21st Century COE Program:
Mathematics of Nonlinear Structure via Singularities

The 6th COE Lecture Series

From the cut-locus via the medial axis to the Voronoi diagram and back

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Contents

Introduction

In July 2005 I gave three lectures to review some related subjects, and to present some of my own results. The subjects were

- Cut-locus and conjugate locus
- Medial axis, symmetry set
- Voronoi diagrams, Delaunay triangulations

The cut-locus and the conjugate locus are the oldest and probably most studied objects in the list. Poincaré called the cut-locus the "ligne de partage", the dividing line. Moving over from one side of the cut-locus to another the minimizing geodesics change drastically and on the cut-locus itself the minimizing geodesic is not unique.

The name "cut-locus" became popular only later through the investigations of Witehead and Myers. Application of singularity theory to differential geometry were instigated by Thom in his "Cut-locus d'une variété plongé": the medial axis of an embedded manifold.

The new results in these lecture notes are mostly the method to calculate Jacobi fields in the first lecture, and in the second lecture there is the notion of a weighted symmetry set. The notion was invented by D. Siersma and I proved the corresponding result. In the third lecture there is a complexity formula for the Delaunay triangulation of a set of points in general position. Other new results which we talked about are in [**SvM05**].

These notes are written in an informal style. I feel that including "the details" would first of all affect the readability, and secondly I did not have enough time to write them all out. Also, the quality of these notes is at best mediocre when compared with the older [**Wal77**].

LECTURE 1

Manifolds: Cut and conjugate loci

1. The energy functional on the space of paths with fixed end-points

Let (M,g) be an m-dimensional Riemannian manifold. Geodesics in M are locally shortest paths between two points. They are the solutions of a system of differential equations that we shall now derive.

Write $g = (g_{ij}(x))_{1\le i,j\le m}$. The energy of a path $\gamma\colon [0,1] \to M$ is the value of the integral

$$E(\gamma) = \frac{1}{2}\int_{t=0}^{1} \sum_{1\le i,j\le m} g_{ij}(\gamma(t))\frac{\partial\gamma_i}{\partial t}\frac{\partial\gamma_j}{\partial t}\,\mathrm{d}t = \frac{1}{2}\int_{t=0}^{1} \|\dot\gamma\|^2\,\mathrm{d}t = \frac{1}{2}\int_{t=0}^{1} \mathcal{L}(\gamma,\dot\gamma,t)\,\mathrm{d}t$$

So the energy is half of the squared length of a path: Paths of minimal energy are also paths of minimal length. One finds the paths of minimal length using the *Euler Lagrange equations*. These are

$$\frac{\partial\mathcal{L}}{\partial\gamma_k} = \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot\gamma_k}\right) \qquad k = 1,\cdots,n$$

Instead of directly using this equation we will ponder a little over how it is derived in this special case.

We view E as a function on the space of paths with fixed end points $\gamma(0) = u_0$ and $\gamma(L) = u_1$. So all the variations of γ will have the same endpoints u_0 and u_1.

The "coordinates" on the space of paths are γ and $\dot\gamma$. The first variation δE is

$$\begin{aligned}\delta E =& \frac{\partial E}{\partial\gamma}\delta\gamma + \frac{\partial E}{\partial\dot\gamma}\delta\dot\gamma\\ =& \frac{1}{2}\int_{t=0}^{l}\left(\sum_{i,j=1}^{m}\frac{\partial g_{ij}}{\partial x_k}\dot\gamma_i\dot\gamma_j\delta\gamma_k + \sum_{i=1}^{m} g_{ki}\dot\gamma_i\delta\dot\gamma_k\right)\mathrm{d}t\end{aligned}$$

One calculates the second term separately using partial integration:

$$\begin{aligned}\frac{\partial E}{\partial\dot\gamma}\delta\dot\gamma =& \frac{\partial E}{\partial\dot\gamma}\delta\left(\frac{\partial\gamma}{\partial t}\right) = \frac{\partial E}{\partial\dot\gamma}\frac{\partial}{\partial t}(\delta\gamma)\\ \Rightarrow& \int_{t=0}^{1}\frac{\partial E}{\partial\dot\gamma}\delta\dot\gamma\,\mathrm{d}t = \left|g_{ij}\gamma_i\gamma_j\right|_{t=0}^{L} - \int_{t=0}^{L}\frac{\partial}{\partial t}\left(\sum_{i=1}^{m} g_{ki}\dot\gamma_i\right)\delta\gamma_k\,\mathrm{d}t\end{aligned}$$

Hence the end result is

$$\frac{1}{2}\sum_{i,j=1}^{m}\frac{\partial g_{ij}}{\partial x_k}\dot\gamma_i\dot\gamma_j = \sum_{i=1}^{m} g_{ki}\ddot\gamma_i + \sum_{i,j=1}^{m}\frac{\partial g_{ki}}{\partial x_j}\dot\gamma_i\dot\gamma_j \tag{1}$$

With the usual definition of the *Christoffel symbols*

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x_k} + \frac{\partial g_{kj}}{\partial x_i} - \frac{\partial g_{ik}}{\partial x_j}\right) \quad \Gamma^k_{ij} = \sum_{l=1}^{m} g^{kl}[ij,l]$$

the geodesic equation is:

$$\ddot{\gamma}_k = -\sum_{i,j} \Gamma^k_{ij}\dot{\gamma}_i\dot{\gamma}_j \tag{2}$$

From an example it is clear that a locally shortest path is not necessarily globally a shortest path. The simplest example is of course the sphere. But on any closed surface there are many geodesics, that are not shortest. A *minimizing geodesic* is a geodesic between two points u and u' on a Riemannian manifold M is a geodesic from u to u' such that there is no geodesic that is shorter, or equivalently whose value of the energy functional is lower. Minimizing geodesics are global minima of the energy functional.

2. The cut-locus

It might happen that for some $u, u' \in M$ there are two or minimizing geodesics from u to u'. Here the simplest example is again the sphere. Between the North pole and the south pole there are infinitely many minimizing geodesics.

For a Riemannian manifold M and a point $u \in M$ the *cut-locus* of u in M consists of the closure in M of those points $u' \in M$ such that there are at least two minimizing geodesics from u to u'.

Examples:

(1) The cut-locus of any point on a sphere is just the opposite point.
(2) The cut-locus of a generic point on a torus is a figure 8, see figure 1.
(3) The cut-locus of a point that is not an umbilic on an ellipsoid is a line segment.

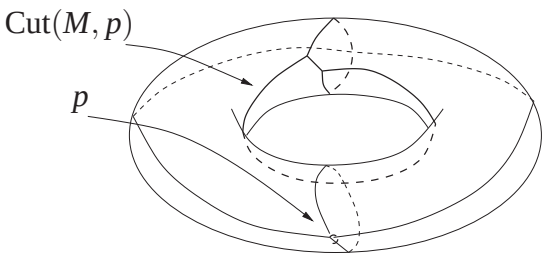


FIGURE 1. A cut-locus on a torus.

The importance of cut-loci in topology is because of the following theorem

THEOREM 1. *Denote $M \setminus \{p\}$ a compact manifold with the point p removed. Denote* $\mathrm{Cut}(M,p)$ *the cut-locus of $p \in M$. Then*

(1) $\mathrm{Cut}(p)$ *is a strong deformation retract of $M \setminus \{p\}$.*
(2) *The inclusion $\iota\colon \mathrm{Cut}(M,p) \hookrightarrow M$ induces isomorphisms in homology $\iota_*\colon H_i(\mathrm{Cut}(M,p)) \to H_i(M)$ and homotopy $\iota_*\colon \pi_i(\mathrm{Cut}(M,p)) \to \pi_i(M)$ for $i \leq -2 + \dim(M)$.*

Cut-loci are notoriously hard to calculate: the geodesic equation can only be solved explicitly in some special cases. Of these we mention the surfaces of revolution. These are surface of the form

$$x = f(r)\cos(\theta) \quad y = f(r)\sin(\theta) \quad z = g(r)$$

Surfaces of revolution are often used to produce counter-examples in differential geometry.

In most cases though we can only use a computer. Here a warning is in place. The geodesic equation is solvable for some interval $t \in [0, \varepsilon]$, according to the general existence theorem for ordinary differential equations. But it is by no means assured that the solutions exist for all $t \in \mathbb{R}$. If they do then we say that the manifold M is *complete*.

Recall that a metric space is complete if every Cauchy sequence converges. The manifold M is a metric space if we put the distance between two points to be $\sqrt{2E(\gamma)}$, with E the energy of a minimizing geodesic. Here's the Hopf-Rinow theorem:

THEOREM 2. *A Riemannian manifold is complete iff. it is complete as a metric space. If the solutions of the geodesic equation at one point in M exist for all t, then they exist at all points in M for all t.*

We will assume from here that all manifolds we consider are compact and hence complete.

The computation of cut-loci is done in [**IS04**].

The cut-locus $\mathrm{Cut}(M, p)$ can also be considered as a subset of T_pM. For each $q \in \mathrm{Cut}(M, p)$ take all $v \in T_p(M)$ such that there is a geodesic γ with $\gamma(0) = p$, $\dot{\gamma}(0) = v$ and $\gamma(1) = q$. Plotting all those vectors in T_pM and taking the closure we get the curve that we call the *tangential cut-locus*.

3. The geodesic flow

For computer implementation the geodesic equation can not be applied directly: there is no coordinate patch that covers the whole manifold. A technique of Perelomov can be used to calculate geodesics in an important special case.

Assume that the manifold M is given as the zero set of a $\mathcal{C}^\infty$ function $F\colon \mathbb{R}^n \to \mathbb{R}$. In $T\mathbb{R}^n$ the equations for a geodesic in $\mathbb{R}^n$ with the Euclidean metric are

$$\dot{x} = v \quad \dot{v} = 0 \tag{3}$$

where v are the coordinates in the fiber of $T\mathbb{R}^n \to \mathbb{R}^n$. Equation (3) produces curves that have

$$\|v\|^2 \equiv \text{Constant}$$

because $\frac{\partial}{\partial t}\left(\|v\|^2\right) = 2\langle v, \dot{v}\rangle = 0$. However they do not remain on the manifold $F(x) = 0$. To get the right equations we have to use a Lagrange multiplier technique. The functional that has to be minimized is not

$$\int_0^L \frac{1}{2}\|v\|^2 \,\mathrm{d}t$$

but

$$\int_0^L \left(\frac{1}{2}\|v\|^2 - \lambda F(x)\right) \mathrm{d}t \tag{4}$$

where we vary over paths with fixed end-points.

THEOREM 3. *Geodesics on a manifold $\mathbb{R}^n \supset M = \{x \mid F(x) = 0\}$, where M has the metric induced from the Euclidean one on $\mathbb{R}^n$, are found by integrating:*

$$\dot{x} = v \quad \dot{v} = -\frac{\frac{\partial^2 F}{\partial x^2} vv}{\|\frac{\partial F}{\partial x}\|^2} \frac{\partial F}{\partial x} \tag{5}$$

PROOF. The functional that has to be minimized is (4), where we vary over paths with fixed end-points. We get

$$\delta\left(\int_0^L \left(\frac{1}{2}\|v\|^2 - \lambda F(x)\right) \mathrm{d}t\right) = \int_0^L \left(v\delta v - \lambda \frac{\partial F}{\partial x}\delta x\right) \mathrm{d}t$$
$$\int_0^L \left(v\delta\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) - \lambda \frac{\partial F}{\partial x}\delta x\right) \mathrm{d}t = \int_0^L \left(-\dot{v}\delta x - \lambda \frac{\partial F}{\partial x}\delta x\right) \mathrm{d}t$$

So we get

$$\dot{x} = v \quad \dot{v} = -\lambda \frac{\partial F}{\partial x} \tag{6}$$

It remains to determine λ. We have

$$F(x) = 0 \Rightarrow \frac{\partial F}{\partial x}\dot{x} = 0 \Rightarrow \frac{\partial F}{\partial x} v = 0 \Rightarrow \frac{\partial^2 F}{\partial x^2} vv + \frac{\partial F}{\partial x}\dot{v} = 0$$

from which we derive that equation (6) becomes as in [**Per00**], and in (5). □

REMARK 1. *The statement of 3 is more or less due to Perelomov. The proof is ours, although we do not exclude that this type of reasoning can be found in some very old textbooks.*

Equation (5) has the advantage that it uses "global coordinates". So it can be used to calculate geodesics without having to change coordinates when the geodesics walk off a coordinate patch, or it can be used to calculate geodesics of a surface, whose implicit equations we have, but whose parameterization is not provided.

Equation (5) can be derived in another way. We know that geodesics on an embedded manifold - with the metric induced from the ambient space - are the curves whose geodesic curvature is zero. So the curvature of a geodesic is the normal curvature, and thus the second derivative wrt. to t is everywhere orthogonal to the embedded manifold. In other words, $\dot{v}$ is a multiple of the gradient of F.

For those who remain sceptical about (5) there is an easy way of checking the correctedness. Namely, using the program supplied with [**Gra98**], the geodesic equation (2) is quickly calculated. One can then compare it with (5) if one has both the implicit equation as well as the embedding of a manifold available.

Note also that we can generalize this framework rather easily. In the most general situation we have the manifold as a zero set of a function $F\colon \mathbb{R}^n \to \mathbb{R}^l$ and we use a homogeneous of degree 2 Lagrangian $\mathcal{L}\colon T_{\setminus 0}\mathbb{R}^n \to \mathbb{R}$. We minimize the functional

$$\int_{t=0}^L \frac{1}{2}\mathcal{L}(x,v) - \lambda F(x)\,\mathrm{d}t$$

and we put $\dot{x} = v$. Finding the $\lambda_i = \lambda_i(x,v)$ is done as in the previous case by differentiating $F_i(x) = 0$ with respect to t twice. Solving the resulting equations involves inverting the matrix:

$$\frac{\partial F}{\partial x}\frac{\partial F}{\partial x}^T = \begin{pmatrix} \frac{\partial F_1}{\partial x}\frac{\partial F_1}{\partial x} & \cdots & \frac{\partial F_1}{\partial x}\frac{\partial F_l}{\partial x} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_l}{\partial x}\frac{\partial F_1}{\partial x} & \cdots & \frac{\partial F_l}{\partial x}\frac{\partial F_l}{\partial x} \end{pmatrix}$$

It can easily be proved that this matrix is invertible. But there seems to be no nice expression for the inverse. So, the formulas for $\lambda_i = \lambda_i(x,v)$ are rather ugly.

We only want to work with manageable formulas, so we restrict to the above case, in theorem 3.

4. Covariant differentiation and curvature

Recall the basic theorems about the Levi-Civita connection. In each point $p \in M$ a vectorfield V determines a geodesic γ_V by putting $\dot{\gamma}_V(0) = V(p)$, and $\gamma_V(0) = p$. *Covariant differentiation* is the operation that assigns to two vectorfields V and W in TM a new vectorfield $\nabla_V W$ according to the rule:

$$\nabla_V W = \frac{\partial W \circ \gamma_V}{\partial t}_{|t=0} + \sum_{i,j} \Gamma^k_{ij} W_i V_j$$

Using covariant differentiation define the *Riemann curvature tensor R*, see [**Spi79a**], chapter 6:

$$R(V,W) = [\nabla_V, \nabla_W] - \nabla_{[V,W]} \tag{7}$$

The Riemann curvature tensor is a $(3,1)$-*tensor*. It is a called a tensor because it is linear over the $\mathcal{C}^\infty$ functions on M. It is called a $(3,1)$-tensor because we have to feed it 3 vectors V_1, V_2 and V_3 to get $R(V_1,V_2)V_3$ which is an element of the dual of the tangent space. Its fundamental significance is twofold:

(1) It is preserved under isometries.
(2) If $R = 0$ then the manifold is isometric to Euclidean space.

If we have a geodesic $\gamma \subset M$ we can define the covariant derivative of a vectorfield along a curve using the formula (6).

THEOREM 4. *The covariant derivative of a vectorfield in TM along a geodesic is*

$$\frac{\mathrm{D}}{\mathrm{d}t}V = \dot{V} + \frac{\frac{\partial^2 F}{\partial x^2} vV}{\|\frac{\partial F}{\partial x}\|^2}\frac{\partial F}{\partial x} \tag{8}$$

PROOF. First of all we need to show that $\mathrm{D}V/\mathrm{d}t$ is again a vectorfield in TM. For this we need to proof that

$$\frac{\partial F}{\partial x}\frac{\mathrm{D}V}{\mathrm{d}t} = 0$$

For that we differentiate the identity wrt. to t:

$$\frac{\partial F}{\partial x}V = 0 \quad \Rightarrow \quad \frac{\partial^2 F}{\partial x^2} vV + \frac{\partial F}{\partial x}\dot{V} = 0 \tag{9}$$

whereas

$$\frac{\mathrm{D}V}{\mathrm{d}t}\frac{\partial F}{\partial x} = \dot{V}\frac{\partial F}{\partial x} + \frac{\frac{\partial^2 F}{\partial x^2} vV}{\|\frac{\partial F}{\partial x}\|^2}\|\frac{\partial F}{\partial x}\|^2 \tag{10}$$

So that (10) gives zero because of (9).

To prove that this is really the Levi-Civita connection, we recall that the Levi-Civita connection is the unique operation satisfying:

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{d}t}(fV) &= \frac{\partial f}{\partial t}V + f\frac{\mathrm{D}}{\mathrm{d}t}V\\ \frac{\mathrm{D}}{\mathrm{d}t}(V+W) &= \frac{\mathrm{D}}{\mathrm{d}t}V + \frac{\mathrm{D}}{\mathrm{d}t}W\\ \frac{\partial}{\partial t}\langle VW\rangle &= \langle\frac{\mathrm{D}}{\mathrm{d}t}V, W\rangle + \langle\frac{\mathrm{D}}{\mathrm{d}t}W, V\rangle\end{aligned}$$

and such that the covariant derivative of the tangent along a geodesic is zero. One readily verifies that these properties hold, so that equation (8) indeed represents the Levi-Civita connection on TM, in our special case. □

Though the author found (8) independently it can be found in almost the same form in[**Lin04**]. There the expression (8) is used to get an expression for the base of the tangent space to the unit tangent bundle.

Suppose that the vectorfields V and W in equation (7) are tangent vectors to some patch of a surface embedded in M at p by a map $s\colon \mathbb{R}^2 \to M$, $s(0,0) = p$. Let (t_1, t_2) be coordinates on the left hand side. They correspond to vectorfields $\partial/\partial t_1$ and $\partial/\partial t_2$. We have

$$0 = s_*\left[\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}\right] = \left[s_*\frac{\partial}{\partial t_1}, s_*\frac{\partial}{\partial t_2}\right]$$

So that

$$R(s_*\frac{\partial}{\partial t_1}, s_*\frac{\partial}{\partial t_2})V = \left[\frac{\mathrm{D}}{\mathrm{d}t_1}, \frac{\mathrm{D}}{\mathrm{d}t_2}\right]V \tag{11}$$

The right hand side of equation (11) can be calculated using the extrinsic coordinates, when M is given as the zeroset of a function $F\colon \mathbb{R}^n \to \mathbb{R}$.

5. Jacobi fields

This section contains some horrible calculations. We quote from the book [**Ber03**], page 204: “Do not despair if the curvature tensor does not appeal to you. It is frightening for everybody. We hope that after a while you will enjoy it.”.

Along a geodesic there exist so called *Jacobi vectorfields*. A Jacobi vectorfield W along a geodesic γ has initial values

$$W(\gamma(0)) = W_0 \text{ and } \frac{\mathrm{D}W}{\mathrm{d}t}_{t=0} = W_1$$

and its values for other points on the geodesic are determined by the differential equation

$$\frac{\mathrm{D}^2 W}{\mathrm{d}t^2} = R(\dot\gamma, W)\dot\gamma \tag{12}$$

The Jacobi vectorfields form a $2\dim(M)$ dimensional vector space, because they are completely determined by their initial values. There is also a symplectic structure ω_J on the space of Jacobi vectorfields:

$$\omega_J((W_0, W_1), (W_0', W_1')) = W_0 W_1' - W_1 W_0'$$

Next, remark that if $W(t)$ is a solution to the equation (12) then also $(a+bt)\dot\gamma + W(t)$ is a solution. Indeed,

$$\frac{\mathrm{D}^2 W}{\mathrm{d}t^2} = \frac{\mathrm{D}^2(W + (a+bt)\dot\gamma)}{\mathrm{d}t^2} = R(\dot\gamma, W + (a+bt)\dot\gamma)\dot\gamma = R(\dot\gamma, W)\dot\gamma$$

On a 2-dimensional manifold M the Jacobi equation becomes much simpler. The only interesting Jacobi fields are orthogonal to the geodesic. When the tangent space T_pM is two dimensional there is thus only one interesting Jacobi field. For a geodesic the vector field orthogonal to $\dot\gamma$ and $\dot\gamma$ form a basis of T_pM. We have

$$\frac{\mathrm{D}}{\mathrm{d}t}\dot\gamma = 0$$

so the only interesting solution is the one for which

$$\langle \frac{\mathrm{D}W}{\mathrm{d}t}_{t=0}, \dot\gamma(0)\rangle = 0$$

and that look like $f(t)W$. Then we find the differential equation

$$\frac{\partial^2 f}{\partial t^2} = -K(\gamma(t))\, f(t) \tag{13}$$

where $K(\gamma(t))$ is the Gaussian curvature at $\gamma(t)$.

In accordance with what we have been doing before we want to describe the differential equation using the coordinates on $\mathbb{R}^n$. For the left hand side we apply $\mathrm{D}/\mathrm{d}t$ - equation (8) - twice to a vectorfield. For the right hand side and use the expression for the Riemann curvature tensor (11).

Start of the calculation with

$$\begin{aligned}
\frac{\mathrm{D}^2 W}{\mathrm{d}t^2} &= \frac{\mathrm{D}}{\mathrm{d}t}\left(\dot W + \frac{\frac{\partial^2 F}{\partial x^2} vW}{\|\frac{\partial F}{\partial x}\|^2}\frac{\partial F}{\partial x}\right) \\
&= \ddot W + \frac{\partial}{\partial t}\left(\frac{\partial^2 F}{\partial x^2} vW\right)\frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^2} + \frac{\partial^2 F}{\partial x^2} vW \frac{\partial}{\partial t}\left(\frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^2}\right) \\
&+ \frac{\partial^2 F}{\partial x^2} v\dot W \frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^2} + \left(\frac{\partial^2 F}{\partial x^2} vW\right)\left(\frac{\partial^2 F}{\partial x^2} v \frac{\partial F}{\partial x}\right)\frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^4} = I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned} \tag{14}$$

We calculate the terms I_1 to I_5 separately. Start with I_2.

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial^2 F}{\partial x^2} vW\right) &= \frac{\partial^3 F}{\partial x^3} vvW + \frac{\partial^2 F}{\partial x^2}\dot{v}W + \frac{\partial^2 F}{\partial x^2} v\dot{W} \\
&= \frac{\partial^3 F}{\partial x^3} vvW - \left(\frac{\partial^2 F}{\partial x^2} vv\right)\frac{\partial^2 F}{\partial x^2}\frac{\partial F}{\partial x} W \frac{1}{\|\frac{\partial F}{\partial x}\|^2} + \frac{\partial^2 F}{\partial x^2} v\dot{W} \Rightarrow \\
I_2 &= \frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^2}\left(\frac{\partial^3 F}{\partial x^3} vvW + \frac{\partial^2 F}{\partial x^2} v\dot{W}\right) - \left(\frac{\partial^2 F}{\partial x^2} vv\right)\frac{\partial^2 F}{\partial x^2}\frac{\partial F}{\partial x} W \frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^4} \Rightarrow \\
I_2 + I_4 &= \frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^2}\left(\frac{\partial^3 F}{\partial x^3} vvW + 2\frac{\partial^2 F}{\partial x^2} v\dot{W}\right) - \left(\frac{\partial^2 F}{\partial x^2} vv\right)\frac{\partial^2 F}{\partial x^2}\frac{\partial F}{\partial x} W \frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^4} \qquad (15)
\end{aligned}$$

For I_3:

$$\frac{\partial}{\partial t}\left(\frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^2}\right) = \frac{\frac{\partial^2 F}{\partial x^2} v}{\|\frac{\partial F}{\partial x}\|^2} - 2\frac{\frac{\partial^2 F}{\partial x^2} v \frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^4}\frac{\partial F}{\partial x} \qquad (16)$$

$$I_3 = \frac{\frac{\partial^2 F}{\partial x^2} vW \frac{\partial^2 F}{\partial x^2} v}{\|\frac{\partial F}{\partial x}\|^2} - 2\frac{\frac{\partial^2 F}{\partial x^2} vW \frac{\partial^2 F}{\partial x^2} v \frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^4}\frac{\partial F}{\partial x} \qquad (17)$$

So that

$$I_1 + I_3 + I_5 = \ddot{W} + \frac{\frac{\partial^2 F}{\partial x^2} vW \frac{\partial^2 F}{\partial x^2} v}{\|\frac{\partial F}{\partial x}\|^2} - \frac{\frac{\partial^2 F}{\partial x^2} vW \frac{\partial^2 F}{\partial x^2} v \frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^4}\frac{\partial F}{\partial x}$$

The next step is to calculate

$$\begin{aligned}
\left[\frac{\mathrm{D}}{\mathrm{d}t_1}, \frac{\mathrm{D}}{\mathrm{d}t_2}\right] V &= \frac{\mathrm{D}}{\mathrm{d}t_1}\left(\frac{\partial V}{\partial t_2} + \frac{\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_2} V}{\|\frac{\partial F}{\partial x}\|^2}\frac{\partial F}{\partial x}\right) V - \frac{\mathrm{D}}{\mathrm{d}t_2}\left(\frac{\partial V}{\partial t_1} + \frac{\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_1} V}{\|\frac{\partial F}{\partial x}\|^2}\frac{\partial F}{\partial x}\right) \\
&= \frac{\partial}{\partial t_1}\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_2} V\right)\frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^2} - \frac{\partial}{\partial t_2}\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_1} V\right)\frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^2} \\
&+ \frac{\partial}{\partial t_1}\left(\frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^2}\right)\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_2} V - \frac{\partial}{\partial t_2}\left(\frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^2}\right)\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_1} V \\
&+ \left(\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_1}\frac{\partial V}{\partial t_2} - \frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_2}\frac{\partial V}{\partial t_1}\right)\frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^2} \\
&+ \frac{\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_2} V\right)\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_1}\frac{\partial F}{\partial x}\right) - \left(\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_1} V\right)\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_2}\frac{\partial F}{\partial x}\right)}{\|\frac{\partial F}{\partial x}\|^4}\frac{\partial F}{\partial x} \\
&= I_2' + I_3' + I_4' + I_5'
\end{aligned}$$

Consider I_2', in which we find the terms:

$$\frac{\partial}{\partial t_1}\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_2}V\right)=\frac{\partial^3 F}{\partial x^3}\frac{\partial s}{\partial t_1}\frac{\partial s}{\partial t_2}V+\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 s}{\partial t_1\partial t_2}V+\frac{\partial^2 F}{\partial x^2}\frac{\partial V}{\partial t_1}\frac{\partial s}{\partial t_2}$$

So that in I_2' only the terms

$$\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial V}{\partial t_1}\frac{\partial s}{\partial t_2}-\frac{\partial^2 F}{\partial x^2}\frac{\partial V}{\partial t_2}\frac{\partial s}{\partial t_1}\right)\frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^2}$$

remain. But this is exactly $-I_4'$. So we find $I_2'+I_4'=0$. We focus on I_3'. To calculate it we use again equation (16).

$$\begin{aligned}I_3' &= \frac{\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_2}V}{\|\frac{\partial F}{\partial x}\|^2}\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_1}-\frac{\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_1}V}{\|\frac{\partial F}{\partial x}\|^2}\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_2}\\ &+2\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_1}V\right)\frac{\partial^2 F}{\partial x^2}\frac{\partial F}{\partial x}\frac{\partial s}{\partial t_2}\frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^4}-2\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_2}V\right)\frac{\partial^2 F}{\partial x^2}\frac{\partial F}{\partial x}\frac{\partial s}{\partial t_1}\frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^4}\Rightarrow\end{aligned}\tag{18}$$

$$\begin{aligned}\left[\frac{\mathrm{D}}{\mathrm{d}t_1},\frac{\mathrm{D}}{\mathrm{d}t_2}\right]V &= I_3'+I_5'=\frac{\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_2}V}{\|\frac{\partial F}{\partial x}\|^2}\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_1}-\frac{\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_1}V}{\|\frac{\partial F}{\partial x}\|^2}\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_2}\\ &+\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_1}V\right)\frac{\partial^2 F}{\partial x^2}\frac{\partial F}{\partial x}\frac{\partial s}{\partial t_2}\frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^4}-\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial s}{\partial t_2}V\right)\frac{\partial^2 F}{\partial x^2}\frac{\partial F}{\partial x}\frac{\partial s}{\partial t_1}\frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^4}\end{aligned}\tag{19}$$

Next we need to calculate the Jacobi equation:

$$\frac{\mathrm{D}^2 W}{\mathrm{d}t^2}=R(v,W)v$$

We use equations (19) and (14), (15), (17). In equation (19), we replace,

$$V\Rightarrow v\quad \frac{\partial s}{\partial t_1}\Rightarrow v\quad \frac{\partial s}{\partial t_2}\Rightarrow W$$

to obtain:

$$\begin{aligned}R(v,W)v &= = \frac{\frac{\partial^2 F}{\partial x^2}Wv}{\|\frac{\partial F}{\partial x}\|^2}\frac{\partial^2 F}{\partial x^2}v-\frac{\frac{\partial^2 F}{\partial x^2}vv}{\|\frac{\partial F}{\partial x}\|^2}\frac{\partial^2 F}{\partial x^2}W\\ &+\left(\frac{\partial^2 F}{\partial x^2}vv\right)\frac{\partial^2 F}{\partial x^2}\frac{\partial F}{\partial x}W\frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^4}-\left(\frac{\partial^2 F}{\partial x^2}Wv\right)\frac{\partial^2 F}{\partial x^2}\frac{\partial F}{\partial x}v\frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^4}\end{aligned}$$

and in equation (12):

$$\dot{W}\Rightarrow\mu$$

The conclusion is that we can calculate Jacobi fields on a manifold $\{F=0\}$ in $\mathbb{R}^n$ by

$$\begin{aligned}
\dot{x} &= v \\
\dot{v} &= -\frac{\frac{\partial^2 F}{\partial x^2} vv}{\|\frac{\partial F}{\partial x}\|^2} \frac{\partial F}{\partial x} \\
\dot{W} &= \mu \\
\dot{\mu} &= -\frac{1}{\|\frac{\partial F}{\partial x}\|^2} \left(\left(\frac{\partial^3 F}{\partial x^3} Wvv + 2\frac{\partial^2 F}{\partial x^2} v\mu \right) \frac{\partial F}{\partial x} + \left(\frac{\partial^2 F}{\partial x^2} vv \right) \frac{\partial^2 F}{\partial x^2} W \right) \\
&\quad + 2\left(\frac{\partial^2 F}{\partial x^2} vv \right) \left(\frac{\partial^2 F}{\partial x^2} W \frac{\partial F}{\partial x} \right) \frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^4}
\end{aligned} \tag{20}$$

The geodesic flow is a map ϕ_t from $T_{\backslash 0}M$ to $T_{\backslash 0}M$. What we now contend is:

THEOREM 5. *The exponential map corresponding to the system* (20) *is the derivative of the geodesic flow, i.e.*

$$\mathrm{d}\phi_t \begin{pmatrix} W(0) \\ \mu(0) \end{pmatrix} = \begin{pmatrix} W(t) \\ \mu(t) \end{pmatrix}$$

PROOF. All we need to do to find the total differential of

$$\left(v, -\left(\frac{\partial^2 F}{\partial x^2} vv \right) \frac{\frac{\partial F}{\partial x}}{\|\frac{\partial F}{\partial x}\|^2} \right)$$

with respect to the x and v variables and replace in that expression $\mathrm{d}x$ by W and $\mathrm{d}v$ by μ. It is straightforward to verify that this leads to the equations (20). □

REMARK 2. *With this theorem, it becomes theoretically possible to calculate Jacobi fields in the more general setting sketched in section 3. Theorem 5 is due to Cartan. It can be found in* [**Car88**], *paragraph 160.*

6. The conjugate locus

The manifold we flow out is the point with all of its rays: $T_{x_0}M$. We are only interested in the geodesics itself, and not in the particular parameterization, so we will flow out

$$T_{x_0}M \cap \{\mathcal{L}(x_0,v)=1\} = T_{x_0}M \cap \{\|v\|=1\}$$

If M is two-dimensional there is only one vector tangent to this manifold, and thus only one Jacobi field. In general, when $\dim(M)>2$, then the interesting initial values (20) are those for which $W(0)=0$, and $v(0)$ is orthogonal to $\mu(0)$. The *conjugate locus* $\mathrm{Conj}(M,p)$ consists of the first points along geodesics from p where a Jacobi field with initial values $(0,\mu(0)) \neq (0,0)$ becomes zero. From theorem 5 we see that at a point of the conjugate locus the map

$$\phi_t \colon \mathbb{R}_{>0} \times (TM \cap \{\mathcal{L}(x_0,v)=1\}) \to TM \cap \{\mathcal{L}(x_0,v)=1\}$$

turns vertical. This means that at a point of the conjugate locus the composite map:

$$\pi_M \circ \phi_t \colon \mathbb{R}_{>0} \times (TM \cap \{\mathcal{L}(x_0,v)=1\}) \to TM \cap \{\mathcal{L}(x_0,v)=1\} \to M$$

is not a diffeomorphism. Indeed there is a vector in the kernel of $\mathrm{d}(\pi_M \circ \phi_t)$, namely $(0, \mu(0))$, because Cartan's theorem tells us that the image is $W(t)$ there. So the cut-locus is the set of first points on the geodesics where the map $\phi_t(x_0, v)$ is no longer injective, and the conjugate locus is the set of points where the differential of $D\phi_t(x_0, v)$ is no longer injective.

Similarly to the tangential cut-locus we can define the *tangential conjugate locus*.

Along the geodesic $\mu(t)$ remains orthogonal to the geodesic. Indeed, we write

$$\frac{\partial}{\partial t}\langle \dot{\gamma}(t), \mu(t)\rangle = \langle \dot{\gamma}(t), \frac{\mathrm{D}\mu}{\mathrm{d}t}\rangle = \langle \dot{\gamma}(t), \frac{\mathrm{D}^2 W}{\mathrm{d}t^2}\rangle = \langle R(\dot{\gamma}, W)\dot{\gamma}, \dot{\gamma}\rangle = 0$$

So that $\langle \dot{\gamma}(t), \mu(t)\rangle = \langle \dot{\gamma}(0), \mu(0)\rangle = 0$. It follows that at a smooth point of the conjugate locus the vector $\mu(t)$ is orthogonal to conjugate locus.

If M is a two-dimensional manifold the k-th conjugate locus consists $\mathrm{Conj}^k(M, p)$ of the k-th zeroes of the Jacobi field with initial values $(0, \mu(0)) \neq (0, 0)$. When $\dim(M) > 2$ the k-th conjugate locus $\mathrm{Conj}^k(M, p)$ is not well-defined, because there is more than one interesting Jacobi field.

7. The symplectic side

The geodesic flow on a complete Riemannian manifold form a ray system. The geodesic equation (2) can be written as a Hamiltonian system, in $T^*_{\setminus 0}M$.

$$\dot{x} = \frac{\partial H}{\partial \xi} \quad \dot{\xi} = -\frac{\partial H}{\partial x}$$

The flow-out of $T^*_{\setminus 0}M \cap \{H(x, \xi) = 1\}$ is a Lagrangian manifold $L_p \subset T^*_{\setminus 0}M$. The conjugate locus is contained in the image of the singular points of the projection map $T^*M \supset L_p \to M$. It is important to realize though that the singular values of the the map $L_p \to M$ contain much more points than just the conjugate locus. What we do know, in the 2-dimensional case, is that each singular value is part of some $\mathrm{Conj}^k(M, p)$. The cut-locus is part of the set sweeped out by self-intersections of wavefronts emanating from p.

8. Known theorems about cut and conjugate loci

Before we give examples of the above calculations we list some known theorems about cut and conjugate loci. There are relatively few results, for a more complete list we refer to the book of Berger, [**Ber03**]. The bare necessities for dealing with conjugate loci are for instance explained in [**Spi79b**], chapter 8.

One of the first questions one can ask is: do the (tangential) conjugate locus and the (tangential) cut-locus always meet? Very surprisingly the answer is "no" in general. The sphere S^2 is exceptional because of the following, see [**Wei68**]:

THEOREM 6. *Let M be a compact smooth manifold, not diffeomorphic to S^2, Then there is a Riemannian metric on M and a point $p \in M$ for which the tangential conjugate locus and the tangential cut-locus are disjoint.*

Weinstein gives some nice examples of cut and conjugate loci. The projective plane is the identification space of S^2, where we identify x with $-x$. The projective plane can be given a metric via the map $S^2 \mapsto \mathbb{P}^2$. With this induced metric the projective plane has constant curvature 1. Weinstein states that on this projective plane the tangential cut and conjugate loci are two disjoint circles.

Weinstein's theorem is complemented by the following theorem of Klingenberg

THEOREM 7. *For a compact simply connected even dimensional manifold M all whose sectional curvatures are positive, then there is a point $p \in M$ for which the tangential cut-locus and the tangential conjugate locus intersect.*

About the conjugate locus little is known. The reference [**War65**], though forty years old, is still the standard one. One very nice result is the following. Weinstein's theorem seems to suggest that things are still quite intuitive on spheres S^2. But Margerin proves in [**Mar93**] that there are metrics on S^2 for which the conjugate locus is not a closed curve. In his examples the tangential cut-locus escapes to infinity. Those examples are surfaces of revolution, just as the examples of Gluck and Singer below.

We can ask whether the cut-locus is always a triangulable set. If the manifold is real analytic then this is the case, see [**Buc77**]. In particular for a simply connected compact 2-dimensional manifold the cut locus is a tree. In the smooth case the answer is "no". Intuitively, the counter-examples is simple to construct. Take a wavefront at some distance d from a point p. Then arrange the metric such that a wavefront from another point q touches the wavefront in infinitely many points, as indicated in figure 2. The point q lies in the cut-locus $\mathrm{Cut}(M, p)$. However if we remove q from $\mathrm{Cut}(M, p)$ the complement of the cut-locus is no longer locally finite. Any triangulable set is locally finite so we have a counter example. Again, this is intuitive reasoning, for the details see [**SG76**].

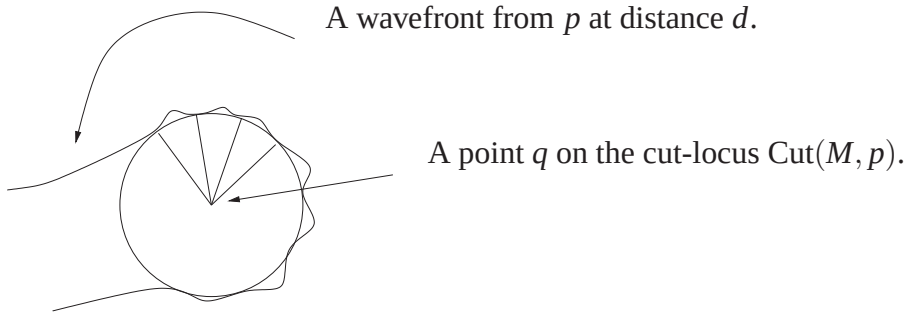


FIGURE 2. Non-triangulable cut-locus example.

Wall has proved in [**Wal77**] for a generic metric the triangulability of the cut-locus. The example of Gluck and Singer is thus an example of a non-generic metric on a surface of revolution. A much stronger statement proved in [**Buc78**] is true in low dimensions $\dim(M) \leq 6$: for a generic metric on a compact manifold of dimension ≤ 6 there is a finite list of singularities of the cut-locus. Combining that with theorem 1 we get a result of Myers, [**Mye35**].

THEOREM 8. *On a simply connected 2-dimensional manifold the cut-locus is a tree.*

The distance from a point p to $\mathrm{Cut}(M,p)$ is called the *injectivity radius*. The length of a geodesic to the cut-locus $\mathrm{Cut}(M,p)$ is a function

$$\iota\colon (T_pM \cap \{\mathcal{L}(p,v)=1\}) \to \mathbb{R}$$

Itoh proves in [**IT01**] that this function is not just continuous (as is proved in most differential geometry books) but that it is actually Lipschitz.

On a surface, that is a 2-dimensional manifold, if M has negative curvature everywhere then the function f in equation (13) is monotonely increasing. Therefore if $K<0$ everywhere then the conjugate locus is empty. In higher dimensions if the sectional curvatures $k_{ij}<0$ everywhere, then the conjugate locus is empty everywhere. We have the following related statements, for complete Riemannian manifolds.

PROPOSITION 1. *If a geodesic from p has a conjugate point q then there lies between p and q a cut point on the geodesic. Every geodesic from $p \in M$ has a cut point iff. M is compact.*

A further special case is that of symmetric spaces. On a symmetric space cut and conjugate loci can be calculated explicitly, see [**Cri62**].

9. A conjecture of Arnol'd

A - now disproved - conjecture of Arnol'd concerns caustics of ray systems close to the ray system of points on a sphere. The conjecture was formulated in [**Arn95**], but the reference [**Arn94**] (lecture 3) is more readily available.

In $T^*\mathbb{R}^n$ consider the ray system that is the flow out of

$$T_0^*\mathbb{R}^n \cap \{\|\xi\|=1\}$$

We get the *Lagrangian cylinder* L_n:

$$\{(x,\xi) \mid x = t\xi\, t \in \mathbb{R}_{>0}\, \|\xi\|=1\}$$

Arnol'd proved that for not too big symplectic perturbations ε (those done with some Hamiltonian) the perturbed manifold $\varepsilon(L_2)$ has a caustic with at least four cusps. He conjectured that those cusps cannot be removed by Hamiltonian isotopies. This was disproved by Entov in [**Ent99**]. However Entov uses $\mathcal{C}^0$ Hamiltonian isotopies, so that the original conjecture of Arnol'd still stands. For instance, in the real analytic category it might well be true. Indeed, there is another paper where Arnol'd states the following far more reasonable conjecture.

CONJECTURE 1 (See [**Arn94**]). *For a residual set of points on a compact strictly convex surface in $\mathbb{R}^3$ each conjugate locus* $\mathrm{Conj}^k(M,p)$ *has at least four cusps.*

As far as the author knows this conjecture still stands. It is not even know whether for a generic strictly convex surface all these tangentially conjugate loci are closed curves of finite length, mutually disjoint.

10. The examples of Markatis

In 1980, a student of Ian Porteous, Stelios Markatis, studied a number of convex surfaces in $\mathbb{R}^3$. His main interest in these surfaces was to study how the pattern of umbilics changes as we pass from one surface to the other. His surfaces though are also interesting for cut and conjugate loci. From Markatis' thesis we already know a lot about these surfaces so they provide nice examples to test conjectures. The examples Markatis studied were the *bumpy spheres*

- The bumpy cube: $x_1^2+x_2^2+x_3^2+\frac{\varepsilon}{3}x_1x_2x_3$.
- The bumpy tennisball: $x_1^2+x_2^2+x_2^3+\frac{\varepsilon}{3}x_1x_2^2$.
- The bumpy sphere of revolution: $x_1^2+x_2^2+x_2^3+\frac{\varepsilon}{3}x_1^3$.
- The bumpy orange: $x_1^2+x_2^2+x_2^3+\frac{\varepsilon}{3}(x_1^3-3x_1x_2^2)$.

For an analysis of these surfaces, we refer to chapter 16 in [**Por94**], or to the papers [**Por83**] and [**Por87**]. In addition to the bumpy spheres there are other nice examples:

- Linner's example: $2x_1^2+3x_2^2+5x_3^2+x_1^3x_3+x_2x_3^2$
- Ellipsoids and "hyperellipsoids":

$$\left(\frac{x_1}{a_1}\right)^4+\left(\frac{x_2}{a_2}\right)^4+\left(\frac{x_3}{a_3}\right)^4$$

The analysis of the ellipsoids is classical and it can be found in [**Kli95**], section 3.5. Their cut and conjugate loci have been determined by [**IK04**]. When a point on the ellipsoids is not an umbilic, the cut-locus is a line segment, and the conjugate locus is a curve with four cusps. Very nice pictures of the conjugate loci, also of higher order order are in [**Sin03**]. The methods used to compute those conjugate loci are similar to the ones we propose here. Our methods though are more exact, and easier to program.

A host of examples can also be made, by adding up two or more of these defining functions. Through a weighted sum you can make a family of surfaces that changes the one example into the other.

REMARK 3. *A manifold with a generic metric is called bumpy. Some authors call a (compact) manifold bumpy if all closed geodesics are non-degenerate. Of course we do not claim that this is the case for the bumpy spheres. It might well be, but we just don't know.*

Here we show one example, the cube $\varepsilon=\frac{9}{10}$. We also show the conjugate locus in the tangent space, and we plot the squared length of a Jacobi field. To produce the figure 3 we look for the zeroes of the Jacobi field. The length of a Jacobi field is $\langle W,W\rangle$. The derivative of that function is $2\langle W,\mu\rangle$. In figure 4 we used the results of numerically integrating the system (20). We plotted both $\langle W,W\rangle$ and $2\langle W,\mu\rangle$ directly from the results, so the graph we see is also a sanity. Other sanity checks, namely checking that everywhere $\langle W,\xi\rangle=0$ and $\langle \mu,W\rangle=0$, and $\langle \xi,\xi\rangle=1$ also give good results. For a geodesic of length 4 times the injectivity radius the numerical error is $\leq 10^{-7}$. We took as an example the cube because the cube is the worst example. There are always at least six cusps on the first conjugate locus. So the perturbation we chose is not generic. Indeed it isn't, the surface has some symmetries, whereas a generic perturbation of the sphere should have no symmetries at all.

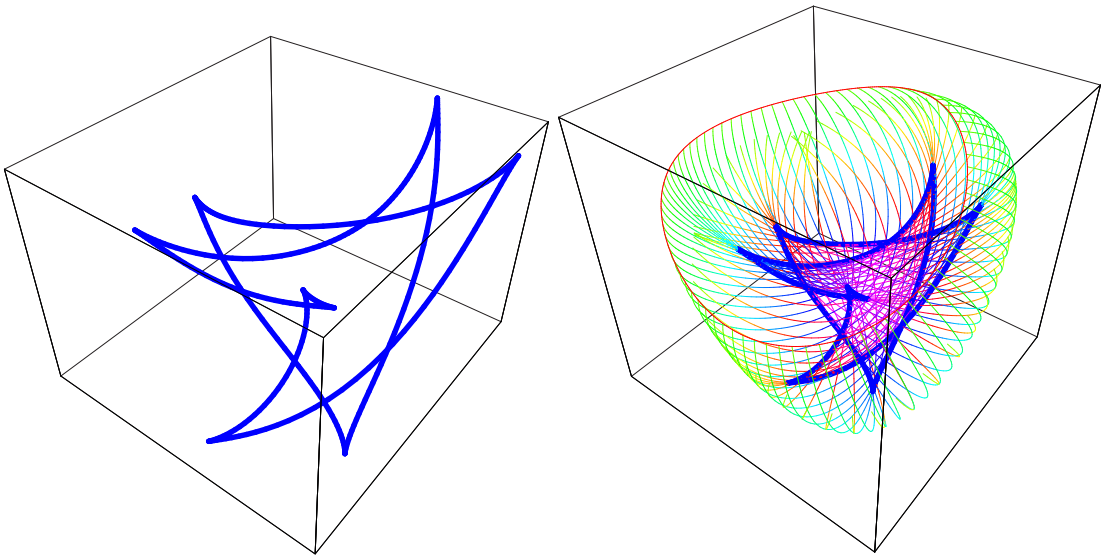


FIGURE 3. A conjugate locus on Markatis' cube. On the left we see the conjugate locus alone. On the right we see the complete picture, where we have drawn the geodesics starting from $t = 2$.

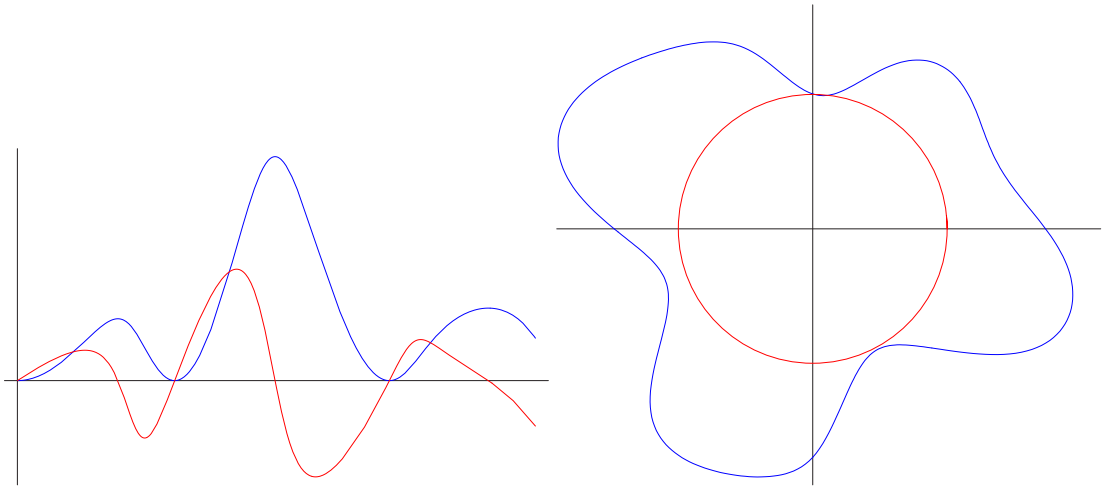


FIGURE 4. Again the Markatis' cube with $\varepsilon = \frac{9}{10}$. On the left we see the squared length of a Jacobi field, together with its derivative. On the right we see the conjugate locus in the tangent space.

For numerical purposes, one finds the kth conjugate locus by looking for the $2k$th zero of the function $\langle \mu, W \rangle$. The programming here was all done with Mathematica. Getting it right requires some Mathematica sophistication. For those who would like to try it themselves: do not use routines such as `FindRoot` or `FindMinimum`. The results are unreliable. Instead, grow your own copies of these routines.

It is also possible to visualize the cut-locus. One needs to find the intersections of the wavefronts. In preparing these notes there was to little time to get everything done so here is a preliminary picture, which we hope, will set the reader thinking.

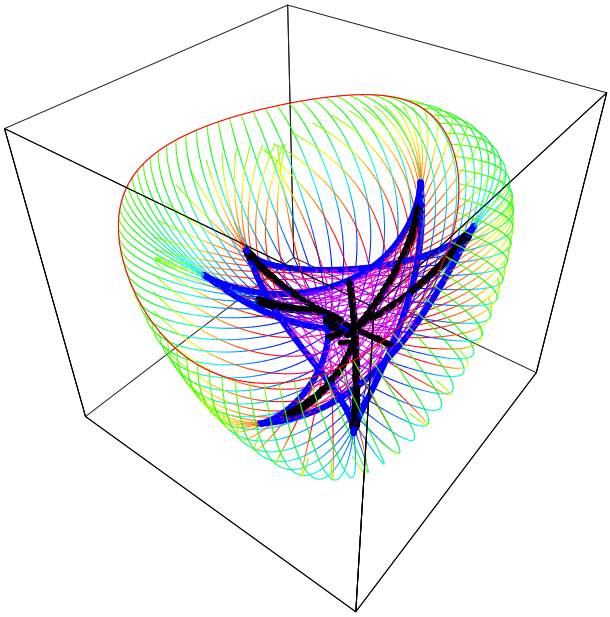


FIGURE 5. The cut-locus?

The hyperellipsoid already shows the limitations of our methods. Consider the pictures in figure 6. Significant parts of the hyperellipsoid are almost flat. If we start with a geodesic there, on such a flat part we can from equation (13) that the differential equation for the Jacobi fields becomes rather unstable.

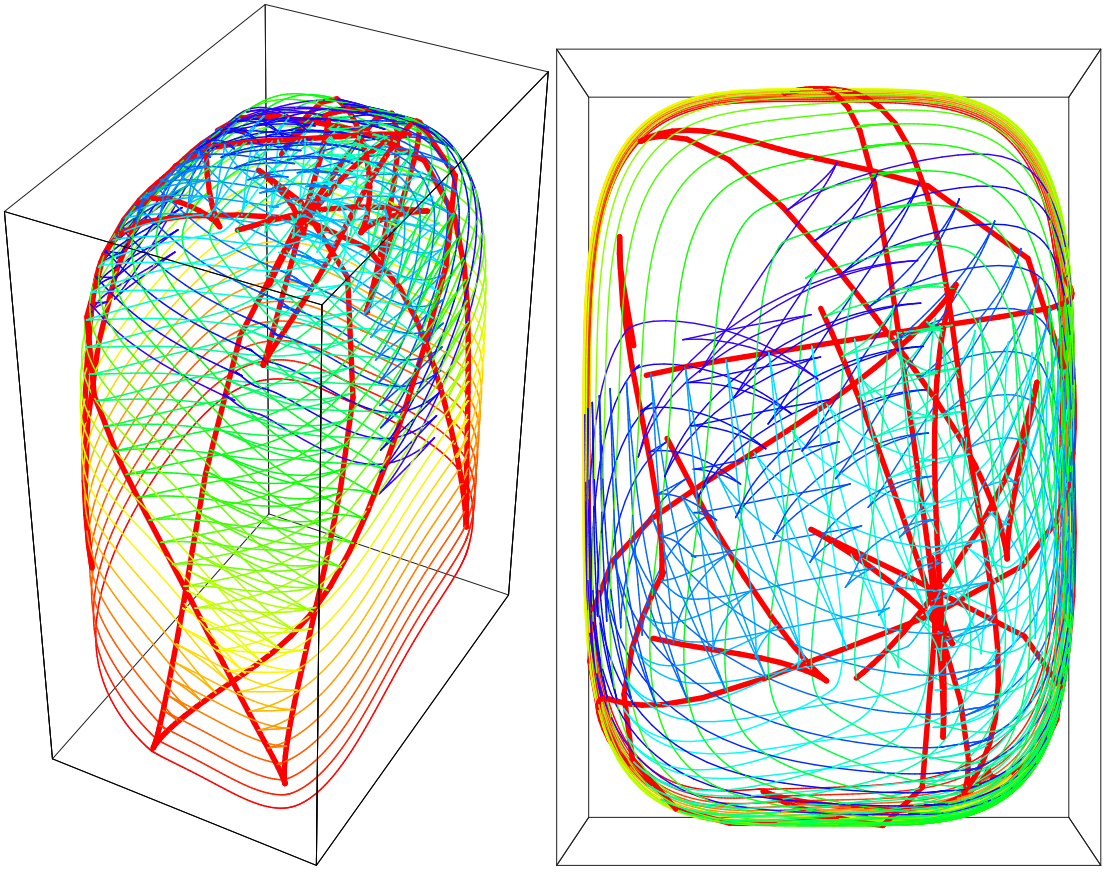


FIGURE 6. Hyperellipsoids show the limitations of our method.

LECTURE 2

Embedded manifolds: medial axis, symmetry sets and caustics

During the first lecture we considered "sets in the middle" using the metric of a compact manifold. We studied "sets in the middle" using the intrinsic geometry. In this lecture we will study the same sort of questions using extrinsic geometry. Let M be a compact closed manifold without boundary.

1. Questions asked by Thom

A function $f \in \mathcal{C}^\infty(M)$ has a critical point at $s_0 \in M$ when the derivative $\mathrm{D}\, f(s_0) = 0$. When in addition, the second derivative $\mathrm{D}^2 f(s_0)$ is a non-degenerate matrix then we say that f has a non-degenerate critical point at s_0. When a function only has non-degenerate critical points and all its critical values are distinct, then we say that f is a *Morse function*. The index of a non-degenerate critical point is the dimension of the maximal subspace for which the second derivative is strictly negative definite. So a non-degenerate minimum has index 0 and a non-degenerate maximum has index $\dim(M)$

Generic functions on a manifold are Morse. However in a family of functions one might encounter other than Morse functions. Thom proposed to construct a family of functions as follows. Take any $\mathcal{C}^\infty$ map ϕ from M to $\mathbb{R}^n$, where $\dim(M) = m < n$. The family of functions to study is

$$F : \mathbb{R}^n \times M \to \mathbb{R} \qquad F(x,s) = \frac{1}{2}\|x - \phi(s)\|^2 \tag{21}$$

We can view F as a map $\mathbb{R}^n \to \mathcal{C}^\infty(M)$. It associates to $x = x_0$ the function $f_{x_0}(s) = F(x_0, s) \in \mathcal{C}^\infty(M)$.

Because M is compact all non-constant elements of $\mathcal{C}^\infty(M)$ have at least two critical points. In the space $\mathcal{C}^\infty(M)$ consider the functions that have at least two identical critical values or one degenerate critical value. If we perturb such functions a little bit around one of the critical points we get a Morse function, because Morse functions are dense in $\mathcal{C}^\infty(M)$. It would be very nice if for the perturbations of a non-Morse function in the family (21) we can take the family itself. But when can this be done? Thom conjectured that for generic embeddings of M in $\mathbb{R}^n$ the family of functions would in fact be representative for most perturbations.

An interesting case is where the map ϕ is an embedding. In that case the values of $x = x_0$ for which $f_{x_0}(s)$ has a degenerate critical value form the *caustic*. The values of $x = x_0$ for which $f_{x_0}(s)$ has a non-unique critical value form the *symmetry set*. The values of $x = x_0$ for which $f_{x_0}(s)$ has a non-unique absolute minimum form the *medial axis*.

The medial axis, the caustic, and the symmetry set have become widely studied notions. So much so that the original questions of Thom have almost been forgotten. Denote $\mathcal{S}$ the set of

functions in $\mathcal{C}^\infty(M)$ such that f is not Morse. One can view $\mathcal{S}$ as a "hypersurface" in $\mathcal{C}^\infty(M)$. Inside the set $\mathcal{S}$ are for instance the following subsets

- $f \in \mathcal{S}$ has a degenerate critical value
- f has a non-unique critical value
- $\mathcal{S}_{i,j}$: f has a unique non-unique critical value $f(s_1) = f(s_2)$ for which we have $\text{index}_f(s_1) = i$ and $\text{index}_f(s_1) = j$

Thom proposed to study the topological properties of these subsets.

The medial axis corresponds to a subset of the closure of the $\mathcal{S}_{0,0}$. subset. In the case of the ellipse (figure 6) all points of the closure of $\mathcal{S}_{0,0}$ correspond to a point on the medial axis. Thom proposes to study the other $\mathcal{S}_{i,j}$ as well. Moreover most of the attention since the publication of [**Tho72**] has been directed at the case where ϕ is an embedding of a codimension one manifold, whereas Thom put forth a more general problem, where ϕ is not even necessarily an embedding.

Denote $\text{Diff}^\infty(M)$ the group of smooth diffeomorphisms of M. Denote $\text{Diff}^\infty(\mathbb{R})$ the group of smooth diffeomorphisms of $\mathbb{R}$. There is an action of $\text{Diff}^\infty(M) \times \text{Diff}^\infty(\mathbb{R})$ on $\mathcal{C}^\infty(M)$. If $\alpha \in \text{Diff}^\infty(M)$, and $\beta \in \text{Diff}^\infty(\mathbb{R})$, then $(\alpha, \beta) \circ f = \beta(f(\alpha))$. Note that the subsets of $\mathcal{C}^\infty(M)$ that we are interested in are invariant under this action.

2. The thesis of Looijenga

In the thesis of Looijenga the main conjecture of Thom stated in his article was found to be true. We copy the definition of canonical stratifications from [**Mat73**]. A *stratification* $\mathcal{W}$ of a closed subset X of a manifold is a subdivision of the set into subsets - called *strata* - such that

(1) Each stratum of $\mathcal{W}$ is locally closed
(2) The subdivision is locally finite

A *Whitney stratification* is a stratification that satisfies conditions A and B of Whitney.

Condition A. Let $W_1 \in \mathcal{W}$ be a stratum and let $\{x_i\}_{i=1}^\infty$ be a sequence that converges to some x_0 in $W_2 \in \mathcal{W}$ and such that $x_i \in W_1$ for all i. The sequence $T_{x_i}W_1$ converges to τ and $T_{x_0}W_2 \subset \tau$.

Condition B. Let $\{x_i\}_{i=1}^\infty$ be a converging sequence, with all elements again in some W_1 and $\{y_i\}_{i=1}^\infty$ a sequence of points in W_2, such that they both converge to a point $x_0 \in W_1$. If the lines $\ell_i = x_i y_i$ converge, they converge to some limit $\ell \subset \lim T_{x_i}W_1$.

REMARK 4. *The Whitney conditions and the concept "regular stratification" were first formulated in* [**Whi65b**] *and* [**Whi65a**].

We refer to [**GWdPL76**], chapter 1, for an exact definition of *canonical stratification*. Informally speaking: a canonical stratification is the coarsest subdivision of X in strata such that the stratification is Whitney.

The results of Looijenga concern a stratification of $\mathcal{C}^\infty(M, \mathbb{R})$. Of course, $\mathcal{C}^\infty(M, \mathbb{R})$ is an infinite dimensional "manifold". Note that it is not a Banach manifold. It is only a Fréchet space: the space is not normed. The notion of a derivative of a map $\mathcal{C}^\infty(M, \mathbb{R}) \to \mathbb{R}^N$ is well-defined though.

A weak codimension k submanifold in a Fréchet space is a subset $V \subset \mathcal{C}^\infty(M, \mathbb{R})$ of a Fréchet space such that for all $f \in V$ there exists an open neighborhood $\mathcal{U}$ of f and a submersion ψ: $\mathcal{U} \to \mathbb{R}^k$ such that $\psi^{-1}(0) \cap \mathcal{U} = V \cap \mathcal{U}$. A weak stratification of a subset X of a

Fréchet space $\mathcal{F}$ is a partition of X into weak codimension k submanifolds. A weak Whitney stratification is a weak stratification of X such that any map Ψ from $\mathbb{R}^N \mapsto \mathcal{F}$ that is transversal to all strata of X pulls back to a Whitney stratification of $\Psi^{-1}(X)$.

In the previous section we asked when a family of functions is "good". We have to define what that exactly means. A family of functions $F \in \mathcal{C}^\infty(M \times \mathbb{R}^n)$ is called *topologically stable* (resp. *smoothly stable*) if for any F' sufficiently near to F, there are homeomorphisms (resp. $\mathcal{C}^\infty$-diffeomorphisms) h, h', h'' such that the diagram (22) commutes.

$$\begin{array}{ccccc} M \times U & \xrightarrow{F \times \mathrm{Id}} & \mathbb{R} \times U & \xrightarrow{\mathrm{Id}} & U \\ \downarrow h & & \downarrow h' & & \downarrow h'' \\ M \times U & \xrightarrow{F' \times \mathrm{Id}} & \mathbb{R} \times U & \xrightarrow{\mathrm{Id}} & U \end{array} \tag{22}$$

THEOREM 9 (Looijenga). *There exists a subset $W(M,\mathbb{R})$ of $\mathcal{C}^\infty(M,\mathbb{R})$ such that the complement of $W(M,\mathbb{R})$ in $\mathcal{C}^\infty(M,\mathbb{R})$ has a weak stratification $\mathcal{S}$. The weak stratification has the following properties: If a map $F\colon \mathbb{R}^N \to \mathcal{C}^\infty(M,\mathbb{R})$ is transversal to $\mathcal{S}$ then*

1. *$F\colon \mathbb{R}^N \times M \to \mathbb{R}$ defines a topologically stable family of functions.*
2. *The weak stratification of the image of F in $\mathcal{C}^\infty(M,\mathbb{R})$ pulls back to a canonical stratification of $\mathbb{R}^n$.*
3. *Mappings transversal to $\mathcal{S}$ form an open and dense subset of $\mathcal{C}^\infty(\mathbb{R}^N, \mathcal{C}^\infty(M,\mathbb{R}))$.*
4. *The strata of $\mathcal{S}$ are invariant under the action of the group $\mathrm{Diff}^\infty(M) \times \mathrm{Diff}^\infty(\mathbb{R})$ on $\mathcal{C}^\infty(M,\mathbb{R})$.*
5. *If $\mathcal{S}'$ is any other weak stratification of $\mathcal{C}^\infty(M,\mathbb{R}) \setminus W(M,\mathbb{R})$ having properties 1, 2, 3 and 4 then it is a refinement of $\mathcal{S}$.*

REMARK 5. *In order to avoid technical points and many definitions we have simplified the formulation of Looijenga's theorem a little. Strictly speaking our formulation is incorrect.*

REMARK 6. *Many authors refer to the thesis of Looijenga. It is often used to prove that in low dimensions a family of functions is smoothly stable. Of course this is a corollary of theorem 9, but the theorem is much stronger than that. It can be used to prove topological stability. Smooth stability of generic families of functions under study usually follows from the results of Mather.*

The theorem of Looijenga simplifies the study of singularities as suggested by Thom significantly. It is fairly easy to prove that for a residual set of embeddings the family of functions

$$\|x - \gamma(s)\|^2 \in \mathcal{C}^\infty(M \times \mathbb{R}^n)$$

is a topologically stable family of functions. It follows that the strata we study form a Whitney stratification, and that the local normal forms are topologically stable for all n, and smoothly stable for $n \leq 5$.

3. The caustic

Let us give the reader some intuition for the different strata that we will study. If the function $s \to \|x_0 - \gamma(s)\|^2$ has a critical point then x lies on the normal pointing out from $\gamma(s)$. In three

dimensions the picture is as in figure 1. The caustic is the conjugate locus of an embedded

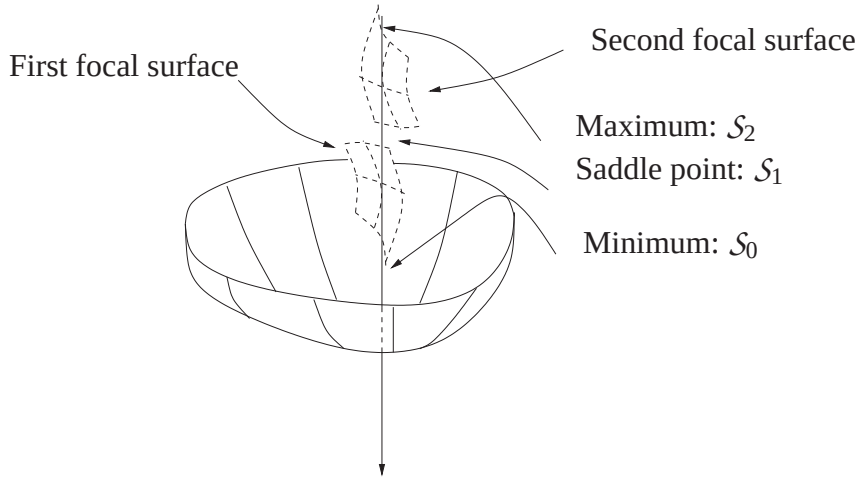


FIGURE 1. Critical points of the distance function.

manifold. If for some $x \in \mathbb{R}^n$ the function $s \to \|x - \phi(s)\|^2$ has degenerate critical value then infinitesimally near normals from the critical point s_0 intersect at x. The same happened with the conjugate locus: geodesics are normals to a point. The envelope of the geodesics are the conjugate locus, the envelope of the normals is the caustic.

At a point where the hypersurface has $n-1$ different principal curvatures $k_1 < k_2 < \cdots < k_{n-1}$, the normal intersects the caustic at $n-1$ different points. At these intersection points the caustic is tangent to the normal, as shown in figure 1. When two of the principal curvatures coincide, two or more of the sheets of the caustic coincide and the caustic has a singularity.

The singularities of the caustic are well-known for $n \leq 5$. They are the ADE-singularities, see [**Arn90**]. In higher dimensions moduli appear, but we still have topological stability, according to the theorem of Looijenga.

In dimensions 2 and 3 a beautiful analysis of the interplay between the geometry of the caustic, and that of the curve or surface is contained in [**Por94**].

4. Singularities of the medial axis

The medial axis is in fact, as Thom remarked, the cut-locus of an embedded manifold. It is not surprising that their generic singularities coincide. The list of local normal forms of the medial axis for a compact submanifold of dimension n is the same as the list of normal forms of cut-loci of points in an $n+1$ dimensional manifold. Indeed both are singularities of a minimal distance function.

The generic singularities of the medial axis of a compact submanifold were classified in two papers, [**Mat83**] and [**Yom81**]. The generic singularities of cut-loci were classified in [**Buc78**].

Let us discuss the theorem of Mather. Mather discusses the singularities of the minimal distance function:

$$x \to \rho_M(x) = \min_{s \in M} \|x - \gamma(s)\|$$

where γ is an embedding of M in $\mathbb{R}^n$.

THEOREM 10. *For a residual set of embeddings of a compact manifold of codimension* 1 *in* $\mathbb{R}^n$*, with* $n \leq 7$ *there exists a finite set* $E_M \subset \mathbb{R}^n$ *such that the distance function* $\rho_M(x)$ *can be reduced to one of the following normal forms.*

- $A_0 : x_1$, $A_1 = \min(x_1, x_2)$, $\cdots$, $A_7 = \min(x_1, \cdots, x_7)$.
- $B_2 = \min_{s\in\mathbb{R}}(s^4 + x_1 s^2 + x_2 s + x_3)$,
- $B_3 = \min(B_2(x_1,x_2,x_3), x_4)$, $\cdots$, $B_7 = \min(B_2(x_1,x_2,x_3), x_4, x_4, x_6, x_7)$.
- $\min(B_2(x_1,x_2,x_3), B_2(x_4,x_5,x_6))$
- $\min(B_2(x_1,x_2,x_3), B_2(x_4,x_5,x_6), x_7)$
- $C_4 = \min_{s\in\mathbb{R}}(s^6 + x_1 s^4 + x_2 s^3 + x_3 s^2 + x_4 s + x_5)$, $\min(C_4(x_1, \cdots, x_5), x_6)$,
- $\min(C_4(x_1, \cdots, x_5), x_6, x_7)$
- $D_6 = \min_{s\in\mathbb{R}}(s^8 + x_1 s^6 + x_2 s^5 + x_3 s^4 + x_4 s^3 + x_5 s^2 + x_6 s + x_7)$

The theorem of Mather thus excludes certain points on the medial axis. In the planar case his theorem is almost void. He excludes the only two singularities, the end-point A_2, and the trivalent vertex A_1.

The singularities of generic medial axes and singularities we get in $\mathbb{R}^3$ are exhibited in figure 2. Those that are not on the finite set of Mather are A_3 and B_2. Let us construct the pictures

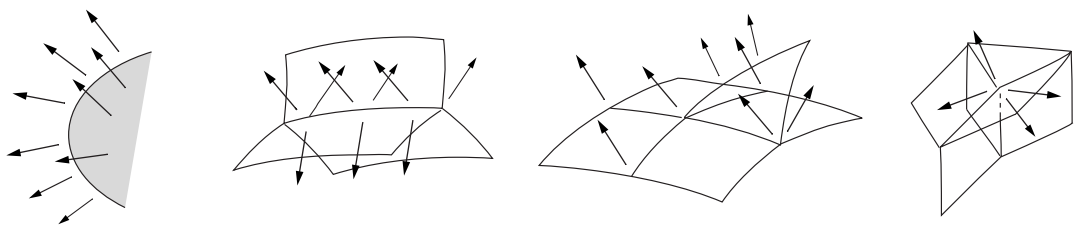


FIGURE 2. Generic singularities of the medial axis of a compact submanifold of dimension 2 in $\mathbb{R}^3$, and also generic singularities of a cut-locus in a 3-dimensional manifold. The picture is shamelessly stolen from [**Dam03**]. It can also be found in [**Buc78**]

associated to A_3. The germ associated to the distance function is locally $\min(x_1, x_2, x_3)$. Thus the non-unique minima are located on

$$\{x_1 = x_2 \leq x_3\} \cup \{x_1 = x_3 \leq x_2\} \cup \{x_2 = x_3 \leq x_1\}$$

These are parameterized by

$$\{(t,t,t+s) \mid s \geq 0\} \cup \{(t,t+s,t) \mid s \geq 0\} \cup \{(t+s,t,t) \mid s \geq 0\} \tag{23}$$

and if we plot equation (23) we get picture 3: This is obviously equal to the second picture from the lhs. in figure 2.

To find the picture belonging to Mather's $B2$ singularity, we consider the polynomial $p(s) = s^4 + x_1 s^2 + x_2 s + x_3$. We have to find the values of (x_1, x_2, x_3) for which the polynomial $p(s)$ has two minima with equal minimal value. In that case $p'(s) = 4s^3 + 2x_1 s + x_2$ has three real roots. Hence, we have $D = 8x_1^3 + 27x_2^2 \leq 0$. If we plot this we see that we deal with a region that is

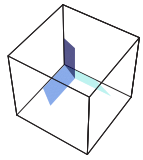


FIGURE 3. Mather's A3

the "interior" of a cuspidal edge, and thus Mather's $B2$ is equal to the first picture from the left in figure 2.

The restriction in the theorem of Mather can be removed. To find all normal forms of the minimal distance functions in dimensions ≤ 6 without excluding a finite set of points take transversal sections of the normal forms one dimension higher. For instance, Mather's normal forms of the medial axis give for an embedding of a curve in $\mathbb{R}^2$ only the smooth part: all the singularities are part of the excluded set. But in $\mathbb{R}^3$ we get exactly the two normal forms on the left hand side in figure 2. If we take a transversal intersection of these stratified sets with a plane we get the endpoint and the trivalent vertex. These are all generic singularities of the medial axis in the plane. It appears that in general one can take the plane $\sum_{i=1}^m x_i = 0$ in the formulae of Mather.

5. Two lesser known theorems in [Tho72]

In this section we present some aspects of the work of Thom that appear to have been largely forgotten, but that we still find interesting.

LEMMA 1. *Let $\phi\colon M \times \mathbb{R}^n$ be some map. Let $\phi(s_0) = x_0 \in \mathbb{R}^n$. The matrix $\mathrm{D}\phi$ has maximal rank in s_0 iff. the second derivative of $s \to \sum_{i=1}^n \|x_0 - \phi(s)\|^2$ is a nondegenerate matrix in $s = s_0$.*

PROOF. It is no restriction to assume that $x_0 = 0$, so $\phi(s_0) = 0$. Then we see that

$$\frac{\partial^2}{\partial s^2}\left(\sum_{i=1}^n \|x_0 - \phi(s)\|^2\right)_{|s=s_0} = \sum_{i=1}^n \frac{\partial \phi_i}{\partial s_j}(s_0)\frac{\partial \phi_i}{\partial s_k}(s_0)$$

This matrix is non-singular iff.

$$\frac{\partial \phi_i}{\partial s_j}(s_0)$$

has maximal rank. □

This elementary lemma has a very nice corollary. Denote $\mathrm{Cut}(\phi) \subset \mathbb{R}^n$ the set of $x_0 \in \mathbb{R}^n$ for which

$$s \to F(\phi, x_0) = \|x_0 - \phi(s)\|^2 \tag{24}$$

does not have a unique global minimum. Denote $\mathrm{Emb}(\phi) \subset M$ the largest open subset of M for which

$$\phi_{|\mathrm{Emb}(\phi)}$$

is an embedding.

PROPOSITION 2. *The sets*

$$\mathrm{Cut}(\phi) \cap \phi(M) \text{ and } \phi(\mathrm{Emb}(\phi))$$

are disjoint. Their union is $\phi(M)$.

PROOF. Let $x_0 \in \phi(M)$. Suppose that $x_0 \notin \mathrm{Cut}(\phi)$. The map (24) has a unique global minimum at s_0, with s_0 being the unique element of M for which $\phi(s_0) = x_0$. According to the previous lemma it follows that the differential of ϕ has maximal rank at s_0. □

We now come to the analogue of theorem 1.

THEOREM 11. *The* $\phi(\mathrm{Emb}(\phi))$ *image is a deformation retract of the complement* $\mathbb{R}^n \setminus \mathrm{Cut}(\phi)$.

Let ϕ now be an embedding of S^{n-1} in $\mathbb{R}^n$. According to the previous lemma $\mathrm{Cut}(\phi)$ does not intersect $\phi(S^{n-1})$. In the interior of $\phi(S^{n-1})$ there lies a point of $\mathrm{Cut}(\phi)$. Thus the *interior medial axis* $\mathrm{IntCut}(\phi)$ is well-defined. From theorem 11 we see $\mathrm{IntCut}(\phi)$ is contractible, so it is a tree. That assertion is the analogue of the theorem of Myers in lecture 1, section 8. Figure 4 illustrates this. One views a cut-locus as the interior medial axis of a wavefront from p. Clearly this can only be done if the manifold $M \setminus \{p\}$ is some $\mathbb{R}^n$, that is M is a sphere S^n.

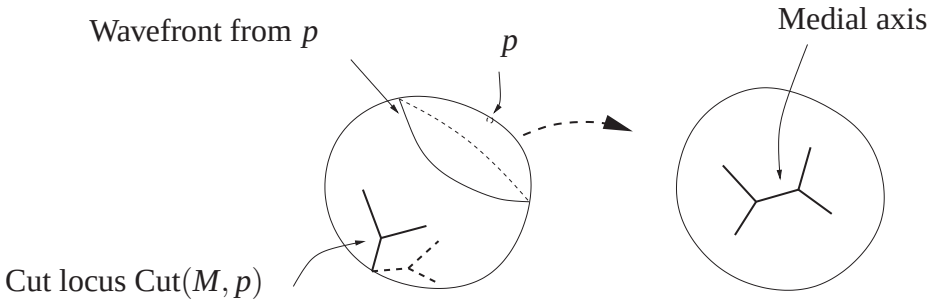


FIGURE 4. Relation between the theorem of Myers on cut-loci and results on the medial axis

In the comments after theorem 11 Thom states: "A simple adaptation of a theorem of Weinstein (see lecture 1, theorem 6) shows that there exist embeddings of S^2 in $\mathbb{R}^3$ such that the interior medial axis does not meet the caustic". It is unclear to the author what Thom exactly means. On S^3 there exists according to the theorem of Weinstein a metric and a point p such that the tangential cut-locus and the tangential conjugate locus are disjoint. An explicit example was constructed in [**Ito84**]. Thom suggests that these examples can be transposed to $\mathbb{R}^3$ to give examples of embeddings of S^2 in $\mathbb{R}^3$ where the medial axis does not meet the caustic.

A wavefront from such a point p is homeomorphic to S_2. The wavefront separates S^3 in two domains: the inner one containing p and the outer one not containing p. The outer domain can be viewed as the interior of a domain bounded by the image of an embedding γ: $S^2 \to \mathbb{R}^3$. The interior medial axis is the cut-locus of the point p, if that diffeomorphism from the outer domain on S^3 to the domain bounded by γ can be made to be an isometry.

In fact, with the euclidean metric on $\mathbb{R}^3$ the caustic and the medial axis cannot be separated: there are always end-points. Is a similar phenomenon true for the special cases of cut and

conjugate loci studied in the first lecture? We suspect that in those cases, with the metric induced from an embedding into $\mathbb{R}^n$, cut and conjugate loci can not be separated for convex surfaces.

Another stratum is the *maximal medial axis* $\mathrm{MaxCut}(\phi)$. The maximal medial axis consists of those $x \in \mathbb{R}^n$ for which $F(\phi,x)(s) \in \mathcal{C}^\infty(M)$ has a non-unique global maximum.

THEOREM 12. *If* $\phi\colon S^{n-1} \to \mathbb{R}^n$ *is an embedding then* $\mathrm{MaxCut}(\phi) \cap \mathrm{IntCut}(\phi) \neq \emptyset$.

PROOF. We will establish a contradiction, so assume $\mathrm{MaxCut}(\phi) \cap \mathrm{IntCut}(\phi) = \emptyset$. Denote $\mathcal{S}_{\min}$ the elements of the form $F(\phi,x)$ in $\mathcal{C}^\infty(S^{n-1})$ for which $F(\phi,x)$ has a non-unique global minimum. Denote $\mathcal{S}_{\max}$ the elements of the form $F(\phi,x)$ in $\mathcal{C}^\infty(S^{n-1})$ On the complement of the medial axis $\mathrm{Cut}(\phi)$ we have a map

$$G_{\min}\colon \mathbb{R}^n \setminus \mathrm{Cut}(\phi) \to S^{n-1}$$

The map $G_{\min}$ associates to x the unique minimum of $\|x-\phi(s)\|^2$. When restricted to S^{n-1} the map is the identity.

On the complement of the maximal medial axis $\mathrm{MaxCut}(\phi)$ we have a map

$$G_{\max}\colon \mathbb{R}^n \setminus \mathrm{MaxCut}(\phi) \to S^{n-1}$$

The map $G_{\max}$ associates to x the unique maximum of $\|x-\phi(s)\|^2$.

Take a small neighborhood U of $\mathrm{IntCut}(\phi)$, that is disjoint of $\mathrm{MaxCut}(\phi)$, and that is contained in the domain bounded by $\phi(S^{n-1})$. On the boundary of U - denoted ∂U - we have two

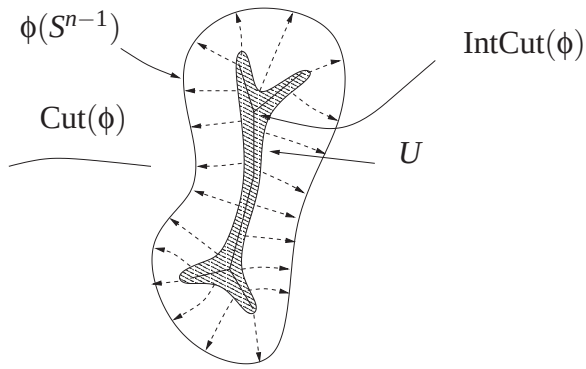


FIGURE 5. Proof of theorem 12.

maps to S^{n-1}: $G_{\max}$ and $G_{\min}$. Suppose that for some $x \in \partial U$ we have $G_{\min}(x) = G_{\max}(x)$. It would follow that for that x the function $\|x-\phi(s)\|^2$ is constant, so we would have that x lies on $\mathrm{IntCut}(\phi)$, which is contrary to our assumption.

The graphs of $G_{\max}$ and $G_{\min}$ can be seen as sections of the fiber bundle $S^{n-1} \times \partial U \mapsto \partial U$. Remove the section $G_{\max}$ from that fiber bundle. We get a homotopy:

$$\begin{tikzcd} S^{n-1} \times \partial U \setminus \mathrm{graph}(G_{\max}) \arrow[rr, tail] \arrow[dr, tail] & & (S^{n-1} \setminus \{p\}) \times \partial U \arrow[dl, tail] \\ & \partial U & \end{tikzcd} \tag{25}$$

It follows that the map $G_{\min}$ is homotopic to the constant map, but from the picture 5 it is immediately clear that the map $G_{\min}$ is also homotopic to a homeomorphism $\partial U \to S^{n-1}$. We have established a contradiction and the proof is complete. □

6. Singularities of the symmetry set

The references for this section are [**BGG85**] and [**BG86**]. Again we have a family of functions: by $x \in \mathbb{R}^n$

$$F(x,s) = \|x - \gamma(s)\|^2$$

As with cut-loci we are looking to minimize the distance. In this case we want to minimize the distance between the point $x \in X$ and M. A non-unique absolute minimum defines the *medial axis*, whilst non-unique critical values define the *symmetry set*. So the symmetry set contains both MaxCut(γ) and Cut(γ) Thus the medial axis is a subset of the symmetry set. Confirm figure 6. Singularities of the symmetry set are classified using the canonical stratification of

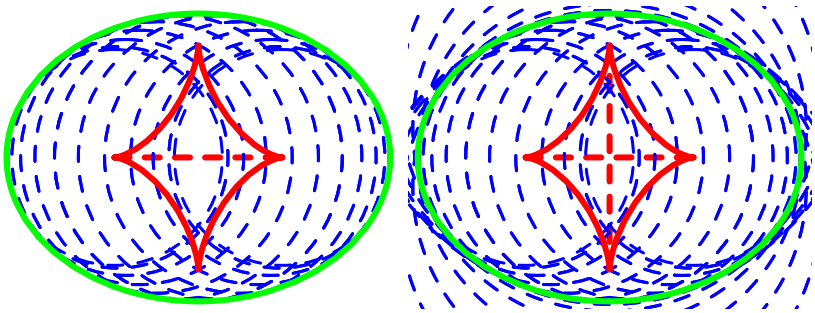


FIGURE 6. Medial axis and symmetry set

Looijenga. Thus the symmetry set Sym(γ) is topologically stable in all dimensions. Its singularities can be derived from the singularities of caustics, medial axis and wavefronts. Let us do this for the plane case. The trivalent vertex of the medial axis becomes a triple crossing as in

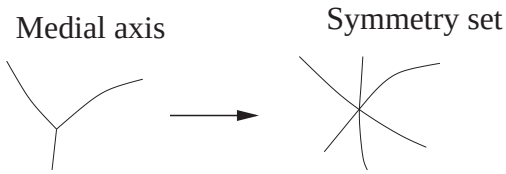


FIGURE 7. Trivalent vertex A_1^3 on the medial axis becomes a triple crossing on the symmetry set.

figure 7. The endpoints of the medial axis are unchanged. Further singularities of the symmetry set all involve other strata that Thom proposed to study. The symmetry set consists of all the self intersections of wavefronts. What is remarkable here is that we do not find all the singularities of the intersections of wavefronts, that was plotted in figure In $\mathbb{R}^3$ a similar analysis can be made. For instance we can have A_1^4, as in figure 9.

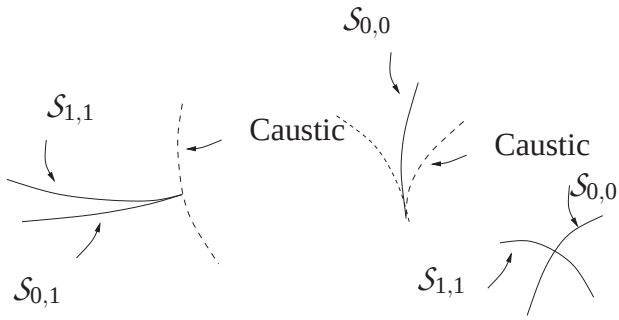


FIGURE 8. Other strata we find on the symmetry set

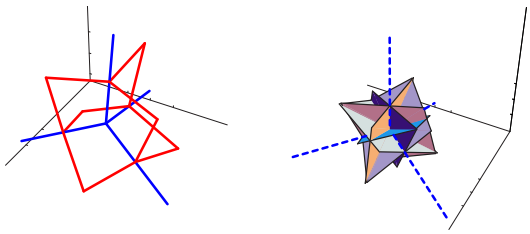


FIGURE 9. Quadruple vertices on the medial axis become sixfold intersections of planes on the symmetry set. This picture is plagiarized from [**vM03**].

7. A weighted symmetry set.

This section contains the answer to a question posed by D. Siersma.

The symmetry set is the closure of the set of points x where the distance function

$$f_x(s) = \|x - \gamma(s)\|^2$$

has two critical points with equal critical value. Take a ratio $[\lambda_1;\lambda_2] \in \mathbb{P}^1$ with $\lambda_1 \neq 0 \neq \lambda_2$. Define the *weighted symmetry set* to be the closure of the set of $x \in \mathbb{R}^n$ such that there exist two critical values s_1 and s_2 of $f_x(s) = \|x - \gamma(s)\|^2$ such that $\lambda_1 f_x(s_1) = \lambda_2 f_x(s_2)$.

THEOREM 13. *Let γ be a generic embedding of a circle in the plane. Fix a compact subset D of $\mathbb{R}^2$. For any $\varepsilon > 0$ there is a δ and a $\lambda = [\lambda_1;\lambda_2] \in \mathbb{P}^1$ with $d_{\mathbb{P}^1}(\lambda, [1;1]) < \delta$ such that for any point $x \in D$ on the smooth part of the symmetry set there are two points x_1 and x_2 in $\mathrm{sym}_\lambda(M)$ such that $\|x - x_1\| < \varepsilon$ and $\|x - x_2\| < \varepsilon$. Furthermore for any point $x \in D$ on the regular part of the focal set there is one $x_1 \in \mathrm{sym}_\lambda(M)$ such that $\|x - x_1\| < \varepsilon$.*

The statement of theorem 13 is best understood in pictures, see figure 10.

PROOF OF THEOREM 13. The proof is fairly trivial. Near the symmetry set the family of function is generic, that means that if we vary x we get a sequence of functions $f_x(s)$ as in figure 12. Similarly near the caustic the family of functions is sketch in figure 11. The convergence is only on a compact subset of $\mathbb{R}^2$. □

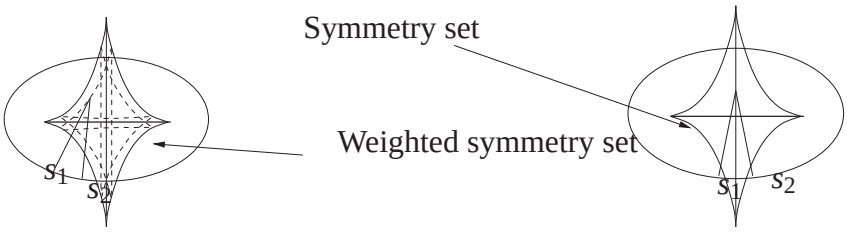


FIGURE 10. The weighted symmetry set is an approximation of both the caustic and the symmetry set

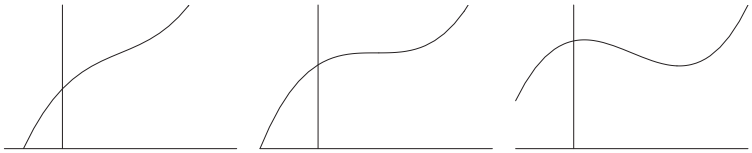


FIGURE 11. The family of functions near the caustic.

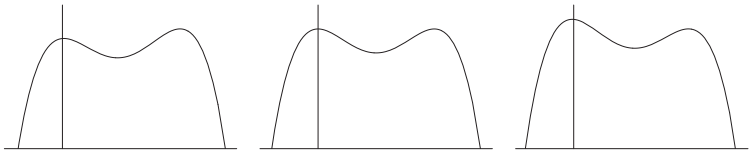


FIGURE 12. The family of functions near the symmetry set.

Of course, a higher dimensional analogue of this theorem is also true. Moreover, the weighted symmetry set is generically the projection of a smooth Legendrian manifold in $\mathbb{P}T^*\mathbb{R}^n$. That last fact can be proved using the techniques developed in [**vM03**].

LECTURE 3

Discrete point sets

In the previous lectures we studied sets in the middle using Riemannian metrics. Here we study "sets in the middle" in the context of discrete point sets. Just as we saw that the medial axis is strongly related to cut-loci, the medial axis is strongly related to the main object under study here: the Voronoi diagram.

Not everything we spoke about is contained in this part of the notes. The reader can consult [**Sie99**] , [**SvM04**] and [**SvM05**]. All we do here is give a few definitions and then we prove a complexity theorem for Voronoi diagrams and Delaunay triangulations. That theorem is not contained in [**SvM05**].

1. Voronoi diagrams and Delaunay triangulations

Let $\{P_1, \cdots, P_N\}$ be a number of distincts points in $\mathbb{R}^n$. We assume throughout that $\dim(\mathrm{CH}(\{P_1, \cdots, P_N\}$ n, so that forcibly $N \geq n+1$.

Denote $g_i(x) = \frac{1}{2}\|x - P_i\|^2$ and $g = \min_{1 \leq i \leq N} g_i(x)$. Denote $\mathrm{Vor}(P_i)$ those $x \in \mathbb{R}^n$ for which $g(x) = g_i(x)$. The set $\mathrm{Vor}(P_i)$ is a polyhedron. The *Voronoi diagram*is the division of $\mathbb{R}^n$ in the cells $\mathrm{Vor}(P_i)$. The cell $\mathrm{Vor}(\{P_{i_1}, \cdots, P_{i_k}\})$ is the closure of

$$x \in \mathbb{R}^n \quad g(x) \begin{cases} = g_j(x) \text{ if } P_j \in \{P_{i_1}, \cdots, P_{i_k}\} \\ < g_j(x) \text{ if } P_j \notin \{P_{i_1}, \cdots, P_{i_k}\} \end{cases}$$

The *Delaunay triangulation* consists of those subsets $\alpha \subset \{P_1, \cdots, P_N\}$ for which $\mathrm{Vor}(\alpha) \neq \emptyset$. The first thing to note is that the Delaunay triangulation is not necessarily a triangulation. As an example consider

$$P_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad P_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad P_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad P_4 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

The Delaunay triangulation is a square. The Voronoi diagram is a cross consisting of the coordinate axes. The Delaunay triangulation and the Voronoi diagram are examples of *polyhedral complexes*.

DEFINITION 1. *A* polyhedral subdivision $\mathcal{T}$ *of a polyhedron* $K \subset \mathbb{R}^n$ *is a subdivision of* K *in polyhedra, called* faces K_i*, such that*

- *The union of all sets* $K_i \in \mathcal{T}$ *is* K*.*
- *If* K_i *and* K_j *are both in* $\mathcal{T}$ *then so is their intersection.*
- *Every compact subset* L *of* K *intersects only a finite number of the* K_i*.*

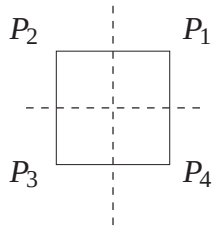


FIGURE 1. The Delaunay triangulation is not necessarily a triangulation.

A polyhedral subdivision becomes a complex if we orient each face. Faces of dimension 0 are called *vertices*. Faces of codimension 1 are called *facets*. Faces of codimension 0 are called cells.

For each subset of length $n+1$ in $\{P_1, \cdots, P_N\}$ define the determinant:

$$\left\| \begin{matrix} 1 & \cdots & 1 \\ P_{i_1} & \cdots & P_{i_{n+2}} \\ \frac{1}{2}\|P_{i_1}\|^2 & \cdots & \frac{1}{2}\|P_{i_{n+2}}\|^2 \end{matrix} \right\| \tag{26}$$

We say that the points $\{P_1, \cdots, P_N\}$ are in *general position* if for any subset of length $n+2$ in $\{P_1, \cdots, P_N\}$ the determinant in equation (26) is not equal to zero.

PROPOSITION 3. *If the determinant in equation* (26) *is never zero then the Delaunay triangulation is a triangulation, i.e. a simplicial complex.*

We defer the proof of proposition 3 to section 3.

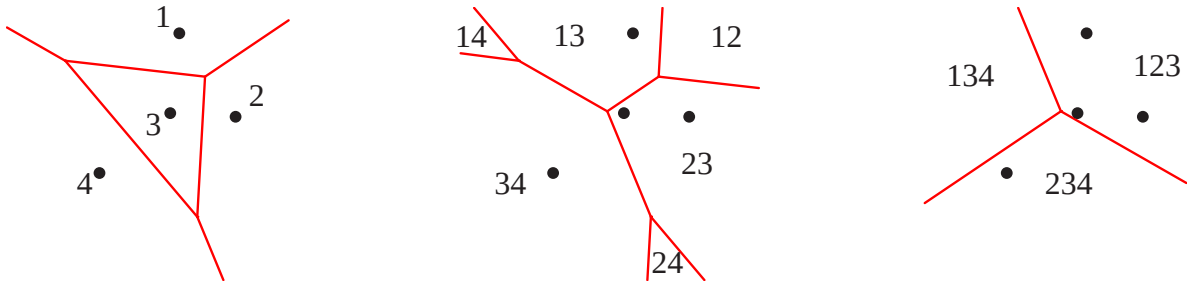
2. Higher order diagrams

We see from the proposition 3 that the vertices of Voronoi diagrams correspond to empty spheres through at least $n+1$ of the N points $\{P_1, \cdots, P_N\}$ such that all other points lie outside the sphere. We define higher order diagrams. The vertices of the kth order Voronoi diagram consist of points Q that are centers of spheres through at least $n+1$ of the N points $\{P_1, \cdots, P_N\}$ such that exactly $k-1$ other points lie inside the sphere. So the first order diagram Voronoi diagram is the ordinary Voronoi diagram that we have just defined. If you put the three pictures in figure 2 together you get a hyperplane arrangement in $\mathbb{R}^2$. For more on that, we refer to [**SvM05**].

3. The lifting transformation

We treat the classical construction called the *lifting transformation*. It is illustrated in figure 3. Let $h(x) = \frac{1}{2}\|x\|^2$ be the paraboloid. We lift each of the points $P_j \in \{P_1, \cdots, P_N\}$, to get new points $Q_j = (P_j, \frac{1}{2}\|P_j\|^2)$. In the point Q_j the tangent plane to the graph of h is the graph of the function:

$$f_j(x) = \langle x, P_j \rangle - \frac{1}{2}\|P_j\|^2$$



First Second. Third.

FIGURE 2. A first, second and third order Voronoi diagram. The picture is again shamelessly stolen. This time from [**Lin02**].

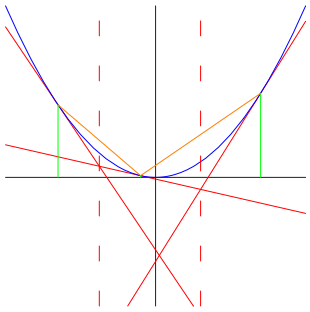


FIGURE 3. The lifting transformation, for three points on a line.

In figure 3 the graph of the paraboloid is the blue curve. The tangent lines - the graphs of the f_j, $j = 1,2,3$ - are the red lines.

Now suppose that two of the tangent planes intersect, then we have

$$f_i(x) = f_j(x) \;\Leftrightarrow\; \frac{1}{2}\|x - P_i\|^2 = \frac{1}{2}\|x - P_j\|^2$$

So the intersection of the two tangent planes is the line at equal distance from both P_i and P_j.

More is true. Take any point $Q \in \mathbb{R}^{n+1}$ on the graph of some f_j. The point Q lies beneath the graph of h. The length of the line segment that goes straight up from Q to the graph of h is

$$\frac{1}{2}\|\pi_n(Q) - P_j\|^2 \text{ with } \pi_n(x_1, \cdots x_{n+1}) = (x_1, \cdots x_n)$$

We see that the Voronoi diagram is the projection of the upper hull of the graphs of $\{f_j \mid 1 \leq j \leq N\}$. Suppose that $\{P_{i_1}, \cdots, P_{i_k}\}$ is a cell of the Delaunay triangulation. That is, $\mathrm{Vor}(\{P_{i_1}, \cdots, P_{i_k}\})$ is not empty. We see from the picture that that means that under the simplex spanned by

$$\{(P_{i_1}, \frac{1}{2}\|P_{i_1}\|^2), \cdots, (P_{i_k}, \frac{1}{2}\|P_{i_k}\|^2)\}$$

there are no other points $(P_j, \frac{1}{2}\|P_j\|^2)$. So the Delaunay triangulation is the projection of the lower faces of the polytope spanned by the lifted points:

$$\{(P_1, \frac{1}{2}\|P_1\|^2), \cdots, (P_N, \frac{1}{2}\|P_N\|^2)\}$$

We come back to the proposition 3. If there are $n+2$ points such that that determinant is zero, then $n+2$ lifted points lie in an n-dimensional plane, and consequently one of the cells in the Delaunay triangulation can be other than a simplex.

4. Complexity theorems for Voronoi diagrams

Complexity theorems estimate the number of faces of a certain dimension in a triangulation. For polytopes the best estimates possible can be derived from the *Dehn-Sommerville equations*. Let P be a polytope of dimension n in $\mathbb{R}^n$. Denote f_i the number of faces of dimension i. In particular f_0 is the number of vertices of P, f_1 is the number of edges of P, and f_{n-1} is the number of facets. It is customary to put $f_{-1} = 1$. The vector $(f_0, \cdots, f_n)$ is called the *f-vector*. A *simplicial polytope* is a polytope all of whose facets are simplices.

THEOREM 14 (Dehn-Sommerville). *The f-vector of a simplicial polytope satisfies:*

$$f_{k-1} = \sum_{i=k}^{n+1} (-1)^{n+1-i} \binom{i}{k} f_{i-1} \quad 0 \le k \le \frac{n+1}{2} \tag{27}$$

If the points we chose are in general position the Delaunay triangulation can be made into a simplicial polytope $\widetilde{\mathrm{Del}}(\{P_1, \cdots, P_N\}) \subset \mathbb{R}^{n+1}$ by adding a point at infinity. If $n = 2$ then we can visualize the construction. We do so in figure 4.

We will apply the Dehn-Sommerville equations to $\widetilde{\mathrm{Del}}(\{P_1, \cdots, P_N\}) \subset \mathbb{R}^{n+1}$ in order to find formulas for the number of faces of the Delaunay triangulation.

Denote

- $e_{-1} = 1, \cdots, e_n$ the number of faces in each dimension of $\mathrm{Del}(\{P_1, \cdots, P_N\}) \subset \mathbb{R}^n$
- $f_{-1} = 1, \cdots, f_n$ the number of faces in each dimension of $\widetilde{\mathrm{Del}}(\{P_1, \cdots, P_N\}) \subset \mathbb{R}^{n+1}$
- $g_{-1} = 1, \cdots, g_{n-1}$ the number of faces in each dimension of $\mathrm{CH}(\{P_1, \cdots, P_N\}) \subset \mathbb{R}^n$

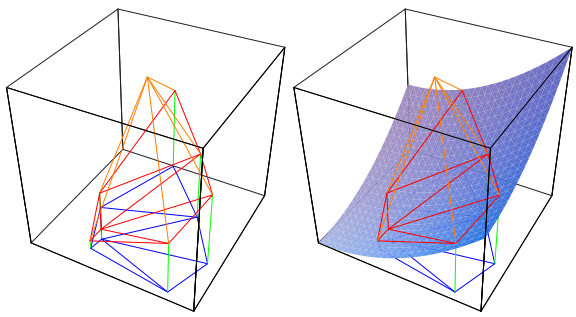


FIGURE 4. The polytopes $\widetilde{\mathrm{Del}}(\{P_1, \cdots, P_N\})$ and $\mathrm{CH}(\{P_1, \cdots, P_N\})$.

THEOREM 15. *With the above notations we have:*

$$e_k - g_k = \sum_{i=k}^{n} (-1)^{n-i} \binom{i+1}{k+1} e_i \quad 1 \leq k \leq \frac{n}{2}$$

REMARK 7. *We do not know whether theorem 15 is known.*

PROOF OF THEOREM 15. Both $\mathrm{CH}(\{P_1, \cdots, P_N\}) \subset \mathbb{R}^n$ and $\widetilde{\mathrm{Del}}(\{P_1, \cdots, P_N\}) \subset \mathbb{R}^{n+1}$ are polytopes. So for both of these we have the Dehn-Sommerville identities. We replace k by $k+1$ in equation (27):

$$f_k = \sum_{i=k+1}^{n} (-1)^{n+1-i} \binom{i}{k+1} f_{i-1} \quad 1 \leq k \leq \frac{n+3}{2} \tag{28}$$

We replace i by $i+1$ in equation (28):

$$f_k = \sum_{i=k}^{n} (-1)^{n-i} \binom{i+1}{k+1} f_i \quad 1 \leq k \leq \frac{n+3}{2} \tag{29}$$

For the g-vector:

$$g_{k-1} = \sum_{i=k}^{n} (-1)^{n-i} \binom{i}{k} g_{i-1} \quad 0 \leq k \leq \frac{n}{2} \tag{30}$$

We replace k by $k+1$.

$$g_k = \sum_{i=k+1}^{n} (-1)^{n-i} \binom{i}{k+1} g_{i-1} \quad 1 \leq k \leq \frac{n+2}{2} \tag{31}$$

Then we use (29) and (31) to compute $e_k = f_k - g_{k-1}$:

$$\begin{aligned} e_k = & \left(\sum_{i=k}^{n} (-1)^{n-i} \binom{i+1}{k+1} f_i \right) - \left(\sum_{i=k}^{n} (-1)^{n-i} \binom{i}{k} g_{i-1} \right) = \\ & \left(\sum_{i=k}^{n} (-1)^{n-i} \binom{i+1}{k+1} (e_i + g_{i-1}) \right) - \left(\sum_{i=k}^{n} (-1)^{n-i} \binom{i}{k} g_{i-1} \right) = \\ & \left(\sum_{i=k}^{n} (-1)^{n-i} \binom{i+1}{k+1} e_i \right) + \left(\sum_{i=k}^{n} (-1)^{n-i} \left(\binom{i+1}{k+1} - \binom{i}{k} \right) g_{i-1} \right) = \\ & \left(\sum_{i=k}^{n} (-1)^{n-i} \binom{i+1}{k+1} e_i \right) + \left(\sum_{i=k}^{n} (-1)^{n-i} \binom{i}{k+1} g_{i-1} \right) \end{aligned} \tag{32}$$

In the last line of equation (32) we put

$$\binom{k}{k+1} = \binom{k+1}{k+1} - \binom{k}{k} = 0 \tag{33}$$

So we can continue the calculation in (32) as

$$
\begin{aligned}
e_k &= \left(\sum_{i=k}^{n}(-1)^{n-i}\binom{i+1}{k+1}e_i\right) + \left(\sum_{i=k+1}^{n}(-1)^{n-i}\binom{i}{k+1}g_{i-1}\right) \Rightarrow \\
e_k - g_k &= \sum_{i=k}^{n}(-1)^{n-i}\binom{i+1}{k+1}e_i \quad 1 \le k \le \frac{n}{2}
\end{aligned}
\tag{34}
$$

The proof is complete. □

COROLLARY 1. *For a point set in* $\mathbb{R}^3$*:*

$$e_1 - g_1 = e_1 - 3e_2 + 6e_3 \Rightarrow g_1 = 3e_2 - 6e_3$$

The corollary can be checked by the reader in the case of five points in 3-space.

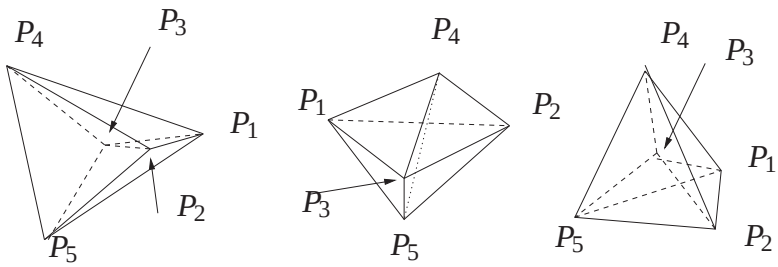


FIGURE 5. Three different Delaunay triangulations on five points in general position in $\mathbb{R}^3$.

$$\{\{P_1,P_2,P_3,P_4\},\{P_1,P_2,P_3,P_5\},\{P_2,P_3,P_4,P_5\}\} \tag{35}$$

$$\{\{P_1,P_2,P_3,P_4\},\{P_1,P_2,P_3,P_5\}\} \tag{36}$$

$$\{\{P_1,P_2,P_3,P_4\},\{P_1,P_2,P_3,P_5\},\{P_1,P_3,P_4,P_5\},\{P_2,P_3,P_4,P_5\}\} \tag{37}$$

In equation (35) we have 9 edges on the convex hull on the convex hull, 3 3-cells in the Delaunay triangulation, and 6 2-cells in the Delaunay triangulation. So $g_1 = 9$, $e_2 = 6$ and $e_3 = 3$. The reader is encouraged to check the other examples, and to write down the equations for the two-dimensional case. After some calculations you should get a lower bound for the number of "different" Voronoi diagrams on a point set. A research question would be: what do we mean when we say that two Voronoi diagrams are "different"? That question was the motivation for the articles [**SvM05**] and [**SvM04**].

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