1$. On the other hand $\phi(t,r)$ must satisfy the *$\bar{\ell}$-equivariant wave map equation*

$$\phi_{tt} - \phi_{rr} - (n-1)\frac{h'(r)}{h(r)}\phi_r + \frac{\bar{\ell}}{h(r)^2}g(\phi)g'(\phi) = 0 \tag{1.7}$$

where $\bar{\ell} = k(k+n-2)$ and for which one considers the Cauchy problem with initial data

$$\phi(0,r) = \phi_0(r), \qquad \phi_t(0,r) = \phi_1(r). \tag{1.8}$$

When $h(r) = r$ the base space is the flat $\mathbb{R}^n$ and (1.7) reduces to the equation originally studied in [18].

In the following statement we use the notation $|D_M| = (-\Delta_M)^{\frac{1}{2}}$, where $\Delta_M$ is the Laplace-Beltrami operator on $M^n$. If $v : M^n \to N^\ell$ is an equivariant map of the form $v = (\phi(r), \chi(\omega))$ with $\chi : \mathbb{S}^{n-1} \to \mathbb{S}^{\ell-1}$ a fixed harmonic map, its Sobolev $H^s(M^n; N^\ell)$ norm can be equivalently expresssed as

$$\|v\|_{H^s(M^n;N^\ell)} \simeq \|\phi\|_{H^s} := \|(1-\Delta_M)^{\frac{s}{2}}\phi\|_{L^2(M^n)}.$$

We define also the weighted Sobolev space $H^s_q(w)$ of radial functions on $M^n$ with norm

$$\|\phi\|_{H^s_q(w)} := \|w^{-1}(|x|)\phi(|x|)\|_{H^s_q(\mathbb{R}^{n+2k})}, \qquad w(r) := r^k \frac{r^{\frac{n-1}{2}}}{h(r)^{\frac{n-1}{2}}}.$$

and we choose the indices $(p,q)$ as

$$p = \frac{4(m+1)}{m+3}, \qquad q = \frac{4m(m+1)}{2m^2-m-5}, \qquad m = n+2k. \tag{1.9}$$

The notation $L^\infty H^s \cap CH^s$ denotes the space of continuous bounded functions from $\mathbb{R}$ to $H^s$, while $L^p H^s_q(w)$ is the space of functions $\phi(t,r)$ which are $L^p$ in time with values in $H^s_q(w)$. Our main result is the following:

**Theorem 1.2** (Global existence in the critical norm)**.** *Let $n \geq 3$, $k \geq 1$, $\bar{\ell} = k(k+n-2)$ and $p,q$ as in (1.9). Assume $M^n$ and $N^\ell$ are two rotationally invariant manifolds with metrics given by (1.6), with $M^n$ admissible, and let $h_\infty$ be the limit of $h^{\frac{1-n}{2}}(h^{\frac{n-1}{2}})''$ as $r \to \infty$. Consider the Cauchy problem (1.7), (1.8).*

*If $h_\infty > 0$ and $\|\phi_0\|_{H^{\frac{n}{2}}} + \|\phi_1\|_{H^{\frac{n}{2}-1}}$ is sufficiently small, the problem has a unique global solution $\phi(t,r) \in L^\infty H^{\frac{n}{2}} \cap CH^{\frac{n}{2}} \cap L^p H_q^{\frac{n-1}{2}}(w)$.*

*If $h_\infty = 0$ and $\||D_M|^{\frac{1}{2}}\phi_0\|_{H^{\frac{n-1}{2}}} + \||D_M|^{-\frac{1}{2}}\phi_1\|_{H^{\frac{n-1}{2}}}$ is sufficiently small, the problem has a unique global solution $\phi(t,r)$ with $|D_M|^{\frac{1}{2}}\phi \in L^\infty H^{\frac{n-1}{2}} \cap CH^{\frac{n-1}{2}}$ and $\phi \in L^p H_q^{\frac{n-1}{2}}(w)$.*

*Remark* 1.1 (Scattering). It is not difficult to prove that the solutions constructed in Theorem 1.2 scatter to solutions of the *linear* equivariant equation

$$\phi_{tt} - \phi_{rr} - (n-1)\frac{h'(r)}{h(r)}\phi_r = 0$$

in $H^{\frac{n}{2}} \times H^{\frac{n}{2}-1}$ as $t \to \pm\infty$, by standard arguments; we omit the details.

*Remark* 1.2 (Local existence with large data). By a simple modification in the proof one can show that the small data assumption can be replaced by the weaker assumption that the linear part of the flow is sufficiently small. This in particular implies existence and uniqueness of a time local solution for large data in the same regularity class.

Thus global existence of small equivariant wave maps on admissible manifolds holds in the critical space $H^{\frac{n}{2}} \times H^{\frac{n}{2}-1}$, as in the case of a flat base manifold. The solution enjoys additional $L^pL^q$ integrability properties, determined by the Strichartz estimates used in the proof. This has the usual drawback that uniqueness holds only in a restricted space. *Unconditional uniqueness* in the critical space without additional restrictions was proved recently for general wave maps on Minkowski space in [14]. We conjecture that a similar result holds also in our situation; as a partial workaround, we prove that if the regularity of the initial data is increased by $\delta = \frac{1}{m+1}$ then uniqueness holds in the space $CH^{\frac{n}{2}+\frac{1}{m+1}}$:

**Theorem 1.3** (Higher regularity and unconditional uniqueness)**.** *Consider* (1.7), (1.8) *under the assumptions of Theorem 1.2, and let* $0 \leq \delta < k$.

*If* $h_\infty > 0$ *and* $\|\phi_0\|_{H^{\frac{n}{2}+\delta}} + \|\phi_1\|_{H^{\frac{n}{2}-1+\delta}}$ *is sufficiently small, the problem has a unique global solution* $\phi \in L^\infty H^{\frac{n}{2}+\delta} \cap CH^{\frac{n}{2}+\delta} \cap L^p H_q^{\frac{n-1}{2}+\delta}(w)$. *Moreover, if* $\delta \geq \frac{1}{m+1}$, *this is the unique solution in* $CH^{\frac{n}{2}+\delta}$.

*If* $h_\infty = 0$ *and* $\||D_M|^{\frac{1}{2}}\phi_0\|_{H^{\frac{n-1}{2}+\delta}(M)} + \||D_M|^{-\frac{1}{2}}\phi_1\|_{H^{\frac{n-1}{2}+\delta}(M)}$ *is sufficiently small, Problem* (1.7), (1.8) *has a unique global solution* $\phi$ *with* $|D_M|^{\frac{1}{2}}\phi \in L^\infty H^{\frac{n-1}{2}+\delta}(M) \cap CH^{\frac{n-1}{2}+\delta}(M)$ *and* $\phi \in L^p H_q^{\frac{n-1}{2}+\delta}(w)$. *Moreover, if* $\delta \geq \frac{1}{m+1}$, *this is the unique solution with* $|D_M|^{\frac{1}{2}}\phi \in CH^{\frac{n-1}{2}+\delta}$.

*Remark* 1.3 (Examples of admissible manifolds). The class of admissibile manifolds is rather large. In particular we can prove that suitable perturbations of admissible manifolds are also admissible; this allows to produce a substantial list of explicit examples. The following manifolds are included in the class:

- The euclidean $\mathbb{R}^n$ and, more generally, rotationally invariant, asymptotically flat spaces of dimension $n \geq 3$,. The precise condition is the following: the radial component of the metric has the form $h_\epsilon(r) = r + \mu(r)$, with $\mu : \mathbb{R}^+ \to \mathbb{R}$ such that for small $\epsilon > 0$
$$|\mu(r)| + r|\mu'(r)| + r^2|\mu''(r)| + r^3|\mu'''(r)| \leq \epsilon r \quad \text{for all} \quad r > 0$$
and
$$|\mu^{(j)}(r)| \lesssim r^{1-j} \quad \text{for} \quad r \gg 1, \quad j \leq [\tfrac{n-1}{2}] + 2.$$
- Real hyperbolic spaces $\mathbb{H}^n$ with $n \geq 3$, for which $h(r) = \sinh r$; more generally, rotationally invariant perturbations of $\mathbb{H}^n$ with a metric $h_\epsilon(r) = \sinh r + \mu(r)$, with $\mu : \mathbb{R}^+ \to \mathbb{R}$ such that for small $\epsilon > 0$
$$|\mu(r)| + |\mu'(r)| + |\mu''(r)| + |\mu'''(r)| \leq \epsilon \langle r \rangle^{-3} \sinh r \quad \text{for all} \quad r > 0$$
and
$$|\mu^{(j)}(r)| \lesssim r^{-1}e^r \quad \text{for} \quad r \gg 1, \quad j \leq [\frac{n-1}{2}] + 2.$$
- Some classes of rotationally invariant manifolds with a metric $h(r)$ of polynomial growth $h(r) \sim r^M$, wher $M$ can be any $M \geq 1$.

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