THE HYDROSTATIC STOKES SEMIGROUP AND WELL-POSEDNESS OF THE PRIMITIVE EQUATIONS ON SPACES OF BOUNDED FUNCTIONS

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Abstract. Consider the 3-d primitive equations in a layer domain $\Omega = G \times (-h, 0)$, $G = (0,1)^2$, subject to mixed Dirichlet and Neumann boundary conditions at $z = -h$ and $z = 0$, respectively, and the periodic lateral boundary condition. It is shown that this equation is globally, strongly well-posed for arbitrary large data of the form $a = a_1 + a_2$, where $a_1 \in C(\overline{G}; L^p(-h,0))$, $a_2 \in L^\infty(G; L^p(-h,0))$ for $p > 3$, and where a_1 is periodic in the horizontal variables and a_2 is sufficiently small. In particular, no differentiability condition on the data is assumed. The approach relies on $L^\infty_H L^p_z(\Omega)$-estimates for terms of the form $t^{1/2}\|\partial_z e^{tA_{\overline{\sigma}}}\mathbb{P}f\|_{L^\infty_H L^p_z(\Omega)} \leq Ce^{t\beta}\|f\|_{L^\infty_H L^p_z(\Omega)}$ for $t > 0$, where $e^{tA_{\overline{\sigma}}}$ denotes the hydrostatic Stokes semigroup. The difficulty in proving estimates of this form is that the hydrostatic Helmholtz projection $\mathbb{P}$ fails to be bounded with respect to the L^∞-norm. The global strong well-posedness result is then obtained by an iteration scheme, splitting the data into a smooth and a rough part and by combining a reference solution for smooth data with an evolution equation for the rough part.

1. Introduction

The primitive equations are a model for oceanic and atmospheric dynamics and are derived from the Navier-Stokes equations by assuming a hydrostatic balance for the pressure term, see [17–19]. These equations are known to be globally and strongly well-posed in the three dimensional setting for arbitrarily large data belonging to H^1 by the celebrated result of Cao and Titi [5]. The latter considers the case of Neumann boundary conditions and this result also holds true for the case mixed Dirichlet and Neumann boundary conditions, again for data in H^1, as shown by Kukavika and Ziane [14].

Several approaches have been developed in the last years aiming for extending the above two results to the case of rough initial data. One approach is based on the theory of weak solutions, see e.g. [13,16,23,24]. Although the existence of weak solutions to the primitive equations for initial data in L^2 is known since the pioneering work by Lions, Temam and Wang [17], its uniqueness remains an open problem until today. Li and Titi [16] proved uniqueness of weak solutions assuming that the initial data are small L^∞-perturbations of continuous data or data belonging to $\{v \in L^6 \colon \partial_z v \in L^2\}$, where z denotes the vertical variable. By a weak-strong uniqueness argument, these unique weak solutions regularize and even become strong solutions. For a survey of known results, see also [15].

A different approach to the primitive equations is based on a semilinear evolution equation for the hydrostatic Stokes operator within the L^p-setting, see [11]. There, the existence of a unique, global, strong solution to the primitive equations for initial data belonging to $H^{2/p,p}$ was proved for the case of mixed Dirichlet-Neumann boundary conditions. This approach was transfered in [8,9] to the case of pure Neumann boundary conditions and global, strong well-posedness of the primitive equations was obtained for data a of the form $a = a_1 + a_2$, where $a_1 \in C(\overline{G}; L^1(-h,0))$ and $a_2 \in L^\infty(G; L^1(-h,0))$ with a_2 being small. These spaces are scaling invariant and represent the anisotropic character of the primitive equations.

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Note that the choice of boundary conditions has a severe impact on the linearized primitive equations. In the setting of layer domains, i.e., $\Omega = G \times (-h, 0) \subset \mathbb{R}^3$ with $G = (0,1)^2$ and $h > 0$, this is illustrated best by the hydrostatic Stokes operator $A_{\overline{\sigma}}$. The latter can be represented formally by the differential expression

$$\mathcal{A}v = \Delta v + \frac{1}{h}\nabla_H(-\Delta_H)^{-1}\mathrm{div}_H\Big(\partial_z v\big|_{z=-h}\Big), \tag{1.1}$$

restricted to hydrostatically solenoidal vector fields, where for $z = -h$ Dirichlet and for $z = 0$ Neumann boundary conditions are imposed and periodicity is assumed horizontally, see [7] for details. In particular, in the case of pure Neumann boundary conditions, the hydrostatic Stokes operator reduces to the Laplacian, i.e. $A_{\overline{\sigma}}v = \Delta v$.

It is the aim of this article to study properties of the hydrostatic Stokes semigroup and terms of the form $\nabla e^{tA_{\overline{\sigma}}}\mathbb{P}$ on spaces of bounded functions. These properties yield then the global, strong well-posedness result of the primitive equations in the case of mixed Dirichlet-Neumann boundary conditions. More precisely, we prove global, strong well-posedness of the primitive equations for initial data of the form

$$a = a_1 + a_2, \quad a_1 \in C(\overline{G}; L^p(-h,0)), \quad \text{and} \quad a_2 \in L^\infty(G; L^p(-h,0)) \quad \text{for} \quad p > 3,$$

where a_1 is periodic in the horizontal variables and a_2 is sufficiently small. Our strategy is to introduce a reference solution for the smoothened part of the initial data and to combine this with an evolution equation approach for the remaining rough part.

The main difficulty when dealing with the primitive equations on spaces of bounded functions is that the hydrostatic Helmholtz projection $\mathbb{P}$ *fails to be bounded* with respect to the L^∞-norm. This is similar to the case of the classical Stokes semigroup, for which L^∞-theory was developed in [1] and [2].

In Sections 6 and 7 we prove that the combination of the three main players, ∇, $\mathbb{P}$, $e^{tA_{\overline{\sigma}}}$, nevertheless give rise to bounded operators on $L^\infty_H L^p_z(\Omega)$, which in addition satisfy typical global, second order parabolic decay estimates of the form

$$\begin{aligned} t^{1/2}\|\partial_i e^{tA_{\overline{\sigma}}}\mathbb{P}f\|_{L^\infty_H L^p_z(\Omega)} &\le Ce^{t\beta}\|f\|_{L^\infty_H L^p_z(\Omega)}, \\ t^{1/2}\|e^{tA_{\overline{\sigma}}}\mathbb{P}\partial_j f\|_{L^\infty_H L^p_z(\Omega)} &\le Ce^{t\beta}\|f\|_{L^\infty_H L^p_z(\Omega)}, \\ t\|\partial_i e^{tA_{\overline{\sigma}}}\mathbb{P}\partial_j f\|_{L^\infty_H L^p_z(\Omega)} &\le Ce^{\beta t}\|f\|_{L^\infty_H L^p_z(\Omega)}, \end{aligned}$$

for $t > 0$, where $\partial_i, \partial_j \in \{\partial_x, \partial_y, \partial_z\}$.

Note that the choice of the boundary conditions involved affects to a very great extent the difficulty in proving these estimates. For the case of mixed Dirichlet-Neumann boundary conditions, our approach relies on the representation (1.1) of the linearized problem. The constraint $p > 3$ arises from embedding properties for the reference solution and estimates for the linearized problem in $L^\infty(G; L^p(-h, 0))$.

Our approach is based on an iteration scheme, which is inspired by the classical schemes to the Navier-Stokes equations. Here, the iterative construction of a unique, local solution relies on $L^\infty_H L^p_z(\Omega)$-estimates for the crucial terms of the form $e^{tA_{\overline{\sigma}}}\mathbb{P}\mathrm{div}\,(u \otimes v)$, where $u = (v, w)$ is the full velocity and v its horizontal component. Let us note that the above linear estimates are of independent interest for further considerations.

The use of a reference solution allows us to obtain the smallness condition on the $L^\infty_H L^p_z$-perturbation a_2 of a_1 by means of an absolute constant, while for Neumann boundary conditions it is needed that a_2 is small compared to a_1, cf. [8]. Also, Li and Titi assume in [16] that a_2 is small compared to the L^4-norm of a_1.

Comparing our result with the one by Li and Titi in [16], which has been obtained for Neumann boundary conditions, we observe that the initial data allowed in our approach are of anisotropic nature and require no conditions on the derivatives of the initial data, such as e.g. $\partial_z v \in L^2$ as in [16].

This article is structured as follows: In Section 2 we collect preliminary facts and fix the notation. In Section 3 we state our main results concerning the global strong well-posedness of the primitive equations for rough data and the crucial estimates for the linearized problem. The proof of our main results starts with a discussion of anisotropic L^p-spaces in Section 4, which is followed in Section 5 by estimates for the Laplacian in anisotropic spaces. The subsequent Sections 6 and 7 are devoted to the development of an $L^\infty(G; L^p(-h,0))$-theory for the hydrostatic Stokes equations and its associated resolvent problem. Finally, in Section 8 we present our iteration scheme yielding the global, strong well-posedness of the primitive equations for rough initial data.

2. Preliminaries

Let $\Omega = G \times (-h, 0)$ where $G = (0,1)^2$. We consider the primitive equations on Ω given by

$$\begin{aligned} \partial_t v - \Delta v + (u \cdot \nabla)v + \nabla_H \pi &= 0 \quad \text{on } \Omega \times (0,\infty), \\ \partial_z \pi &= 0 \quad \text{on } \Omega \times (0,\infty), \\ \mathrm{div}_H \overline{v} &= 0 \quad \text{on } G \times (0,\infty), \\ v(0) &= a \quad \text{on } \Omega, \end{aligned} \tag{2.1}$$

using the notations $\mathrm{div}_H v = \partial_x v_1 + \partial_y v_2$ and $\nabla_H \pi = (\partial_x \pi, \partial_y \pi)^T$, while $\overline{v} = \frac{1}{h}\int_{-h}^0 v(\cdot, z)\, dz$ is the vertical average, $\pi\colon G \to \mathbb{R}$ denotes the surface pressure, $u = (v, w)$ is the velocity field with horizontal and vertical components $v : \Omega \to \mathbb{R}^2$ and $w : \Omega \to \mathbb{R}$ respectively, where $w = w(v)$ is given by the relation

$$w(x, y, z) = -\int_h^z \mathrm{div}_H v(x, y, r)\, dr. \tag{2.2}$$

This is supplemented by mixed Dirichlet and Neumann boundary conditions

$$\partial_z v = 0 \text{ on } \Gamma_u \times (0,\infty), \quad \pi, v \text{ periodic on } \Gamma_l \times (0,\infty), \quad v = 0 \text{ on } \Gamma_b \times (0,\infty), \tag{2.3}$$

where the boundary is divided into $\Gamma_u = G \times \{0\}$, $\Gamma_l = \partial G \times [-h, 0]$ and $\Gamma_b = G \times \{0\}$.

In the following we will be dealing with anisotropic L^p-spaces on cylindrical sets of the type $U = \Omega$ or $U = \mathbb{R}^2 \times \mathbb{R}$. More precisely, if $U = U' \times U_3 \subset \mathbb{R}^2 \times \mathbb{R}$ is a product of measurable sets and $q, p \in [1,\infty]$ we define

$$L^q_H L^p_z(U) := L^q(U'; L^p(U_3)) := \{f : U \to \mathbb{K} \text{ measurable}, \|f\|_{L^q_H L^p_z(U)} < \infty\},$$

for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with norm

$$\|f\|_{L^q_H L^p_z(U)} := \begin{cases} \left(\int_{U'} \|f(x', \cdot)\|^q_{L^p(U_3)}\, dx'\right)^{1/q}, & q \in [1,\infty), \\ \mathrm{ess\,sup}_{x' \in U'} \|f(x', \cdot)\|_{L^p(U_3)}, & q = \infty. \end{cases}$$

Endowed with this norm, $L^q_H L^p_z(U)$ is a Banach space for all $p, q \in [1,\infty]$.

We will denote the $W^{k,p}$-closure of $C^\infty_{\mathrm{per}}(\overline{\Omega})$ by $W^{k,p}_{\mathrm{per}}(\Omega)$, where $C^\infty_{\mathrm{per}}(\overline{\Omega})$ denotes the space of smooth functions v on $\overline{\Omega}$ that such that $\partial_x^\alpha v$ and $\partial_y^\alpha v$ are periodic on Γ_l with period 1 in the variables x and y for all $\alpha \in \mathbb{N}$, but not necessarily periodic with respect to the vertical direction z. Moreover, by $C^{m,\alpha}(\overline{\Omega})$, $C^{m,\alpha}(\overline{G})$ we denote the spaces of m-times differentiable functions with Hölder-continuous derivatives of exponents $\alpha \in (0,1)$ and the subspaces of functions periodic on Γ_l and ∂G will be denoted by $C^{m,\alpha}_{\mathrm{per}}(\overline{\Omega})$ and $C^{m,\alpha}_{\mathrm{per}}(\overline{G})$, respectively. For a Banach space E we denote by $C_{\mathrm{per}}([0,1]^2; E)$ the set of continuous functions $f : [0,1]^2 \to E$ such that $f(0, y) = f(1, y)$ and $f(y, 0) = f(y, 1)$ for all $x, y \in [0,1]$.

In order to include the condition $\mathrm{div}_H \overline{v} = 0$ one defines the *hydrostatic Helmholtz projection* $\mathbb{P}$ as in [7, 11] using the two-dimensional Helmholtz projection Q with periodic boundary conditions given by $Qg = g - \nabla_H \pi$ for $g : G \to \mathbb{R}^2$ solving $\Delta_H \pi = \mathrm{div}_H g$ for π periodic on ∂G, where $\Delta_H g = \partial_x^2 g + \partial_y^2 g$. The hydrostatic Helmholtz projection is then defined as

$$\mathbb{P}f = f - (1 - Q)\overline{f} = f + \frac{1}{h}\nabla_H(-\Delta_H)^{-1}\mathrm{div}_H \overline{f} = f - \nabla_H \pi.$$

The range of $\mathbb{P}\colon L^p(\Omega)^2 \to L^p(\Omega)^2$, $p \in (1,\infty)$, is denoted by $L^p_{\overline{\sigma}}(\Omega)$ and is given by

$$\overline{\{v \in C^\infty_{\mathrm{per}}(\overline{\Omega})^2 : \mathrm{div}_H \overline{v} = 0\}}^{\|\cdot\|_{L^p(\Omega)}}.$$

Further characterizations of $L^p_{\overline{\sigma}}(\Omega)$ are given in [11, Proposition 4.3].

Since $\mathbb{P}$ fails to be bounded on $L^\infty(\Omega)^2$ it is not evident which space is a suitable substitute for $L^p_{\overline{\sigma}}(\Omega)$ in the case $p = \infty$. In this article, we will be considering the spaces

$$X := C_{\text{per}}([0,1]^2; L^p(-h,0))^2 \quad \text{and} \quad X_{\overline{\sigma}} := X \cap L^p_{\overline{\sigma}}(\Omega), \quad p \in (1,\infty). \tag{2.4}$$

The linearization of equation (2.1), called the *hydrostatic Stokes equation*, is given by

$$\partial_t v - \Delta v + \nabla_H \pi = f, \quad \text{div}_H \overline{v} = 0, \quad v(0) = a \tag{2.5}$$

and subject to boundary conditions (2.3). The dynamics of this evolution equation is governed by the hydrostatic Stokes operator, and its $X_{\overline{\sigma}}$-realization $A_{\overline{\sigma}}$ is given by

$$A_{\overline{\sigma}} v := \mathcal{A} v, \quad D(A^{\overline{\sigma}}_\infty) = \{v \in W^{2,p}_{\text{per}}(\Omega)^2 \cap X_{\overline{\sigma}} : \partial_z v\big|_{\Gamma_u} = 0, v\big|_{\Gamma_b} = 0, \mathcal{A} v \in X_{\overline{\sigma}}\},$$

where $\mathcal{A}v$ is defined by (1.1). It wil be proved that $A_{\overline{\sigma}}$ generates a strongly continuous, analytic semigroup $e^{tA_{\overline{\sigma}}}$ on $X_{\overline{\sigma}}$. Information on the linear theory in $L^p_{\overline{\sigma}}(\Omega)$ for $p \in (1,\infty)$ can be found in [7].

3. Main results

Our first main result concerns the global well-posedness of the primitive equations for *arbitrarily large* initial data in $X_{\overline{\sigma}}$, while the second result extends this situation to the case of small perturbations in $L^\infty_H L^p_z(\Omega)$. Here, a *strong solution* means – as in [11] – a solution v to the primitive equations satisfying

$$v \in C^1((0,\infty); L^p(\Omega))^2 \cap C((0,\infty); W^{2,p}(\Omega))^2. \tag{3.1}$$

Our third main result concerns $L^\infty_H L^p_Z$-estimates for the hydrostatic Stokes semigroup. These estimates are essential for proving the above two results on the non-linear problem. They are also of independent interest.

Theorem 3.1. *Let $p \in (3,\infty)$. Then for all $a \in X_{\overline{\sigma}}$ there exists a unique, global, strong solution v to the primitive equations* (2.1) *with $v(0) = a$ satisfying*

$$v \in C([0,\infty); X_{\overline{\sigma}}), \quad t^{1/2}\nabla v \in L^\infty((0,\infty); X), \quad \limsup_{t\to 0+} t^{1/2}\|\nabla v(t)\|_{L^\infty_H L^p_z(\Omega)} = 0.$$

The corresponding pressure satisfies

$$\pi \in C((0,\infty); C^{1,\alpha}([0,1]^2)), \quad \alpha \in (0, 1-3/p)$$

and is unique up to an additive constant.

Theorem 3.2. *Let $p \in (3,\infty)$. Then there exists a constant $C_0 > 0$ such that if $a = a_1 + a_2$ with $a_1 \in X_{\overline{\sigma}}$ and $a_2 \in L^\infty_H L^p_z(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega)$ with*

$$\|a_2\|_{L^\infty_H L^p_z(\Omega)} \le C_0,$$

then there exists a unique, global, strong solution v to the primitive equations (2.1) *with $v(0) = a$ satisfying*

$$v \in C([0,\infty); L^p_{\overline{\sigma}}(\Omega)) \cap L^\infty((0,T); L^\infty_H L^p_z(\Omega))^2$$

as well as

$$t^{1/2}\nabla v \in L^\infty((0,\infty); X), \quad \limsup_{t\to 0+} t^{1/2}\|\nabla v\|_{L^\infty_H L^p_z(\Omega)} \le C\|a_2\|_{L^\infty_H L^p_z},$$

where $C > 0$ does not depend on the data, and the pressure has the same regularity as in Theorem 3.1.

Taking advantage of the regularization of solutions for $t > 0$ one passes into the setting discussed in [11] and [9], and thus we obtain the following corollary.

Corollary 3.3. *For $t > 0$ the solution v, π in Theorem 3.1 and Theorem 3.2 are real analytic in time and space, and the velocity v decays exponentially as $t \to \infty$.*

Our main result on the hydrostatic semigroup acting on $X_{\overline{\sigma}}$ reads as follows.

Theorem 3.4. *Let $p \in (3,\infty)$. Then the following assertions hold true:*
a) $A_{\overline{\sigma}}$ is the generator of a strongly continuous, analytic and exponentially stable semigroup $e^{tA_{\overline{\sigma}}}$ on $X_{\overline{\sigma}}$ of angle $\pi/2$.
b) There exist constants $C>0$, $\beta>0$ such that for $\partial_i, \partial_j \in \{\partial_x, \partial_y, \partial_z\}$

$$\text{(i)} \qquad t^{1/2}\|\partial_j e^{tA_{\overline{\sigma}}} f\|_{L^\infty_H L^p_z(\Omega)} \leq Ce^{\beta t}\|f\|_{L^\infty_H L^p_z(\Omega)}, \quad t>0,\ f \in X_{\overline{\sigma}},$$

$$\text{(ii)} \qquad t^{1/2}\|e^{tA_{\overline{\sigma}}} \mathbb{P}\partial_j f\|_{L^\infty_H L^p_z(\Omega)} \leq Ce^{\beta t}\|f\|_{L^\infty_H L^p_z(\Omega)}, \quad t>0,\ f \in X_{\overline{\sigma}},$$

$$\text{(iii)} \qquad t\|\partial_i e^{tA_{\overline{\sigma}}} \mathbb{P}\partial_j f\|_{L^\infty_H L^p_z(\Omega)} \leq Ce^{\beta t}\|f\|_{L^\infty_H L^p_z(\Omega)}, \quad t>0,\ f \in X_{\overline{\sigma}};$$

c) *For all $f \in X_{\overline{\sigma}}$*

$$\lim_{t\to 0+} t^{1/2}\|\nabla e^{tA_{\overline{\sigma}}} f\|_{L^\infty_H L^p_z(\Omega)} = 0.$$

Remarks 3.5. a) We note that when in the situation of Theorem 3.2 the initial data do not belong to X, i.e. when $a_2 \neq 0$, the solution fails to be continuous at $t=0$ with respect to the $L^\infty_H L^p_z$-norm.
b) The condition $p>3$ is due to the embeddings

$$v_{\mathrm{ref}}(t_0) \in B^{2-2/q}_{pq}(\Omega)^2 \hookrightarrow C^1(\overline{\Omega})^2 \quad \text{and} \quad W^{2,p}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega}) \quad \text{for } p \in (3,\infty),$$

cf. [25, Section 3.3.1]. Since the two-dimensional Helmholtz projection Q fails to be bounded with respect to the L^∞-norm, we instead estimate it in spaces of Hölder continuous functions $C^{0,\alpha}_{\mathrm{per}}([0,1]^2) = C^{0,\alpha}(\mathbb{T}^2)$ for $\alpha \in (0,1)$ where $\mathbb{T}^2$ denotes the two-dimensional torus. In fact Q is bounded with respect to the $C^{0,\alpha}$-norm. This follows by the theory of Fourier multipliers on Besov spaces, compare e.g. [3, Theorem 6.2] for the whole space, and the periodic case follows using periodic extension.
c) In Theorem 3.4 one can even consider $f \in L^\infty_H L^p_z(\Omega)^2$ for $p \in (3,\infty)$. Then the corresponding semigroup is still analytic, but it fails to be strongly continuous. The estimates $(i)-(iii)$ still hold, whereas property $c)$ in Theorem 3.4 has to be replaced by

$$\limsup_{t\to 0+} t^{1/2}\|\nabla e^{tA_{\overline{\sigma}}} v\|_{L^\infty_H L^p_z(\Omega)} \leq C\|v\|_{L^\infty_H L^p_z(\Omega)}$$

for some $C>0$, where with a slight abuse of notation $e^{tA_{\overline{\sigma}}}$ denotes also the hydrostatic Stokes semigroup on $L^\infty_H L^p_z(\Omega)$.
d) Some words about our strategy for proving the global well-posedness results are in order:

(i) We will first construct a local, mild solution to the problem (2.1), i.e. a function satisfying the relation

$$v(t) = e^{tA_{\overline{\sigma}}} a + \int_0^t e^{(t-s)A_{\overline{\sigma}}} \mathbb{P} F(v(s))\, ds, \quad t \in (0,T) \tag{3.2}$$

for some $T>0$, where $F(v) = -(u\cdot\nabla)v$. We will then show that v regularizes for $t_0>0$ and using the result of [11, Theorem 6.1] or [9, Theorem 3.1], we may take $v(t_0)$ as a new initial value to extend the mild solution to a global, strong solution on (t_0,∞) and then on $(0,\infty)$ by uniqueness. The additional regularity for $t\to 0+$ results form the construction of the mild solutions.

(ii) In order to construct a mild solution we decompose $a = a_{\mathrm{ref}} + a_0$ such that a_{ref} is sufficiently smooth and a_0 can be taken to be arbitrarily small.

(iii) Using previously established results concerning the existence of solutions to the primitive equations for *smooth* data, we obtain a reference solution v_{ref} and construct then $V := v - v_{\mathrm{ref}}$ via an iteration scheme using L^∞-type estimates for terms of the form $\nabla e^{tA_{\overline{\sigma}}}\mathbb{P}$ given in Theorem 3.4.

4. Properties of anisotropic spaces

In this section, we will discuss properties of anisotropic L^q-L^p-spaces. We will write $C(U'; L^p(U_3))$ for the set of continuous $L^p(U_3)$-valued functions on U' and likewise

$$L^q(U'; C(U_3)) := \{f \in L^q_H L^\infty_z(U) : f(x',\cdot) \in C(U_3) \text{ for almost all } x' \in U'\},$$

and $C_c(U';L^p(U_3))$ and $L^q(U';C_c(U_3))$ for the subsets of functions with compact support in horizontal and vertical variables, respectively. For $p,q\in[1,\infty)$ the space $C_c^\infty(U)$ is dense in these spaces as well as in $L^q_HL^p_z(U)$, and furthermore we have

$$\overline{C_c^\infty(\mathbb{R}^3)}^{\|\cdot\|_{L^\infty_H L^p_z}} = C_0(\mathbb{R}^2;L^p(\mathbb{R})), \quad \overline{C_c^\infty(\mathbb{R}^3)}^{\|\cdot\|_{L^q_H L^\infty_z}} = L^q(\mathbb{R}^2;C_0(\mathbb{R})),$$

as well as

$$\overline{C^\infty_{\mathrm{per}}(\overline{\Omega})^2}^{\|\cdot\|_{L^\infty_H L^p_z}} = X, \quad \overline{C^\infty_{\mathrm{per}}(\overline{\Omega})}^{\|\cdot\|_{L^q_H L^\infty_z}} = L^q(G;C[-h,0]).$$

Observe thst even $C^\infty_{\mathrm{per}}([0,1]^2;C_c^\infty(-h,0))^2$ is dense in X and $L^q_HL^p_z(\Omega)^2$. If $p=q=\infty$, then

$$\overline{C_c^\infty(\mathbb{R}^3)}^{\|\cdot\|_{L^\infty_H L^\infty_z}} = C_0(\mathbb{R}^3), \quad \overline{C^\infty_{\mathrm{per}}(\overline{\Omega})}^{\|\cdot\|_{L^\infty_H L^\infty_z}} = C_{\mathrm{per}}([0,1]^2;C[-h,0]).$$

Here $C_0(\mathbb{R}^d)$ denotes the set of functions vanishing at infinity. These density results follow from the fact that if E is a Banach space over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, then the linear space generated by elementary tensor functions $f\otimes e$ for measurable $f:U'\to\mathbb{K}$ and $e\in E$ is dense in $L^q(U';E)$ for $q\in[1,\infty)$, since it contains the simple E-valued functions. It is also dense in $C_0(U';E)$, if one only considers continuous functions f, due to a generalization of the Stone-Weierstrass theorem, see e.g. [12].

In the case that $U\subset\mathbb{R}^3$ is bounded, we also have

$$L^{q_1}_HL^{p_1}_z(U)\hookrightarrow L^{q_2}_HL^{p_2}_z(U)$$

whenever $q_1\ge q_2$ and $p_1\ge p_2$. See [11, Section 5] for more details.

Another important property of the $L^q_HL^p_z$-norm is its behaviour under operations like multiplication and convolution. For the former one, we obviously obtain

$$\|fg\|_{L^q_HL^p_z(U)} \le \|f\|_{L^{q_1}_HL^{p_1}_z(U)}\|g\|_{L^{q_2}_HL^{p_2}_z(U)}$$

whenever $1/p_1+1/p_2=1/p$ and $1/q_1+1/q_2=1/q$. For the latter one, the following variant of Young's inequality holds true.

Lemma 4.1. *[10, Theorem 3.1]. Let $f\in L^q_HL^p_z(\mathbb{R}^3)$ for $p,q\in[1,\infty]$ and $g\in L^1(\mathbb{R}^3)$. Then*

$$\|g*f\|_{L^q_HL^p_z(\mathbb{R}^3)} \le \|g\|_{L^1(\mathbb{R}^3)}\|f\|_{L^q_HL^p_z(\mathbb{R}^3)}.$$

5. Linear estimates for the Laplace operator

In this section we establish resolvent and semigroup estimates for Laplace operators with a focus on anisotropic $L^q_HL^p_z$-spaces, where $p,q\in[1,\infty]$.

First, we consider the resolvent problem for the Laplacian on the full space for

$$\lambda\in\Sigma_\theta=\{\lambda\in\mathbb{C}\setminus\{0\}:|\arg(\lambda)|<\theta\}, \quad \theta\in(0,\pi),$$

i.e.

$$\Delta v-\lambda v=f \text{ on } \mathbb{R}^3, \quad f\in C_c^\infty(\mathbb{R}^3), \tag{5.1}$$

and for $\partial_j\in\{\partial_x,\partial_y,\partial_z\}$

$$\Delta w-\lambda w=\partial_j f \text{ on } \mathbb{R}^3, \quad f\in C_c^\infty(\mathbb{R}^3). \tag{5.2}$$

It is well known that the solution to problem (5.1) is given by the convolution $v=K_\lambda*f$ and the one to problem (5.2) by $v=\partial_jK_\lambda*f$, where K_λ is explicitly given by

$$K_\lambda(x)=\frac{1}{4\pi}\frac{e^{-\lambda^{1/2}|x|}}{|x|}, \quad x\in\mathbb{R}^3\setminus\{0\}.$$

Using this representation one easily obtains the following uniform $L^1(\mathbb{R}^3)$-estimates.

Lemma 5.1. *For all $\theta\in(0,\pi)$ there exists $C_\theta>0$ such that for all $\lambda\in\Sigma_\theta$ one has*

$$|\lambda|\cdot\|K_\lambda\|_{L^1(\mathbb{R}^3)}+|\lambda|^{1/2}\|\nabla K_\lambda\|_{L^1(\mathbb{R}^3)}\le C_\theta.$$

Proof. Set $\psi := \arg(\lambda) \in (-\theta, \theta)$. Since K_λ is radially symmetric we use spherical coordinates to obtain

$$\int_{\mathbb{R}^3} |K_\lambda(x)|\, dx = \int_0^\infty r e^{-|\lambda|^{1/2}\cos(\psi/2) r}\, dr$$

as well as

$$\int_{\mathbb{R}^3} |\nabla K_\lambda(x)|\, dx \leq \int_0^\infty (1 + |\lambda|^{1/2} r) e^{-|\lambda|^{1/2}\cos(\psi/2) r}\, dr.$$

So, $|\lambda| \cdot \|K_\lambda\|_{L^1(\mathbb{R}^3)} = \sec(\psi/2)^2$ and $|\lambda|^{1/2}\|\nabla K_\lambda\|_{L^1(\mathbb{R}^3)} \leq \sec(\psi/2) + \sec(\psi/2)^2$, and thus we obtain the desired result. □

From this and Young's inequality for convolutions in anisotropic spaces, cf. Lemma 4.1, one immediately obtains suitable $L^q_H L^p_z$-estimates for the resolvent problems (5.1) and (5.2) for $q, p \in [1, \infty]$.

Corollary 5.2. *Let $\lambda \in \Sigma_\theta$ for some $\theta \in (0, \pi)$. Assume one of the following cases:*

(i) *$p, q \in [1, \infty)$ and $f \in L^q_H L^p_z(\mathbb{R}^3)$, or*
(ii) *$p \in [1, \infty)$, $q = \infty$, and $f \in L^q_H L^p_z(\mathbb{R}^3)$ with compact support in horizontal direction, or*
(iii) *$p = \infty$, $q \in [1, \infty)$, and $f \in L^q_H L^p_z(\mathbb{R}^3)$ with compact support in vertical direction.*

Then the functions

$$v = K_\lambda * f \quad \text{and} \quad w = \partial_j K_\lambda * f$$

are the unique solutions to the problems (5.1) and (5.2) in $L^q_H L^p_z(\mathbb{R}^3)$, respectively, and there exists a constant $C_\theta > 0$ such that

$$|\lambda| \cdot \|v\|_{L^q_H L^p_z(\mathbb{R}^3)} + |\lambda|^{1/2}\|\nabla v\|_{L^q_H L^p_z(\mathbb{R}^3)} + \|\Delta v\|_{L^q_H L^p_z(\mathbb{R}^3)} \leq C_\theta \|f\|_{L^q_H L^p_z(\mathbb{R}^3)}, \tag{5.3}$$

$$|\lambda|^{1/2}\|w\|_{L^q_H L^p_z(\mathbb{R}^3)} \leq C_\theta \|f\|_{L^q_H L^p_z(\mathbb{R}^3)}. \tag{5.4}$$

Remark 5.3. In the case $q, p \in [1, \infty)$ we have that $C_c^\infty(\mathbb{R}^3)$ is dense in $L^q_H L^p_z(\mathbb{R}^3)$, so we may assume that f is essentially bounded and has compact support, yielding $\partial_i(K_\lambda * f) = (\partial_i K_\lambda) * f$. In the cases where q and/or p is infinite we add this as an assumption.

We now investigate for the Laplacian on Ω with boundary conditions (2.3) the resolvent problems

$$\lambda v - \Delta v = f \text{ on } \Omega, \tag{5.5}$$

and for $\partial_i \in \{\partial_x, \partial_y, \partial_z\}$

$$\lambda w - \Delta w = \partial_i f \text{ on } \Omega. \tag{5.6}$$

Lemma 5.4. *Let $\theta \in (0, \pi)$ and $f \in L^q_H L^p_z(\Omega)$ for $q \in [1, \infty], p \in [1, \infty)$. Then there exists $\lambda_0 > 0$ such that for $\lambda \in \Sigma_\theta$ with $|\lambda| \geq \lambda_0$ the problems (5.5) and (5.6) have unique solutions $v \in L^q_H L^p_z(\Omega)$ and $w \in L^q_H L^p_z(\Omega)$, respectively, and there exists a constant $C_\theta > 0$ such that*

$$|\lambda| \cdot \|v\|_{L^q_H L^p_z(\Omega)} + |\lambda|^{1/2}\|\nabla v\|_{L^q_H L^p_z(\Omega)} + \|\Delta v\|_{L^q_H L^p_z(\Omega)} \leq C_\theta \|f\|_{L^q_H L^p_z(\Omega)}, \tag{5.7}$$

$$|\lambda|^{1/2}\|w\|_{L^q_H L^p_z(\Omega)} \leq C_\theta \|f\|_{L^q_H L^p_z(\Omega)}. \tag{5.8}$$

In particular for $q = \infty$ and $p \in (2, \infty)$ one can chose $\lambda_0 = 0$.

To prove this lemma, we will need some facts concerning isotropic L^p-spaces. So, for $p \in (1, \infty)$ denote by Δ_p the Laplace operator on $L^p(\Omega)$ defined by

$$\Delta_p v = \Delta v, \quad D(\Delta_p) = \{v \in W^{2,p}_{\text{per}}(\Omega) : \partial_z v\big|_{\Gamma_u} = 0, v\big|_{\Gamma_b} = 0\}.$$

One has $\rho(-\Delta_p) \subset \mathbb{C} \setminus [\delta, \infty)$, for some $\delta > 0$, i.e. $0 \in \rho(-\Delta_p)$, cf. [21, Remark 8.23], and the resolvent satisfies for some $C_{\theta,p} > 0$ the estimate

$$|\lambda| \cdot \|(\Delta_p - \lambda)^{-1} f\|_{L^p(\Omega)} + \|\Delta_p(\Delta_p - \lambda)^{-1} f\|_{L^p(\Omega)} \leq C_{\theta,p}\|f\|_{L^p(\Omega)}, \quad f \in L^p(\Omega), \tag{5.9}$$

where $\lambda \in \Sigma_\theta$, $\theta \in (0,\pi)$. Furthermore, $-\Delta_p$ possesses a bounded $\mathcal{H}^\infty$-calculus of angle 0, see e.g. [21], and therefore

$$D((-\Delta_p)^\vartheta) = [L^p(\Omega), D(\Delta_p)]_\vartheta \subset W^{2\vartheta,p}(\Omega), \quad \vartheta \in [0,1], \tag{5.10}$$

where $[\cdot,\cdot]$ denotes the complex interpolation functor. In particular $\partial_j(-\Delta_p)^{-1/2}$ is bounded on $L^p(\Omega)$ for $\partial_j \in \{\partial_x, \partial_y, \partial_z\}$ and by taking adjoints the same holds true for the closure of $(-\Delta_p)^{-1/2}\partial_j$. This yields the estimates

$$\begin{aligned} |\lambda|^{1/2}\|\partial_j(\Delta_p - \lambda)^{-1} f\|_{L^p(\Omega)} + |\lambda|^{1/2}\|(\Delta_p-\lambda)^{-1}\partial_j f\|_{L^p(\Omega)} &\le C_{\theta,p}\|f\|_{L^p(\Omega)}, \quad f \in L^p(\Omega), \\ \|\partial_j(\Delta_p-\lambda)^{-1}\partial_i f\|_{L^p(\Omega)} &\le C_{\theta,p}\|f\|_{L^p(\Omega)}, \quad f \in L^p(\Omega) \end{aligned} \tag{5.11}$$

for $\lambda \in \Sigma_\theta$, $\theta \in (0,\pi)$, and some $C_{\theta,p} > 0$.

Proof of Lemma 5.4. First, we apply the following density arguments:

(i) For $q,p \in [1,\infty)$ and $f \in L^q_H L^p_z(\Omega)$ we assume that $f \in C^\infty_{\text{per}}([0,1]^2; C^\infty_c(-h,0))$ since $C^\infty_{\text{per}}([0,1]^2; C^\infty_c(-h,0))$ is a dense subspace of $L^q_H L^p_z(\Omega)$.
(ii) For $q = \infty$ and $f \in L^\infty(G; L^p(-h,0))$, we assume that $f \in L^\infty(G; C^\infty_c(-h,0))$, as the latter space is dense in $L^\infty_H L^p_z(\Omega)$.

In particular, in either case we may assume that $f = 0$ on $\Gamma_u \cup \Gamma_b$ and $f \in L^\infty(\Omega)$. The existence of a unique solution to the problems (5.5) and (5.6) in $L^q_H L^p_z(\Omega)$ for such smooth f follows from the properties of the mappings $(\lambda-\Delta)^{-1}$ and $(\lambda-\Delta)^{-1}\partial_i$ in $L^r(\Omega)$ for $\lambda \in \lambda \in \Sigma_\theta$ since

$$v \in W^{2,r}(\Omega) \hookrightarrow L^\infty(\Omega) \hookrightarrow L^q_H L^p_z(\Omega) \quad \text{and} \quad w \in W^{1,r}(\Omega) \hookrightarrow L^\infty(\Omega) \hookrightarrow L^q_H L^p_z(\Omega), \quad r > 3.$$

It therefore suffices to prove the estimates (5.7) and (5.8). This is done in the following by localizing the results of Lemma 5.2.

For this purpose we first make use of the extension operator

$$E = E_z^{\text{even,odd}} \circ E_H^{\text{per}}$$

where E_H^{per} is the periodic extension operator from G to $\mathbb{R}^2$ in horizontal direction and $E_z^{\text{even,odd}}$ extends from $(-h,0)$ to $(-2h,h)$ in vertical direction via even and odd reflexion at the top and bottom part of the boundary respectively.

Second, we utilize a family of cut-off-functions $\chi_r \in C^\infty_c(\mathbb{R}^3)$ for $r \in (0,\infty)$ of the form $\chi_r(x,y,z) = \varphi_r(x,y)\psi_r(z)$ where $\varphi_r \in C^\infty_c(\mathbb{R}^2)$ and $\psi_r \in C^\infty_c(\mathbb{R})$ satisfy

$$\begin{aligned} \varphi_r &\equiv 1 \quad \text{on } [-1/4, 5/4]^2, \quad \varphi_r \equiv 0 \quad \text{on } \left((-\infty,-r-1/4] \cup [5/4+r,\infty)\right)^2, \\ \psi_r &\equiv 1 \quad \text{on } [-5h/4, h/4], \quad \psi_r \equiv 0 \quad \text{on } (-\infty,-r-5h/4] \cup [h/4+r,\infty), \end{aligned}$$

and there is a constant $M > 0$ independent of r such that

$$\|\varphi_r\|_\infty + \|\psi_r\|_\infty + r\left(\|\nabla_H\varphi_r\|_\infty + \|\partial_z\psi_r\|_\infty\right) + r^2\left(\|\Delta_H\varphi_r\|_\infty + \|\partial_z^2\psi_r\|_\infty\right) \le M.$$

Here, we consider $0 < 4r < 3\min\{1,h\}$ which implies that φ_r and ψ_r are supported on $(-1,2)$ and $(-2h,h)$ respectively. We now define an extension of v from Ω onto the whole space $\mathbb{R}^3$ via

$$u(x,y,z) = \chi_r(x,y,z)(Ev)(x,y,z)$$

for a suitable value of r which we will specify later on. If v solves problem (5.5) then u solves the problem

$$\lambda u - \Delta u = F \text{ on } \mathbb{R}^3, \quad F := \chi_r Ef - 2(\nabla\chi_r)\cdot E(\nabla v) - (\Delta\chi_r)Ev.$$

Here we made use of the fact that E commutes with derivatives of v.

Note that not only does F have compact support, but we also have $F \in L^\infty(\mathbb{R}^3)$ since we may assume that $f \in L^\infty(\Omega)$ and $v \in W^{1,\infty}(\Omega)$ by the above approximation argument. Thus we may now apply Lemma 5.2, and estimate (5.3) yields

$$|\lambda|\cdot\|u\|_{L^q_H L^p_z(\mathbb{R}^3)} + |\lambda|^{1/2}\|\nabla u\|_{L^q_H L^p_z(\mathbb{R}^3)} \le C_\theta\|F\|_{L^q_H L^p_z(\mathbb{R}^3)}.$$

To estimate F we use that χ_r is supported on $(-1,2)^2\times(-2h,h)$, and therefore

$$\begin{aligned}\|\chi_r Ef\|_{L^q_H L^p_z(\mathbb{R}^3)} &\leq 27M^2\|f\|_{L^q_H L^p_z(\Omega)},\\ \|(\nabla\chi_r)\cdot E(\nabla v)\|_{L^q_H L^p_z(\mathbb{R}^3)} &\leq 27M^2 r^{-1}\|\nabla v\|_{L^q_H L^p_z(\Omega)},\\ \|(\Delta\chi_r)Ev\|_{L^q_H L^p_z(\mathbb{R}^3)} &\leq 27M^2 r^{-2}\|v\|_{L^q_H L^p_z(\Omega)}.\end{aligned}$$

Next, we set $r=\eta|\lambda|^{-1/2}$ to obtain

$$\|F\|_{L^q_H L^p_z(\mathbb{R}^3)} \leq 27M^2\left(\|f\|_{L^q_H L^p_z(\Omega)} + 2\eta^{-1}|\lambda|^{1/2}\|\nabla v\|_{L^q_H L^p_z(\Omega)} + \eta^{-2}|\lambda|\cdot\|v\|_{L^q_H L^p_z(\Omega)}\right).$$

Now assume that $\eta>0$ is sufficiently large enough such that $54C_\theta M^2\eta^{-1}<1/2$, $27C_\theta M^2\eta^{-2}<1/2$ and then assume that $\lambda_0>0$ is large enough such that $4\eta\lambda_0^{-1/2}<3\min\{1,h\}$. This and the fact that u is an extension of v then yields

$$|\lambda|\cdot\|v\|_{L^q_H L^p_z(\Omega)} + |\lambda|^{1/2}\|\nabla v\|_{L^q_H L^p_z(\Omega)} \leq 54C_\theta M^2\|f\|_{L^q_H L^p_z(\Omega)} \quad \text{for} \quad |\lambda|\geq\lambda_0.$$

In the case $q=\infty$, $p\in(2,\infty)$ we obtain the estimate for the full range of $\lambda\in\Sigma_\theta$ by setting $\lambda_1:=\frac{\lambda_0}{|\lambda|}\lambda$ for $0<|\lambda|<\lambda_0$. Then $f\in L^\infty_H L^p_z(\Omega)\hookrightarrow L^p(\Omega)$ yields

$$|\lambda|\cdot\|v\|_{L^p(\Omega)} + |\lambda|^{1/2}\|\nabla v\|_{L^p(\Omega)} + \|\Delta v\|_{L^p(\Omega)} \leq C_{\theta,p}\|f\|_{L^p(\Omega)}$$

by (5.9) and since $\lambda_1 v-\Delta v = f+(\lambda_1-\lambda)v$ we obtain

$$|\lambda_1|\cdot\|v\|_{L^\infty_H L^p_z(\Omega)} + |\lambda_1|^{1/2}|\nabla v|_{L^\infty_H L^p_z(\Omega)} + \|\Delta v\|_{L^\infty_H L^p_z(\Omega)} \leq C_{\theta,p}\left(\|f\|_{L^\infty_H L^p_z(\Omega)} + |\lambda_1-\lambda|\cdot\|v\|_{L^\infty_H L^p_z(\Omega)}\right)$$

where we can further estimate $|\lambda_1-\lambda|<\lambda_0$, and $p\in(1,\infty)$ yields

$$\|v\|_{L^\infty_H L^p_z(\Omega)} \leq C_p\|v\|_{W^{2,p}_H L^p_z(\Omega)} \leq C_p\|v\|_{W^{2,p}(\Omega)} \leq C_p\|\Delta v\|_{L^p(\Omega)} \leq C_p\|f\|_{L^p(\Omega)} \leq C_p\|f\|_{L^\infty_H L^p_z(\Omega)}$$

where we used $W^{2,p}(G)\hookrightarrow L^\infty(G)$ and that Δ_p is invertible on $L^p(\Omega)$. Since $|\lambda_1|=\lambda_0>|\lambda|$, this yields the desired result for the full range of $\lambda\in\Sigma_\theta$, $\theta\in(0,\pi)$.

If v instead solves problem (5.6) with $\partial_i=\partial_z$ then u solves the problem

$$\lambda u-\Delta u = G \text{ on } \mathbb{R}^3, \quad G:=\chi_r E(\partial_z f) - 2(\nabla\chi_r)\cdot E(\nabla v) - (\Delta\chi_r)Ev.$$

We rewrite

$$-2(\nabla\chi_r)\cdot E(\nabla v) - (\Delta\chi_r)Ev = -2\mathrm{div}(\nabla\chi_r Ev) + (\Delta\chi_r)Ev, \quad \chi_r E(\partial_z f) = \partial_z(\chi_r sEf) - (\partial_z\chi_r)sEf$$

where

$$s(z)=\begin{cases}1, & z\in(-2h,0),\\ -1, & x\in(0,h).\end{cases}$$

Here, by the density argument above, we may assume $f=0$ on $\Gamma_u\cup\Gamma_b$. This yields $u=u_1+u_2$ where

$$\begin{aligned}\lambda u_1-\Delta u_1 &= \partial_z G_1 + \mathrm{div}_H G_2 && \text{on } \mathbb{R}^3, \quad G_1:=\chi_r sEf, \quad G_2:=-2(\nabla\chi_r)Ev,\\ \lambda u_2-\Delta u_2 &= G_3 && \text{on } \mathbb{R}^3, \quad G_3:=-(\partial_z\chi_r)sEf + (\Delta\chi_r)Ev.\end{aligned}$$

Since G_i, $i\in\{1,2,3\}$, are bounded and have compact support, we may apply Lemma 5.2 to obtain the estimate

$$|\lambda|^{1/2}\|u\|_{L^q_H L^p_z(\mathbb{R}^3)} \leq C_\theta\left(\|G_1\|_{L^q_H L^p_z(\mathbb{R}^3)} + \|G_2\|_{L^q_H L^p_z(\mathbb{R}^3)} + |\lambda|^{-1/2}\|G_3\|_{L^q_H L^p_z(\mathbb{R}^3)}\right).$$

Proceeding as above we obtain

$$\begin{aligned}\|G_1\|_{L^q_H L^p_z(\mathbb{R}^3)} &\leq 27M^2\|f\|_{L^q_H L^p_z(\Omega)},\\ \|G_2\|_{L^q_H L^p_z(\mathbb{R}^3)} &\leq 54M^2\eta^{-1}|\lambda|^{1/2}\|v\|_{L^\infty_H L^p_z(\Omega)},\\ \|G_3\|_{L^\infty_H L^p_z(\mathbb{R}^3)} &\leq 27M^2\eta^{-1}|\lambda|^{1/2}\|f\|_{L^\infty_H L^p_z(\Omega)} + 27M^2\eta^{-2}|\lambda|\cdot\|v\|_{L^\infty_H L^p_z(\Omega)}.\end{aligned}$$

The above assumptions on η and λ_0 then yield the desired result for $|\lambda|>\lambda_0$. The case $\partial_i\in\{\partial_x,\partial_y\}$ is analogous where for $f\in L^\infty(G;C_c^\infty(-h,0))$ horizontal derivatives are understood in the sense of distributions, and otherwise derivatives can be treated using smooth approximations as above.

For the case $q = \infty$ and $p \in (2, \infty)$, to extend this estimate to the full range of $\lambda \in \Sigma_\theta$ one proceeds as above to obtain

$$|\lambda|^{1/2} \cdot \|v\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\Omega)} \leq C_{\theta,p} \|f\|_{L^p(\Omega)}$$

from (5.11), as well as

$$|\lambda_1|^{1/2} \cdot \|v\|_{L^\infty_H L^p_z(\Omega)} + \|\nabla v\|_{L^\infty_H L^p_z(\Omega)} \leq C_\theta \left(\|f\|_{L^\infty_H L^p_z(\Omega)} + |\lambda_1|^{-1/2} |\lambda_1 - \lambda| \cdot \|v\|_{L^\infty_H L^p_z(\Omega)} \right).$$

Since we have $|\lambda_1|^{-1/2}|\lambda_1 - \lambda| \leq \lambda_0^{1/2}$ and $p \in (2, \infty)$ this yields

$$\|v\|_{L^\infty_H L^p_z(\Omega)} \leq C_p \|v\|_{W^{1,p}_H L^p_z(\Omega)} \leq C_p \|v\|_{W^{1,p}(\Omega)} \leq C_p \|\nabla v\|_{L^p(\Omega)} \leq C_p \|f\|_{L^p(\Omega)} \leq C_p \|f\|_{L^\infty_H L^p_z(\Omega)},$$

where we used the embedding $W^{1,p}(G) \hookrightarrow L^\infty(G)$ and the Poincaré inequality $\|v\|_{L^p(\Omega)} \leq C_p \|\nabla v\|_{L^p(\Omega)}$ for v with $v\big|_{\Gamma_b} = 0$. □

Remark 5.5. The results of Lemma 5.4 also hold true if the condition $\partial_z v\big|_{\Gamma_u} = 0$ is replaced by $v\big|_{\Gamma_u} = 0$ or if $L^q_H L^p_z(\Omega)$ is replaced by $C_{\text{per}}([0,1]^2; L^p(-h, 0))$. For pure Dirichlet boundary conditions one extends by an odd reflexion at both $z = 0$ and $z = -h$ replacing $E_z^{\text{even,odd}}$ by $E_z^{\text{odd,odd}}$ and setting $s(z) \equiv 1$ in the proof.

Since $\Omega = G \times (-h, 0)$ is a cylindrical domain the semigroup generated by the Laplacian with the above boundary conditions satisfies

$$e^{t\Delta}(f \otimes g) = e^{t\Delta_H} f \otimes e^{t\Delta_z} g, \quad f : G \to \mathbb{R}^2, \quad g : (-h, 0) \to \mathbb{R},$$

where $(f \otimes g)(x, y, z) := f(x, y)g(z)$ is an elementary tensor, $\Delta_H := \partial_x^2 + \partial_y^2$ is the Laplacian on G with periodic boundary conditions and Δ_z is defined by

$$\Delta_z v := \partial_z^2 v, \quad D(\Delta_z) = \{f \in W^{2,p}(-h, 0) : f(-h) = \partial_z f(0) = 0\}.$$

We now investigate these operators separately, starting with the vertical one, cf. [6, 21].

Lemma 5.6. *Let $p \in (1, \infty)$. Then the operator Δ_z generates a strongly continuous, exponentially stable, analytic semigroup on $L^p(-h, 0)$.*

Lemma 5.7. *Let $\theta \in (0, \pi/2)$. Then there exists a constant $C_\theta > 0$ such that for all $\tau \in \Sigma_\theta$ we have*

$$|\tau|^{1/2} \|\nabla_H e^{\tau \Delta_H} Q f\|_{L^\infty(G)} \leq C_\theta \|f\|_{L^\infty(G)}, \quad f \in L^\infty(G).$$

Remark 5.8. Note that although the two-dimensional Helmholtz projector with periodic boundary conditions Q is unbounded on $L^\infty(G)$, the composition $\nabla_H e^{\tau \Delta_H} Q$ defines a bounded operator for $\tau \in \Sigma_\theta$.

Proof of Lemma 5.7. Let $Q_{\mathbb{R}^2}$ and Q be the Helmholtz projection on $\mathbb{R}^2$ and $\mathbb{T}^2$, respectively, and E_H^{per} be the periodic extension operator from G onto $\mathbb{R}^2$. Then $E_H^{\text{per}} Q f = Q_{\mathbb{R}^2} E_H^{\text{per}} f$ for all $f : G \to \mathbb{R}^2$ and

$$E_H^{\text{per}} |\tau|^{1/2} \nabla_H e^{\tau \Delta_H} Q f = |\tau|^{1/2} \nabla_H e^{\tau \Delta_H} E_H^{\text{per}} Q f = |\tau|^{1/2} \nabla_H e^{\tau \Delta_H} Q_{\mathbb{R}^2} E_H^{\text{per}} f.$$

Since $\|E_H^{\text{per}} f\|_{L^\infty(\mathbb{R}^2)} = \|f\|_{L^\infty(G)}$ it therefore suffices to consider the operator Δ_H on the full space $\mathbb{R}^2$. Recall that $1 - Q_{\mathbb{R}^2}$ is given by $(R_j R_k)_{1 \leq j,k \leq 2}$ where R_j is the Riesz transform in the j-th direction. We therefore investigate the family of Fourier multipliers

$$m_{\tau,j,k,l}(\xi) = \begin{cases} |\tau|^{1/2} \xi_l \left(\delta_{j,k} - \frac{\xi_j \xi_k}{|\xi|^2} \right) e^{-\tau |\xi|^2}, & \xi \in \mathbb{R}^2 \setminus \{0\}, \\ 0, \quad \xi = 0, \end{cases} \quad \text{for} \quad 1 \leq j, k, l \leq 2.$$

Using the invariance under rescaling and replacing ξ with $|\tau|^{-1/2}\xi$, we may assume that $\tau = e^{i\psi}$ where $|\psi| < \theta$. We show that for each of these symbols we have $m = \hat{g}$ for some $g \in L^1(\mathbb{R}^2)$ such that $\|g\|_{L^1(\mathbb{R}^2)} \leq C_\theta$. The desired estimate then follows from Young's inequality. Since this family of symbols belongs to $C(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus \{0\})$ we verify the Mikhlin condition

$$\max_{|\alpha| \leq 2} \sup_{\xi \in \mathbb{R}^2 \setminus \{0\}} |\xi|^{|\alpha| + \delta} |D^\alpha m(\xi)| < M < \infty, \tag{5.12}$$

for some $\delta > 0$. Elementary calculations using the homogeneity of the first factor show that for an arbitrary multi-index $\alpha \in \mathbb{N}^2$ we have

$$\sup_{\xi\in\mathbb{R}^2\setminus\{0\}} |\xi|^\alpha \left| D^\alpha \frac{\xi_j\xi_k}{|\xi|^2} \right| < M_\alpha < \infty, \qquad \sup_{\xi\in\mathbb{R}^2\setminus\{0\}} |\xi|^{\alpha+\delta} \left| D^\alpha \xi_l e^{-e^{i\psi}|\xi|^2} \right| < M_{\alpha,\delta,\psi} \leq M_{\alpha,\delta,\theta} < \infty$$

for $\delta \in (0,1)$ which together with the product rule yield that (5.12) is satisfied. Analogously we verify the condition

$$|\xi|^{|\alpha|} |D^\alpha m(\xi)| \leq C_\alpha |\xi|, \quad |\xi| \leq 1, \xi \neq 0 \tag{5.13}$$

for $0 < |\alpha| \leq 2$ by noting that

$$|\xi|^{|\alpha|} \left| D^\alpha \frac{\xi_j\xi_k\xi_j}{|\xi|^2} \right| \leq C_\alpha |\xi|, \quad |\xi| \leq 1, \xi \neq 0$$

and

$$|\xi|^{|\alpha|} \left| D^\alpha e^{-e^{i\psi}|\xi|^2} \right| + |\xi|^{|\alpha|} \left| D^\alpha \xi_l e^{-e^{i\psi}|\xi|^2} \right| \leq C_{\alpha,\delta,\psi} \leq C_{\alpha,\delta,\theta} \quad |\xi| \leq 1, \xi \neq 0.$$

We now split the symbol into $m = \varphi m + (1-\varphi)m$ where $\varphi \in C_c^\infty(\mathbb{R}^2)$ is a cut-off function satisfying $\varphi(\xi) = 1$ for $|\xi| \leq 2$. Applying [4, Lemma 8.2.3 and 8.2.4] to the terms $(1-\varphi)m$ and φm respectively then yields the desired results. □

6. Linear estimates for the hydrostatic Stokes operator: part 1

A key element in the proof of our global existence results are the estimates for the hydrostatic Stokes semigroup in $X_{\overline{\sigma}}$. To this end, we prove first estimates in the larger space X, where we make use of representation (1.1). We thus define the operator A by

$$Av := \mathcal{A}v, \quad D(A) = \{v \in W^{2,p}_{\mathrm{per}}(\Omega)^2 \cap X \colon \partial_z v\big|_{\Gamma_u} = 0, v\big|_{\Gamma_b} = 0, \mathcal{A}v \in X\}.$$

It is the aim of this section to prove the following claim.

Claim 6.1. *Let $p \in (3,\infty)$. Then*
a) A is the generator of a strongly continuous, analytic semigroup on X.
b) There exist constants $C > 0$, $\beta \in \mathbb{R}$ such that for $\partial_i \in \{\partial_x, \partial_y, \partial_z\}$, $t > 0$ and $f \in X$ one has that

$$t^{1/2} \|\partial_i e^{tA} f\|_{L^\infty_H L^p_z(\Omega)} \leq C e^{\beta t} \|f\|_{L^\infty_H L^p_z(\Omega)}, \quad t > 0,\ f \in X, \tag{i}$$

for $\partial_j \in \{\partial_x, \partial_y\}$

$$t^{1/2} \|\partial_j e^{tA} \mathbb{P} f\|_{L^\infty_H L^p_z(\Omega)} \leq C e^{\beta t} \|f\|_{L^\infty_H L^p_z(\Omega)}, \quad t > 0,\ f \in X, \tag{ii}$$

$$t^{1/2} \|e^{tA} \mathbb{P} \partial_j f\|_{L^\infty_H L^p_z(\Omega)} \leq C e^{\beta t} \|f\|_{L^\infty_H L^p_z(\Omega)}, \quad t > 0,\ f \in X, \tag{iii}$$

$$t \|\partial_i e^{tA} \mathbb{P} \partial_j f\|_{L^\infty_H L^p_z(\Omega)} \leq C e^{\beta t} \|f\|_{L^\infty_H L^p_z(\Omega)}, \quad t > 0,\ f \in X. \tag{iv}$$

c) $X_{\overline{\sigma}}$ is an invariant subspace of A, and its restriction is $A_{\overline{\sigma}}$. The semigroup e^{tA} restricts to an exponentially stable, strongly continuous, analytic semigroup of angle $\pi/2$ on $X_{\overline{\sigma}}$.
d) Furthermore, for all $v \in X_{\overline{\sigma}}$

$$\lim_{t\to 0+} t^{1/2} \|\nabla e^{tA} v\|_{L^\infty_H L^p_z(\Omega)} = 0.$$

In order to solve equation (2.5) in $X_{\overline{\sigma}}$, we collect first several facts concerning the corresponding theory in $L^p_{\overline{\sigma}}(\Omega)$. To this end, let $p \in (1,\infty)$, and define $A_{p,\overline{\sigma}} \colon D(A_{p,\overline{\sigma}}) \to L^p_{\overline{\sigma}}(\Omega)$ by

$$A_{p,\overline{\sigma}} v := \mathbb{P}\Delta v, \quad D(A_{p,\overline{\sigma}}) = \{v \in W^{2,p}_{\mathrm{per}}(\Omega)^2 : \mathrm{div}_H \overline{v} = 0, \partial_z v\big|_{\Gamma_u} = 0, v\big|_{\Gamma_b} = 0.\}.$$

Consider furthermore $A_p \colon D(A_p) \to L^p(\Omega)^2$ defined by

$$A_p v := \Delta_p v + Bv, \quad D(A_p) := D(\Delta_p)^2, \quad Bv := \frac{1}{h}(1-Q)\partial_z v\big|_{\Gamma_b},$$

where Δ_p denotes the Laplacian in $L^p(\Omega)^2$ as in the last section. By [7], the operator A_p is an extension of $A_{p,\overline{\sigma}}$. The idea is that the pressure term may be recovered by applying the vertical average and horizontal divergence to (2.5), yielding

$$\Delta_H \pi = \mathrm{div}_H \overline{f} - \mathrm{div}_H \frac{1}{h} \partial_z v\big|_{\Gamma_b}, \tag{6.1}$$

or equivalently since $1-Q$ agrees with $\nabla_H(-\Delta_H)^{-1}\mathrm{div}_H$ one has $\nabla_H \pi = (1-Q)\overline{f} - Bv$.

Note that the following inclusions hold

$$A \subset A_p \quad \text{and} \quad A_{\overline{\sigma}} \subset A_{p,\overline{\sigma}}, \tag{6.2}$$

and that $e^{tA_{p,\overline{\sigma}}}$, e^{tA_p}, e^{tA} and $e^{tA_{\overline{\sigma}}}$ are consistent semigroups.

Proof of Claim 6.1. Let $\lambda_0 > 0$ with $\lambda_0 \in \rho(A_p)$, $\theta \in (0, \pi/2)$, and

$$\lambda \in \Sigma_{\theta+\pi/2} \cap B_{\lambda_0}(0)^c \subset \rho(A_p).$$

By (6.2) it follows that $\lambda - A$ is injective for $\lambda \in \rho(A_p)$ and likewise $\lambda - A_{\overline{\sigma}}$ is injective for $\lambda \in \rho(A_p^{\overline{\sigma}})$. Since $X \hookrightarrow L^p(\Omega)^2$ the existence of a unique $v \in D(A_p)$ for $p \in (1,\infty)$ follows from the L^p-theory for A_p, cf. [7], and since $W^{2,p}_{\mathrm{per}}(\Omega)^2 \hookrightarrow X$ for $p \in (3/2, \infty)$ it follows that $v \in D(A)$. Since $(A_p - \lambda)^{-1}$ further leaves $L^p_{\overline{\sigma}}(\Omega)$ invariant, $f \in X_{\overline{\sigma}}$ implies $v \in D(A_{\overline{\sigma}})$. Hence,

$$\rho(A_p) \subset \rho(A) \quad \text{and} \quad \rho(A_{p,\overline{\sigma}}) \subset \rho(A_{\overline{\sigma}}). \tag{6.3}$$

In particular the resolvent sets are non-empty and thus the operators are closed.

Since the semigroup estimates follow from resolvent estimates by arguments involving the inverse Laplace transform, it now remains to prove suitable resolvent estimates in X. To this end we observe first that $v = (\lambda - A)^{-1} f$ is equivalent to

$$v = (\lambda - \Delta_p)^{-1}(f + Bv), \tag{6.4}$$

and second, using the fact that Q is continuous on $C^{0,\alpha}_{per}([0,1]^2)$ for $\alpha \in (0,1)$, that

$$\|Bv\|_{L^\infty_H L^p_z(\Omega)} \le h^{1/p} \|Bv\|_{L^\infty(\Omega)} \le h^{1/p} \|Bv\|_{C^{0,\alpha}([0,1]^2)} \le C \|\partial_z v\big|_{\Gamma_b}\|_{C^{0,\alpha}([0,1]^2)} \le C \|v\|_{C^{1,\alpha}(\overline{\Omega})}.$$

Assuming $p \in (3, \infty)$ we have $W^{2,p}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$ for some $\alpha = \alpha_p \in (0, 1-3/p)$. Using the resolvent estimate for A_p in $L^p(\Omega)^2$ we obtain

$$\|v\|_{C^{1,\alpha}(\overline{\Omega})} \le C_p \|v\|_{W^{2,p}(\Omega)} \le C_p \left(\|v\|_{L^p(\Omega)} + \|Av\|_{L^p(\Omega)} \right) \le C_p (1 + |\lambda|^{-1}) \|f\|_{L^p(\Omega)}.$$

This and $|\lambda| > \lambda_0$ yield $\|Bv\|_{L^\infty_H L^p_z(\Omega)} \le C_p(1 + \lambda_0^{-1}) \|f\|_{L^\infty_H L^p_z(\Omega)}$. So, using Lemma 5.4 we obtain

$$|\lambda| \cdot \|v\|_{L^\infty_H L^p_z(\Omega)} + |\lambda|^{1/2} \|\nabla v\|_{L^\infty_H L^p_z(\Omega)} + \|Av\|_{L^\infty_H L^p_z(\Omega)} \le C_{\theta,p,\lambda_0} \|f\|_{L^\infty_H L^p_z(\Omega)}, \tag{6.5}$$

where we used that for λ as above and $p \in (3, \infty)$ one has

$$\|Av\|_{L^\infty_H L^p_z(\Omega)} \le \|\Delta v\|_{L^\infty_H L^p_z(\Omega)} + \|Bv\|_{L^\infty_H L^p_z(\Omega)} \le C_{\theta,p,\lambda_0} \|f\|_{L^\infty_H L^p_z(\Omega)}.$$

Note that if one instead considers $f \in X_{\overline{\sigma}}$, then $\lambda_0 > 0$ can be taken to be arbitrarily small and θ arbitrarily close to $\pi/2$ by [7, Theorem 3.1]. Since $0 \in \rho(A_{p,\overline{\sigma}}) \subset \rho(A_{\overline{\sigma}})$, compare [11, Theorem 3.1] and (6.3) it follows that the spectral bound

$$\beta := \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A_{\overline{\sigma}})\}$$

is negative implying exponential decay, and estimate (6.5) is valid for all $\lambda \in \Sigma_\theta$, $\theta \in (0, \pi)$ and $f \in X_{\overline{\sigma}}$.

To verify that $D(A)$ and $D(A_{\overline{\sigma}})$ are dense in X and $X_{\overline{\sigma}}$ respectively, observe that the space

$$C^\infty_{\mathrm{per}}([0,1]^2; C^\infty_c((-h,0)))^2$$

is contained in $D(A)$ and dense in X, so the semigroup generated by A is strongly continuous on X. Since it leaves $L^p_{\overline{\sigma}}(\Omega)$ invariant, the restriction of the semigroup on $X \cap L^p_{\overline{\sigma}}(\Omega) = X_{\overline{\sigma}}$ is strongly continuous as well and generated by the restriction of A onto $D(A) \cap L^p_{\overline{\sigma}}(\Omega) = D(A_{\overline{\sigma}})$, i.e. $A_{\overline{\sigma}}$, which is therefore densely defined on $X_{\overline{\sigma}}$. Thus we have proven *a*), *c*) and estimate (*i*) in *b*).

To prove the remaining semigroup estimates in b) we consider the corresponding resolvent estimates. Since $X \hookrightarrow L^p(\Omega)^2$ and $\mathbb{P}$ is bounded on $L^p(\Omega)^2$ the existence of

$$v := (\lambda - A_{p,\overline{\sigma}})^{-1}\mathbb{P}f \in D(A_{p,\overline{\sigma}}) \hookrightarrow W^{2,p}_{\text{per}}(\Omega)^2 \hookrightarrow X$$

for $f \in X$ follows from the L^p-theory for $A_{p,\overline{\sigma}}$, and it suffices to extend the L^p-estimate

$$|\lambda|^{1/2}\|\partial_i(\lambda - A_{p,\overline{\sigma}})^{-1}f\|_{L^p(\Omega)} + |\lambda|^{1/2}\|(\lambda - A_{p,\overline{\sigma}})^{-1}\partial_i f\|_{L^p(\Omega)} \le C_{\theta,p}\|f\|_{L^p(\Omega)}, \quad f \in L^p_{\overline{\sigma}}(\Omega), \tag{6.6}$$

where $\partial_i \in \{\partial_x, \partial_y\}$, $\theta \in (0, \pi)$, $C_{\theta,p} > 0$, to X, i.e. to prove the estimate

$$|\lambda|^{1/2}\|\nabla_H v\|_{L^\infty_H L^p_z(\Omega)} \le C_{\theta,p}\|f\|_{L^\infty_H L^p_z(\Omega)}, \quad \lambda \in \Sigma_\theta. \tag{6.7}$$

Recall that $\mathbb{P}f = f - (1 - Q)\overline{f} = \tilde{f} + Q\overline{f}$, and that if $f \in X$ then $\overline{f} \in C_{\text{per}}([0, 1]^2)^2$ satisfies $\|\overline{f}\|_\infty \le C\|f\|_{L^\infty_H L^p_z(\Omega)}$ for any $p \in [1, \infty]$. Using (6.4) we rewrite

$$v = (\lambda - A_{\overline{\sigma}})^{-1}\mathbb{P}f = (\lambda - \Delta)^{-1}(\tilde{f} + Bv + Q\overline{f}),$$

and since the term $\tilde{f} + Bv$ can be dealt with as before, it suffices to show the estimate

$$|\lambda|^{1/2}\|\nabla_H(\lambda - \Delta)^{-1}Q\overline{f}\|_{L^\infty_H L^p_z(\Omega)} \le C_\theta\|\overline{f}\|_{L^\infty(G)}. \tag{6.8}$$

Since $Q\overline{f}$ does not depend on z we can write $Q\overline{f} = Q\overline{f}\otimes 1$, and so for $\lambda = |\lambda|e^{i\psi}$ with $\psi \in (-\pi/2+\varepsilon, \pi/2-\varepsilon)$ for small $\varepsilon > 0$ we have

$$|\lambda|^{1/2}\nabla_H(\lambda - \Delta)^{-1}\left(Q\overline{f} \otimes 1\right) = |\lambda|^{1/2}\int_0^\infty e^{-\lambda t}\left(\nabla_H e^{t\Delta_H}Q\overline{f} \otimes e^{t\Delta_z}1\right)\, dt,$$

where $e^{t\Delta_z}$ denotes the semigroup from Lemma 5.6. Applying the estimates in Lemma 5.7 and 5.6 yields

$$\begin{aligned}
|\lambda|^{1/2}\|\nabla_H(\lambda - \Delta)^{-1}\left(Q\overline{f} \otimes 1\right)\|_{L^\infty_H L^p_z(\Omega)} &\le |\lambda|^{1/2}\int_0^\infty e^{-\lambda t}\|\nabla_H e^{t\Delta_H}Q\overline{f}\|_{L^\infty(G)}\|e^{t\Delta_z}1\|_{L^p(-h,0)}\, dt \\
&\le C|\lambda|^{1/2}\left(\int_0^\infty e^{-|\lambda|\cos(\psi)t}t^{-1/2}\, dt\right)\|\overline{f}\|_{L^\infty(G)} \\
&\le C\frac{\sqrt{\pi}}{\sqrt{\cos(\pi/2 - \varepsilon)}}\|\overline{f}\|_{L^\infty(G)}.
\end{aligned}$$

To include the full range of angles ψ one simply replaces Δ_H and Δ_z with $e^{i\theta}\Delta_H$ and $e^{i\theta}\Delta_z$ respectively where $\theta \in (-\pi/2, \pi/2)$ is a suitable angle.

Since an elementary calculation shows that ∇_H commutes with A and $\mathbb{P}$ we obtain

$$\partial_i(\lambda - A)^{-1}f = (\lambda - A)^{-1}\partial_i f, \quad \partial_i(\lambda - A)^{-1}\mathbb{P}f = (\lambda - A)^{-1}\mathbb{P}\partial_i f$$

for horizontal derivatives $\partial_i \in \{\partial_x, \partial_y\}$ and $f \in C^\infty_{\text{per}}([0, 1]^2; C^\infty_c[-h, 0])^2$. Note that for any $v \in W^{2,p}_{\text{per}}(\Omega)$ the horizontal derivatives $\partial_x v$ and $\partial_y v$ are periodic on Γ_l as well. This yields suitable estimates for the right-hand sides.

To verify d), we first make use of the density of the domains of the generators. So, let $\varepsilon > 0$ and $v' \in D(A_{\overline{\sigma}})$ such that $\|v - v'\|_{L^\infty_H L^p_z(\Omega)} < \varepsilon/2C_0$. By b) (i) we have

$$t^{1/2}\|\nabla e^{tA}v\|_{L^\infty_H L^p_z(\Omega)} \le C_0\|v\|_{L^\infty_H L^p_z(\Omega)}$$

for all $v \in X$ and $t > 0$. Then

$$t^{1/2}\|\nabla e^{tA}v\|_{L^\infty_H L^p_z(\Omega)} \le \frac{\varepsilon}{2} + t^{1/2}\|\nabla e^{tA}v'\|_{L^\infty_H L^p_z(\Omega)}$$

and we can further estimate

$$\|\nabla e^{tA}v'\|_{L^\infty_H L^p_z(\Omega)} \le h^{1/p}\|e^{tA}v'\|_{C^1(\overline{\Omega})} \le C_p\|e^{tA}v'\|_{D(A_{p,\overline{\sigma}})}.$$

This and the invertibility of $A_{p,\overline{\sigma}}$ on $L^p_{\overline{\sigma}}(\Omega)$ yield

$$t^{1/2}\|\nabla e^{tA_{p,\overline{\sigma}}}v'\|_{L^p_{\overline{\sigma}}(\Omega)} \le C_p t^{1/2}\|A_{p,\overline{\sigma}}e^{tA_{p,\overline{\sigma}}}v'\|_{L^p_{\overline{\sigma}}(\Omega)} = C_p t^{1/2}\|e^{tA_{p,\overline{\sigma}}}A_{p,\overline{\sigma}}v'\|_{L^p_{\overline{\sigma}}(\Omega)} \le C_p t^{1/2}\|A_{p,\overline{\sigma}}v'\|_{L^p_{\overline{\sigma}}(\Omega)}$$

and since $A_{p,\overline{\sigma}}v' \in L^p_{\overline{\sigma}}(\Omega)$ the claim follows. □

7. Linear estimates for the hydrostatic Stokes operator: part 2

This section is devoted to prove that the estimates of Claim 6.1 in the case of vertical derivatives, i.e. that the estimates (ii), (iii) and (iv) in Claim 6.1 are valid even for $\partial_j = \partial_z$.

Claim 7.1. *Under the assumptions of Claim 6.1 there exist constants $C > 0$ and $\beta \in \mathbb{R}$ such that*

$$t^{1/2}\|\partial_z e^{tA}\mathbb{P}f\|_{L^\infty_H L^p_z(\Omega)} \le Ce^{t\beta}\|f\|_{L^\infty_H L^p_z(\Omega)}, \tag{7.1}$$

$$t^{1/2}\|e^{tA}\mathbb{P}\partial_z f\|_{L^\infty_H L^p_z(\Omega)} \le Ce^{t\beta}\|f\|_{L^\infty_H L^p_z(\Omega)}, \tag{7.2}$$

$$t\|\partial_i e^{tA}\mathbb{P}\partial_j f\|_{L^\infty_H L^p_z(\Omega)} \le Ce^{\beta t}\|f\|_{L^\infty_H L^p_z(\Omega)}, \tag{7.3}$$

where $\partial_i, \partial_j \in \{\partial_x, \partial_y, \partial_z\}$, for all $t > 0$ and $f \in X$.

As in the last section, these semigroup estimates follow from suitable resolvent estimates and standard arguments involving the inverse Laplace transform.

Before investigating the estimate for $\partial_z(\lambda - A)\mathbb{P}$ we present an anisotropic version of an interpolation inequality. We use the notation $(x, y, z) =: (x', z)$ and let $B(x'_0; r) = \{x' \in \mathbb{R}^2 : |x' - x'_0| < r\}$ denote a disk in $\mathbb{R}^2$.

Lemma 7.2. *Let $p \in (2, \infty)$, $q \in [1, \infty]$, $r > 0$, and $x'_0 \in \mathbb{R}^2$. Then, for $v \in W^{1,p}(B(x'_0; r); L^q_z)$, $L^q_z = L^q(-h, 0)$ we have*

$$\|v\|_{L^\infty(B(x'_0;r);L^q_z)} \le Cr^{-2/p}(\|v\|_{L^p(B(x'_0;r);L^q_z)} + r\|\nabla_H v\|_{L^p(B(x'_0;r);L^q_z)}),$$

where the constant $C = C_{\Omega,p,q} > 0$ is independent of r and x'_0.

Proof. We put $w(x') := (\int_{-h}^0 |v(x', z)|^q \, dz)^{1/q}$ and apply a two-dimensional interpolation inequality, compare [20, Lemma 3.1.4] to have

$$\|w\|_{L^\infty(B(x'_0;r))} \le Cr^{-2/p}(\|w\|_{L^p(B(x'_0;r))} + r\|\nabla_H w\|_{L^p(B(x'_0;r))}). \tag{7.4}$$

One sees that $\|w\|_{L^p(B(x'_0;r))} = \|v\|_{L^p(B(x'_0;r);L^q_z)}$. To estimate the second term we compute $\partial_i w$ for $\partial_i \in \{\partial_x, \partial_y\}$ as follows:

$$\partial_i w(x') = \left(\int_{-h}^0 |v(x', z)|^q \, dz\right)^{1/q-1} \int_{-h}^0 |v(x', z)|^{q-2}(\partial_i v(x', z) \cdot v(x', z)) \, dz.$$

Using Hölder's inquality we obtain

$$|\partial_i w(x')| \le \left(\int_{-h}^0 |v(x', z)|^q \, dz\right)^{1/q-1} \int_{-h}^0 |v(x', z)|^{q-1}|\partial_i v(x', z)| \, dz \le \left(\int_{-h}^0 |\partial_i v(x', z)|^q \, dz\right)^{1/q}$$

and substituting this into (7.4) proves the estimate for $q < \infty$. The case $q = \infty$ is a straightforward result of (7.4). □

It is well known that $1 - Q = -\nabla_H(-\Delta_H)^{-1}\mathrm{div}_H = \nabla_H\Delta_H^{-1}\mathrm{div}_H$ with periodic boundary conditions is a singular integral operator which fails to be bounded in $L^\infty(G)^2$. However, if one allows for a logarithmic (and therefore divergent) factor, some L^∞-type estimate are still available. In this spirit we give a local L^p-estimate for the operator $\nabla_H(-\Delta_H)^{-1}\mathrm{div}_H$ corresponding to the scale of the L^∞-norm.

Proposition 7.3. *Let $p \in (1, \infty)$, $x'_0 \in G$. Then there exists $r_0 > 0$ such that for all $r \in (0, r_0)$ the weak solution of*

$$\Delta_H \pi = \mathrm{div}_H F \quad \text{in} \quad G, \qquad \pi|_{\partial G} : \text{ periodic}, \qquad \int_G \pi \, dx' = 0, \tag{7.5}$$

for $F \in L^\infty(G)^2$ satisfies

$$\|\nabla_H \pi\|_{L^p(B(x'_0;r))} \le Cr^{2/p}(1 + |\log r|)\|F\|_{L^\infty(G)}.$$

Here the constant $C = C_{G,p} > 0$ is independent of x'_0 and r.

Proof. By applying a periodic extension we may assume that (7.5) holds in a larger square $G' := (-2,3)^2$. We choose $r_0 < 1/8$ to obtain $B(x_0'; 4r_0) \subset (-1/2, 3/2)^2$ and utilize two cut-off functions $\omega, \theta \in C_c^\infty(\mathbb{R}^2)$, $\theta = \theta_r$, satisfying the following properties:

$$\begin{aligned} &\omega \equiv 1 \text{ on } [-1,2]^2, && \operatorname{supp}(\omega) \subset G', && \|\nabla_H^k \omega\|_{L^\infty(\mathbb{R}^2)} \leq C, \\ &\theta \equiv 1 \text{ on } B(x_0'; 2r), && \operatorname{supp}(\theta) \subset B(x_0'; 4r), && \|\nabla_H^k \theta\|_{L^\infty(\mathbb{R}^2)} \leq Cr^{-k} \end{aligned}$$

for $k = 0, 1, 2$; compare the proof of Lemma 5.4. From (7.5) we see that $\omega\pi$ satisfies

$$\Delta_H(\omega\pi) = \operatorname{div}_H(\omega F) - \nabla_H\omega \cdot F + 2\operatorname{div}_H\big((\nabla_H\omega)\pi\big) - (\Delta_H\omega)\pi \quad \text{in} \quad \mathbb{R}^2.$$

Then, letting $\Psi(x', y') := \frac{1}{2\pi}\log|x' - y'|$ be the Green's function for the Laplacian in $\mathbb{R}^2$, we obtain

$$(\omega\pi)(x') = -\int_{\mathbb{R}^2} (\nabla_{y'}\Psi)(x', y') \cdot \big[\omega F + 2(\nabla_{y'}\omega)\pi\big](y')\, dy' - \int_{\mathbb{R}^2} \Psi(x', y')\big[(\nabla_H\omega)\cdot F + (\Delta_H\omega)\pi\big](y')\, dy'.$$

Therefore, for $x' \in B(x_0'; r)$ we have the representation

$$\begin{aligned} \nabla_H\pi(x') &= -\int_{\mathbb{R}^2} (\nabla_{x'}\nabla_{y'}\Psi)(x', y')\big[\omega F + 2(\nabla_{y'}\omega)\pi\big](y')\, dy' - \int_{\mathbb{R}^2} (\nabla_{x'}\Psi)(x', y')\big[(\nabla_H\omega)\cdot F + (\Delta_H\omega)\pi\big](y')\, dy' \\ &= -\int_{\mathbb{R}^2} (\nabla_{x'}\nabla_{y'}\Psi)(x', y')\big[\theta F + \omega(1-\theta)F + 2(\nabla_{y'}\omega)\pi\big](y')\, dy' \\ &\quad - \int_{\mathbb{R}^2} (\nabla_{x'}\Psi)(x', y')\big[(\nabla_H\omega)\cdot F + (\Delta_H\omega)\pi\big](y')\, dy' \\ &=: \Pi_1(x') + \Pi_2(x') + \Pi_3(x') + \Pi_4(x') + \Pi_5(x') \end{aligned}$$

where in the second step we used $\omega\theta = \theta$. We derive $L^p(B(x_0'; r))$-estimates for each of the above terms as follows: By the Calderón–Zygmund inequality we have

$$\|\Pi_1\|_{L^p(\mathbb{R}^2)} \leq C\|\theta F\|_{L^p(\mathbb{R}^2)} \leq C\|\theta\|_{L^p(\mathbb{R}^2)}\|F\|_{L^\infty(G')} \leq Cr^{2/p}\|F\|_{L^\infty(G)}.$$

For the second term note that we have $|\nabla_{x'}\nabla_{y'}\Psi(x', y')| \leq C|x' - y'|^{-2}$ and

$$\operatorname{supp}(\omega(1-\theta)) = \operatorname{supp}(\omega - \theta) \subset \operatorname{supp}(\omega) \setminus B(x_0'; 2r)$$

yields $\operatorname{supp}(\omega(1-\theta)) \subset \{r \leq |x' - y'| \leq 4\}$ and therefore

$$\begin{aligned} \|\Pi_2\|_{L^p(B(x_0';r))} &\leq \|1\|_{L^p(B(x_0';r))}\left(\sup_{x' \in B(x_0';r)} \int_{r \leq |x'-y'| \leq 4} C|x' - y'|^{-2}\, dy'\right)\|\omega(1-\theta)F\|_{L^\infty(G')} \\ &\leq Cr^{2/p}(1 + |\log r|)\|F\|_{L^\infty(G)}. \end{aligned}$$

The condition $\operatorname{supp}(\nabla_H\omega) \subset G' \setminus [-1, 2]$ yields

$$\|\Pi_3\|_{L^p(B(x_0';r))} \leq \|1\|_{L^p(B(x_0';r))}\left(\sup_{1/2 \leq |x'-y'| \leq 3} C|x' - y'|^{-2}\right)\|2(\nabla_H\omega)\pi\|_{L^1(G')} \leq Cr^{2/p}\|\pi\|_{L^1(G)}.$$

It follows from Poincaré's inequality and the L^2-theory for (7.5) that

$$\|\pi\|_{L^1(G')} \leq C\|\pi\|_{L^2(G)} \leq C\|\nabla_H\pi\|_{L^2(G)} \leq C\|F\|_{L^2(G)} \leq C\|F\|_{L^\infty(G)}$$

and therefore $\|\Pi_3\|_{L^p(B(x_0';r))} \leq Cr^{2/p}\|F\|_{L^\infty(G)}$. Similarly to Π_3, we have

$$\begin{aligned} \|\Pi_4 + \Pi_5\|_{L^p(B(x_0';r))} &\leq \|1\|_{L^p(B(x_0';r))}\Big(\sup_{1/2 \leq |x'-y'| \leq 3} C|x' - y'|^{-1}\Big)(\|F\|_{L^1(G)} + \|\pi\|_{L^1(G)}) \\ &\leq Cr^{2/p}\|F\|_{L^\infty(G)}. \end{aligned}$$

Combining these estimates yields the desired estimate. □

Remark 7.4. Note that the Calderón–Zygmund inequality we have used to estimate Π_1 does not hold for $p \in \{1, \infty\}$ while the arguments of Section 6 can be adapted to cover the case $p = \infty$.

We now turn to prove the estimate $|\lambda|^{1/2}\|\partial_z(\lambda-A)^{-1}\mathbb{P}\|_{L^\infty_H L^p_z(\Omega)} \le C_{\theta,p,\lambda_0}\|f\|_{L^\infty_H L^p_z(\Omega)}$ for $\lambda\in\Sigma_\theta$, $|\lambda|>\lambda_0$ and for $\theta\in(0,\pi)$, $p>3$. For this purpose we observe that the solution v to the resolvent problem

$$\lambda v - Av = \mathbb{P}f \quad \text{on} \quad \Omega$$

with boundary conditions (2.3) is decomposed as $v = v_1+v_2$, where (v_1,π_1) and (v_1,π_1) solve

$$\lambda v_1 - \Delta v_1 + \nabla_H \pi_1 = f \text{ on } \Omega, \quad \Delta_H\, \pi_1 = -h^{-1}\mathrm{div}_H(\partial_z v|_{\Gamma_b}) \text{ on } G, \tag{7.6}$$

and

$$\lambda v_2 - \Delta v_2 + \nabla_H \pi_2 = 0 \text{ on } \Omega, \quad \Delta_H\, \pi_2 = \mathrm{div}_H\, \bar{f} \text{ on } G, \tag{7.7}$$

respectively, both equipped with the boundary conditions (2.3) and periodic boundary conditions for π_i on ∂G, as $\pi := \pi_1+\pi_2$ satisfies (6.1). Since (7.6) is equivalent to $v_1 = (\lambda-\Delta)^{-1}(f+Bv)$ we obtain

$$|\lambda|^{1/2}\|\partial_z v_1\|_{L^\infty_H L^p_z(\Omega)} \le |\lambda|^{1/2}\|\nabla v_1\|_{L^\infty_H L^p_z(\Omega)} \le C_{\theta,p,\lambda_0}\|f\|_{L^\infty_H L^p_z(\Omega)} \tag{7.8}$$

for $|\lambda|>\lambda_0$ by the same argument used to derive (6.5). This, $\nabla_H v_2 = \nabla_H v - \nabla_H v_1$, and estimate (6.7) yield

$$|\lambda|^{1/2}\|\nabla_H v_2\|_{L^\infty_H L^p_z(\Omega)} \le C_{\theta,p,\lambda_0}\|f\|_{L^\infty_H L^p_z(\Omega)}, \quad \lambda\in\Sigma_\theta. \tag{7.9}$$

In order to prove estimate (7.1) it thus remains to establish the following.

Proposition 7.5. *Let $p\in(3,\infty)$ and $\theta\in(0,\pi)$. Then there exists constants $\lambda_0>0$ and $C_{\theta,p,\lambda_0}>0$ such that for all $\lambda\in\Sigma_\theta$ with $|\lambda|>\lambda_0$ and $f\in X$ the solution v_2 of (7.7) satisfies*

$$|\lambda|^{1/2}\|\partial_z v_2\|_{L^\infty_H L^p_z(\Omega)} \le C_{\theta,p,\lambda_0}\|f\|_{L^\infty_H L^p_z(\Omega)}.$$

Remark 7.6. The estimate

$$|\lambda|^{1/2}\|\partial_z(\lambda-A)^{-1}\mathbb{P}f\|_{L^\infty_H L^p_z(\Omega)} \le C_{\theta,p}\|f\|_{L^\infty_H L^p_z(\Omega)}, \quad f\in X$$

actually holds for the full range of $\lambda\in\Sigma_\theta$, $\theta\in(0,\pi)$, i.e. one can take $\lambda_0=0$. This is obtained by using that $\mathbb{P}f\in L^p_{\overline{\sigma}}(\Omega)$ yields $v:=(\lambda-A)^{-1}\mathbb{P}f\in D(A_{p,\overline{\sigma}})$ and therefore

$$\|v\|_{L^\infty_H L^p_z(\Omega)} \le C_p\|v\|_{W^{2,p}(\Omega)} \le C_p\|Av\|_{L^p_{\overline{\sigma}}(\Omega)} \le C_p\|\mathbb{P}f\|_{L^p_{\overline{\sigma}}(\Omega)} \le C_p\|f\|_{L^p_{\overline{\sigma}}(\Omega)} \le C_p\|f\|_{L^\infty_H L^p_z(\Omega)},$$

so the same argument as in the proof of Lemma 5.4 applies.

Proof of Proposition 7.5. We will simply write (v,π) instead of (v_2,π_2) for the solution of (7.7). By applying a periodic extension in the horizontal variables we may assume that (7.7) holds in a larger domain allowing us to replace Ω and G by $\Omega' := G'\times(-h,0)$ and $G' := (-2,3)^2$ respectively. We decompose the boundary of Ω' into $\Gamma'_u = G'\times\{0\}$, $\Gamma'_l := \partial G'\times[-h,0]$ and $\Gamma'_b = G\times\{-h\}$. For simplicity we continue to denote the periodic extensions of v, π and f in the same manner.

Let $\eta>1$ be a parameter to be fixed later, and let λ_0 be a positive number such that

$$r_0 := \eta\,\lambda_0^{-1/2} < \min\{1/8, h/4\}. \tag{7.10}$$

We fix arbitrary $\lambda\in\Sigma_\theta$, $|\lambda|>\lambda_0$, put $r:=\eta|\lambda|^{-1/2}<r_0$, and introduce two cut-off functions $\alpha=\alpha_r$, $\beta=\beta_r$, satisfying

$$\begin{aligned}
&\alpha\in C^\infty([-h,0]), \quad \alpha\equiv 0 \text{ on } [-h,-h+r], \quad \alpha\equiv 1 \text{ on } [-h+2r,0], \quad |\partial_z^k\alpha(z)|\le Cr^{-k},\\
&\beta\in C^\infty([-h,0]), \quad \beta\equiv 1 \text{ on } [-h,-h+2r], \quad \beta\equiv 0 \text{ on } [-h+3r,0], \quad |\partial_z^k\beta(z)|\le Cr^{-k}
\end{aligned}$$

for $k=0,1,2$, compare the proof of Lemma 5.4. We then split the estimate for $\partial_z v$ into the "upper" and "lower" parts in Ω as

$$\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)} \le \|\partial_z(\alpha v)\|_{L^\infty_H L^p_z(\Omega)} + \|\partial_z(\beta v)\|_{L^\infty_H L^p_z(\Omega)}. \tag{7.11}$$

Step 1. Let us first focus on $\partial_z(\alpha v)$. By Lemma 7.2 with radius $|\lambda|^{-1/2}$ and $p = q$ we have

$$|\lambda|^{1/2}\|\partial_z(\alpha v)\|_{L^\infty_H L^p_z(\Omega)} \leq C_p|\lambda|^{1/p} \sup_{x_0'\in G}\Big(|\lambda|^{1/2}\|\partial_z(\alpha v)\|_{L^p(C(x_0';|\lambda|^{-1/2}))} + \|\nabla_H\partial_z(\alpha v)\|_{L^p(C(x_0';|\lambda|^{-1/2}))}\Big), \tag{7.12}$$

where $C(x_0';|\lambda|^{-1/2})$ denotes the cylinder $B(x_0';|\lambda|^{-1/2})\times(-h,0)$ and we used that

$$\|f\|_{L^\infty_H L^p_z(\Omega)} = \sup_{x_0'\in G}\|f\|_{L^\infty(B(x_0',R);L^p_z)}, \quad R>0.$$

In the following we fix arbitrary $x_0'\in G$ and introduce a cut-off function $\theta=\theta_r\in C_c^\infty(\mathbb{R}^2)$ such that

$$\theta\equiv 1 \text{ in } \overline{B(x_0';|\lambda|^{-1/2})}, \quad \operatorname{supp}\theta\subset B(x_0';r), \quad \|\nabla_H^k\theta\|_{L^\infty(\mathbb{R}^2)}\leq Cr^{-k}$$

for $k=0,1,2$. Then $\theta\alpha v$ solves

$$\begin{aligned}\lambda(\theta\alpha v)-\Delta(\theta\alpha v) &= -\theta\alpha\nabla_H\pi - 2\nabla(\theta\alpha)\cdot\nabla v - (\Delta(\theta\alpha))v \quad \text{on} \quad \Omega',\\ \partial_z(\theta\alpha v)|_{\Gamma_u'\cup\Gamma_b'} &= 0, \quad \theta\alpha v \text{ periodic on } \Gamma_l'.\end{aligned}$$

We further differentiate this equation with respect to z to obtain

$$\lambda(\theta\partial_z(\alpha v)) - \Delta(\theta\partial_z(\alpha v)) = F_1 + \partial_z F_2 \quad \text{on} \quad \Omega', \quad \theta\partial_z(\alpha v)\big|_{\Gamma_u'\cup\Gamma_b'} = 0, \quad \theta\partial_z(\alpha v) \text{ periodic on } \Gamma_l'.$$

where

$$\begin{aligned}F_1 &:= -\theta(\partial_z\alpha)(\nabla_H\pi) - (\Delta_H\theta)(\partial_z\alpha)v - (\Delta_H\theta)\alpha(\partial_z v),\\ F_2 &:= -2(\nabla_H\theta)\alpha\cdot(\nabla_H v) - 2\theta(\partial_z\alpha)(\partial_z v) - \theta(\partial_z^2\alpha)v.\end{aligned}$$

By (5.7) and (5.8) for Ω' in the case $q=p$, we obtain the estimate

$$|\lambda|^{1/2}\|\partial_z(\theta\alpha v)\|_{L^p(\Omega')} + \|\nabla\partial_z(\theta\alpha v)\|_{L^p(\Omega')} \leq C_\theta\left(|\lambda|^{-1/2}\|F_1\|_{L^p(\Omega')} + \|F_2\|_{L^p(\Omega')}\right). \tag{7.13}$$

and since $\theta\equiv 1$ on $C(x_0';|\lambda|^{-1/2})\subset\Omega'$ by (7.10), we further have

$$\begin{aligned}\|\partial_z(\alpha v)\|_{L^p(C(x_0';|\lambda|^{-1/2}))} &\leq \|\partial_z(\theta\alpha v)\|_{L^p(\Omega')},\\ \|\nabla\partial_z(\alpha v)\|_{L^p(C(x_0';|\lambda|^{-1/2}))} &\leq \|\nabla\partial_z(\alpha v)\|_{L^p(C(x_0';|\lambda|^{-1/2}))}.\end{aligned} \tag{7.14}$$

Let us estimate each term on this right-hand side of (7.13) as follows: Denoting $\|\cdot\|_{L^p_H} := \|\cdot\|_{L^p(B(x_0';r))}$ and $\|\cdot\|_{L^p_z} := \|\cdot\|_{L^p(-h,0)}$, we first observe that the cut-off functions satisfy

$$\|\theta\|_{L^p_H}\leq Cr^{2/p}, \quad \|\nabla_H\theta\|_{L^p_H}\leq Cr^{2/p-1}, \quad \|\Delta_H\theta\|_{L^p_H}\leq Cr^{2/p-2}$$

as well as

$$\|\partial_z\alpha\|_{L^p_z}\leq Cr^{1/p-1}, \quad \|\partial_z^2\alpha\|_{L^p_z}\leq Cr^{1/p-2}.$$

By Proposition 7.3 we then have

$$\|\theta(\partial_z\alpha)(\nabla_H\pi)\|_{L^p(\Omega')} \leq \|\theta\|_\infty\|\partial_z\alpha\|_{L^p_z}\|\nabla_H\pi\|_{L^p_H} \leq C_p r^{3/p-1}(1+|\log r|)\|f\|_{L^\infty_H L^p_z(\Omega)}.$$

We further have the Poincaré inequality

$$\|f\|_{L^\infty(G';L^p(-h,-h+d))} \leq d\|\partial_z f\|_{L^\infty_H L^p_z}, \quad 0\leq d\leq h, \quad f\big|_{\Gamma_b'}=0 \tag{7.15}$$

and hence using Hölder's inequality yields

$$\|(\Delta_H\theta)(\partial_z\alpha)v\|_{L^p(\Omega')} \leq \|\Delta_H\theta\|_{L^p_H}\|\partial_z\alpha\|_\infty\|v\|_{L^\infty(G';L^p(-h,-h+2r))} \leq Cr^{2/p-2}\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)}.$$

For the third term in F_1 we simply have

$$\|(\Delta_H\theta)\alpha(\partial_z v)\|_{L^p(\Omega')} \leq \|\Delta_H\theta\|_{L^p_H}\|\alpha\|_\infty\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)} \leq Cr^{2/p-2}\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)}.$$

The first term in F_2 is estimated via (7.9), yielding

$$\|(\nabla_H\theta)\alpha(\nabla_H v)\|_{L^p(\Omega')} \leq \|\nabla_H\theta\|_{L^p_H}\|\alpha\|_\infty\|\nabla_H v\|_{L^\infty_H L^p_z(\Omega)} \leq C_{\theta,p,\lambda_0} r^{2/p-1}|\lambda|^{-1/2}\|f\|_{L^\infty_H L^p_z(\Omega)},$$

whereas for the second term in F_2 we simply have

$$\|\theta(\partial_z\alpha)(\partial_z v)\|_{L^p(\Omega')} \le \|\theta\|_\infty \|\partial_z\alpha\|_\infty \|\partial_z v\|_{L^\infty_H L^p_z(\Omega)} \le Cr^{2/p-1}\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)},$$

and by the Poincaré inequality (7.15) we estimate the last term by

$$\|\theta(\partial_z^2\alpha)v\|_{L^p(\Omega')} \le \|\theta\|_{L^p_H}\|\partial_z^2\alpha\|_\infty\|v\|_{L^\infty_H L^p_z(\Omega)} \le Cr^{2/p-1}\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)}.$$

Collecting the above estimates, using (7.12), (7.13) and (7.14), as well as $r = \eta|\lambda|^{-1/2}$, we obtain that

$$\begin{aligned} |\lambda|^{1/2}\|\partial_z(\alpha v)\|_{L^\infty_H L^p_z(\Omega)} &\le C_{\theta,p,\lambda_0}\left(\eta^{2/p-2} + \eta^{3/p-2}|\lambda|^{-1/2p} + \eta^{2/p-1}r^{1/p}|\log(r)|\right)\|f\|_{L^\infty_H L^p_z(\Omega)} \\ &\quad + C_{\theta,p}(\eta^{2/p-1}+\eta^{2/p-2})|\lambda|^{1/2}\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)}. \\ &\le C_{\theta,p,\lambda_0}\eta^{2/p-1}\big(1 + r^{1/p}|\log r|\big)\|f\|_{L^\infty_H L^p_z(\Omega)} \\ &\quad + C_{\theta,p}(\eta^{2/p-1}+\eta^{2/p-2})|\lambda|^{1/2}\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)}. \end{aligned} \tag{7.16}$$

Step 2: Now we shall estimate $\partial_z(\beta v)$. We apply Lemma 7.2 as in the previous step to obtain

$$|\lambda|^{1/2}\|\partial_z(\beta v)\|_{L^\infty_H L^p_z(\Omega)} \le C_p|\lambda|^{1/p}\sup_{x_0'\in G}\left(|\lambda|^{1/2}\|\partial_z(\beta v)\|_{L^p(C(x_0';|\lambda|^{-1/2}))} + \|\nabla_H\partial_z(\beta v)\|_{L^p(C(x_0';|\lambda|^{-1/2}))}\right). \tag{7.17}$$

In the following we fix an arbitrary point $x_0' \in G$. With the same cut-off function $\theta \in C_c^\infty(\mathbb{R}^2)$ as in Step 1, we find that $\theta\beta v$ solves

$$\lambda(\theta\beta v) - \Delta(\theta\beta v) = F_3 \quad \text{in} \quad \Omega', \quad \partial_z(\theta\beta v)|_{\Gamma_u'} = 0, \quad \theta\beta v\big|_{\Gamma_b'} = 0, \quad \theta\beta v \text{ periodic on } \Gamma_l'$$

where

$$F_3 := -\theta\beta(\nabla_H\pi) - 2(\nabla_H\theta)\beta\cdot(\nabla_H v) - 2\theta(\partial_z\beta)(\partial_z v) - (\Delta_H\theta)\beta v - 2\theta(\partial_z^2\beta)v.$$

We apply estimate (5.7) on Ω' with $q = p$ to obtain

$$|\lambda|^{1/2}\|\nabla(\theta\beta v)\|_{L^p(\Omega')} + \|\Delta(\theta\beta v)\|_{L^p(\Omega')} \le C_\theta\|F_3\|_{L^p(\Omega')} \tag{7.18}$$

where we further have, compare (7.14), that

$$\begin{aligned} \|\partial_z(\beta v)\|_{L^p(C(x_0';|\lambda|^{-1/2}))} &\le \|\nabla(\theta\beta v)\|_{L^p(\Omega')}, \\ \|\nabla_H\partial_z(\beta v)\|_{L^p(C(x_0';|\lambda|^{-1/2}))} &\le \|\nabla_H\partial_z(\theta\beta v)\|_{L^p(\Omega')} \le \|\theta\beta v\|_{W^{2,p}(\Omega')} \le C_p\|\Delta(\theta\beta v)\|_{L^p(\Omega')} \end{aligned} \tag{7.19}$$

by the invertibility of the Laplace operator with mixed Neumann and Dirichlet boundary conditions, compare Section 5.

We now estimate the right-hand side of (7.18) as follows: Note that β satisfies the estimates

$$\|\beta\|_{L^p_z} \le Cr^{1/p}, \quad \|\partial_z\beta\|_{L^p_z} \le Cr^{1/p-1}, \quad \|\partial_z^2\beta\|_{L^p_z} \le Cr^{1/p-2}$$

since $\operatorname{supp}(\beta) \subset [-h, -h+3r]$. It follows from Proposition 7.3 that

$$\|\theta\beta(\nabla_H\pi)\|_{L^p(\Omega')} \le \|\theta\|_\infty\|\beta\|_{L^p_z}\|\nabla_H\pi\|_{L^p_H} \le C_p r^{3/p}(1+|\log r|)\|f\|_{L^\infty_H L^p_z(\Omega)}.$$

The estimate (7.9) implies that

$$\|(\nabla_H\theta)\beta\cdot(\nabla_H v)\|_{L^p_H} \le \|\nabla_H\theta\|_{L^p_H}\|\beta\|_\infty\|\nabla_H v\|_{L^\infty_H L^p_z(\Omega)} \le C_{\theta,p,\lambda_0}r^{2/p-1}|\lambda|^{-1/2}\|f\|_{L^\infty_H L^p_z(\Omega)},$$

and for the term containing vertical derivatives we have

$$\|\theta(\partial_z\beta)(\partial_z v)\|_{L^p(\Omega')} \le \|\theta\|_{L^p_H}\|\partial_z\beta\|_\infty\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)} \le Cr^{2/p-1}\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)}.$$

By the Poincaré inequality (7.15) we have

$$\|(\Delta_H\theta)\beta v\|_{L^p(\Omega')} \le \|\Delta_H\theta\|_{L^p_H}\|\beta\|_\infty\|v\|_{L^\infty(G;L^p(-h,-h+3r))} \le Cr^{2/p-1}\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)}$$

as well as

$$\|\theta(\partial_z^2\beta)\,v\|_{L^p(\Omega')} \le \|\theta\|_{L^p_H}\|\partial_z^2\beta\|_\infty\|v\|_{L^\infty(G;L^p(-h,-h+3r))} \le Cr^{2/p-1}\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)}.$$

Combining the above estimates with (7.17), (7.18) and (7.19) as well as $r = \eta|\lambda|^{-1/2}$ then yields

$$\begin{aligned}|\lambda|^{1/2}\|\partial_z(\beta v)\|_{L^\infty_H L^p_z(\Omega)} &\le C_{\theta,p,\lambda_0}\left(\eta^{2/p-1} + \eta^{3/p}|\lambda|^{-1/2p}\big(1 + |\log(\eta|\lambda|^{-1/2})|\big)\right)\|f\|_{L^\infty_H L^p_z(\Omega)} \\ &\quad + C_{\theta,p}\eta^{2/p-1}|\lambda|^{1/2}\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)}.\end{aligned} \tag{7.20}$$

We now substitute (7.16) and (7.20) into (7.11). Since all constants $C > 0$ do not depend on the parameter $\eta > 0$, we can take it to be sufficiently large and so similarly to the proof of Lemma 5.4 we obtain

$$|\lambda|^{1/2}\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)} \le C_{\theta,p,\lambda_0}\left(\eta^{2/p-1}(1 + r^{1/p}|\log(r)|) + \eta^{3/p}|\lambda|^{-1/2p}(1 + |\log(|\lambda|)|)\right)\|f\|_{L^\infty_H L^p_z(\Omega)}.$$

Since

$$\sup_{0<r<r_0} r^{1/p}|\log(r)| < \infty, \qquad \sup_{|\lambda|>\lambda_0} |\lambda|^{-1/2p}(1 + |\log(|\lambda|)|) < \infty,$$

for any $r_0, \lambda_0 > 0$ and $p \in (1,\infty)$, this implies the desired estimate $|\lambda|^{1/2}\|\partial_z v\|_{L^\infty_H L^p_z(\Omega)} \le C\|f\|_{L^\infty_H L^p_z(\Omega)}$ for $|\lambda| \ge \lambda_0$. □

We now turn to the problem

$$\lambda v - Av = \mathbb{P}\partial_z f \text{ on } \Omega \tag{7.21}$$

with boundary conditions (2.3) for $f \in X$. Since

$$\mathbb{P}\partial_z f = \partial_z f - (1 - Q)\overline{\partial_z f} = \partial_z f, \tag{7.22}$$

whenever $f = 0$ on $\Gamma_u \cup \Gamma_b$ and $C^\infty_{\text{per}}([0,1]^2; C^\infty_c(-h,0))^2$ is dense in X we may assume without loss of generality that (7.22) holds. Moreover, in view of periodic extension we may assume that (7.21) holds in a larger domain $\Omega' := G' \times (-h,0)$, $G' := (-2,3)^2$. Since the problem is well-posed in $L^p_{\overline{\sigma}}(\Omega)$ by (6.6), estimate (7.2) then follows from the following:

Proposition 7.7. *Let $p \in (2,\infty)$ and $\theta \in (0,\pi)$. Then there exists constants $\lambda_0 > 0$ and $C_{\theta,p,\lambda_0} > 0$ such that for all $\lambda \in \Sigma_\theta$ with $|\lambda| > \lambda_0$ and $f \in X$ the solution to the problem* (7.21) *satisfies*

$$|\lambda|^{1/2}\|v\|_{L^\infty_H L^p_z(\Omega)} \le C_{\theta,p,\lambda_0}\|f\|_{L^\infty_H L^p_z(\Omega)}.$$

To prove this estimate, we adopt a duality argument combined with the use of a regularized delta function, which is based on the methodology known in L^∞-type error analysis of the finite element method, cf. [22].

In order to prove this estimate we first introduce some notation. Using periodicity, one sees that for any $\varepsilon \in (0,1)$ we have $B(x'_0,\varepsilon) \subset G'$ for $x'_0 \in G$ and

$$\|v\|^p_{L^\infty_H L^p_z(\Omega)} = \sup_{x'_0\in G}\sup_{x'\in B(x'_0;\varepsilon)} \int_{-h}^0 |v(x',z)|^p\,dz,$$

where by $B(x'_0;\varepsilon)$ we continue to denote a disk in $\mathbb{R}^2$, compare Lemma 7.2. In the following we fix arbitrary $x'_0 \in G$, $x' \in B(x'_0;\varepsilon)$ and choose $\varepsilon = |\lambda|^{-\frac{p}{2(p-2)}}$ for λ as above.

Letting $\delta \ge 0$ be a smooth nonnegative function in the variables $(x,y) =: x'$ such that $\operatorname{supp}\delta \subset B(0;1)$ and $\int_{\mathbb{R}^2}\delta\,dx' = 1$, we introduce a rescaled function as

$$\delta_\varepsilon(x') := \frac{1}{\varepsilon^2}\delta\left(\frac{x'}{\varepsilon}\right), \qquad \delta_{\varepsilon,x'_0}(x') := \delta_\varepsilon(x' - x'_0). \tag{7.23}$$

We then obtain

$$\int_{-h}^0 |v(x',z)|^p\,dz = \int_{-h}^0\int_{G'}\big(|v(x',z)|^p - |v(y',z)|^p\big)\delta_{\varepsilon,x'_0}(y')\,dy'dz + (v, \delta_{\varepsilon,x'_0}|v|^{p-2}v^*)_{\Omega'} =: I_1(x') + I_2, \tag{7.24}$$

where v^* means the complex conjugate of v and $(\cdot,\cdot)_{\Omega'}$ denotes the inner product on $L^2(\Omega')^2$. In the following we estimate the two terms on the right-hand side separately, beginning with I_1.

Lemma 7.8. *Under the assumptions of Proposition 7.7 we have for all for all $x_0' \in G$ and $x' \in B(x_0';\varepsilon)$, $\varepsilon = |\lambda|^{-\frac{p}{2(p-2)}}$, that*

$$|I_1(x')| = \left| \int_{\Omega'} \big(|v(x',z)|^p - |v(y',z)|^p\big)\delta_{\varepsilon,x_0'}(y')\, dy' dz \right| \leq C_{\theta,p}|\lambda|^{-1/2}\|f\|_{L^\infty_H L^p_z(\Omega)}\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)}.$$

Proof. Since $\int_{\mathbb{R}^2} \delta_{\varepsilon,x_0'}(y')\, dy' = 1$ and $\operatorname{supp} \delta_{\varepsilon,x_0'} \subset B(x_0';\varepsilon)$ we obtain

$$\begin{aligned} |I_1(x')| &\leq \sup_{y'\in B(x_0';\varepsilon)} \int_{-h}^0 \big||v(x',z)|^p - |v(y',z)|^p\big|\, dz \\ &\leq C \sup_{y'\in B(x_0';\varepsilon)} \int_{-h}^0 (|v(x',z)|^{p-1} + |v(y',z)|^{p-1})\big|v(x',z) - v(y',z)\big|\, dz, \end{aligned}$$

where we have used the elementary inequality

$$|a^p - b^p| \leq p \max\{a,b\}^{p-1}|a-b| \leq p(a+b)^{p-1}|a-b| \leq p2^{p-2}(a^{p-1} + b^{p-1})|a-b|$$

for all $a, b \geq 0$, where we used that $p \in [2,\infty)$ implies that $x \mapsto x^{p-1}$ is a convex function. Hölder's inequality then implies that

$$\int_{-h}^0 (|v(x',z)|^{p-1} + |v(y',z)|^{p-1})\big|v(x',z) - v(y',z)\big|\, dz \leq (\|v(x')\|^{p-1}_{L^p_z} + \|v(y')\|^{p-1}_{L^p_z})\|v(x') - v(y')\|_{L^p_z}.$$

Hence we have

$$\sup_{x'\in B(x_0';\varepsilon)} |I_1(x')| \leq C \sup_{y'\in B(x_0';\varepsilon)} \|v(y')\|^{p-1}_{L^p_z} \sup_{y'\in B(x_0';\varepsilon)} \|v(x') - v(y')\|_{L^p_z} \leq C\|v\|^{p-1}_{L^\infty_H L^p_z}\varepsilon^\alpha \|v\|_{C^\alpha_H L^p_z(\Omega)},$$

where $\alpha := 1-2/p > 0$ and $\|v\|_{C^\alpha_H L^p_z(\Omega)}$ denotes the space of $L^p(-h,0)$-valued Hölder continuous functions of exponent α on $\overline{G}$.

The assumption $\varepsilon = |\lambda|^{-\frac{p}{2(p-2)}}$ then yields $\varepsilon^\alpha = |\lambda|^{-1/2}$. We now use the Sobolev embedding $W^{1,p}(G) \hookrightarrow C^\alpha(\overline{G})$ to obtain the estimate $\|v\|_{C^\alpha_H L^p_z} \leq C\|v\|_{W^{1,p}(\Omega)}$. In addition, the Poincaré inequality yields

$$\|v\|_{W^{1,p}(\Omega)} \leq C_p\|\nabla v\|_{L^p(\Omega)} = C_p\|\nabla(\lambda - A_{p,\overline{\sigma}})^{-1}\partial_z f\|_{L^p(\Omega)} \leq C_{\theta,p}\|f\|_{L^p(\Omega)} \leq C_{\theta,p}\|f\|_{L^\infty_H L^p_z(\Omega)},$$

where we used that $\nabla(-A_{p,\overline{\sigma}})^{-1/2}$, $A_{p,\overline{\sigma}}(\lambda - A_{p,\overline{\sigma}})^{-1}$ and $(-A_{p,\overline{\sigma}})^{-1/2}\partial_z$ are (uniformly) bounded on $L^p_{\overline{\sigma}}(\Omega)$ for $\lambda \in \Sigma_\theta$ by [7]. Combining these results then gives the desired estimate. □

In order to estimate I_2 we perform a duality argument. For this purpose we introduce an auxiliary problem corresponding to (7.21) as follows:

$$\begin{aligned} \lambda^* w - \Delta w + \nabla_H \Pi &= \delta_{\varepsilon,x_0'}|v|^{p-2}v^* \quad \text{in} \quad \Omega', \\ \partial_z \Pi &= 0 \quad \text{in} \quad \Omega', \\ \operatorname{div}_H \bar{w} &= 0 \quad \text{in} \quad G', \\ \partial_z w|_{\Gamma_u'} = 0, \quad w|_{\Gamma_b'} &= 0, \quad w, \Pi \text{ periodic on } \Gamma_l', \end{aligned} \tag{7.25}$$

where the upper script $*$ means complex conjugate as before. We establish an $L^1_H L^q_z$-estimate to this problem, where $q := p/(p-1)$ is the dual index of p.

Proposition 7.9. *Let $p \in (2,\infty)$, $1/p + 1/q = 1$ and $\theta \in (0,\pi)$. Then there exists a sufficiently large $\lambda_0 > 0$ and a constant $C_{p,\lambda_0,\theta} > 0$ such that the solution of* (7.25) *satisfies*

$$|\lambda|^{1/2}\|\partial_z w\|_{L^1_H L^q_z(\Omega')} \leq C_{\theta,p}\left(1 + |\lambda|^{-1/2q}\varepsilon^{2/s-2}\right)\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)},$$

for all $\varepsilon \in (0,1)$, $s \in (1,q]$, $x_0' \in G$, $\lambda \in \Sigma_\theta$, $|\lambda| > \lambda_0$, and $v \in X$.

Remark 7.10. If one even has $p \in (3,\infty)$ then this result can be extended to the full range of $\lambda \in \Sigma_\theta$ by a similar argument as in the proof of Lemma 5.4, compare Remark 7.6.

For simplicity, we write $L^p_H L^q_z$ to refer to $L^p_H L^q_z(\Omega') = L^p(G'; L^q(-h,0))$ when there is no ambiguity. First we introduce the following result.

Lemma 7.11. *Let $\varepsilon \in (0,1)$, $x'_0 \in G$, $p \in (1,\infty)$, $1/p + 1/q = 1$ and $v \in X$ be arbitrary. Then, for $\delta_{\varepsilon,x'_0}$ defined as in (7.23) and $s \in [1,q]$ we have*

$$\|\delta_{\varepsilon,x'_0}|v|^{p-2}v^*\|_{L^s(\Omega')} \le C\|\delta_{\varepsilon,x'_0}|v|^{p-2}v^*\|_{L^s_H L^q_z} \le C\varepsilon^{2/s-2}\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)}$$

for a constant $C > 0$ not depending on ε, x'_0 and v.

Proof. We set $F := \delta_{\varepsilon,x'_0}|v|^{p-2}v^*$. Noting that $|F|^q = \delta^q_{\varepsilon,x'_0}|v|^p$ and that $\delta_{\varepsilon,x'_0}$ is independent of z, we obtain

$$\begin{aligned}\|F\|_{L^s_H L^q_z} &= \left[\int_{G'}\left(\int_{-h}^0 \delta_\varepsilon(x'-x'_0)^q|v(x',z)|^p\,dz\right)^{s/q} dx'\right]^{1/s}\\ &\le \left(\int_{G'}\delta_\varepsilon(x'-x'_0)^s\,dx'\right)^{1/s}\left[\sup_{x'\in G'}\left(\int_{-h}^0 |v(x',z)|^p\,dz\right)^{1/p}\right]^{p/q}\\ &\le C\varepsilon^{2/s-2}\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)},\end{aligned}$$

where we used the periodicity of v in the last step. This completes the proof. ☐

Proof of Proposition 7.9. We set $r := \eta|\lambda|^{-1/2}$, where $\eta > 0$ is a large number to be fixed later and $|\lambda| > \lambda_0$, where $\lambda_0 > 0$ is sufficiently large such that $\eta\lambda_0^{-1/2} < 1$. We introduce two cut-off functions $\alpha = \alpha_r$, $\beta = \beta_r$ in the vertical direction as follows:

$$\begin{aligned}&\alpha \in C^\infty([-h,0]), \quad \alpha \equiv 0 \text{ in } [-h,-h+r], \quad \alpha \equiv 1 \text{ in } [-h+2r,0], \quad |\partial_z^k\alpha(z)| \le Cr^{-k},\\ &\beta \in C^\infty([-h,0]), \quad \beta \equiv 1 \text{ in } [-h,-h+2r], \quad \beta \equiv 0 \text{ in } [-h+3r,0], \quad |\partial_z^k\beta(z)| \le Cr^{-k}\end{aligned}$$

for $k = 0,1,2$. Then we may split the estimate for $\partial_z w$ into the "upper" and "lower" parts in Ω' as

$$\|\partial_z w\|_{L^1_H L^q_z} \le \|\partial_z(\alpha w)\|_{L^1_H L^q_z} + \|\partial_z(\beta w)\|_{L^1_H L^q_z}. \tag{7.26}$$

Step 1. We consider αw, which satisfies

$$\begin{aligned}\lambda^*\alpha w - \Delta(\alpha w) &= \alpha F - \alpha(\nabla_H\Pi) - 2(\partial_z\alpha)(\partial_z w) - (\partial_z^2\alpha)w,\\ \partial_z(\alpha w) &= 0 \quad \text{on} \quad \Gamma'_u \cup \Gamma'_b, \qquad \alpha w \text{ periodic on } \Gamma'_l\end{aligned}$$

where $F := \delta_{\varepsilon,x'_0}|v|^{p-2}v^*$ as in the proof of Lemma7.11. Differentiating this with respect to z yields

$$\begin{aligned}\lambda^*\partial_z(\alpha w) - \Delta(\partial_z(\alpha w)) &= \partial_z\left[\alpha F - 2(\partial_z\alpha)(\partial_z w) - (\partial_z^2\alpha)w\right] - (\partial_z\alpha)(\nabla_H\Pi) \quad \text{in} \quad \Omega',\\ \partial_z(\alpha w) &= 0 \quad \text{on} \quad \Gamma'_u \cup \Gamma'_b, \qquad \partial_z(\alpha w) \text{ periodic on } \Gamma'_l.\end{aligned}$$

Applying Lemma 5.4 in $L^1_H L^q_z(\Omega')$ we obtain

$$\begin{aligned}|\lambda|^{1/2}\|\partial_z(\alpha w)\|_{L^1_H L^q_z} \le& C\left(\|\alpha F\|_{L^1_H L^q_z} + \|(\partial_z\alpha)(\partial_z w)\|_{L^1_H L^q_z} + \|(\partial_z^2\alpha)w\|_{L^1_H L^q_z}\right)\\ &+C|\lambda|^{-1/2}\|(\partial_z\alpha)(\nabla_H\Pi)\|_{L^1_H L^q_z}.\end{aligned}$$

We now estimate each term on the right-hand side. By Lemma 7.11 with $s = 1$ we have

$$\|\alpha F\|_{L^1_H L^q_z} \le \|\alpha\|_\infty\|F\|_{L^1_H L^q_z} \le C\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)}.$$

Using the estimate on derivatives of α we obtain

$$\|(\partial_z\alpha)(\partial_z w)\|_{L^1_H L^q_z} \le Cr^{-1}\|\partial_z w\|_{L^1_H L^q_z},$$

and by the Poincaré inequality we have

$$\|(\partial_z^2\alpha)w\|_{L^1_H L^q_z} \le Cr^{-2}\|w\|_{L^1_H L^q_z(G'\times(-h,-h+2r))} \le Cr^{-1}\|\partial_z w\|_{L^1_H L^q_z}.$$

Using $L^s(G') \hookrightarrow L^1(G')$ as well as the estimate on the pressure term, cf. [11, Theorem 3.1.], in $L^s(\Omega)$ for $s \in (1, q]$, we obtain

$$\|(\partial_z \alpha)(\nabla_H \Pi)\|_{L^1_H L^q_z} \le C\|\partial_z \alpha\|_{L^q_z}\|\nabla_H \Pi\|_{L^s(G')} \le Cr^{1/q-1}\|F\|_{L^s} \le Cr^{1/q-1}\varepsilon^{2/s-2}\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)},$$

Collecting the above estimates and plugging in $r = \eta|\lambda|^{-1/2}$ yields

$$|\lambda|^{1/2}\|\partial_z(\alpha w)\|_{L^1_H L^q_z} \le C(1 + \eta^{1/q-1}|\lambda|^{-1/2q}\varepsilon^{2/s-2})\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)} + C\eta^{-1}|\lambda|^{1/2}\|\partial_z w\|_{L^1_H L^q_z}. \tag{7.27}$$

Step 2. We consider βw, which satisfies

$$\begin{aligned} \lambda^*\beta w - \Delta(\beta w) &= \beta F - \beta\nabla_H \Pi - 2(\partial_z \beta)(\partial_z w) - (\partial_z^2 \beta) w \quad \text{in} \quad \Omega', \\ \partial_z(\beta w) = 0 \text{ on } \Gamma'_u, &\quad \beta w = 0 \text{ on } \Gamma'_b, \quad \partial_z(\beta w) \text{ periodic on } \Gamma'_l. \end{aligned}$$

Applying Lemma 5.4 in $L^1_H L^q_z$ we obtain

$$|\lambda|^{1/2}\|\partial_z(\beta w)\|_{L^1_H L^q_z} \le C(\|\beta F\|_{L^1_H L^q_z} + \|(\partial_z\beta)(\partial_z w)\|_{L^1_H L^q_z} + \|(\partial_z^2\beta)w\|_{L^1_H L^q_z} + \|\beta(\nabla_H \Pi)\|_{L^1_H L^q_z}).$$

A calculation similar to Step 1 then gives

$$|\lambda|^{1/2}\|\partial_z(\beta w)\|_{L^1_H L^q_z} \le C(1 + \eta^{1/q}|\lambda|^{-1/2q}\varepsilon^{2/s-2})\|v\|^{p-1}_{L^\infty_H L^p_z} + C\eta^{-1}|\lambda|^{1/2}\|\partial_z w\|_{L^1_H L^q_z}. \tag{7.28}$$

Substituting (7.27) and (7.28) into (7.26) and choosing sufficiently large η enable us to absorb the term $|\lambda|^{1/2}\|\partial_z w\|_{L^1_H L^q_z}$ from the right-hand side, which leads to

$$|\lambda|^{1/2}\|\partial_z w\|_{L^1_H L^q_z} \le C(1 + |\lambda|^{-1/2q}\varepsilon^{2/s-2})\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)}$$

This completes the proof. ☐

With the preparations above, we are now in the position to prove Proposition 7.7.

Proof of Proposition 7.7. By (7.24) and Lemma 7.8 we have

$$\|v\|^p_{L^\infty_H L^p_z(\Omega)} \le C|\lambda|^{-1/2}\|f\|_{L^\infty_H L^p_z(\Omega)}\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)} + I_2, \tag{7.29}$$

with I_2 as defined in (7.24). Substituting (7.25) and integrating by parts, we find that

$$\begin{aligned} I_2 &= (v, \delta_{\varepsilon,x'_0}|v|^{p-2}v^*)_{\Omega'} = (v, \lambda^* w - \Delta w + \nabla_H \Pi)_{\Omega'} = (\lambda v - \Delta v + \nabla_H \pi, w)_{\Omega'} = (\partial_z f, w)_{\Omega'} \\ &= -(f, \partial_z w)_{\Omega'}, \end{aligned}$$

where we have used that $(v, \nabla_H \Pi)_{\Omega'} = 0 = (\nabla_H \pi, w)_{\Omega'}$ since $\mathrm{div}_H \overline{v} = 0 = \mathrm{div}_H \overline{w}$ for the third and $f|_{\Gamma_u \cup \Gamma_b} = 0$ for the last equality. Using $1/p + 1/q = 1$ and applying Proposition 7.9 we obtain

$$|I_2| \le \|f\|_{L^\infty_H L^p_z(\Omega)}\|\partial_z w\|_{L^1_H L^q_z} \le C|\lambda|^{-1/2}\|f\|_{L^\infty_H L^p_z(\Omega)}\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)}\left(1 + |\lambda|^{-1/2q}\varepsilon^{2/s-2}\right).$$

We set $\varepsilon = |\lambda|^{-\frac{p}{2(p-2)}}$ for $|\lambda| > 1$ and $s = \min\{\frac{4p}{3p+2}, \frac{p}{p-1}\} \in (1, q]$. This yields

$$-\frac{1}{2q} + \left(1 - \frac{1}{s}\right)\frac{p}{p-2} = -\frac{1}{2} + \frac{1}{2p} + \left(1 - \frac{1}{s}\right)\frac{p}{p-2} \le -\frac{1}{4} + \frac{1}{2p} < 0$$

which implies that $1 + |\lambda|^{-1/2q}\varepsilon^{2/s-2} \le 2$ for $|\lambda| > 1$ and therefore

$$|I_2| \le C|\lambda|^{-1/2}\|f\|_{L^\infty_H L^p_z(\Omega)}\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)}, \quad |\lambda| > 1. \tag{7.30}$$

The desired estimate then follows from (7.29) and (7.30) after dividing by $\|v\|^{p-1}_{L^\infty_H L^p_z(\Omega)}$. ☐

Proof of Claim 7.1. Estimate (7.1) now follows from (7.8) and Proposition 7.5, whereas estimate (7.2) follows from Proposition 7.7. Estimate 7.3 follows from (7.1), (7.2) and Claim 6.1. ☐

8. Proof of the main results

Theorem 3.4 is a direct consequence of Claims 6.1 and 7.1.

For the non-linear problem in the space X we will make use of the following estimates.

Lemma 8.1. *Let $p > 3$. Then exists a constant $C > 0$ such that for all $t > 0$ and $v_i \in X_{\overline{\sigma}}$ satisfying $\nabla v_i \in X$ and $v_i\big|_{\Gamma_b} = 0$ with $u_i = (v_i, w_i)$ as in (2.2) for $i = 1, 2$ we have*

$$\text{(i)} \qquad \|e^{tA}\mathbb{P}(u_1 \cdot \nabla)v_2\|_{L^\infty_H L^p_z} \leq Ct^{-1/2}\|\nabla v_1\|_{L^\infty_H L^p_z}\|v_2\|_{L^\infty_H L^p_z},$$

$$\text{(ii)} \qquad \|\nabla e^{tA}\mathbb{P}(u_1 \cdot \nabla)v_2\|_{L^\infty_H L^p_z} \leq Ct^{-1/2}\|\nabla v_1\|_{L^\infty_H L^p_z}\|\nabla v_2\|_{L^\infty_H L^p_z},$$

$$\text{(iii)} \qquad \|\nabla e^{tA}\mathbb{P}(u_1 \cdot \nabla)v_2\|_{L^\infty_H L^p_z} \leq Ct^{-1}\|\nabla v_1\|_{L^\infty_H L^p_z}\|v_2\|_{L^\infty_H L^p_z},$$

as well as

$$\text{(iv)} \qquad \|e^{tA}\mathbb{P}(u_1 \cdot \nabla)v_2\|_{L^\infty_H L^p_z} \leq C\left(t^{-1/2}\|\nabla v_i\|_{L^\infty_H L^p_z}\|v_j\|_{L^\infty_H L^p_z} + \|\nabla v_1\|_{L^\infty_H L^p_z}\|\nabla v_2\|_{L^\infty_H L^p_z}\right)$$

where $\{i, j\} = \{1, 2\}$.

Proof. We begin by noting that

$$\|(u_1 \cdot \nabla)v_2\|_{L^\infty_H L^p_z} \leq \left(\|v_1\|_{L^\infty(\Omega)} + \|w_1\|_{L^\infty(\Omega)}\right)\|\nabla v_2\|_{L^\infty_H L^p_z}.$$

So, using Sobolev embeddings, the Poincaré inequality and $X \hookrightarrow L^p(\Omega)^2$ we obtain

$$\|v_i\|_{L^\infty(\Omega)} \leq C\|v_i\|_{W^{1,p}(\Omega)} \leq C\|\nabla v_i\|_{L^p(\Omega)} \leq C\|\nabla v_i\|_{L^\infty_H L^p_z}.$$

Similarly one has

$$\|w_i\|_{L^\infty(\Omega)} \leq C\|\mathrm{div}_H v_i\|_{L^\infty_H L^p_z} \leq C\|\nabla v_i\|_{L^\infty_H L^p_z}.$$

This allows us to obtain (ii) via Claim 6.1 and 7.1 as well as

$$\|\nabla e^{tA}\mathbb{P}(u_1 \cdot \nabla)v_2\|_{L^\infty_H L^p_z} \leq Ct^{-1/2}\|(u_1 \cdot \nabla)v_2\|_{L^\infty_H L^p_z} \leq Ct^{-1/2}\|\nabla v_1\|_{L^\infty_H L^p_z}\|\nabla v_2\|_{L^\infty_H L^p_z}.$$

To prove (i) we proceed analogously as above to obtain

$$\|v_1 \otimes v_2\|_{L^\infty_H L^p_z} \leq C\|\nabla v_i\|_{L^\infty_H L^p_z}\|v_j\|_{L^\infty_H L^p_z}, \quad \|w_1 v_2\|_{L^\infty_H L^p_z} \leq C\|\nabla v_1\|_{L^\infty_H L^p_z}\|v_2\|_{L^\infty_H L^p_z}$$

where $\{i, j\} = \{1, 2\}$ and since $\mathrm{div}\, u_i = 0$ we can write

$$(u_1 \cdot \nabla)v_2 = \nabla \cdot (u_1 \otimes v_2) = \nabla_H \cdot (v_1 \otimes v_2) + \partial_z(w_1 v_2)$$

which allows us to apply Claim 6.1 and 7.1 yielding

$$\begin{aligned}
\|e^{tA}\mathbb{P}(u_1 \cdot \nabla)v_2\|_{L^\infty_H L^p_z} &= \|e^{tA}\mathbb{P}\nabla \cdot (u_1 \otimes v_2)\|_{L^\infty_H L^p_z} \\
&\leq \|e^{tA}\mathbb{P}\nabla_H \cdot (v_1 \otimes v_2)\|_{L^\infty_H L^p_z} + \|e^{tA}\mathbb{P}\partial_z(w_1 v_2)\|_{L^\infty_H L^p_z} \\
&\leq Ct^{-1/2}\left(\|v_1 \otimes v_2\|_{L^\infty_H L^p_z} + \|w_1 v_2\|_{L^\infty_H L^p_z}\right) \\
&\leq Ct^{-1/2}\|\nabla v_1\|_{L^\infty_H L^p_z}\|v_2\|_{L^\infty_H L^p_z},
\end{aligned}$$

and estimate (iii) is obtained analogously via

$$\|\nabla e^{tA}\mathbb{P}(u_1 \cdot \nabla)v_2\|_{L^\infty_H L^p_z} \leq Ct^{-1}\left(\|v_1 \otimes v_2\|_{L^\infty_H L^p_z} + \|w_1 v_2\|_{L^\infty_H L^p_z}\right) \leq Ct^{-1}\|\nabla v_1\|_{L^\infty_H L^p_z}\|v_2\|_{L^\infty_H L^p_z}.$$

To prove (iv) we observe that $w_i = 0$ on $\Gamma_u \cup \Gamma_b$ implies that

$$\mathbb{P}\partial_z(w_1 v_2) = \partial_z(w_1 v_2) = -(\mathrm{div}_H v_1)v_2 + w_1 \partial_z v_2$$

and the right-hand side is further estimated via

$$\|(\mathrm{div}_H v_1)v_2\|_{L^\infty_H L^p_z} \leq C\|\nabla v_1\|_{L^\infty_H L^p_z}\|v_2\|_{L^\infty(\Omega)} \leq C\|\nabla v_1\|_{L^\infty_H L^p_z}\|\nabla v_2\|_{L^\infty_H L^p_z},$$

and

$$\|w_1 \partial_z v_2\|_{L^\infty_H L^p_z} \leq \|w_1\|_{L^\infty(\Omega)}\|\partial_z v_2\|_{L^\infty_H L^p_z} \leq C\|\nabla v_1\|_{L^\infty_H L^p_z}\|\nabla v_2\|_{L^\infty_H L^p_z}.$$

Applying Claim 6.1 then yields that for $\{i,j\}=\{1,2\}$ we have

$$\|e^{tA}\mathbb{P}\nabla_H\cdot(v_1\otimes v_2)\|_{L^\infty_H L^p_z}\le Ct^{-1/2}\|v_1\otimes v_2\|_{L^\infty_H L^p_z}\le Ct^{-1/2}\|\nabla v_i\|_{L^\infty_H L^p_z}\|v_j\|_{L^\infty_H L^p_z},$$

as well as

$$\|e^{tA}\mathbb{P}\partial_z(w_1v_2)\|_{L^\infty_H L^p_z}\le C\|\nabla v_1\|_{L^\infty_H L^p_z}\|\nabla v_2\|_{L^\infty_H L^p_z}$$

which implies (iv) and completes the proof. □

It has been proven in [7] that the operator $A_{p,\overline{\sigma}}$ possesses maximal L^q-regularity. In [9] the authors applied this to develop a solution theory for initial data

$$a\in X_\gamma:=(L^p_{\overline{\sigma}}(\Omega),D(A_p))_{1-1/q,q}\subset B^{2-2/q}_{pq}(\Omega)^2\cap L^p_{\overline{\sigma}}(\Omega)$$

where $p,q\in(1,\infty)$ satisfy $1/p+1/q\le 1$. In particular, one has the following result.

Lemma 8.2. *Let $a\in X_\gamma$. Then there exists a unique strong solution to the primitive equations* (2.1) *with boundary conditions* (2.3) *satisfying*

$$v\in C([0,\infty);X_\gamma).$$

This enables a key step in the proof of our main result as it guarantees the existence of smooth reference solutions v_{ref} to the primitive equations given sufficiently smooth reference data a_{ref}. In order to construct v as a solution to problem (2.1) with initial data a we construct $V:=v-v_{\text{ref}}$ by an iterative method using initial data $a_0:=a-a_{\text{ref}}$. Before we do so, we establish an auxiliary lemma.

Lemma 8.3. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that*

$$a_{m+1}\le a_0+c_1a_m^2+c_2a_m \quad \text{for all } m\in\mathbb{N}$$

and constants $c_1>0$ and $c_2\in(0,1)$ such that $4c_1a_0<(1-c_2)^2$. Then $a_m<\frac{2}{1-c_2}a_0$ for all $m\in\mathbb{N}$.

Proof. Let x_0 be the smallest solution to the equation $x=a_0+c_1x^2+c_2x$. Then

$$0<x_0=\frac{(1-c_2)-\sqrt{(1-c_2)^2-4c_1a_0}}{2c_1}=\frac{1}{2c_1}\frac{4c_1a_0}{(1-c_2)+\sqrt{(1-c_2)^2-4c_1a_0}}<\frac{2}{1-c_2}a_0,$$

and since $p(x)=a_0+c_1x^2+c_2x$ is an increasing function on $[0,\infty)$ it follows that $p(x)\le x_0$ for $x\in[0,x_0]$. The condition $c_2\in(0,1)$ further yields

$$(1-c_2)+\sqrt{(1-c_2)^2-4c_1a_0}<2$$

from which it follows that $a_0<x_0$ and thus the claim is easily derived by induction. □

We now prove our main result.

Proof of Theorem 3.1. **Step 1:** *Decomposition of data.*

Given an initial value $a\in X_{\overline{\sigma}}$ we will split it into a smooth part a_{ref} and a small rough part a_0, where $a=a_{\text{ref}}+a_0$, as follows: Since $A_{\overline{\sigma}}$ is densely defined on $X_{\overline{\sigma}}$ we take $a_{\text{ref}}\in D(A_{\overline{\sigma}})$ such that $a_0:=a-a_{\text{ref}}$ can be assumed to be arbitrarily small in $X_{\overline{\sigma}}$. Now let $q\in(1,\infty)$ be such that $1/q+1/p\le 1$ and $2/q+3/p<1$. The latter condition on q then yields the embedding $X_\gamma\hookrightarrow C^1(\overline{\Omega})^2$. Due to $D(A_{\overline{\sigma}})\subset D(A_{p,\overline{\sigma}})\subset X_\gamma$ it follows from Lemma 8.2 that taking a_{ref} as initial data of the primitive equations, there exists a function $v_{\text{ref}}\in C([0,\infty);X_\gamma)$ solving the primitive equations with initial data $v_{\text{ref}}(0)=a_{\text{ref}}$.

Step 2: *Estimates for the construction of a local solution.*

We will show that there exists a constant $C_0>0$ such that if $a_0\in X_{\overline{\sigma}}$ satisfies $\|a_0\|_{L^\infty_H L^p_z}<C_0$ then there exists a time $T>0$ and a unique function

$$V\in\mathcal{S}(T):=\{V\in C([0,T];X_{\overline{\sigma}}):\|\nabla V(t)\|_{L^\infty_H L^p_z}=o(t^{-1/2})\},$$

where

$$\|V\|_{\mathcal{S}(T)}=\max\left\{\sup_{0<t<T}\|V(t)\|_{L^\infty_H L^p_z},\ \sup_{0<t<T}t^{1/2}\|\nabla V(t)\|_{L^\infty_H L^p_z}\right\}$$

such that $v = v_{\mathrm{ref}} + V$ solves problem (2.1) on $(0,T)$ with initial value $v(0) = a$. In order to construct V we define the iterative sequence of functions $(V_m)_{m\in\mathbb{N}}$ via

$$V_0(t) = e^{tA}a_0, \quad V_{m+1}(t) = e^{tA}a_0 + \int_0^t e^{(t-s)A}F_m(s)\,ds \tag{8.1}$$

where

$$F_m := -\mathbb{P}\left((U_m\cdot\nabla)V_m + (U_m\cdot\nabla)v_{\mathrm{ref}} + (u_{\mathrm{ref}}\cdot\nabla)V_m\right)$$

and $U_m = (V_m, W_m)$, $u_{\mathrm{ref}} = (v_{\mathrm{ref}}, w_{\mathrm{ref}})$ with the vertical component w given by the horizontal component v via the relation (2.2). We will now estimate this sequence in $\mathcal{S}(T)$ for some value $T > 0$ to be fixed later on. Since $\mathbb{P}a_0 = a_0$ we have

$$\|V_0\|_{\mathcal{S}(T)} \le C\|a_0\|_{L^\infty_H L^p_z}, \quad T\in(0,\infty)$$

by Lemma 6.1. For $m \ge 1$ we will first consider the gradient estimates. We have already estimated the term $\nabla e^{tA}a_0$, whereas for the convolution integrals we have

$$\begin{aligned}\left\|\int_0^{t/2}\nabla e^{(t-s)A}\mathbb{P}\left((U_m(s)\cdot\nabla)V_m(s)\right)\,ds\right\|_{L^\infty_H L^p_z} &\le C\left(\int_0^{t/2}(t-s)^{-1}s^{-1/2}\,ds\right)K_m(t)H_m(t)\\ &= Ct^{-1/2}K_m(t)H_m(t)\end{aligned}$$

by Lemma 8.1 (iii) where

$$K_m(t) := \sup_{0<s<t} s^{1/2}\|\nabla V_m(s)\|_{L^\infty_H L^p_z}, \quad H_m(t) := \sup_{0<s<t}\|V_m(s)\|_{L^\infty_H L^p_z}$$

and via Lemma 8.1 (ii) we obtain

$$\begin{aligned}\left\|\int_{t/2}^{t}\nabla e^{(t-s)A}\mathbb{P}\left(U_m(s)\cdot\nabla V_m(s)\right)\,ds\right\|_{L^\infty_H L^p_z} &\le C\left(\int_{t/2}^{t}(t-s)^{-1/2}s^{-1}\right)K_m(t)^2\\ &\le Ct^{-1/2}K_m(t)^2.\end{aligned}$$

Finally applying Lemma 8.1 (ii) to the two remaining mixed terms yields

$$\begin{aligned}\left\|\int_0^{t}\nabla e^{(t-s)A}\mathbb{P}\left(U_m(s)\cdot\nabla)v_{\mathrm{ref}}(s)\right)\,ds\right\|_{L^\infty_H L^p_z} &\le C\left(\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds\right)\sup_{0<s<t}\|\nabla v_{\mathrm{ref}}(s)\|_{L^\infty_H L^p_z}K_m(t)\\ &= C\sup_{0<s<t}\|\nabla v_{\mathrm{ref}}(s)\|_{L^\infty_H L^p_z}K_m(t),\\ \left\|\int_0^{t}\nabla e^{(t-s)A}\mathbb{P}\left(u_{\mathrm{ref}}(s)\cdot\nabla)V_m(s)\right)\,ds\right\|_{L^\infty_H L^p_z} &\le C\sup_{0<s<t}\|\nabla v_{\mathrm{ref}}(s)\|_{L^\infty_H L^p_z}K_m(t).\end{aligned}$$

We set $R := \sup_{0\le t\le T_0}\|\nabla v_{\mathrm{ref}}(t)\|_{L^\infty_H L^p_z}$ and note that $0 < R < \infty$ by Lemma 8.2, since $v_{\mathrm{ref}} \in C([0,\infty); X_\gamma)$ and $2/q + 3/p < 1$ implies that $X_\gamma \subset B^{2-2/q}_{pq}(\Omega)^2 \hookrightarrow C^1(\overline{\Omega})^2$ via embedding theory, cf. [25, Section 3.3.1]. Taking these estimates together yields

$$t^{1/2}\|\nabla V_{m+1}(t)\|_{L^\infty_H L^p_z} \le C_1\Big(\|a_0\|_{L^\infty_H L^p_z} + K_m(t)H_m(t) + K_m(t)^2 + Rt^{1/2}K_m(t)\Big). \tag{8.2}$$

To estimate $\|V_{m+1}(t)\|_{L^\infty_H L^p_z}$ we apply Lemma 8.1 (i) to obtain

$$\begin{aligned}\left\|\int_0^{t} e^{(t-s)A}\mathbb{P}\left((U_m(s)\cdot\nabla)V_m(s)\right)\,ds\right\|_{L^\infty_H L^p_z} &\le C\left(\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds\right)K_m(t)H_m(t)\\ &= CK_m(t)H_m(t)\end{aligned}$$

whereas for the mixed terms Lemma 8.1 (iv) yields

$$\|e^{(t-s)A}\mathbb{P}(U_m(s)\cdot\nabla)v_{\text{ref}}(s)\|_{L_H^\infty L_z^p} \le C\Big((t-s)^{-1/2}\|\nabla v_{\text{ref}}(s)\|_{L_H^\infty L_z^p}\|V_m(s)\|_{L_H^\infty L_z^p} + \|\nabla V_m(s)\|_{L_H^\infty L_z^p}\|\nabla v_{\text{ref}}(s)\|_{L_H^\infty L_z^p}\Big)$$

and therefore

$$\Big\|\int_0^t e^{(t-s)A}\mathbb{P}((U_m(s)\cdot\nabla)v_{\text{ref}}(s))\,ds\Big\|_{L_H^\infty L_z^p} \le C\left(\int_0^t (t-s)^{-1/2}\,ds\right)RH_m(t) + C\left(\int_0^t s^{-1/2}\,ds\right)RK_m(t) = CRt^{1/2}(H_m(t)+K_m(t)),$$

and the other mixed term can be treated analogously due to the symmetry of the right-hand side in (iv). Taking these estimates together yields

$$\|V_{m+1}(t)\|_{L_H^\infty L_z^p} \le C_1\Big(\|a_0\|_{L_H^\infty L_z^p} + K_m(t)H_m(t) + t^{1/2}H_m(t) + t^{1/2}K_m(t)\Big). \tag{8.3}$$

Since the right-hand sides of (8.2) and (8.3) are increasing functions we obtain for $t>0$ that

$$\begin{aligned} K_{m+1}(t) &\le C_1\Big(\|a_0\|_{L_H^\infty L_z^p} + K_m(t)H_m(t) + K_m(t)^2 + Rt^{1/2}K_m(t)\Big),\\ H_{m+1}(t) &\le C_1\Big(\|a_0\|_{L_H^\infty L_z^p} + K_m(t)H_m(t) + Rt^{1/2}H_m(t) + Rt^{1/2}K_m(t)\Big). \end{aligned} \tag{8.4}$$

Now let $T\in(0,T_0)$ where $T_0>0$ is chosen in such a way that

$$8C_1RT_0^{1/2} < 1.$$

Then for all $0<t\le T<T_0$ we have

$$\|V_{m+1}\|_{\mathcal{S}(t)} \le C_1\|a_0\|_{L_H^\infty L_z^p} + 2C_1\|V_m\|_{\mathcal{S}(t)}^2 + \frac{1}{4}\|V_m\|_{\mathcal{S}(t)}.$$

By Lemma 8.3 it follows that if $8C_1^2\|a_0\|_{L_H^\infty L_z^p} < (1-1/4)^2$, then for all $m\in\mathbb{N}$ we have

$$\|V_m\|_{\mathcal{S}(t)} \le \frac{8}{3}C_1\|a_0\|_{L_H^\infty L_z^p}, \quad t\in(0,T]. \tag{8.5}$$

The property $\lim_{t\to0+} t^{1/2}\|\nabla V_m(t)\|_{L_H^\infty L_z^p} = 0$ is then easily obtained via induction and Claim 6.1 (e).

Step 3: *Convergence.*

We now show that $(V_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}(T)$ if $\|a_0\|_{L_H^\infty L_z^p}$ is sufficiently small. For this purpose we consider the new sequence

$$\tilde{V}_m := V_{m+1} - V_m, \quad m\ge 0.$$

Using the previous estimates we already know that $\|\tilde{V}_0\|_{\mathcal{S}(T)} < \infty$. To estimate this sequence further we use

$$F_m - F_{m-1} = \left(\tilde{U}_{m-1}\cdot\nabla\right)V_m + (U_{m-1}\cdot\nabla)\,\tilde{V}_{m-1} + \left(\tilde{U}_{m-1}\cdot\nabla\right)v_{\text{ref}} + (U_{\text{ref}}\cdot\nabla)\,\tilde{V}_{m-1}$$

and proceed as above to obtain

$$\begin{aligned} t^{1/2}\|\nabla\tilde{V}_m(t)\|_{L_H^\infty L_z^p} \le C_2\Big(&2H_m(t)\tilde{K}_{m-1}(t) + 2\tilde{H}_{m-1}(t)K_{m-1}(t) + K_m(t)\tilde{K}_{m-1}(t)\\ &+ K_{m-1}(t)\tilde{K}_{m-1}(t) + 2Rt^{1/2}\tilde{K}_{m-1}\Big) \end{aligned} \tag{8.6}$$

as well as

$$\|\tilde{V}_m(t)\|_{L_H^\infty L_z^p} \le C_2\Big(\tilde{K}_{m-1}(t)\left[H_m(t)+H_{m-1}(t)\right] + Rt^{1/2}\left[\tilde{K}_{m-1}(t)+\tilde{H}_{m-1}(t)\right]\Big), \tag{8.7}$$

where
$$\tilde{K}_m(t) := \sup_{0<s<t} s^{1/2}\|\nabla \tilde{V}_m(t)\|_{L^\infty_H L^p_z}, \quad \tilde{H}_m(t) := \sup_{0<s<t} \|\tilde{V}_m(t)\|_{L^\infty_H L^p_z}.$$
By (8.5) it follows that if
$$\max\{2RT_0^{1/2}, 16C_1\|a_0\|_{L^\infty_H L^p_z}\} < 1/4C_2,$$
then for $m \geq 1$ and $0 < t \leq T < T_0$ we have
$$\|\tilde{V}_m(t)\|_{\mathcal{S}(t)} \leq C_2\left(16C_1\|a_0\|_{L^\infty_H L^p_z} + 2Rt^{1/2}\right)\|\tilde{V}_{m-1}(t)\|_{\mathcal{S}(t)} < \frac{1}{2}\|\tilde{V}_{m-1}(t)\|_{\mathcal{S}(t)}.$$
Therefore, since $\mathcal{S}(T)$ is a Banach space, $(V_m)_{m\in\mathbb{N}}$ converges in $\mathcal{S}(T)$. We denote the limit by V and see that it satisfies
$$V(t) = e^{tA}a_0 - \int_0^t e^{(t-s)A}\mathbb{P}\Big((U(s)\cdot\nabla)V(s) + (U(s)\cdot\nabla)v_{\mathrm{ref}}(s) + (u_{\mathrm{ref}}(s)\cdot\nabla)V(s)\Big)\,ds \tag{8.8}$$
for $t \in (0,T)$ and thus $v := V + v_{\mathrm{ref}}$ is a solution to the primitive equations (2.1).

Step 4: *Extending to a global solution.*

Using $V \in \mathcal{S}(T)$, the embedding $L^\infty_H L^p_z(\Omega) \hookrightarrow L^p(\Omega)$, as well as the semigroup estimates
$$t^\vartheta\|e^{tA}\mathbb{P}f\|_{D((-A_{p,\overline{\sigma}})^\vartheta)} \leq C\|f\|_{L^p(\Omega)}, \quad t^{1/2}\|e^{tA}\mathbb{P}\nabla\cdot f\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \quad t>0, \quad \vartheta\in[0,1]$$
compare [11, Lemma 4.6] and [7, Theorem 3.7], one easily obtains that $V(t_0) \in D((-A_{p,\overline{\sigma}})^\vartheta)$ for $t_0 > 0$, and thus $v(t_0) \in D((-A_{p,\overline{\sigma}})^{1/p})$ as well, so v can be extended to a global solution that is strong on (t_0,∞).

Step 5: *Uniqueness.*

To see that v is a unique solution and thus strong on $(0,t_0)$ as well, we consider $v^{(1)}$ and $v^{(2)}$ both to be solutions in the sense of Theorem 3.1 with initial value a and set
$$t^* := \inf\{t\in[0,\infty) : v^{(1)}(t) \neq v^{(2)}(t)\}.$$
Assume that $t^* \in (0,\infty)$. Then using continuity of the solutions
$$a^* := v^{(1)}(t^*) = v^{(2)}(t^*) = a^*_{\mathrm{ref}} + a^*_0$$
where $a^*_0 \in X_{\overline{\sigma}}$ is sufficiently small and $a^*_{\mathrm{ref}} \in D(A_{\overline{\sigma}})$. Let v^*_{ref} be the reference solution to the initial data a^*_{ref} and
$$V^{(i)}(t) := v^{(i)}(t^*+t) - v^*_{\mathrm{ref}}(t^*), \quad i = 1,2.$$
Then $V^{(1)}, V^{(2)} \in \mathcal{S}(T^*)$ both satisfy the condition (8.8) for arbitrary $t\in(0,T^*)$, $T^*\in(0,\infty)$. We set $\tilde{V} := V^{(1)} - V^{(2)}$ and observe that proceeding analogously as before one obtains
$$\tilde{H}(t) \leq C_3\Big(t^{1/2}(\tilde{H}(t)+\tilde{K}(t)) + H^{(1)}(t)\tilde{K}(t) + K^{(2)}(t)\tilde{H}(t)\Big),$$
$$\tilde{K}(t) \leq C_3\Big(2t^{1/2}\tilde{K}(t) + H^{(1)}(t)\tilde{K}(t) + K^{(2)}(t)\tilde{H}(t) + K^{(1)}\tilde{K}(t) + K^{(2)}\tilde{K}(t)\Big),$$
where $\tilde{H}, H^{(i)}, \tilde{K}, K^{(i)}$ are defined analogously to above. This yields
$$\|\tilde{V}\|_{\mathcal{S}(t)} \leq C_3\left(t^{1/2} + H^{(1)}(t) + H^{(2)}(t) + K^{(1)}(t) + K^{(2)}(t)\right)\|\tilde{V}\|_{\mathcal{S}(t)}, \quad t\in(0,T^*). \tag{8.9}$$
By taking $T^* > 0$ to be small the terms $(T^*)^{1/2}$ and $K^{(1)}(T)^*, K^{(2)}(T^*)$ can be taken to be arbitrarily small due to $\|\nabla V^{(i)}(t)\| = o(t^{-1/2})$, which in the case $t^* = 0$ follows from the regularity of v and in the case $t^* > 0$ this follows from $\|\nabla v(t^*) \in L^\infty_H L^p_z(\Omega)^2$.

As for $H^{(1)}$ and $H^{(2)}$, using the same arguments that derived (8.3) one obtains for $t\in(0,T^*)$ that
$$H^{(i)}(t) \leq C_1\left(\|a^*_0\|_{L^\infty_H L^p_z} + K^{(i)}(t)H^{(i)}(t) + R^*t^{1/2}H^{(i)}(t) + R^*t^{1/2}K^{(i)}(t)\right), \tag{8.10}$$
where $R^* := \sup_{0\leq t\leq T^*}\|\nabla v^*_{\mathrm{ref}}(t)\|_{L^\infty_H L^p_z}$. Now, we choose $T\in(0,T^*)$ so small that
$$K^{(i)}(T)H^{(i)}(T^*) + R^*T^{1/2}H^{(i)}(T^*) + R^*T^{1/2}K^{(i)}(T^*) \leq \|a^*_0\|_{L^\infty_H L^p_z}.$$

Now, taking $\|a_0^*\|_{L^\infty_H L^p_z}$ to be sufficiently small, using that the constants $C_i > 0$, $i = 1, 2, 3$, are independent of $\|a_0^*\|_{L^\infty_H L^p_z}$, we obtain that the pre-factor in (8.9) is smaller 1. Hence, it follows that $\|\tilde{V}\|_{\mathcal{S}(t)} = 0$ for $t \in (0, T)$ and thus $v^{(1)} = v^{(2)}$ on $[0, t^* + T)$ which is a contradiction.

Step 6: Additional regularity.

By [11, Theorem 6.1] we thus have

$$v \in C^1((0,\infty); L^p_{\overline{\sigma}}(\Omega)) \cap C((0,\infty); W^{2,p}(\Omega))^2, \quad \pi \in C((0,\infty; W^{1,p}(G)).$$

The additional regularity $v \in C([0,\infty]; X_{\overline{\sigma}})$ follows from the strong continuity of the semigroup on $X_{\overline{\sigma}}$.

For the pressure we have $\pi(t) \in W^{1,p}(G) \hookrightarrow C^{0,\alpha}([0,1]^2)$ for $\alpha \in (0, 1 - 2/p)$. To obtain the regularity of $\nabla_H \pi$, observe that

$$\nabla_H \pi = -Bv - (1 - \mathbb{P})(u \cdot \nabla)v = -Bv - (1 - Q)\overline{(u \cdot \nabla)v}$$

where we used that $(1 - \mathbb{P})f = (1 - Q)\overline{f}$. In the proof of Claim 6.1 we have already proven that $Bv(t) \in C^{0,\alpha}([0,1]^2)$ for $\alpha \in (0, 1 - 3/p)$ if $v(t) \in W^{2,p}(\Omega)^2$. Likewise, since $1 - Q$ is continuous on $C^{0,\alpha}_{\text{per}}([0,1]^2)^2$ and $v \in C((0,\infty); W^{2,p}(\Omega))^2$, we obtain that the remaining terms belong to $C((0,\infty); C^{0,\alpha}([0,1]^2))^2$. ☐

Proof of Theorem 3.2. Here, we make use of the fact that the relevant estimates in Claim 6.1 and Claim 7.1 can also be applied in $L^\infty_H L^p_z(\Omega)^2$, compare Remark 3.5 (c).

Let $a = a_1 + a_2$ be as in Theorem 3.2. Next, we introduce a decomposition setting $a_0 := a_2 + (a_1 - a_{\text{ref}})$ where

$$a_{\text{ref}} \in D(A_{\overline{\sigma}}), \quad a_1 \in X_{\overline{\sigma}} \quad \text{and} \quad a_2 \in L^\infty_H L^p_z(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega),$$

where a_{ref} is such that a_0 satisfies the smallness condition of Theorem 3.1.

Then the same iteration scheme as in the previous proof can be used to construct V for the initial value a_0 and, in turn, v to the initial value a.

The property

$$v \in C([0,\infty); L^p_{\overline{\sigma}}(\Omega)) \cap L^\infty((0,T); L^\infty_H L^p_z(\Omega))^2$$

follows from the boundedness and exponential stability of the semigroup on $L^p_{\overline{\sigma}}(\Omega)$ and $L^\infty_H L^p_z(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega)$, as well as the strong continuity on $L^p_{\overline{\sigma}}(\Omega)$. Since the solution regularizes at $t_0 > 0$, compare Step 4 in the previous proof, we further obtain $v \in C((0,\infty); X_{\overline{\sigma}})$ from the strong continuity on $X_{\overline{\sigma}}$.

The condition

$$\limsup_{t \to 0+} t^{1/2} \|\nabla v\|_{L^\infty_H L^p_z(\Omega)} \le C \|a_2\|_{L^\infty_H L^p_z}, \tag{8.11}$$

is verified as follows. Since Claim 6.1 yields

$$\limsup_{t \to 0+} t^{1/2} \|\nabla e^{tA}(a_1 - a_{\text{ref}})\|_{L^\infty_H L^p_z} = 0, \quad t^{1/2} \|\nabla e^{tA} a_2\|_{L^\infty_H L^p_z} \le C_4 \|a_2\|_{L^\infty_H L^p_z}, \quad t > 0,$$

one obtains

$$\limsup_{t \to 0+} t^{1/2} \|\nabla V_0(t)\|_{L^\infty_H L^p_z} \le C_4 \|a_2\|_{L^\infty_H L^p_z}.$$

We now prove $\limsup_{t \to 0+} t^{1/2} \|\nabla V_m(t)\|_{L^\infty_H L^p_z} \le 2C_4 \|a_2\|_{L^\infty_H L^p_z}$ by induction. Assuming the claim holds for $m \in \mathbb{N}$ we obtain

$$\limsup_{t \searrow 0} t^{1/2} \|\nabla V_{m+1}(t)\|_{L^\infty_H L^p_z} \le \Big(1 + 2C_1 \|a_0\|_{L^\infty_H L^p_z} + 4C_4 \|a_2\|_{L^\infty_H L^p_z}\Big) C_4 \|a_2\|_{L^\infty_H L^p_z}$$

in the same manner as (8.2). Assuming that $\|a_0\|_{L^\infty_H L^p_z} < 1/4C_1$ and $\|a_2\|_{L^\infty_H L^p_z} < 1/8C_4$ it follows that the claim holds for all $m \in \mathbb{N}$ and by taking the limit the same estimate holds for V. Using $v_{\text{ref}} \in C([0,\infty); C^1(\overline{\Omega})^2)$, we obtain that $v = V + v_{\text{ref}}$ satisfies (8.11).

To prove uniqueness we make the following modifications. If $v^{(1)}$ and $v^{(2)}$ are both solutions in the sense of Theorem 3.2, we again define $t^* := \inf\{t \in [0,\infty) : v^{(1)}(t) \ne v^{(2)}\}$.

In the case $t^* > 0$ we have $a^* = v^{(1)}(t^*) = v^{(2)}(t^*) \in D((-A_{p,\overline{\sigma}})^\vartheta)$ for any $\vartheta \in [0,1]$, compare Step 4 of the proof of Theorem 3.1. Choosing $2\vartheta - 3/p > 0$ we have that $D((-A_{p,\overline{\sigma}})^\vartheta) \hookrightarrow X_{\overline{\sigma}}$ and thus we can decompose $a^* = a^*_{\text{ref}} + a^*_0$ as before and the same argument applies.

If we instead have $t^* = 0$ we continue to use the decomposition $a = a_{\mathrm{ref}} + a_0$ where $a_0 = a_2 + (a_1 - a_{\mathrm{ref}})$. In this case we have $\lim_{t\to 0+} K^{(i)}(t) \le C\|a_2\|_{L^\infty_H L^p_z}$ for an absolute constant $C > 0$ and thus the quantities on the right-hand side of (8.9) can again be taken to be sufficiently small, where on the right-hand side of (8.10) one has $\|a_0\|_{L^\infty_H L^p_z}$ instead of $\|a_0^*\|_{L^\infty_H L^p_z}$, which again yields uniqueness.

This completes the proof. □

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