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Discussion on size effect of footing in ultimate bearing capacity of sandy soil using rigid plastic finite element method

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Abstract

Currently, there are many formulas used to calculate the ultimate bearing capacity. However, the formula has disadvantages in application to practice since it is only applied in calculating simple footing shape and uniform grounds. Most formulas don't take into account the size effect of footing on ultimate bearing capacity except for the formula by Architectural Institute of Japan. The advantage of finite element method is the application to non-uniform grounds, which are for example multi-layered ground and improved ground, and complicated footing shape in three dimensional condition. It greatly improves the accuracy in estimating ultimate bearing capacity. The objective of this study is proposing a rigid plastic constitutive equation using non-linear shear strength property against the confining pressure. The constitutive equation was built based on the experiment regarding non-linear shear strength property against confining pressure reported by Tatsuoka and other researchers. The obtained results from experiment on Toyoura sand and various kinds of sands indicated that although internal friction angle differs among sandy soils, the normalized internal friction angle decreased with the increase in the normalized first stress invariant for various sands despite of dispersion in data. This property always holds irrespective of the reference value of the confining pressure in normalization of internal friction angle. Applicability of proposed rigid plastic equation was proved by comparing with the ultimate bearing capacity formula by Architectural Institute of Japan, which is an experimental formula to take into account the size effect of footing. The results of RPFEM with the proposed constitutive equation were obtained similar to the results by Architectural Institute of Japan. It is clear that RPFEM with the use of non-linear shear strength against the confining pressure provides good estimations to the ultimate bearing capacity of footing by taking account of size effect of footing.

Keywords: *Ultimate Bearing capacity, size effect, stress dependent shear strength, finite element method*

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1. Introduction

In design of buildings, the assessment for ultimate bearing capacity of footing is an important task in order to examine the stability of building - ground system. The pioneering works were conducted by Prandtl (1921)[29] and Reissner (1924)[30]. Prandtl considered a rigid-perfectly plastic half space loaded by a strip punch. The punch-soil interface can be frictional or smooth, and the material is set as weightless. The stress boundary condition is zero traction on the surface of the haft space, except for the strip punch. Prandtl proposed the bearing capacity factor N_c by analytical consideration. Reissner (1924) analyzed a similar problem, but there are two conditions different from those of Prandtl. The material is set as purely frictional (c=0), and a uniformly distributed pressure is loaded at the surface of half space. Reissner applied the hyperbolic type equations to solve the boundary value problem and introduced the bearing capacity factor N_q. In case of frictional-cohesive material, the analyzed slip-line is obtained similarly to the slip-line field. The bearing capacity factors N_q, N_c, are adopted to many ultimate bearing capacity formulae. The ultimate bearing capacity formula of footing by Terzaghi (1943) has been widely employed in practice. It takes account of the effects of cohesion, surcharge and soil weight [40]. The ultimate bearing capacity formula is typically expressed as below:

$$q = cN_c + \tfrac{1}{2}\gamma BN_\gamma + \gamma D_f N_q \tag{1}$$

where N_c, N_γ, N_q are the bearing capacity factors, which are functions of internal friction angle of the soil, ϕ. The other indexes are as follows.

γ: unit weight of soil $(\mathrm{kN/m^3})$

D_f : depth of footing (m)

B: footing width (m)

Since this approach has been proposed, various studies regarding bearing capacity factors have been conducted. Bearing capacity factors N_q and N_c were provided by Prandtl (1921) and Reissner (1924)

$$N_q = \mathrm{e}^{\pi \tan\phi} \tan^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \tag{2}$$

$$N_c = (N_q - 1)\cot\phi \tag{3}$$

With regards to N_γ factor, several formulations have been proposed but no formula is totally accurate. For example, the formula of Meyerhof (1963) is expressed in the following way:

$$N_\gamma = (N_q - 1)\tan(1.4\phi) \tag{4}$$

Meyerhof (1951, 1963) [25] introduced the other factors such as semi - empirical inclination factors i_c, i_γ, i_q. The ultimate bearing capacity formula is described as follows:

$$q = i_c c N_c + \frac{1}{2} i_\gamma \gamma_1 B N_\gamma + i_q \gamma_2 D_f N_q \tag{5}$$

$$i_c = i_q = \left(1 - \frac{\theta}{90^\circ}\right)^2 \tag{6}$$

$$i_\gamma = \left(1 - \frac{\theta}{\phi}\right)^2 \tag{7}$$

where θ : the inclination angle of load with respect to the vertical plane.

Architectural Institute of Japan (AIJ, 1988, 2001) [1] developed the ultimate bearing capacity formula and now is widely used in Japan. It was developed semi-experimentally. By using factors N_c, N_q given by Prandtl and N_γ described by Meyerhof, the ultimate bearing capacity formula is expressed as follows:

$$q = i_c \alpha c N_c + i_\gamma \gamma_1 \beta B \eta N_\gamma + i_q \gamma_2 D_f N_q \tag{8}$$

In the above equation, α and β express the shape coefficient and α = 1 and β = 0.5 are recommended by De Beer[5], respectively. There, η is the size effect factor defined in the following.

$$\eta = \left(\frac{B}{B_o}\right)^m \tag{9}$$

where, B_0: reference value in footing width

m: coefficient determined from the experiment, m = -1/3 is recommended in practice.

The ultimate bearing capacity formula by AIJ successfully takes into account of the size effect of footing which has not been considered in the past formulae employing the Mohr-Coulomb criteria for soils strength. Since the past formulae overestimate the ultimate bearing capacity with the increase in footing width, this effect needs to be examined for intensive practical request. Ueno et al. [42] expressed that the size effect on ultimate bearing capacity was mainly attributed to the stress level effect on shear strength of soils. Their research indicated that the mean stress ranged from $2\gamma B$ to $10\gamma B$ beneath the footing and it caused the change in internal friction angle of ground widely due to the mean stress. This study attempts to discuss the size effect on ultimate bearing capacity by using the finite element analysis with the rigid plastic constitutive equation which simulates the non-linear shear strength property of sandy soil against the confining pressure.

In recent years, the finite element method (FEM) is widely accepted as one of the well-established and convenient technique for solving complex problems in various fields of engineering and mathematical physics. The latest four decades have observed a growing use of finite element method in geotechnical engineering. FEM has been applied to estimate the bearing capacity of strip footing on cohesionless soils such as Sloan and Randolph, 1982 [32]; Griffiths [11], 1982; Frydman and Burd, 1997 [10]. The rigid-plastic finite element method (RPFEM) has been developed for geotechnical engineering by Tamura et al. (1984, 1987) [37]. In this process, the limit load is calculated without the assumption on the potential failure mode. The method is effective in calculating the ultimate bearing capacity of footing against the three dimensional boundary value problems where the soil condition is varied as multi-layered ground. Although RPFEM is originally developed based on the upper bound theorem in plasticity, Tamura et al. proved that it could be derived directly using the rigid plastic constitutive equation. The advantage of rigid plastic constitutive equation is the scalability for considering the material property of soils as the non-associated flow rule. This study improves RPFEM by using the non-linear shear strength property of soils and introduces the rigid plastic constitutive equation of parabolic yield function regarding the confining pressure.

Tatsuoka et al. (1986) [38] and other researchers [16] reported the effects of confining pressure on the internal friction angle for sandy soils by experiments. The obtained results from experiment on Toyoura sand, Degebo sand, Eastern Scheldt sand, and Darmstadt sand indicated that although internal friction angles are different for soils, the normalized internal friction angle shows the same trend for all case studies. In this study, non-linear shear strength property against confining pressure is introduced into RPFEM in order to assess the ultimate bearing capacity of sandy soils by taking account of the size effect of footing. The agreement in ultimate bearing capacity between RPFEM and AIJ formula shows the applicability of RPFEM.

Size effect of footing in ultimate bearing capacity can be observed for not only uniform grounds, but also multi-layered grounds. Since the ultimate bearing capacity formula is developed for uniform grounds, the applicability of the method is severely limited in design practice. The results in both ultimate bearing capacity and failure mode are shown appropriately obtained for the prescribed footing width. Through the examination on the computed results, the developed rigid plastic FEM is proved to afford a rational assessment for the problems in which the ultimate bearing capacity is difficult to be assessed by using the current bearing capacity formulas.

2. Rigid plastic constitutive equation for finite element method

Rigid plastic finite element method is basically developed based on the upper bound theorem in the limit analysis.

It is widely employed for the stability assessment of soil structures in geotechnical engineering. Tamura et al. [30] derived the rigid plastic constitutive equation and proved FEM with the rigid plastic constitutive equation to match RPFEM developed by the upper bound theorem. The advantage of rigid plastic constitutive equation exists in the extensibility to more complicate material properties such as the non-associated flow rule. In this chapter, the rigid plastic constitutive equation for the Drucker-Prager yield function is exhibited. Hoshina et al. (2011) [19] derived the rigid plastic constitutive equation by introducing the dilatancy condition explicitly modelled with the use of penalty method.

2.1 Rigid Plastic constitutive equation for Drucker –Prager yield function

Tamura (1991) developed the rigid plastic constitutive equation for frictional material [36]. The Drucker-Prager's yield function is expressed as follows:

$$\mathrm{f}(\sigma) = aI_1 + \sqrt{J_2} - b = 0 \tag{10}$$

where $I_1 = \mathrm{tr}(\sigma_{ij})$: first stress invariant

$J_2 = \frac{1}{2}\mathrm{s}_{ij}\mathrm{s}_{ij}$: second invariant of deviator stress $\mathrm{s}_{ij} = \sigma_{ij} - \left(\mathrm{I}_1/3\right)\delta_{ij}$ where δ_{ij} is the Kronecker's operator.

The coefficients $a = \frac{\tan\phi}{\sqrt{9+12\tan^2\phi}}$ and $b = \frac{3c}{\sqrt{9+12\tan^2\phi}}$ express the soil constants corresponding to the internal friction angle and cohesion, respectively.

The volumetric strain rate is expressed as follows:

$$\dot{\varepsilon}_v = tr(\dot{\boldsymbol{\varepsilon}}) = tr\left(\lambda\frac{\partial f(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}\right) = tr\left(\lambda\left(\alpha\mathbf{I} + \frac{\mathbf{s}}{2\sqrt{J_2}}\right)\right) = \frac{3a}{\sqrt{3a^2 + 1/2}}\dot{e} \tag{11}$$

where λ : the plastic multiplier, and $\dot{e}$: the norm of strain rate. **I** and **s** express the unit and the deviatoric stress tensors. The strain rate $\dot{\boldsymbol{\varepsilon}}$, which is purely plastic component, should satisfy the volumetric constraint condition which is derived by Eq. (11) as follows:

$$h(\dot{\boldsymbol{\varepsilon}}) = \dot{\varepsilon}_v - \frac{3a}{\sqrt{3a^2 + 1/2}}\dot{e} = \dot{\varepsilon}_v - \hat{\eta}\dot{e} = 0 \tag{12}$$

Any strain rate which is compatible with Drucker-Prager's yield criterion must satisfy the kinematical constraint conditions of Eq. (12). $\hat{\eta}$ is a coefficient determined by Eq. (12) which is on the dilation characteristics. The

rigid plastic constitutive equation is expressed by Lagragian method after Tamura (1991) as follows:

$$\boldsymbol{\sigma} = \frac{b}{\sqrt{3a^2 + \frac{1}{2}}}\frac{\dot{\boldsymbol{\varepsilon}}}{\dot{e}} + \hat{\beta}\left(\mathbf{I} - \frac{3a}{\sqrt{3a^2 + \frac{1}{2}}}\frac{\dot{\boldsymbol{\varepsilon}}}{\dot{e}}\right) \tag{13}$$

The first term expresses the stress component uniquely determined for the yield function, and the second term expresses the indeterminate stress component along the yield function. The indeterminate stress parameter $\hat{\beta}$ still remains unknown until the boundary value problem with Eq. (12) is solved.

In this study, the constrain condition on strain rate is introduced into the constitutive equation directly with the use of penalty method (Hoshina et al., 2011) [19].

$$\boldsymbol{\sigma} = \frac{b}{\sqrt{3a^2 + \frac{1}{2}}}\frac{\dot{\boldsymbol{\varepsilon}}}{\dot{e}} + \kappa(\dot{\varepsilon}_v - \hat{\eta}\dot{e})\left(\mathbf{I} - \frac{3a}{\sqrt{3a^2 + \frac{1}{2}}}\frac{\dot{\boldsymbol{\varepsilon}}}{\dot{e}}\right) \tag{14}$$

where, κ is a penalty constant. This technique makes the computation more stable and faster. FEM with this constitutive equation provides the same formulation of the upper bound theorem in plasticity [36] so that this method is called as RPFEM in this study. In RPFEM, the occurrence of zero energy modes has been pointed out and some numerical techniques to avoid it have been introduced into FEM. However, zero energy modes have not been observed in computation with the rigid plastic constitutive equation using the Penalty method.

2.2 Ultimate bearing capacity of footing under plane strain condition

In this study, the input parameters for ultimate bearing capacity analysis under plane strain condition are derived from triaxial compression tests in the same way with the conventional methods. If the computed results show the good agreement between the RPFEM and the conventional formulas, it indicates RPFEM can provide a good estimation for ultimate bearing capacity since the conventional formulas are developed semi-empirically. In this study, ultimate bearing capacity of strip footing subjected to uniform vertical load is investigated by RPFEM. The load is applied at the center of footing with the width *B*. This footing is modeled by a solid element, the strength of which is set large to be rigid. The typical finite element mesh and the boundary condition employed for RPFEM are shown in Fig. 1.

Ultimate bearing capacity is computed for B=10m and ϕ=30deg. The obtained velocity field is shown in Fig. 2 which indicates the typical failure mode of ground. The norm of strain rate, $\dot{e}$ is presented by contour lines. It

is illustrated by the range between $\dot{e}_{max}$ and $0(\dot{e}_{min})$ since it is basically indeterminate and the relative magnitude in $\dot{e}$ affects the magnitude of ultimate bearing capacity. The slip-line assumed in the conventional bearing capacity formula is also plotted in the figure. The failure mode that is inferred by computation result is similar with the slip-line assumed in the conventional formula. It is difficult to determine the slip-line by RPFEM since FEM is based on the continuum theory. However, it can be seen to provide the similar slip-line although it is slightly smaller than that of the conventional formula. In case of rigid footing, stress concentration is widely known to generate at edge of footing. It causes a problem of singularity in stress distribution of ground. Since finite element analysis is based on continuous function for shape function, it can't analyze the singularity problem directly. Thus, it analyzes the problem approximately. In sandy soil, the shear strength at edge of footing is affected by free stress condition of ground surface outside the footing. The degree of singularity in stress distribution is, therefore, comparatively moderate in case of sandy soil since the shear strength depends on confining stress. In this study, no special numerical technique to analyze the ultimate bearing capacity is employed as the past references (Ukritchon et al., (2003) and Lyamin et al., (2002)). As shown in Fig. 2, the velocity field of ground at edge of footing is obtained greatly from the viewpoint of total balance in velocity field. It seems to reflect the above-mentioned problem, but it is due to the limitation of regular finite element method. This problem is partly resolved by using finer finite elements. The applicability of rigid plastic finite element method is examined through the comparison with the past bearing capacity formulas and finite element analysis. Fig. 3 expresses the comparison of bearing capacity factor N_γ among the various methods for the change in internal friction angle. It proves the rigid plastic finite element method gives a good estimation for ultimate bearing capacity although the defect in treatment of singularity problem.

Ultimate bearing capacity is computed for various footing widths from 1m to 100m at internal friction angles of 20 and 30deg. The results are presented in Figs.4a and 4b. The bigger the footing width is, the higher the ultimate bearing capacity. The values obtained from RPFEM with Drucker-Prager (DP) yield function are coincident with the results from the formulas of Meyerhof and Euro-code 7 when the footing width is less than 30m. Since the Euro-code formula employs different concepts regarding the bearing capacity factor, it leads to the ultimate bearing capacity values in a different way than the other formula. Thus, the discrepancies among them become larger at the footing width of 100m. This width seems too large in practice, but it is considered clearly to discuss the size effect of footing on ultimate bearing capacity.

In preliminary analysis, the effect of mesh size on ultimate bearing capacity was investigated by comparing

bearing capacities computed for 1640 and 3423 element meshes which produces ultimate bearing capacity of 201.9 kPa, 504.9 kPa, 1530.7 kPa, 3822.1 kPa and 13691.2 kPa. The finite element meshes in this study produce ultimate bearing capacity of 201.8 kPa, 503.8 kPa, 1528.8 kPa, 3821.7 kPa and 13685.4 kPa with footing widths: 1m, 3m, 10m, 30m and 100m, respectively. The obtained results are almost coincident for all cases where the footing width is varied from 1m to 100m. Thus, the employed finite element meshes provide good estimation for various cases in this study.

AIJ formula takes into account the size effect of footing on ultimate bearing capacity. Fig. 5 indicates the comparison in ultimate bearing capacity among AIJ formula and others. The results from AIJ formula are smaller than those from others that don't consider the size effect of footing. A great discrepancy can be seen in ultimate bearing capacity at footing width of 100m. Since AIJ formula is developed semi-experimentally, it implies RPFEM needs to take into account the size effect of footing in ultimate bearing capacity assessment.

3. Rigid plastic constitutive equation of sandy soils

3.1 Strength tests of Toyoura sand by Tatsuoka et al.

As mentioned above, the effect of confining pressure on shear strength is clearly presented in Fig. 6 through experiments by Tatsuoka et al. on Toyoura sand. This figure shows that the internal friction angle decreases with the increase in confining pressure for constant void ratio. In this study, in order to estimate the influence of pressure level on ϕ in triaxial compression, the relationship between internal friction angle and first stress invariant is arranged in the normalization form. The general property in internal friction angle is surveyed against confining pressure. Fig. 6 indicates that the internal friction angle ϕ can be inferred by confining pressure for various void ratios. Fig. 7 demonstrates the relationship between internal friction angle ϕ and first stress invariant I_1 at failure. In reality, the friction angle decreases with an increase in the first stress variant in a logarithmic function. The range of the first stress variant is chosen according to test results. The secant friction angle corresponding to the peak of each first stress variant was larger than the approximated value obtained from the Mohr-Coulomb approach. Although the relationship is different depending on the void ratio, the figure shows the internal friction angle decreased with an increase in first stress invariant, irrespective of void ratio. Fig. 8 indicates the relationship between normalized internal friction angle and normalized first stress invariant. ϕ_0 and I_{10} are the reference values of internal friction angle and first stress invariant. The figure shows that the normalized internal friction angles display a similar trend irrespective of void ratio, which means that the obtained relationship exhibits the common property of Toyoura sand.

Hettler and Gudehus (1988) [16] used three different types of sands which are: Degebo sand, Eastern Scheldt sand and Darmstadt sand. The normalized internal friction angle ϕ/ϕ_0 and first stress invariant I_1/I_{10} for all types of soils show the same trends in the figure. It persuaded that the obtained relationship in the figure can be applied not only to Toyoura sand but also to various kinds of sands. Hettler and Gudehus (1988) proposed the formula showing the relationship between internal friction angle ϕ and ϕ^* as below:

$$\phi = \arcsin \frac{\sin\phi^*}{\left(\frac{\sigma_2}{\sigma_{20}}\right)^{\zeta} + \sin\phi^* \left[1 - \left(\frac{\sigma_2}{\sigma_{20}}\right)^{\zeta}\right]} \tag{15}$$

where, σ_2 : lateral stress, ζ estimated from triaxial tests.

ϕ^*: internal friction angle for the reference lateral stress σ_{20}.

Hettler and Gudehus (1988) also indicated that ζ is close to 0.1 and keep unchanged for various sands and densities as Table 1.

Regarding Fig. 9, the references I_{10} and ϕ_0 are chosen depended on the examiner in the laboratory. However, the property of the normalization between internal friction angle and first stress invariant always holds irrespective of the reference value of the confining pressure in the standardization of internal friction angle. Tatsuoka et al. (1986) and Ueno et al. (1998) [42] indicated that the effect of confining pressure is considerable. Therefore, this study improves the rigid plastic finite element method by introducing the non-linear shear strength property against the confining pressure.

3.2 Proposal of rigid plastic constitutive equation for non-linear strength property

In this study, the higher order hyperbolic function is introduced into the yield function of sandy soils as follows:

$$\mathrm{f}(\boldsymbol{\sigma}) = aI_1 + (J_2)^n - b = 0 \tag{16}$$

where *a* and *b* are the soil constants. The index *n* expresses the degree in non-linearity in shear strength against the first stress invariant. Eq. (16) is identical with Drucker-Prager yield function in case of n=1/2. The non-linear parameters *a*, *b* and *n* are identified by the testing data. In the figure, the results by triaxial compression test are plotted for various confining stresses. Fig. 10 shows an example of how the parameter *n* in Eq. (16) influences the internal friction angle for confining pressure. This figure indicated that the parameter n affects the non-linear property in shear strength of soils intensively. It means that parameter *n* increase when internal friction angle

decrease at various confining pressure.

Based on the associated flow rule, the strain rate is obtained as follows for the yield function of Eq. (16)

$$\dot{\boldsymbol{\varepsilon}} = \lambda \frac{\partial \mathrm{f}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = \lambda \frac{\partial}{\partial \boldsymbol{\sigma}} \left(aI_1 + (J_2)^n - b \right) = \lambda \left(a\mathbf{I} + \mathrm{n}\mathrm{J}_2^{\mathrm{n}-1}\mathbf{s} \right) \tag{17}$$

In the above equation, λ is the plastic multiplier. The volumetric strain rate is expressed as follows:

$$\dot{\varepsilon}_v = tr\dot{\boldsymbol{\varepsilon}} = tr\left(\lambda\left(aI + nJ_2^{n-1}s\right)\right) = 3a\lambda = \frac{3a}{\sqrt{3a^2 + 2n^2(b - aI_1)^{2-1/n}}}\dot{e} \tag{18}$$

The first stress invariant I_1 is identified from Eq. (16) to Eq. (18) as the following equation:

$$I_1 = \frac{b}{a} - \frac{1}{a}\left\{ \frac{1}{2n^2}\left[\left(3a\frac{\dot{\mathrm{e}}}{\dot{\varepsilon}_{\mathrm{v}}} \right)^2 - 3a^2 \right] \right\}^{\frac{n}{2n-1}} \tag{19}$$

In this study, the non-linear rigid plastic constitutive equation for confining pressure is finally obtained as follows:

$$\boldsymbol{\sigma} = \frac{3a}{n}\left\{ \frac{1}{2n^2}\left[\left(3a\frac{\dot{\mathrm{e}}}{\dot{\varepsilon}_v} \right)^2 - 3a^2 \right] \right\}^{\frac{1-n}{2n-1}} \frac{\dot{\boldsymbol{\varepsilon}}}{\dot{\varepsilon}_v} + \left(\frac{b}{3a} - \frac{1}{3a}\left[\frac{1}{2n^2}\left(3a\frac{\dot{\mathrm{e}}}{\dot{\varepsilon}_v} \right)^2 - 3a^2 \right]^{\frac{n}{2n-1}} - \frac{a}{n}\left[\frac{1}{2n^2}\left(3a\frac{\dot{\mathrm{e}}}{\dot{\varepsilon}_v} \right)^2 - 3a^2 \right]^{\frac{1-n}{2n-1}} \right)\mathbf{I} \tag{20}$$

In this equation, stress is uniquely determined for plastic strain rate and it is different from Eq. (14) for Drucker-Prager yield function.

4. Discussion on size effect of footing on ultimate bearing capacity

The conventional RPFEM with Drucker-Prager function does not take into account the size effect on ultimate bearing capacity, which is considered in the AIJ formula, because RPFEM is based on the same framework with the other conventional ultimate bearing capacity formulae. This study improves RPFEM by using the non-linear shear strength property of soils and introduces the rigid plastic constitutive equation of parabolic yield function regarding the confining pressure. This study has shown that internal friction angle is not constant and decreases with the increase in confining pressure in sandy soils. It implies the confining pressure dependency in soil shear strength may be one of the most important factors affecting the size effect of footing.

In bearing capacity problem, the larger the footing width is, the higher the confining pressure will be. This leads the internal friction angle to be decreased as discussed above. It is, therefore, necessary to apply the non-linear

shear strength property against the confining pressure to take into account the size effect of footing on ultimate bearing capacity. On the other hand, the internal friction angle is set constant in RPFEM in case of the Drucker-Prager yield function. Therefore, the ultimate bearing capacity calculated using the non-linear rigid-plastic constitutive equation becomes smaller than that obtained from the Drucker-Prager yield function. This means that the size effect of footing is properly taken into account in computation. Non-linear yield function (Eq.16) is defined by the parameters a, b, and n which are derived from the experiment. In this study, a series of numerical simulation are conducted for Toyoura sand based on the experiment of Tatsuoka (1986). Through the case studies, the non-linear shear strength parameters of Toyoura sand are set as a=0.24, b=2.4 (kPa) and n=0.56, respectively.

Fig. 11 shows the deformation of ground at the limit state computed by multiplying arbitrary time increment to the velocity field obtained by RPFEM for B=10m.The obtained failure mode of ground is similar to that in Fig. 2 for the linear shear strength of Drucker-Prager yield function. However, the deformation area in the case of linear shear strength is obtained larger than that in the case of non-linear shear strength, especially around the edge of footing. Fig. 12 shows the results of RPFEM with non-linear shear strength in the case the internal friction angle of 20 and 30 deg. In the figure, these results are clearly identical with those of AIJ. This means that the results obtained by employing non-linear shear strength property are rational and shows that the size effect of footing in ultimate bearing capacity can be expressed well by considering the non-linear shear strength against the confining pressure.

The computed results are utilized to determine the bearing capacity factor N_γ for various internal friction angles from 0 deg to 40 deg. The obtained bearing capacity factor N_γ is compared with these factors defined based on empirical method by Meyerhof (1963 - Semi-empirical), Muhs and Weiss (1969-Euro-code7, Semi-empirical).

Although the cohesion of soils (c = 1 kN/m^2) is introduced into the analysis to make the computation process stable, it does not affect the ultimate bearing capacity too much. Therefore, Eqs. (21) and (22) are applied to approximately define N_γ.

The bearing capacity factor N_γ of RPFEM for Drucker –Prager is calculated by the following equation:

$$N_\gamma^{DP} = \frac{2q^{DP}}{\gamma_1 B} \tag{21}$$

On the other hand, the bearing capacity factor N_γ for non-linear shear strength is determined by the equation:

$$N_\gamma^{NL} = \frac{2q^{NL}}{\gamma_1 B} \tag{22}$$

The bearing capacity factor N_γ was compared among the bearing capacity formulas of AIJ, Euro-code 7 and Meyerhof with RPFEM. Fig. 13 shows the comparison in bearing capacity factor by changing internal friction angle from 0 to 40 deg. As shown in the figure, the bearing capacity factor by RPFEM employing non-linear shear strength against the confining pressure match those by AIJ formula in the wide range of internal friction angle. It is obtained smaller than that by the formulas of Euro-code 7, and Meyerhof. When the internal friction angle is less than 30 deg, there is no much difference in the bearing capacity factor among them. But, the difference becomes greater at the internal friction angle of 40 deg.

5. Conclusions

Terzaghi (1943) and others (e.g. Meyerhof, 1951, 1963) have proposed many formulas to evaluate ultimate bearing capacity. However, the application of formulas is limited due to their disadvantages. Rigid plastic finite element method is effective to solve the complex problems such as multi-layered soil and footing shape in the three dimensional condition. Moreover, limit state analysis is possible to be conducted without the assumption on potential failure modes. In this study, RPFEM is employed for the assessment of ultimate bearing capacity. The applicability of the method is presented through the comparison with those by the semi-experimental ultimate bearing capacity formulas.

Size effect of footing is observed in ultimate bearing capacity, but basically it is not accounted in the ultimate bearing capacity formulas. In this study, discussion on the size effect was conducted in case of a uniform sandy ground. On sandy soils, a rigid plastic constitutive equation is proposed by considering the experiments, where the secant internal friction angle reduces with the increase in confining pressure. This equation is expressed by the higher order parabolic function and easily applied to RPFEM. The obtained ultimate bearing capacity shows a good agreement with that of the ultimate bearing capacity formula by the Architectural Institute of Japan (AIJ, 1998, 2001), which takes into account the size effect of footing. It is clear that RPFEM with the use of proposed constitutive equation provides a good estimation in ultimate bearing capacity assessment by considering the size effect of footing.

On the other hand, all the numerical calculations are for the vertical loading cases of rigid flat footing under the plane strain condition. In case inclined load is considered, the vertical load at failure decreases with the increase in inclination angle. It causes the decrease in confining pressure and the change in internal friction angle in the ground. Therefore, the limit state in vertical and horizontal load space is not so simple as the previous work as Meyerhof due to the variance in internal friction angle. The assessment of ultimate bearing capacity for inclined

load is a subject for future study, but the analytical method will provide the reliable computation results to this problem.

Through the case studies for various footing widths, the change in both ultimate bearing capacity and failure mode due to footing width is shown properly simulated. The obtained conclusions are summarized as follows:

(1) On sandy soils, the size effect of footing in ultimate bearing capacity was well simulated by RPFEM with the use of proposed constitutive equation. It was proved by the comparison in ultimate bearing capacity between the semi-experimental bearing capacity formula of AIJ and RPFEM.

(2) A rigid plastic constitutive equation was proposed for sandy soils based on the experiments by Tatsuoka and other researchers for various soils. The relationship between the secant internal friction angle and first stress invariant was uniquely expressed in normalized form although some scatters existed. The yield function was modeled into the higher order parabolic function regarding the first stress invariant.

(3) Bearing capacity factor N_γ was compared among the bearing capacity formulas of AIJ, Euro-code 7 and Meyerhof with RPFEM by changing internal friction angle from 0 to 40 deg. The bearing capacity factor by RPFEM employing non-linear shear strength against the confining pressure, matched those by AIJ formula in the wide range of internal friction angle. It was obtained smaller than that by the formulas of Euro-code 7 and Meyerhof. The difference in bearing capacity factor was shown greater at the internal friction angle of 40 deg.

(4) Wide applicability of developed RPFEM to the assessment of ultimate bearing capacity was shown through the case studies.

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Table Captions

Table 1. Data for different sands [16]

Sand	ϕ^* (°)	σ_{20} **(kPa)**	ζ
Toyoura	41	10	0.1
Degebo	40	50	0.1
Eastern Scheldt	38	50	0.08
Darmstadt	43.8	50	0.1

Figures

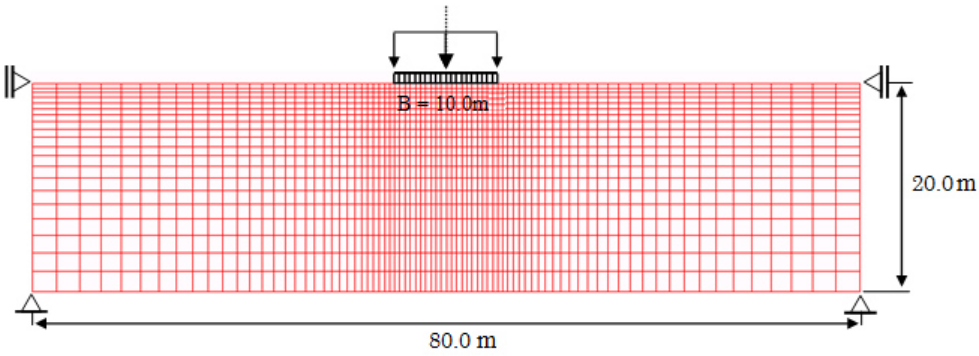


Figure1. Typical finite element mesh and boundary condition in case of B=10m

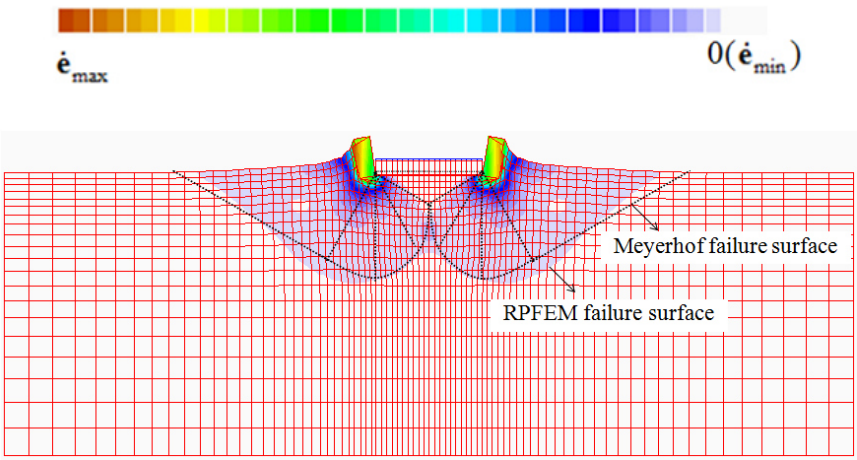


Figure 2. Deformation diagrams of the Drucker-Prager yield function with B=10m in case $\phi = 30^{o}$

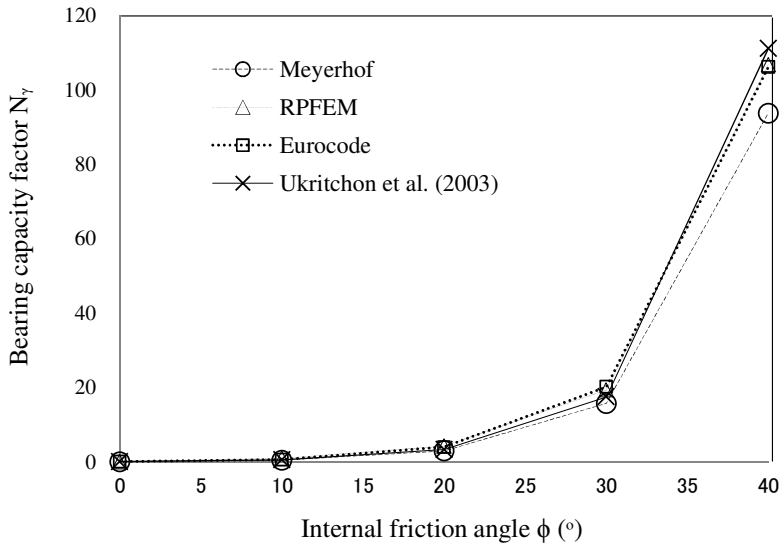
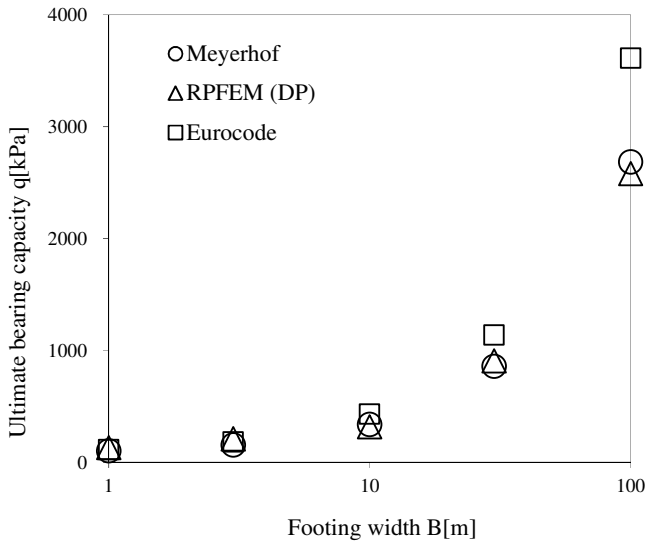
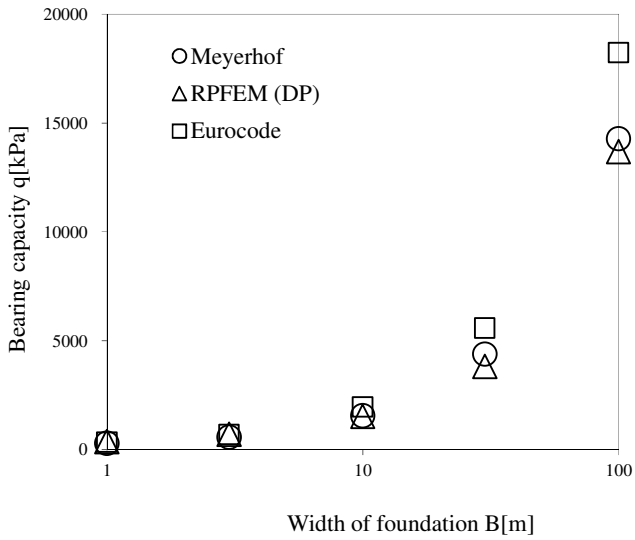


Figure 3. Comparison of bearing capacity factor N_γ among the various methods

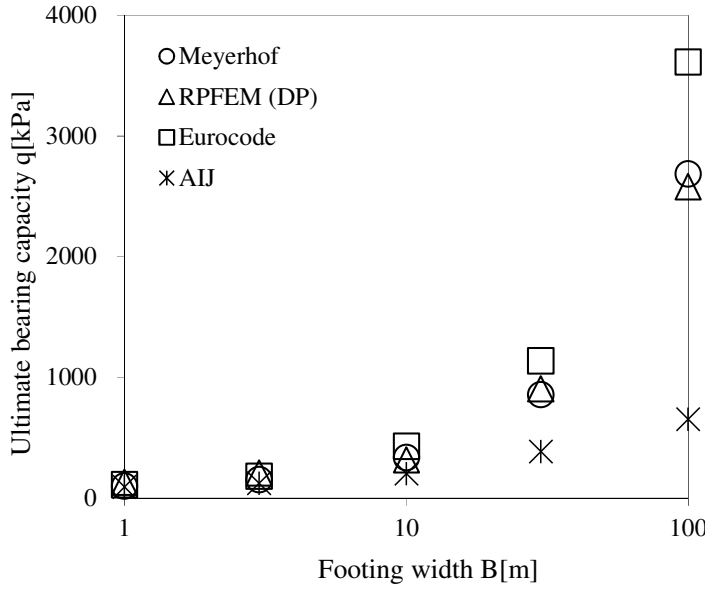


(a) ϕ = 20deg

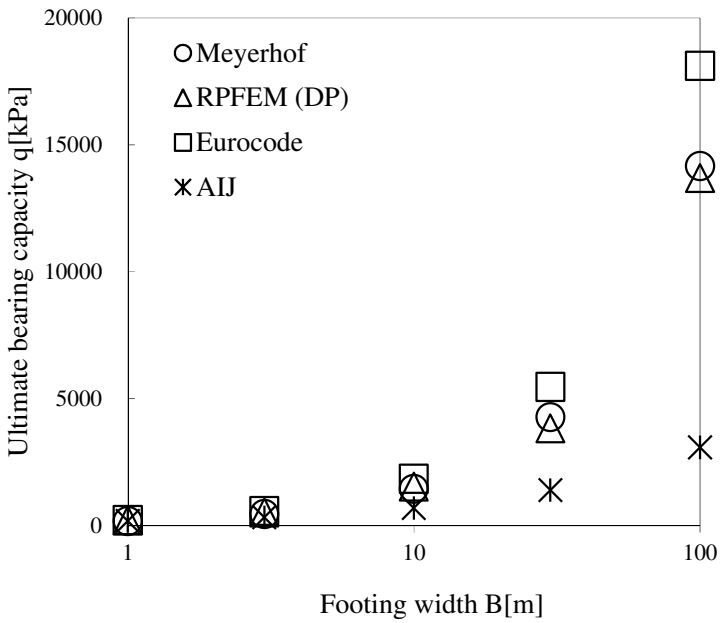


(b) ϕ = 30deg.

Figure 4. Ultimate bearing capacity for vertical load application in case (a) ϕ = 20deg and (b) ϕ = 30deg.



ϕ = 20deg



(b) ϕ = 30deg

Figure 5. Effect of footing width on ultimate bearing capacity for vertical load application

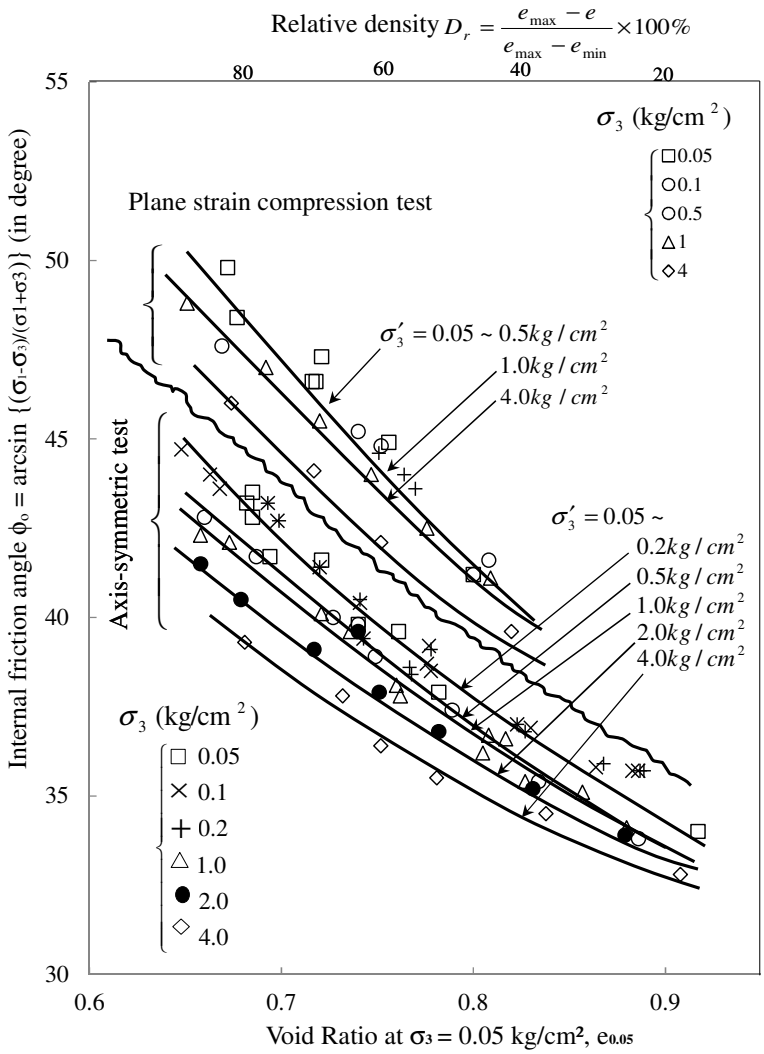


Figure 6. Experimental result of Toyoura sand (Tatsuoka et al., 1986)

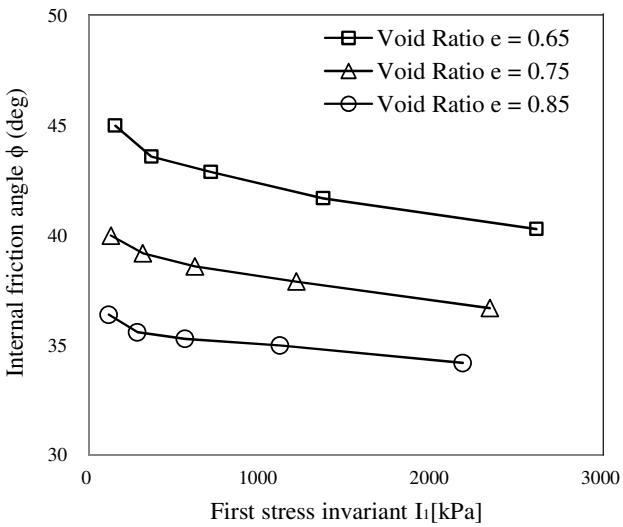


Figure 7. Relationship between internal friction angle and first stress invariant for Toyoura sand

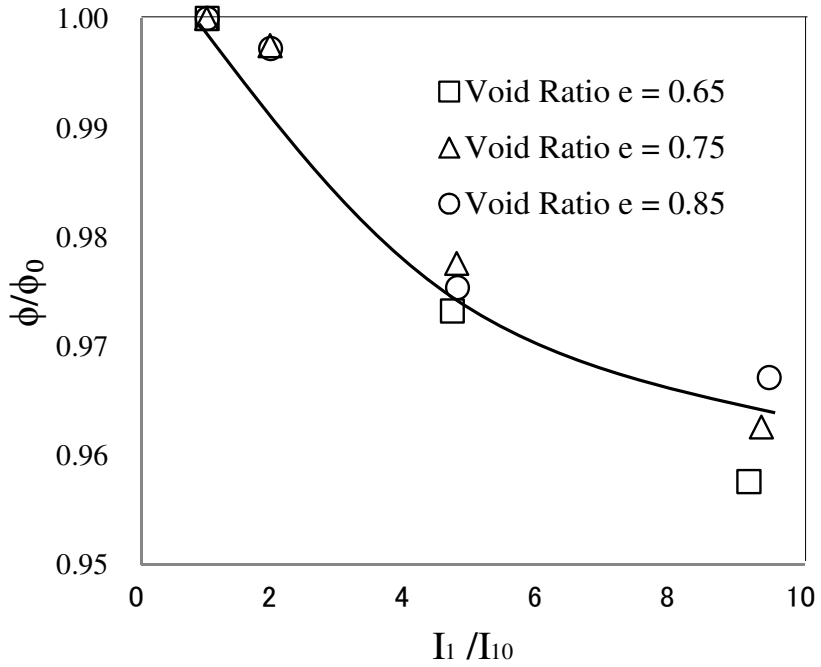


Figure 8. Relationship between normalized internal friction angle ϕ/ϕo and normalized first stress invariant I_1/I_{10} for Toyoura sand

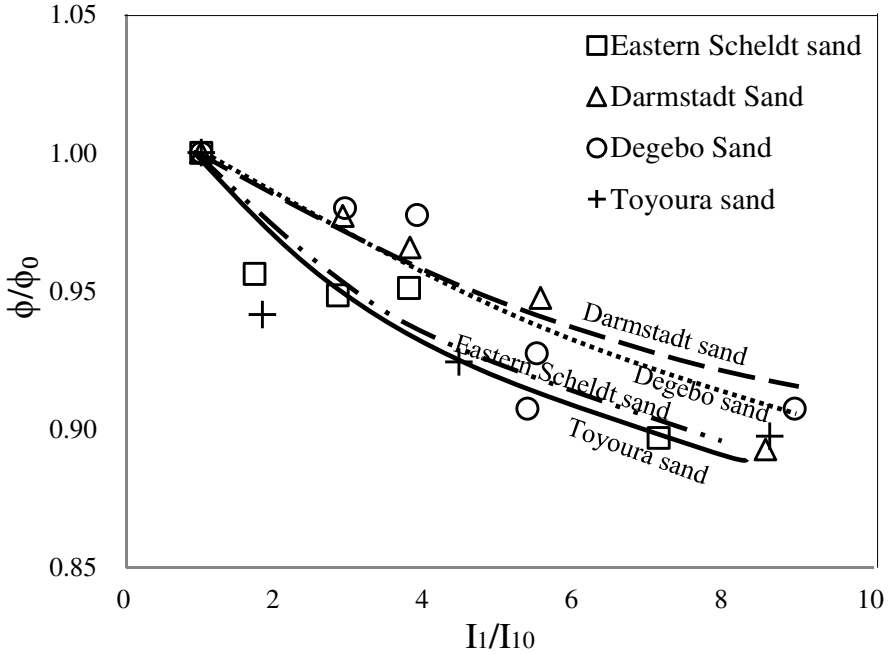


Figure 9. Relationship between ϕ/ϕo and I_1/I_{10} for various kinds of sand

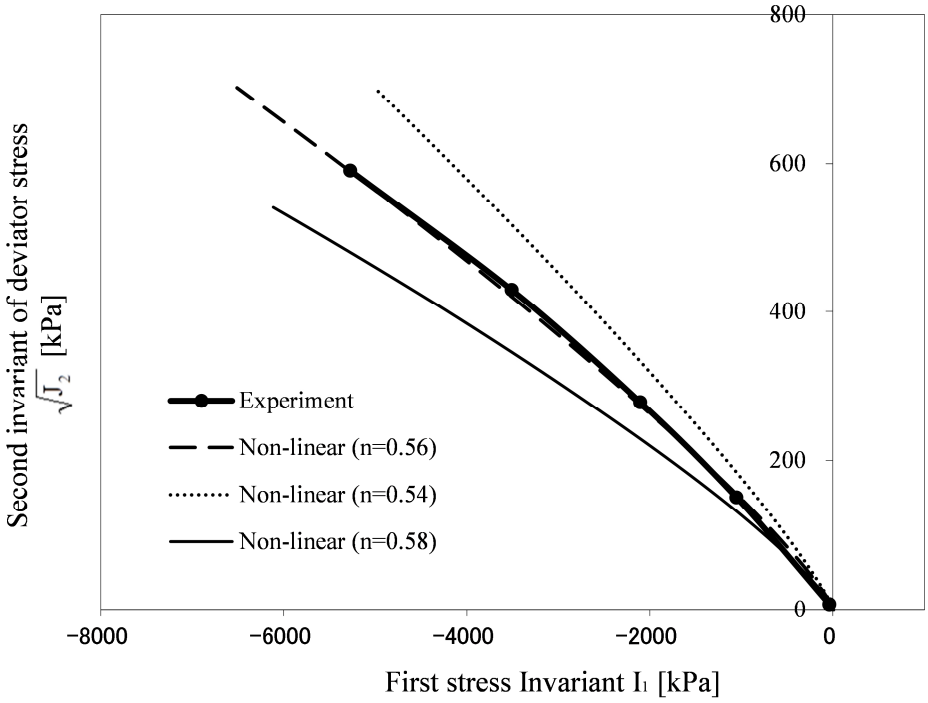


Figure 10. Non-linear parameter *n* affects the non-linear property in shear strength of soils in case ϕ_o = 30deg

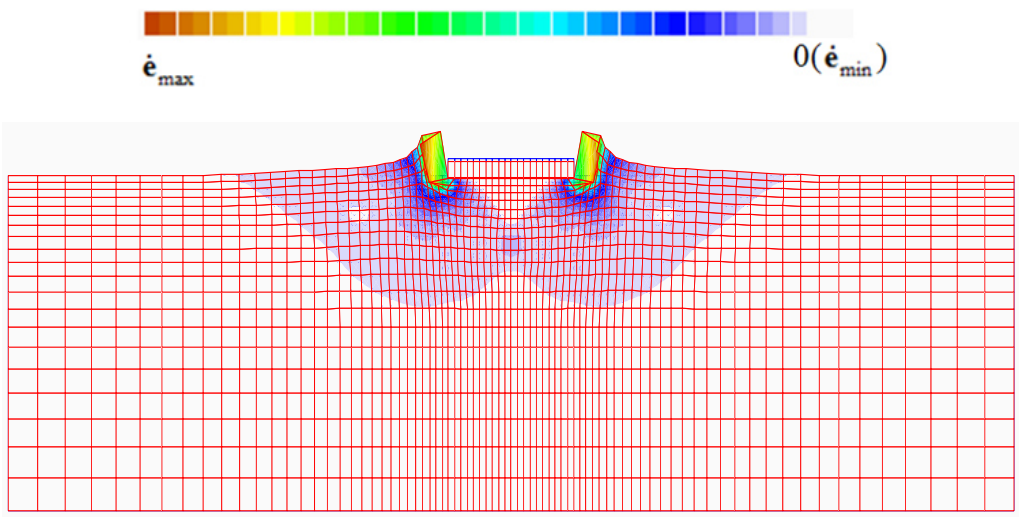
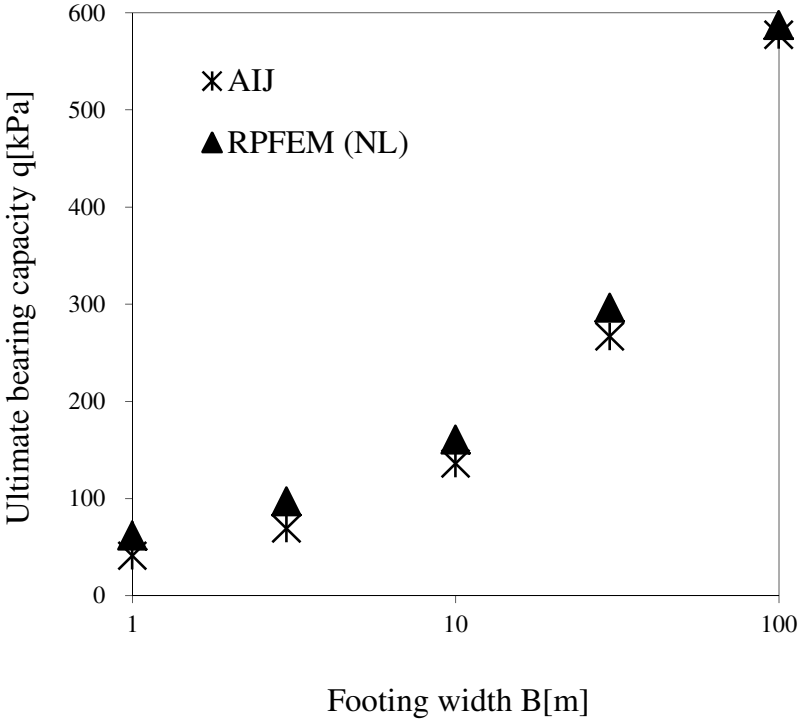
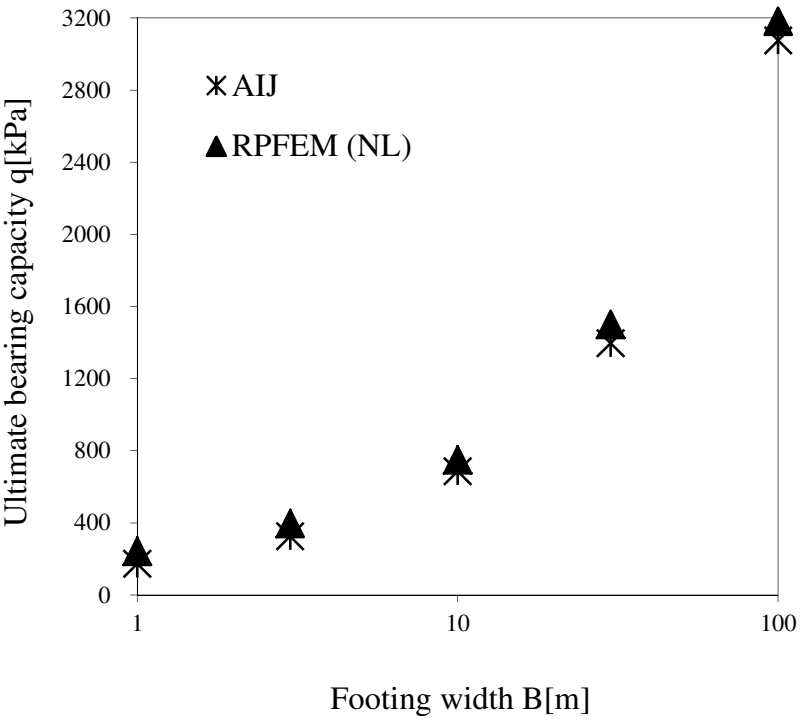


Figure 11. Deformation diagram of the non-linear shear strength with B=10m



(a) ϕ_o = 20deg



(b) ϕ_o = 30deg.

Figure 12. Ultimate bearing capacity with non-linear shear strength in case (a) ϕ_o = 20deg and (b) ϕ_o = 30deg.

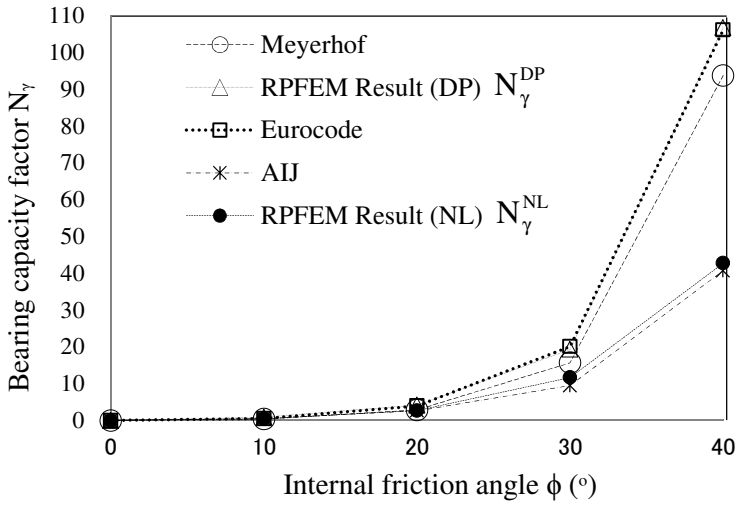


Figure 13. Relationship between bearing capacity factor N_γ and internal friction angle ϕ