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Sapporo Guest House Symposium on Mathematics 15

Evolution Equations

Edited by H. Kubo and T. Ozawa

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SAPPORO GUEST HOUSE SYMPOSIUM ON MATHEMATICS 15

Evolution Equations

日 時：2003 年 11 月 15 日（土曜日）10:00 ~ 11 月 16 日（日曜日）15:50

November 15, 2003 (Saturday) ~ November 16 (Sunday)

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8 minutes walk from Sumikawa (Subway Station)

November 15 (Saturday)

10:00–11:00	中西 賢次 (Kenji Nakanishi) (名大) [Nagoya Univ.] Scattering in the energy space for nonlinear Schrödinger equations with small solitary waves
11:20–12:20	横山 和義 (Kazuyoshi Yokoyama) (道工大) [Hokkaido Institute Tech.] Global small amplitude solutions to systems of nonlinear wave equations with mulitple speeds
14:00–15:00	Jürgen Saal (北大) [Hokkaido Univ.] H^∞ -calculus for the Stokes operator on L_q -spaces
15:20–16:20	小林 孝行 (Takayuki Kobayashi) (佐賀大) [Saga Univ.] Asymptotic behavior of solutions to the Compressible Navier-Stokes equations on the half space
16:40–17:40	黒木場 正城 (Masaki Kurokiba) (福岡大) [Fukuoka Univ.] Maximal attractor and inertial sets for Eguchi-Oki-Matsumura equation
18:30–	懇親会 (Welcome Party)

November 16 (Sunday)

9:30–10:30	加藤 淳 (Jun Kato) (東北大) [Tohoku Univ.] On some generalization of the weighted Strichartz estimates for the wave equation and application to self-similar solutions
10:50–11:50	小川 卓克 (Takayoshi Ogawa) (九大) [Kyushu Univ.] TBA
13:30–14:30	石毛 和弘 (Kazuhiro Ishige) (名大) [Nagoya Univ.] Blow-up problem for semilinear heat equations under Neumann boundary conditions
14:50–15:50	星賀 彰 (Akira Hoshiga) (静岡大) [Shizuoka Univ.] Existence and blowing up of solutions to systems of quasilinear wave equations in two space dimensions

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**SCATTERING IN THE ENERGY SPACE
FOR NONLINEAR SCHRÖDINGER EQUATIONS
WITH SMALL SOLITARY WAVES**

STEPHEN GUSTAFSON, KENJI NAKANISHI, TAI-PENG TSAI

We consider nonlinear Schrödinger equations of the form

$$i\dot{u} = (-\Delta + V)u + f(u), \quad (1)$$

and study long-time behavior of solutions close to solitary wave solutions. The wave function u is defined on $\mathbb{R}^{1+3} \rightarrow \mathbb{C}$, $V = V(x)$ is a given potential and $f(u)$ is a given nonlinearity.

Solitary waves are those solutions of the form $u = e^{-itE}Q(x)$. Then (1) is satisfied if $Q(x)$ solves the following elliptic equation:

$$(-\Delta + V)Q + f(Q) = EQ. \quad (2)$$

Under the gauge covariance condition $f(e^{i\alpha}u) = e^{i\alpha}f(u)$, $e^{i\alpha}Q(x)$ gives another solitary wave. Thus the family of solitary waves have at least two parameters E and α . If $V \equiv 0$, translation and Galilei invariance give rise to more parameters.

Thus we have a family of solitary waves $Q = Q[z]$, which depends continuously on the parameter z in some finite dimensional space. As for long-time behavior of solutions close to those $Q[z]$, there are the following two types of results:

- (1) Lyapunov stability [4, 7, 9, 14, 23]: For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|u(0) - Q[z_0]\|_{H^1} < \delta$ for some z_0 , then we have

$$\sup_{t \in \mathbb{R}} \inf_z \|u(t) - Q[z]\|_{H^1} < \varepsilon. \quad (3)$$

- (2) Asymptotic stability [1, 2, 3, 5, 6, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22]: There exists $\delta > 0$ such that if $\|u(0) - Q[z_0]\|_{H^1 \cap L^1} < \delta$, then for some time-dependent parameter $z(t)$, we have

$$\|u(t) - Q[z(t)]\|_{L^\infty} \rightarrow 0 \quad (t \rightarrow \infty). \quad (4)$$

In some cases L^1 is replaced with some weighted space $\langle x \rangle^{-s}L^2 \subset L^1$. The behavior of parameter $z(t)$ is also described asymptotically.

The second statement gives finer description on the asymptotic behavior, but it requires a stronger assumption on the initial data, which can not remain valid for long time. Our aim of this study is to derive “asymptotic stability” in the H^1 framework as it is compatible with variational arguments used for the Lyapunov stability.

We assume the following on the potential $V = V(x)$ and the nonlinearity $f(u)$.

- (1) $V \in L^2 + L^\infty$, its negative part decays in the sense that $\|V_-\|_{L^2 + L^\infty(|x|>R)} \rightarrow 0$ as $R \rightarrow \infty$. and V has only one eigenvalue $e_0 < 0$. We denote the normalized

ground state by $\phi_0 > 0$ and the projection onto the continuous spectrum by $P_c = 1 - \phi_0 \langle \phi_0 \rangle$.

- (2) The linear propagator $e^{it(\Delta-V)}$ satisfies the Strichartz estimate for H^1 on the continuous spectrum in the following sense: denoting $X = L_t^\infty H^1 \cap L_t^2 W^{1,6} \cap L_t^2 L^{6,2}$, we have

$$\begin{aligned} \|e^{it(\Delta-V)} P_c \varphi\|_X &\lesssim \|\varphi\|_{H^1}, \\ \left\| \int_{-\infty}^t e^{i(t-s)(\Delta-V)} P_c g(s) ds \right\|_X &\lesssim \|g\|_{L_t^2 W^{1,6/5}}. \end{aligned} \quad (5)$$

- (3) $f(u) = N(u) + (\Phi * |u|^2)u$, where $N : \mathbb{C} \rightarrow \mathbb{C}$, $N(e^{i\alpha} u) = e^{i\alpha} N(u)$, $N(\mathbb{R}) \subset \mathbb{R}$, $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}$ and

$$|N''(s)| \lesssim |s|^{1/3} + |s|^3, \quad \Phi \in L^1 + L^{3/2,\infty}. \quad (6)$$

Our main result is the following.

Theorem 1. *Under the above assumptions (1)–(3), there exists $\delta > 0$ such that*

- (1) *For any $\varphi \in H^1$ satisfying $\|\varphi\|_{H^1} < \delta$, we have a unique global solution $u(t)$ for (1), which is decomposed as*

$$u(t) = Q[z(t)] + \eta(t), \quad (7)$$

where $z : \mathbb{R} \rightarrow \mathbb{C}$, $Q[z]$ satisfies (2) with $E = E[z]$, $(Q, \phi_0)_{L^2} = z$ and we have

$$\begin{aligned} \|\eta\|_X + \|z\|_{L_t^\infty} &\lesssim \|\varphi\|_{H^1}, \\ \|\dot{z} + iE[z]z\|_{L_t^1 \cap L_t^2} &\lesssim \|\varphi\|_{H^1}^2. \end{aligned} \quad (8)$$

Moreover, there exist $m_\infty \in [0, \delta]$ and $\eta_\infty \in P_c(H^1)$ such that

$$|z(t)| \rightarrow m_\infty, \quad \|\eta(t) - e^{it(\Delta-V)} \eta_\infty\|_{H^1} \rightarrow 0, \quad (t \rightarrow \infty). \quad (9)$$

- (2) *For any $m_\infty \in [0, \delta]$ and any $\varphi \in P_c(H^1)$ satisfying $\|\varphi\|_{H^1} < \delta$, we have a global solution $u(t)$ of (1) satisfying the above asymptotic behavior.*

The crucial tool in our proof is the endpoint Strichartz estimate. First we use the simple projection P_c to estimate the dispersive part η . Then we get a linear term of η from the nonlinearity, for which we need the endpoint estimate and smallness of the solitary waves. Because of that linear term, we can get only L_t^2 estimate on $\partial_t|z|$ by this argument. Thus we employ time-dependent decomposition where the linear term of η vanish in the ODE for z so that we can obtain L^1 bound on $\partial_t|z|$, which guarantees that the parameter $z(t)$ remains small for all time.

The second statement in the theorem is proved by assuming the final state at finite large time T and taking weak limits as $T \rightarrow \infty$. Therefore, we do not get uniqueness for u and we do not expect uniqueness since the asymptotic data (m_∞, η_∞) seems insufficient to determine u . It would be interesting if one could define the set of final states such that the correspondence to the initial state $u(0)$ becomes one-to-one.

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Global small amplitude solutions to systems of nonlinear wave equations with multiple speeds

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§1. Introduction.

We consider the Cauchy problem for systems of m -nonlinear wave equations in three space dimensions:

$$(1) \quad (\partial_t^2 - c_i^2 \Delta) u_i(t, x) = F_i(u, \partial u, \partial^2 u) \quad \text{for } t > 0, x \in \mathbf{R}^3, \\ i = 1, 2, \dots, m.$$

$\partial u = (\partial_a u_i)_{0 \leq a \leq 3}$ stands for space-time gradient of $u = (u_i)_{1 \leq i \leq m}$. We are interested in the case where the system (1) has multiple speeds. Then this system is considered to be a simplified model of the nonlinear elastic wave equations, the Klein-Gordon-Zakharov equations, for example. For simplicity we assume that the propagation speeds are distinct:

$$(2) \quad c_i \neq c_j \quad \text{if } i \neq j.$$

We assume that the nonlinear terms $F_i(u, \partial u, \partial^2 u)$ are quadratic with respect to their arguments. More precisely, the nonlinear terms are written in the following forms:

$$(3) \quad F_i(u, \partial u, \partial^2 u) = \sum_{j=1}^m \sum_{a,b=0}^3 C_{ij}^{ab}(u, \partial u) \partial_a \partial_b u_j + B_i(u, \partial u),$$

$$(4) \quad C_{ij}^{ab}(u, \partial u) = \sum_{k=1}^m \sum_{c=0}^3 C_{ijk}^{abc} \partial_c u_k + \sum_{k=1}^m C_{ijk}^{ab} u_k,$$

$$(5) \quad B_i(u, \partial u) = \sum_{j,k=1}^m \sum_{a,b=0}^3 B_{ijk}^{ab} \partial_a u_j \partial_b u_k + \sum_{j,k=1}^m \sum_{a=0}^3 B_{ijk}^a u_j \partial_a u_k.$$

In order to use the energy method, we also assume symmetry conditions

$$(6) \quad C_{ij}^{ab}(u, \partial u) = C_{ij}^{ba}(u, \partial u) = C_{ji}^{ab}(u, \partial u).$$

We are interested in generalizing nonlinear terms $F_i(u, \partial u, \partial^2 u)$ so that the Cauchy problem (1) has a global solution in space and time. Since we are concerned with small amplitude solutions, higher degree nonlinear terms are favorable for global existence. However, solutions of nonlinear wave equations with quadratic nonlinearities can develop singularities in finite time, however small the initial data are (see [4]). So in order to construct global solutions, we need some assumptions on the quadratic nonlinear terms.

To describe the condition for the global existence, we define Q_0 and Q_{ab} by the following forms:

$$(7) \quad Q_0(\phi, \psi; c_i) = \partial_t \phi \partial_t \psi - c_i^2 \sum_{j=1}^3 \partial_j \phi \partial_j \psi, \quad Q_{ab}(\phi, \psi) = \partial_a \phi \partial_b \psi - \partial_b \phi \partial_a \psi.$$

In the nonlinear terms, we mainly pay attention to the terms which consist of products of the same component of u , such as $\partial_a u_i \partial_b u_i$. We call them self-interaction terms. The effects of self-interaction terms are weakened by taking the null forms. If the nonlinear terms $F_i(u, \partial u, \partial^2 u)$ depend only on $\partial u, \partial^2 u$ and not on u , then the condition is known as the null condition ([2, 3, 8, 14, 15]).

We consider the case where the nonlinear terms $F_i(u, \partial u, \partial^2 u)$ depend also on u . Katayama obtained a global existence result in [7] when the nonlinear terms are written in the following forms:

$$(8) \quad F_i(u, \partial u, \partial^2 u) = \sum_{j=1}^m \sum_{|\alpha| \leq 1} {}' Q_0(u_j, \partial^\alpha u_j; c_j) + \sum_{j=1}^m \sum_{a,b,c=0}^3 {}' Q_{ab}(u_j, \partial_c u_j) \\ + \sum_{\substack{|\alpha| \leq 1 \\ 1 \leq |\beta| \leq 2}} \sum_{j \neq k} {}' \partial^\alpha u_j \partial^\beta u_k.$$

Here and in what follows, the expression

$$(9) \quad f = \sum_\lambda {}' f_\lambda$$

means that f is a linear combination of f_λ .

If the nonlinear terms $F_i(u, \partial u, \partial^2 u)$ do not depend on u , the null condition prescribes the null forms to only the terms concerning i -th component in the i -th equation. However if u is involved in the nonlinear terms, all self-interaction terms need to be written by the null forms, by Ohta's example ([12]).

We generalize the result of Katayama further, by adding other self-interaction terms $u_j \partial_a u_j$:

$$(10) \quad F_i(u, \partial u, \partial^2 u) = \sum_{j \neq i} \sum_{a=0}^m {}' u_j \partial_a u_j + \sum_{j=1}^m \sum_{|\alpha| \leq 1} {}' Q_0(u_j, \partial^\alpha u_j; c_j) \\ + \sum_{j=1}^m \sum_{a,b,c=0}^3 {}' Q_{ab}(u_j, \partial_c u_j) + \sum_{\substack{|\alpha| \leq 1 \\ 1 \leq |\beta| \leq 2}} \sum_{j \neq k} {}' \partial^\alpha u_j \partial^\beta u_k.$$

Now we state our main result.

Theorem 1 Assume that the nonlinear terms (3) satisfy (6) and take the form (10). The Cauchy problem for the following system of nonlinear wave equations

$$(11) \quad \begin{cases} (\partial_t^2 - c_i^2 \Delta) u_i(t, x) = F_i(u, \partial u, \partial^2 u) & \text{for } t > 0, x \in \mathbf{R}^3, i = 1, \dots, m, \\ u_i(0, x) = \varepsilon f_i(x), \quad \partial_t u_i(0, x) = \varepsilon g_i(x) & \text{for } x \in \mathbf{R}^3, i = 1, \dots, m, \end{cases}$$

have a unique global smooth solution if $\varepsilon > 0$ is sufficiently small. Here, f_i and g_i are C^∞ -functions which decrease sufficiently fast as $|x| \rightarrow \infty$.

§2. Notation.

We define the scaling operator S and the angular-momentum operators Ω_{jk} :

$$(12) \quad S = t\partial_t + \sum_{j=1}^3 x_j\partial_j \quad \text{and} \quad \Omega_{jk} = x_j\partial_k - x_k\partial_j \quad \text{for } 1 \leq j < k \leq 3.$$

We will denote them along with the usual partial derivatives by $\Gamma = (\Gamma_0, \dots, \Gamma_7)$, when we do not need to distinguish between them:

$$(13) \quad \Gamma_0 = S, \quad \Gamma_1 = \Omega_{12}, \quad \Gamma_2 = \Omega_{13}, \quad \Gamma_3 = \Omega_{23}, \quad \Gamma_k = \partial_{k-4} \quad (4 \leq k \leq 7).$$

Using these differential operators, we next introduce two norms for a non-negative integer s and a smooth function v :

$$(14) \quad |v(t, x)|_s = \sum_{|\alpha| \leq s} |\Gamma^\alpha v(t, x)|$$

and

$$(15) \quad \|v(t, \cdot)\|_s = \sum_{|\alpha| \leq s} \|\Gamma^\alpha v(t, \cdot)\|_{L^2(\mathbf{R}^3)}.$$

Here we use the usual multi-index notation.

Finally, we introduce two linear operators. For each $i \in \{1, \dots, m\}$, we write $U_i^*[f, g]$ for the solution to the Cauchy Problem

$$(16) \quad \begin{cases} \square_i U_i^*[f, g](t, x) = 0 & \text{in } (0, \infty) \times \mathbf{R}^3, \\ U_i^*[f, g](0, x) = f(x), \quad \partial_t U_i^*[f, g](0, x) = g(x) & \text{for } x \in \mathbf{R}^3. \end{cases}$$

Similarly, $U_i[\Phi]$ stands for the solution to the Cauchy problem

$$(17) \quad \begin{cases} \square_i U_i[\Phi](t, x) = \Phi(t, x) & \text{in } (0, \infty) \times \mathbf{R}^3, \\ U_i[\Phi](0, x) = \partial_t U_i[\Phi](0, x) = 0 & \text{for } x \in \mathbf{R}^3. \end{cases}$$

§3. Outline of proof.

We explain the method of our proof. We know that there is a local solution by a classical theorem. So our plan is to continue the local solution. We first set a certain quantity $E[u](t)$ defined by the local solution u . If the classical solution cannot be continued globally, then $E[u](t)$ must tend to infinity at a finite time.

So we estimate the size of $E[u](t)$. To show a contradiction, we assume that there exists a positive number T such that the following situation occurs:

$$(18) \quad E[u](t) \leq C_0 \varepsilon \quad \text{for } 0 \leq t \leq T,$$

$$(19) \quad \sup_{T < t < T+\delta} E[u](t) > C_0 \varepsilon \quad \text{for any } \delta > 0.$$

C_0 is taken to be sufficiently large, depending on f_i , g_i and the nonlinear terms. ε is the parameter of amplitude. We use the information (18) to estimate the effects of nonlinear terms. From the estimate of nonlinear terms, we evaluate $E[u](t)$ again, and find that

$$(20) \quad E[u](t) < C_0 \varepsilon \quad \text{for } 0 \leq t \leq T.$$

This is a contradiction. So there is no such T , and (18) must hold for any length of time.

$E[u](t)$ is defined as follows:

$$(21) \quad E[u](t) = \sup_{0 < s < t} \left\{ \sup_{x \in \mathbf{R}^3} \sum_{i=1}^m \langle r \rangle \langle c_i s - r \rangle |u_i(s, x)|_{K+2} \right. \\ + \sup_{x \in \mathbf{R}^3} \sum_{i=1}^m \langle r \rangle \langle c_i s - r \rangle w(s, r)^\nu |\partial u_i(s, x)|_{K+3} \\ + \langle s \rangle^{-\lambda} \sum_{i=1}^m \|\langle c_i s - r \rangle |\partial u_i(s, \cdot)|_{2K-1}\|_{L^2} \\ \left. + \langle s \rangle^{-\lambda} \|u(s, \cdot)\|_{2K} + \langle s \rangle^{-\lambda} \|\partial u(s, \cdot)\|_{2K} \right\}.$$

Here, $r = |x|$, $\langle \rho \rangle = \sqrt{1 + \rho^2}$,

$$(22) \quad w(s, r) = \left(\langle r \rangle^{-1} + \sum_{i=1}^m \langle c_i s - r \rangle^{-1} \right)^{-1},$$

and $0 < \nu < 1$, $0 < \lambda < 1$, $\lambda + \nu < 1$. The weights in L^∞ -norms reflect the behaviors of the solutions. It is important to notice that the partial derivatives of u behave a little bit better than u . Compared with Katayama [7], the weights are weakened, both inside the cones and along the cones. In [7], $\langle r \rangle$ in the first L^∞ -norm was $\langle s + r \rangle$ and $w(t, r)$ in the second L^∞ -norm was $\langle c_i s - r \rangle$.

In the present problem, we need to evaluate the quantities written as $\partial U_i[\partial \Phi]$ carefully. In order to take advantage of the divergent structure $\partial \Phi$, we notice the following: the first derivatives of u behave better than u , and the second derivatives behave better than the first derivatives. We can understand these facts from the estimates on L^2 -norms (see [7]):

$$(23) \quad \|U_i[\Phi](t, \cdot)\|_1 \leq C \int_0^t \|\langle s + |\cdot|\rangle \Phi(s, \cdot)\|_{L^2} ds,$$

$$(24) \quad \|\partial U_i[\Phi](t, \cdot)\|_{L^2} \leq C \int_0^t \|\Phi(s, \cdot)\|_{L^2} ds,$$

$$(25) \quad \|\langle c_i t - |\cdot| \rangle \partial U_i[\Phi](t, \cdot)\|_{L^2} \leq C \int_0^t \|\langle s + |\cdot| \rangle \Phi(s, \cdot)\|_{L^2} ds,$$

$$(26) \quad \|\langle c_i t - |\cdot| \rangle \partial^2 U_i[\Phi](t, \cdot)\|_{L^2} \leq C \int_0^t \|\Phi(s, \cdot)\|_1 ds + C \|\langle t + |\cdot| \rangle \Phi(t, \cdot)\|.$$

We use this idea also to pointwise estimates. Klainerman and Sideris [9] proved that

$$(27) \quad \begin{aligned} & \langle t + |x| \rangle \left\{ |\Delta U_i[\Phi](t, x)| + |\partial_t \partial_x U_i[\Phi](t, x)| + |\partial_t^2 U_i[\Phi](t, x)| \right\} \\ & \leq C |\partial U_i[\Phi](t, x)|_1 + C \langle t + |x| \rangle |\Phi(t, x)|. \end{aligned}$$

We also use

$$(28) \quad \langle |x| \rangle |\partial_j \partial_k U_i(t, x)| \leq \langle |x| \rangle |\Delta U_i(t, x)| + C |\partial U_i(t, x)|_1.$$

Combining these inequalities, we can evaluate $\partial U_i[\partial \Phi]$.

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H^∞ -calculus for the Stokes operator on L_q -spaces

André Noll Jürgen Saal

Abstract

It is proved that the Stokes operator on a bounded domain, an exterior domain, or a perturbed half-space Ω admits a bounded H^∞ -calculus on $L_{q,\sigma}(\Omega)$ if $q \in (1, \infty)$ and Ω is of class C^3 .

Let $1 < q < \infty$, Ω be a domain in \mathbb{R}^n of class C^2 , and denote by $C_{c,\sigma}^\infty(\Omega)$ the space of all C^∞ vector fields u with compact support satisfying $\operatorname{div} u = 0$. Then the Stokes operator A_Ω is defined in $L_{q,\sigma}(\Omega) := \overline{C_{c,\sigma}^\infty(\Omega)}^{L^q}$ by

$$\begin{aligned} A_\Omega u &:= -P_\Omega \Delta u \quad \text{on} \\ D(A_\Omega) &:= W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L_{q,\sigma}(\Omega), \end{aligned}$$

where $P_\Omega : L_q(\Omega) \rightarrow L_{q,\sigma}(\Omega)$ is the well known Helm-Holtz projection.

Our main result reads as follows

Theorem 0.1 *Let $n \geq 3$ and let $\Omega \subset \mathbb{R}^n$ be a C^3 -domain which is either bounded, exterior, or a perturbed half-space. Then the Stokes operator A_Ω admits a bounded H^∞ -calculus in $L_{q,\sigma}(\Omega)$ if $1 < q < \infty$.*

Let $\phi \in (0, \pi)$ and $\Sigma_\phi := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \phi\}$. The class of all operators that admit a bounded H^∞ -calculus on a Banachspace X , denoted by $\mathcal{H}(X)$, contains all operators that satisfy the estimate

$$\|h(A)\|_{\mathcal{L}(X)} \leq C \|h\|_\infty \tag{1}$$

for all $h \in H^\infty(\Sigma_\phi) := \{f : \Sigma_\phi \rightarrow \mathbb{C} : f \text{ is holomorphic and bounded}\}$, where $h(A)$ is defined by the Cauchy integral in the usual way. Since $s \mapsto e^{is} \in H^\infty(\Sigma_\phi)$, the class $\mathcal{H}(X)$ is a subclass of $\text{BIP}(X)$, the class of all operators that have bounded imaginary powers, i.e.

$$\|A^{is}\|_{\mathcal{L}(X)} \leq C, \quad s \in (-1, 1).$$

Hence, $\mathcal{H}(X)$ enjoys all properties of this larger class. For instance, the domain of fractional powers can be determined in terms of a complex interpolation space. Another reason is the maximal L_q -regularity of the associated evolution equation $u_t + Au(t) = f(t)$ if X is a UMD space and A has power angle less than $\pi/2$. The maximal regularity was proved by Solonnikov for the Stokes operator in $L_{q,\sigma}(\Omega)$ by direct methods, see [Sol77].

In [Gig81] Giga proved that the Stokes operator on a bounded C^∞ -domain has bounded imaginary powers and in [GS89] Giga and Sohr verified this result also for exterior domains. The proof in [Gig81] makes use of pseudodifferential operators and Seeley's theory on the description of fractional powers of an elliptic system. Our approach, however, is different as it relies on perturbation methods of the class $\mathcal{H}^\infty(X)$ and it includes domains with merely C^3 boundary. Moreover, it is known that the class of all operators admitting a bounded H^∞ -calculus coincides with the (a priori smaller) class of all operators admitting an R-bounded H^∞ -calculus if the underlying

Banach space has property (α) , see [KW01]. Since the space $L_{q,\sigma}(\Omega)$ is known to enjoy this property for any domain Ω and any $q \in [1, \infty]$, we can immediately conclude that A_Ω even admits an R-bounded H^∞ -calculus for the domains treated in this article. This is relevant for handling perturbations of linear operators in view of the results in [KW01].

The proof of our result is based on two main ingredients. Firstly, on an abstract perturbation result of the class $\mathcal{H}(X)$ of Prüss (see [DDH⁺02]), secondly, on a localization procedure. The starting point is the bounded H^∞ -calculus for the Stokes operator on the space $L_{q,\sigma}(\mathbb{R}_+^n)$, which is proved in [DHP01] by Desch, Hieber, and Prüss. The transference of that result to bent half-spaces leads first to a perturbed Stokes operator A_R in the half-space \mathbb{R}_+^n . One problem in comparing this operator with the Stokes operator in \mathbb{R}_+^n is that these two operators act on different Banach spaces. To overcome this problem we consider the operator $A_T := TA_R T^{-1}$, where T is a transformation which maps the domain of A_R isomorphic on $D(A_{\mathbb{R}_+^n})$. The price which one has to pay for this conjugation with T is that we have to assume the treated domains Ω to be of class C^3 (instead of C^2). Although the operator A_T now fits into the setting of the perturbation result of Prüss, another problem is to verify the assumptions of that result for A_T . Besides the relative boundedness of the perturbation one has also to prove a certain estimate which involves A_T and fractional powers of $A_{\mathbb{R}_+^n}$.

By localizing the Stokes resolvent problem

$$(SRP) \left\{ \begin{array}{rcl} \lambda u - \Delta u + \nabla p & = & f \quad \text{on } \Omega, \\ \nabla \cdot u & = & 0 \quad \text{on } \Omega, \\ u & = & 0, \quad \text{on } \partial\Omega, \end{array} \right.$$

i.e. by covering Ω by finite many balls and choosing the balls small enough we then can reduce the problem on Ω to \mathbb{R}^n (which is also a well known case) or to the bent half-space. This leads to the bounded H^∞ -calculus for the Stokes operator on $L_{q,\sigma}(\Omega)$ for Ω bounded, exterior, or a perturbed half-space. However, also the localization is not straight forward, since the solution and right hand side of the localized equations are no more solenoidal. Thus we can not apply the results in \mathbb{R}^n , \mathbb{R}_+^n , and the bent half-space directly. This problem can be solved by splitting the localized solution in several parts and by treating each term separately. For this purpose we frequently make use of estimates for the generalized Stokes resolvent problem proved in [FS94] by Farwig and Sohr.

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Asymptotic behavior of solutions to the Compressible Navier-Stokes Equations on the half space

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In this talk, I am going to talk about the asymptotic behavior of solutions to the initial boundary value problem for the compressible Navier-Stokes equation on the half space of \mathbf{R}_+^n , $n \geq 2$. The results in this talk were obtained in a joint work with Yoshiyuki KAGEI (Kyushu Univ.).

We consider the initial boundary value problem for the compressible Navier-Stokes equation in $\mathbf{R}_+^n = \{x = (x', x_n); x' \in \mathbf{R}^{n-1}, x_n > 0\}$:

$$(1.1) \quad \begin{aligned} \partial_t \rho + \operatorname{div} m &= 0, \\ \partial_t m + \operatorname{div} \left(\frac{m \otimes m}{\rho} \right) + \nabla P(\rho) &= \nu \Delta \left(\frac{m}{\rho} \right) + (\nu + \tilde{\nu}) \nabla \operatorname{div} \left(\frac{m}{\rho} \right), \end{aligned}$$

$$(1.2) \quad m|_{x_n=0} = 0, \quad \rho(0, x) = \rho_0(x), \quad m(0, x) = m_0(x).$$

Here $\rho = \rho(t, x)$ and $m = (m_1(t, x), \dots, m_n(t, x))$ denote the unknown density and momentum, respectively; $P = P(\rho)$ is the pressure; ν and $\tilde{\nu}$ are the viscosity coefficients that satisfy $\nu > 0$, $\frac{2}{n}\nu + \tilde{\nu} \geq 0$. The aim of this talk is to study the asymptotic behavior of solutions to problem (1.1), (1.2) around a constant equilibrium $(\rho, m) = (\rho^*, 0)$, where ρ^* is a given positive number.

ASSUMPTIONS. In this talk we assume the following A1 and A2.

- A1. The pressure $P = P(\rho)$ is a smooth function of ρ in a neighborhood of ρ^* and $p_1 \equiv \frac{\partial P}{\partial \rho}(\rho^*) > 0$.
- A2. The initial perturbation $(\rho_0 - \rho^*, m_0)$ belongs to H^3 and (ρ_0, m_0) satisfies the compatibility condition:

$$m_0|_{x_n=0} = 0$$

and

$$-\operatorname{div} \left(\frac{m_0 \otimes m_0}{\rho_0} \right) - \nabla P(\rho_0) + \nu \Delta \left(\frac{m_0}{\rho_0} \right) + (\nu + \tilde{\nu}) \nabla \operatorname{div} \left(\frac{m_0}{\rho_0} \right) \Big|_{x_n=0} = 0.$$

Theorem 1. Let $n \geq 2$. Then under the assumptions (A1) and (A2) the following assertions hold.

(i) There exists a positive number δ_1 such that if the initial perturbation $(\rho_0 - \rho^*, m_0) \in H^{[\frac{n}{2}]+2} \cap L^1$ and if $\|(\rho_0 - \rho^*, m_0)\|_{H^{[\frac{n}{2}]+2}} + \|(\rho_0 - \rho^*, m_0)\|_{L^1} \leq \delta_1$, then the perturbation $U(t) \equiv (\rho(t) - \rho^*, m(t))$ satisfies

$$\|U(t)\|_{L^p} = O(t^{-\frac{n}{2}(1-\frac{1}{p})})$$

and

$$\|\partial_x U(t)\|_{L^2} = \begin{cases} O(t^{-\frac{n}{4}-\frac{3}{8}}) & (n=2) \\ O(t^{-\frac{n}{4}-\frac{1}{2}}) & (n \geq 3) \end{cases}$$

for $2 \leq p \leq \infty$ as $t \rightarrow \infty$.

(ii) For $u_0 = (\bar{\rho}_0, \bar{m}_0)$ with $\bar{\rho}_0 \in H^1$ and $\bar{m}_0 = (\bar{m}_{0,1}, \dots, \bar{m}_{0,n}) \in L^2$, let $\bar{U}(t)[u_0](x) = (\bar{\rho}(t,x), \bar{m}(t,x))$ denote the solution of the linearized problem at $(\rho^*, 0)$:

$$\begin{aligned} \partial_t \bar{\rho} + \operatorname{div} \bar{m} &= 0 \\ \partial_t \bar{m} - \hat{\nu} \Delta \bar{m} - (\hat{\nu} + \hat{\tilde{\nu}}) \nabla \operatorname{div} \bar{m} + p^* \nabla \bar{\rho} &= 0, \\ \bar{m}|_{x_n=0} &= 0, \quad (\bar{\rho}(0,x), \bar{m}(0,x)) = u_0(x), \end{aligned}$$

where $\hat{\nu} = \nu/\rho^*$, $\hat{\tilde{\nu}} = \tilde{\nu}/\rho^*$, $p_1 = \partial_\rho P(\rho^*)$. Then, under the same assumptions on $(\rho_0 - \rho^*, m_0)$ in (i),

$$\|U(t) - \bar{U}(t)[u_0]\|_{L^2} = O(t^{-\frac{n}{4}-\frac{1}{4}} L(t))$$

as $t \rightarrow \infty$, where $u_0 = (\rho_0 - \rho^*, m_0)$, and $L(t) = \log t$ for $n = 2$ and $L(t) = 1$ for $n \geq 3$.

(iii) If, in addition, the initial perturbation $(\rho_0 - \rho^*, m_0) \in H^{[\frac{n}{2}]+3}$, then

$$\|u(t) - \bar{U}(t)[u_0]\|_{L^\infty} = \begin{cases} O(t^{-\frac{n}{2}-\frac{3}{8}} \log t) & (n=2) \\ O(t^{-\frac{n}{2}-\frac{1}{2}}) & (n \geq 3) \end{cases}$$

as $t \rightarrow \infty$, where $u_0 = (\rho_0 - \rho^*, m_0)$.

Theorem 2. Let $n \geq 2$ and let (\tilde{p}, \tilde{v}) be the solution of the nonstationary Stokes problem with $\tilde{p}(t) \in L^{\tilde{r}}(\mathbf{R}_+^n)$ for all $t > 0$ and some $2 < \tilde{r} < \infty$. Assume (A1) and (A2). Then, for $2 < p \leq \infty$,

$$\|U(t) - (\tilde{p}(t), \tilde{v}(t))\|_{L^p} = O(t^{-\frac{n}{2}(1-\frac{1}{p})-\delta(n,p)})$$

as $t \rightarrow \infty$ for some $\delta(n,p) > 0$.

MAXIMAL ATTRACTOR AND INERTIAL SET FOR EGUCHI-OKI-MATSUMURA EQUATION

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The results of this talk were obtained in a joint work with Naoto Tanaka (Fukuoka University) and Atsushi Tani (Keio University). In this talk we consider following system of equations which was proposed by Eguchi-Oki-Matsumura ([1]):

$$(1) \quad \begin{cases} \frac{\partial u}{\partial t} = \Delta(-\Delta u + 2u + uv^2), & (x, t) \in Q_T \equiv \Omega \times (0, T), \\ \frac{\partial v}{\partial t} = \beta\Delta v + \alpha v(a^2 - u^2 - b^2v^2), & (x, t) \in Q_T, \\ \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0, \quad \frac{\partial v}{\partial n} = 0, & (x, t) \in \Gamma_T \equiv \Gamma \times (0, T), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega, \end{cases}$$

where Ω is a bounded domain in \mathbf{R}^3 with smooth boundary $\partial\Omega \equiv \Gamma$ and $T > 0$. Here $u(x, t)$ is the local concentration of the solute atoms, $v(x, t)$ is the local degree of order, respectively. α , β , a , b are positive constants and $\frac{\partial}{\partial n}$ is exterior normal derivative to Γ .

It is well known that phase separation is described by so-called Cahn-Hilliard equation which is fourth order parabolic type, while the order-disorder transition is described by Allen-Cahn equation. The system (1) is a model of simultaneous order-disorder and phase separation in binary alloys. In our previous work [2], it was proved that there exist a unique local and global solution to problem (1).

In this talk we show the dynamics of the problem, namely the existence of a maximal attractor and of an inertial set to problem (1). The main theorems are as follows:

Theorem 1. Let $H_{\bar{u}} = \{(u, v) \in (H^1(\Omega))' \times L^2(\Omega); \frac{1}{|\Omega|} \int_{\Omega} u(x) dx = \bar{u}\}$. For any $\delta \geq 0$, the semi-group $S(t)$ associated with problem (1) posesses in $\mathcal{H}_{\delta} = \bigcup_{|\bar{u}| \leq \delta} H_{\bar{u}}$ a maximal attractor \mathcal{A}_{δ} that is connected.

Theorem 2. Let B_{δ} be the absorbing set in $(H^1(\Omega))' \times L^2(\Omega)$ and $X_{\delta} = \overline{\bigcup_{t \geq t_0} S(t)B_{\delta}}$. Then there exists an inertial set M_{δ} for $(S(t)_{t \geq 0}, X_{\delta})$ which has fractal dimension.

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On some generalization of the weighted Strichartz estimates for the wave equation and application to self-similar solutions

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1 Weighted Strichartz estimates

This talk is based on our recent paper [5]. Let w be a solution to the following Cauchy problem of the inhomogeneous wave equation with zero data,

$$\partial_t^2 w - \Delta w = F, \quad (t, x) \in (0, \infty) \times \mathbf{R}^n \equiv \mathbf{R}_+^{1+n}, \quad (1.1)$$

$$w|_{t=0} = 0, \quad w_t|_{t=0} = 0, \quad x \in \mathbf{R}^n. \quad (1.2)$$

We consider the time-space weighted L^q - $L^{q'}$ estimates for the solution w of the form

$$\||t^2 - |x|^2|^a w\|_{L^q(\mathbf{R}_+^{1+n})} \leq C \||t^2 - |x|^2|^b F\|_{L^{q'}(\mathbf{R}_+^{1+n})}, \quad 2 \leq q \leq \frac{2(n+1)}{n-1}, \quad (1.3)$$

which is called the weighted Strichartz estimates. Here, q' is the conjugate exponent to q . Estimates (1.3) have been proved under the following conditions:

- Georgiev-Lindblad-Sogge [2],

$$a < \frac{n-1}{2} - \frac{n}{q}, \quad b > \frac{1}{q}, \quad \text{supp } F \subset \{(t, x); |x| < t - 1\},$$

(D'Ancona-Georgiev-Kubo [1] removed the assumption on the support of F .)

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- Tataru [9], the *scale invariant* case,

$$a - b + \frac{n+1}{q} = \frac{n-1}{2}, \quad b < \frac{1}{q}, \quad \text{supp } F \subset \{(t, x); |x| < t\}.$$

The purpose of this talk is to show the estimates (1.3) without the assumption on the support of F in the *scale invariant* case, which have application to the existence of the self-similar solutions to nonlinear wave equations as we shall see below. In this case, it is known that the estimates (1.3) hold if F is radial in space variables without the assumption on the support of F :

- Kato-Ozawa [6, 7],

$$a - b + \frac{n+1}{q} = \frac{n-1}{2}, \quad \frac{n}{q} - \frac{n-1}{2} < b < \frac{1}{q}, \quad F(t, x) = \tilde{F}(t, |x|).$$

We intended to remove the assumption of radial symmetry on F of the above result and obtained the following result.

Theorem 1.1. *Let $n \geq 2$. For $2 < q < \frac{2(n+1)}{n-1}$, we assume $a, b \in \mathbf{R}$ satisfy*

$$a - b + \frac{n+1}{q} = \frac{n-1}{2}, \quad \frac{n}{q} - \frac{n-1}{2} < b < \frac{1}{q}.$$

Then, for the solution w to (1.1), (1.2),

$$\| |t^2 - |x|^2|^a w \|_{L_{t,r}^q L_\omega^2} \leq C \| |t^2 - |x|^2|^b F \|_{L_{t,r}^{q'} L_\omega^2} \quad (1.4)$$

holds. In particular, if $n \geq 3$ is odd and b further satisfy $b > \frac{n+1}{2q} - \frac{n-1}{4}$, then

$$\| |t^2 - |x|^2|^a w \|_{L_{t,r}^q H_\omega^{1/2}} \leq C \| |t^2 - |x|^2|^b F \|_{L_{t,r}^{q'} L_\omega^2}. \quad (1.5)$$

Here, for $G = G(t, x)$, the norm $\|\cdot\|_{L_{t,r}^p L_\omega^2}$ is defined by

$$\|G\|_{L_{t,r}^p L_\omega^2} = \left\{ \int_0^\infty \int_0^\infty \|G(t, r\cdot)\|_{L^2(S^{n-1})}^p r^{n-1} dr dt \right\}^{1/p},$$

using polar coordinates $x = r\omega$ ($r > 0$, $\omega \in S^{n-1}$), and H_ω^s denotes the Sobolev space on S^{n-1} .

2 Existence of self-similar solutions

As an application of Theorem 1.1, we are able to show the existence of self-similar solutions to the nonlinear wave equation

$$\partial_t^2 u - \Delta u = |u|^p, \quad (t, x) \in (0, \infty) \times \mathbf{R}^n. \quad (2.1)$$

The solution u to (2.1) is called *self-similar solution* if u satisfy

$$u(t, x) = \lambda^{-\frac{2}{p-1}} u(\lambda t, \lambda x) \quad (2.2)$$

for all $\lambda > 0$. Letting $\lambda = 1/t$, $u(1, \cdot) = W(\cdot)$, we observe that self-similar solution is the solution of the following form

$$u(t, x) = t^{-\frac{2}{p-1}} W(x/t).$$

From such scaling properties, it is known that self-similar solutions are useful to investigate the asymptotic behavior of the time-global solutions as $t \rightarrow \infty$.

It is known that there is a close connection between the existence of the self-similar solutions to (2.1) and the power of the nonlinear term p . In fact, in three space dimensions, Pecher [8] proved that if $p > 1 + \sqrt{2}$, there exist self-similar solutions, and if $p \leq 1 + \sqrt{2}$, self-similar solutions do not exist.

We intended to extend such sharp existence results of self-similar solutions to higher dimensions. We denote $p_0(n)$ the positive root of

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

Then, $p_0(3) = 1 + \sqrt{2}$ and we expect $p_0(n)$ to be the critical power concerning the existence of self-similar solutions to the equation (2.1). We notice that $p_0(n)$ is the critical exponent concerning the existence of time-global solutions to the Cauchy problem of the equation (2.1) with compactly supported, small, smooth initial data. (See John [4], ..., Georgiev-Lindblad-Sogge [2].) So, it is natural to expect $p_0(n)$ to be the one because self-similar solutions are also the time-global solutions. In fact, the following result are known.

- Hidano [3], $n = 2, 3$, $p > p_0(n)$, existence of self-similar solutions,
- Kato-Ozawa [6, 7], $n \geq 2$, $p > p_0(n)$, existence of *radial* self-similar solutions.

As an application of Theorem 1.1, we obtain the following result.

Theorem 2.1. *Let $2 \leq n \leq 5$. For $p_0(n) < p < \frac{n+3}{n-1}$, we assume that $\phi, \psi \in C^\infty(\mathbf{R}^n)$ are homogeneous of degree $-\frac{2}{p-1}, -\frac{2}{p-1} - 1$, respectively. Then, if $\varepsilon > 0$ is sufficiently small, there exists a unique time-global solution u to (2.1) with data*

$$u(0, x) = \varepsilon \phi(x), \quad \partial_t u(0, x) = \varepsilon \psi(x) \quad (2.3)$$

satisfying

$$\| |t^2 - |x|^2 |^\gamma u; \mathcal{L}_{t,r}^{p+1} H_\omega^{\frac{n-1}{2} + \delta} \| \leq C\varepsilon,$$

where, $\gamma = \frac{1}{p-1} - \frac{n+1}{2(p+1)}$, δ is small, \mathcal{L}^q denotes the weak Lebesgue space.

Remark 2.2. By the homogeneity of the data (2.3) and the uniqueness of the solution, the solution obtained in Theorem 2.1 is to be the self-similar solution. That is, we obtain the self-similar solution to (2.1) when $2 \leq n \leq 5$, $p > p_0(n)$.

Remark 2.3. The regularity on the unit sphere assures $H^{\frac{n-1}{2}+\delta}(S^{n-1}) \hookrightarrow L^\infty(S^{n-1})$, which enables us to estimate the nonlinear term. While this regularity also causes the restriction $n \leq 5$.

3 Outline of the proof of Theorem 1.1

The proof of Theorem 1 is based on the expansion of the solution w with respect to the spherical harmonics. Here, we describe the outline of the proof of (1.4).

For $k \geq 0$, let \mathcal{H}_k denote the space of spherical harmonics of degree k on S^{n-1} . We set its dimension α_k and set $\{Y_1^k, \dots, Y_{\alpha_k}^k\}$ the orthonormal basis of \mathcal{H}_k . It is well known that $L^2(S^{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$.

Denoting $F(t, x) = F(t, r\theta)$ by using the polar coordinates, F has the expansion

$$F(t, r\theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{\alpha_k} F_l^k(t, r) Y_l^k(\theta).$$

Then, we have $\|F(t, r\cdot)\|_{L^2(S^{n-1})} = (\sum_{k,l} |F_l^k(t, r)|^2)^{1/2}$. And applying this expansion, we obtain the expansion on w ,

$$\begin{aligned} w(t, r\theta) &= \int_0^t (-\Delta)^{-\frac{1}{2}} \sin[(-\Delta)^{\frac{1}{2}}(t-s)] F(s) ds \\ &= \sum_{k,l} S_k(F_l^k)(t, r) Y_l^k(\theta). \end{aligned} \tag{3.1}$$

For example, in odd space dimensions, S_k is given by

$$S_k(G)(t, r) = r^{-\frac{n-1}{2}} \int_0^t \int_{|t-s-r|}^{t-s+r} P_{k+\frac{n-3}{2}}(\mu) \lambda^{\frac{n-1}{2}} G(s, \lambda) d\lambda ds,$$

where P_m is the Legendre polynomial of degree m , and $\mu = (r^2 + \lambda^2 - (t-s)^2)/2r\lambda$. The coefficients of the expansion on w have the following estimates.

Lemma 3.1. *Let $n \geq 2$. We suppose q , a , and b satisfy the same condition as Theorem 1.1. Then, there exists a constant $C > 0$ which is independent of k such that*

$$\| |t^2 - r^2|^a r^{\frac{n-1}{q}} S_k(G) \|_{L_{t,r}^q} \leq C \| |t^2 - r^2|^b r^{\frac{n-1}{q'}} G \|_{L_{t,r}^{q'}}.$$

Then, the estimates (1.4) are obtained as follows. By the expansion (3.1) and Lemma 3.1,

$$\begin{aligned}
\| |t^2 - |x|^2|^a w \|_{L_{t,r}^q L_\theta^2} &= \| |t^2 - r^2|^a r^{\frac{n-1}{q}} \left(\sum_{k,l} |S_k(F_l^k)|^2 \right)^{\frac{1}{2}} \|_{L_{t,r}^q} \\
&\leq \left(\sum_{k,l} \| |t^2 - r^2|^a r^{\frac{n-1}{q}} S_k(F_l^k) \|_{L_{t,r}^q}^2 \right)^{\frac{1}{2}} \\
&\leq C \left(\sum_{k,l} \| |t^2 - r^2|^b r^{\frac{n-1}{q'}} F_l^k \|_{L_{t,r}^{q'}}^2 \right)^{\frac{1}{2}} \\
&\leq C \| |t^2 - r^2|^b r^{\frac{n-1}{q'}} \left(\sum_{k,l} |F_l^k|^2 \right)^{\frac{1}{2}} \|_{L_{t,r}^{q'}} \\
&\leq C \| |t^2 - |x|^2|^b F \|_{L_{t,r}^{q'} L_\theta^2}
\end{aligned}$$

where we have used the Minkowski's integral inequality repeatedly, since $q > 2$ and $q' < 2$.

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Blow-up problem for semilinear heat equations under Neumann boundary conditions

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In this talk we consider the blow-up problem of the Cauchy-Neumann problem

$$(1) \quad \begin{cases} u_t = D\Delta u + u^p & \text{in } \Omega \times (0, T_D), \\ \frac{\partial u}{\partial \nu}(x, t) = 0 & \text{on } \partial\Omega \times (0, T_D), \\ u(x, 0) = \phi(x) \geq 0 & \text{in } \Omega, \end{cases}$$

where $D > 0$, $p > 1$, $T_D > 0$, Ω is a bounded smooth domain in \mathbf{R}^N , and ν is the exterior unit normal vector to $\partial\Omega$. We say that a solution u of (1) is said to blow up at $t = T_D < \infty$ if there are sequences $\{t_n\} \subset (0, T_D)$, $\{a_n\} \subset \bar{\Omega}$ and a point $a \in \bar{\Omega}$ with $t_n \rightarrow T_D$ and $a_n \rightarrow a$ as $n \rightarrow \infty$ such that $u(a_n, t_n) \rightarrow \infty$ as $n \rightarrow \infty$. Then we call T_D and $a \in \mathbf{R}^N$ the blow-up time of u and a blow-up point of u , respectively. Furthermore the blow-up set of u means the set of all blow-up points of u . We denote by $B_D(\phi)$ the blow-up set of the solution u of (1).

We investigate the location of blow-up set $B_D(\phi)$ of the solution u of (1) with large D , and give the following theorem.

Theorem 0.1 (with Noriko Mizoguchi [1] and with Hiroki Yagishita [2])
Let P_2 be the projection from $L^2(\Omega)$ onto the second Neumann eigenspace.
Assume that $P_2\phi \not\equiv 0$ in Ω , and put

$$\mathcal{M}(\phi) = \{x \in \bar{\Omega} : (P_2\phi)(x) = \max_{y \in \bar{\Omega}}(P_2\phi)(y)\}.$$

Then there holds

$$\lim_{D \rightarrow \infty} \sup\{|x - y| : x \in B_D(\phi), y \in \mathcal{M}(\phi)\} = 0.$$

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EXISTENCE AND BLOWING UP OF SOLUTIONS TO SYSTEMS OF QUASILINEAR WAVE EQUATIONS IN TWO SPACE DIMENSIONS

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In this talk, we consider the Cauchy problem:

$$\square_i u^i \equiv \partial_t^2 u^i - c_i^2 \Delta u^i = \sum_{l=1}^m \sum_{\alpha, \beta=0}^2 a_{il}^{\alpha\beta} (\partial u) \partial_\alpha \partial_\beta u^l \quad \text{in } \mathbf{R}^2 \times (0, \infty), \quad (1)$$

$$u^i(x, 0) = \varepsilon f^i(x), \quad \partial_t u^i(x, 0) = \varepsilon g^i(x) \quad \text{in } \mathbf{R}^2, \quad (2)$$

where $i = 1, \dots, m$ and $u(x, t) = {}^t(u^1(x, t), \dots, u^m(x, t))$. We denote $\partial = (\partial_0, \partial_1, \partial_2)$ with $\partial_0 = \partial_t = \partial/\partial t$ and $\partial_j = \partial/\partial x_j$ ($j = 1, 2$). Let $\varepsilon > 0$ be a small parameter and assume that $f^i, g^i \in C_0^\infty(\mathbf{R}^2)$, $|f^i| + |g^i| \not\equiv 0$, $\text{supp}\{f^i\}, \text{supp}\{g^i\} \subset \{x \in \mathbf{R}^2 : |x| \leq M\}$, $a_{il}^{\alpha\beta} \in C^\infty(\mathbf{R}^{3m})$, $a_{il}^{\alpha\beta}(0) = 0$ and

$$a_{il}^{\alpha\beta}(\partial u) = a_{li}^{\alpha\beta}(\partial u) = a_{li}^{\beta\alpha}(\partial u) \quad (3)$$

for any $i, l = 1, \dots, m$ and $\alpha, \beta = 0, 1, 2$. We also denote

$$a_{il}^{\alpha\beta}(\partial u) = \sum_{j=1}^m \sum_{\gamma=0}^2 Q_{ijl}^{\alpha\beta\gamma} \partial_\gamma u^j + \sum_{j,k=1}^m \sum_{\gamma,\delta=0}^2 C_{ijkl}^{\alpha\beta\gamma\delta} \partial_\gamma u^j \partial_\delta u^k + O(|\partial u|^3) \quad (4)$$

near the origin. Here $Q_{ijl}^{\alpha\beta\gamma}$ and $C_{ijkl}^{\alpha\beta\gamma\delta}$ are constants.

Furthermore, we assume that the propagation speeds of system (1) are distinct, namely we assume

$$0 < c_1 < c_2 < \dots < c_m. \quad (5)$$

Our purpose in this talk is to obtain a precise estimate for the *lifespan* T_ε . Here T_ε is defined by the supremum of all $T > 0$ for which C^∞ -solutions to (1) and (2) exist in $\mathbf{R}^2 \times [0, T)$. For this purpose, we prepare some notations. For $X = (X_0, X_1, X_2) \in \mathbf{R}^3$, define $\Psi(X) = (\Psi_1(X), \dots, \Psi_m(X))$ and $\Phi(X) = (\Phi_1(X), \dots, \Phi_m(X))$ by

$$\begin{aligned}\Psi_i(X) &= \sum_{\alpha, \beta, \gamma=0}^2 Q_{iij}^{\alpha\beta\gamma} X_\alpha X_\beta X_\gamma, \\ \Phi_i(X) &= \sum_{\alpha, \beta, \gamma, \delta=0}^2 C_{iiji}^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma X_\delta.\end{aligned}$$

Moreover, for a vector valued function $\phi(X) = (\phi_1(X), \dots, \phi_m(X))$, we say $\phi = 0$ (or $\phi \neq 0$) if and only if $\phi_i(X) \equiv 0$ for any $i = 1, \dots, m$ (or $\phi_i(X) \not\equiv 0$ for some $i = 1, \dots, m$) on the hypersurface $X_0^2 = c_i^2(X_1^2 + X_2^2)$. Then we have already known the following results.

When $m = 1$, S. Alinhac proved in [2] that if $\Psi = 0$ and $\Phi = 0$, then $T_\varepsilon = \infty$ for small ε , and if $\Psi = 0$ and $\Phi \neq 0$, then there exists a constant $K_* > 0$ such that

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log T_\varepsilon \geq K_*.$$

He also proved in [1] that if $\Psi \neq 0$, there exists a constant $H_* > 0$ such that

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \sqrt{T_\varepsilon} \geq H_*.$$

On the other hand, when $m \geq 2$, [7] proved that if $Q_{ijl}^{\alpha\beta\gamma} = 0$ for any $i, j, l = 1, \dots, m$, $\alpha, \beta, \gamma = 0, 1, 2$ and $\Phi = 0$, then $T_\varepsilon = \infty$ for small ε . Also [6] proved that if $Q_{ijl}^{\alpha\beta\gamma} = 0$ for any $i, j, l = 1, \dots, m$, $\alpha, \beta, \gamma = 0, 1, 2$ and $\Phi \neq 0$, then there exists a constant $K > 0$ such that

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log T_\varepsilon \geq K.$$

Here the constant K_* , H_* and K are described explicitly by f^i, g^i, Ψ_i and Φ_i .

However, there is no result when $m \geq 2$ and $Q_{ijl}^{\alpha\beta\gamma} \neq 0$ for some $i, j, l = 1, \dots, m$, $\alpha, \beta, \gamma = 0, 1, 2$. Thus, in this talk, we deal with this case. To state our results, we introduce the *Friedlander radiation field* \mathcal{F}^i ($i = 1, \dots, m$). Let $u_0^i(x, t)$ be the solution of the linear homogenous wave equation:

$$\square u_0^i = 0 \quad \text{in } \mathbf{R}^2 \times (0, \infty), \tag{6}$$

$$u_0^i(x, 0) = f^i(x), \quad \partial_t u_0^i(x, 0) = g^i(x) \quad \text{in } \mathbf{R}^2. \tag{7}$$

Then, for any $\rho_i \in \mathbf{R}$ and $\omega \in S^1$, we define \mathcal{F}^i by

$$\mathcal{F}^i(\rho_i, \omega) = \lim_{\substack{r \rightarrow \infty \\ r - c_i t = \rho}} r^{\frac{1}{2}} u_0^i(x, t) \quad \text{with } r = |x|, \omega = \frac{x}{r}, \rho_i = r - c_i t.$$

\mathcal{F}^i is expressed by

$$\mathcal{F}^i(\rho_i, \omega) = \frac{1}{2\sqrt{2}\pi} \int_{\rho_i}^{\infty} (s - \rho_i)^{-\frac{1}{2}} \{R_{g^i}(s, \omega) - \partial_s R_{f^i}(s, \omega)\} ds,$$

where R_h is the Radon transform of $h \in C_0^\infty(\mathbf{R}^2)$, i.e.,

$$R_h(s, \omega) = \int_{\omega \cdot y = s} h(y) dS_y.$$

Note that \mathcal{F}^i satisfies

$$\mathcal{F}^i(\rho_i, \omega) = 0 \quad \text{for } \rho_i \geq M, \quad (8)$$

$$|\partial_{\rho_i}^l \mathcal{F}^i(\rho_i, \omega)| \leq C(1 + |\rho_i|)^{-\frac{1}{2}}, \quad l = 0, 1, 2 \quad (9)$$

and

$$|r^{\frac{1}{2}} \partial_t^l u_0^i(x, t) - (-c_i)^l \partial_{\rho_i}^l \mathcal{F}^i(\omega, r - c_i t)| \leq \frac{C(1 + |r - c_i t|)^{\frac{1}{2}-l}}{t}, \quad l = 0, 1, 2 \quad (10)$$

for $r \geq c_i t / 2 \geq 1$. (For the proof, see Lemma 2.1.1 and Theorem 2.1.2 in L. Hörmander [4].)

Now, we can state our results.

Theorem 1 Let T_ε be the lifespan of (1) and (2). Assume that (3) and (5) hold. Set

$$H_i = \max_{\substack{\rho \in \mathbf{R} \\ \omega \in S^1}} \left\{ -c_i^{-\frac{3}{2}} \Psi_i(-c_i, \omega) \partial_\rho^2 \mathcal{F}^i(\omega, \rho) \right\}$$

and

$$H = \max\{ H_i \mid 1 \leq i \leq m \}.$$

Then, if $H > 0$, we have

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \sqrt{T_\varepsilon} \geq \frac{1}{H}. \quad (11)$$

By (8) and (9), we know that each H_i is well-defined and nonnegative. In what follows, we assume that $H = H_{i_0}$ and the maximum in the definition of H_{i_0} is attained at a point $(\rho_0, \omega_0) \in \mathbf{R} \times S^1$.

Since Theorem 1 claims only a lower bound for the lifespan T_ε , we can not say whether the evaluation (11) is optimal or not. To make it clear, we next consider upper bounds of the lifespan of the following Cauchy problem:

$$\partial_t^2 u^i - \mathcal{C}_i^2(\partial_t u) \Delta u^i = 0 \quad \text{in } \mathbf{R}^2 \times (0, \infty), \quad (12)$$

$$u^i(x, 0) = \varepsilon \phi^i(|x|), \quad \partial_t u^i(x, 0) = \varepsilon \psi^i(|x|) \quad \text{in } \mathbf{R}^2, \quad (13)$$

where $i = 1, 2, \dots, m$. Assume $\phi^i(|x|), \psi^i(|x|) \in C_0^\infty(\mathbf{R}^2)$, $\mathcal{C}_i \in C^\infty(\mathbf{R}^{3m})$ and

$$\mathcal{C}_i(\partial_t u) = c_i + \frac{1}{2c_i} \sum_{j=1}^m a_{ij} \partial_t u^j + O(|\partial_t u|^2) \quad \text{near } \partial_t u = 0, \quad (14)$$

i.e.,

$$\mathcal{C}_i^2(\partial_t u) = c_i^2 + \sum_{j=1}^m a_{ij} \partial_t u^j + O(|\partial_t u|^2) \quad \text{near } \partial_t u = 0, \quad (15)$$

where a_{ij} are constants and c_i are propagation speeds satisfying (5). By (15), we can rewrite the equation (12) as

$$\square_i u^i = \sum_{j=1}^m a_{ij} \partial_t u^j \Delta u^i + (\text{higher order terms}).$$

Thus we find that (12) and (13) is a special case of (1) and (2) and that

$$H_i = \max_{\rho \in \mathbf{R}} \left\{ \frac{1}{\sqrt{c_i}} a_{ii} (\mathcal{F}^i)''(\rho) \right\} \quad (16)$$

for (12) and (13). Here $\mathcal{F}^i(\rho)$ stands for the Friedlander radiation field for ϕ^i and ψ^i . Note that \mathcal{F}^i is independent of ω , since $\phi^i(|x|)$ and $\psi^i(|x|)$ are radially symmetric. Then we can show an upper bound for the lifespan.

Theorem 2 *Let T_ε be the lifespan of the Cauchy problem (12) and (13). Assume that there is a number i ($= 1, \dots, m$) such that*

$$a_{ii} \neq 0 \quad \text{and} \quad |\phi^i| + |\psi^i| \not\equiv 0. \quad (17)$$

Then we have

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \sqrt{T_\varepsilon} \leq \frac{1}{H}. \quad (18)$$

If $|\phi^i| + |\psi^i| \not\equiv 0$, we find that $\mathcal{F}^i \not\equiv 0$. (See [4].) Thus it follows from (8), (9) and (17) that $H > 0$. Furthermore, Theorems 1 and 2 guarantee that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \sqrt{T_\varepsilon} = \frac{1}{H}$$

holds for the lifespan T_ε of (12) and (13).

The condition $H > 0$ in Theorems 1 and 2 is equivalent to $\Psi \neq 0$. Therefore, it remains the case where $\Psi = 0$ and $Q_{ijl}^{\alpha\beta\gamma} \neq 0$ for some $i, j, l = 0, 1, \dots, m$, $\alpha, \beta, \gamma = 0, 1, 2$. In this case, we obtain the following.

Theorem 3 *Let T_ε be the lifespan of the Cauchy problem (1) and (2). Assume that $\Psi = 0$, $\Phi = 0$ and $Q_{ijl}^{\alpha\beta\gamma} = 0$ when $(j, l) \neq (i, i)$. Then, there exists an $\varepsilon_0 > 0$ such that $T_\varepsilon = \infty$ for $0 < \varepsilon < \varepsilon_0$.*

For the proof of Theorem 3, we use the *Ghost weight energy* which Alinhac used for single wave equation.

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