RECOGNITION OF PLANE-TO-PLANE MAP-GERMS

YUTARO KABATA

Abstract. We present a complete set of criteria for determining $\mathcal{A}$-types of plane-to-plane map-germs of corank one with $\mathcal{A}$-codimension ≤ 6, which provides a new insight into the $\mathcal{A}$-classification theory from the viewpoint of recognition problem. As an application to generic differential geometry, we discuss about projections of smooth surfaces in 3-space.

1. Introduction

We revisit the $\mathcal{A}$-classification of local singularities of plane-to-plane maps. Here $\mathcal{A}$ denotes the group of diffeomorphism germs of source and target planes preserving the origin. The classification has been achieved by J. H. Rieger, M. A. S. Ruas [15, 16, 18] – for instance, Table 1 below shows the list of all corank one map-germs with $\mathcal{A}$-codimension ≤ 6. When we apply the classification to some specific geometric situation, it often becomes a cumbersome task to detect which $\mathcal{A}$-type a given map-germ belongs to, that is referred to as "$\mathcal{A}$-recognition problem" (cf. [6]). In fact, Rieger's algorithm frequently uses Mather's Lemma to reduce the jet to some nicer form, at which the coordinate changes are not explicitly given (dotted lines in the recognition trees Fig. 1–5 in [15] indicate such processes). To fill up the process is not easy: the task is essentially related to deeper understanding on a filtered structure of the $\mathcal{A}$-tangent space of the germ, as T. Gaffney pointed out in an earlier work [6].

In this paper, we present a complete set of criteria for detecting $\mathcal{A}$-types of corank one germs with $\mathcal{A}$-codimension ≤ 6 (Theorem 3.1). That is a useful package consisting of two-phased criteria (Table 3 and Table 4), which would easily be implemented in computer. The first one is about geometric conditions on 'specified jets' for *topological $\mathcal{A}$-types* in terms of *intrinsic derivatives* [23, 19, 20, 14, 9], and the second is about algebraic conditions on *Taylor coefficients* of germs with each specified jet, which are obtained by describing explicitly all the required coordinate changes of source and target of map-germs which are hidden in the classification process (Proposition 3.3, 3.5, 3.6, and 3.8).

For example, look at the cases of the butterfly $(x, xy + y^5 \pm y^7)$ and the elder butterfly $(x, xy+y^5)$, which are combined into a single topological $\mathcal{A}$-type. Suppose that a map-germ $f = (f_1, f_2) : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ with corank one is given. Put $\lambda(x, y) := \frac{\partial(f_1,f_2)}{\partial(x,y)}$, and take an arbitrary vector field $\eta := \eta_1(x, y)\frac{\partial}{\partial x} + \eta_2(x, y)\frac{\partial}{\partial y}$ near the origin of the source space so that η spans $\ker df$ on $\lambda = 0$ (Saji [19]). Denote $\eta^k g := \eta(\eta^{k-1}g)$. We show that the corresponding weighted homogeneous

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specified jet (see section 2.2) is characterized in terms of λ and η:

$$j^5 f(0) \sim_{\mathcal{A}^5} (x, xy + y^5) \iff \begin{cases} d\lambda(0) \neq 0, \\ \eta\lambda(0) = \eta^2\lambda(0) = \eta^3\lambda(0) = 0, \quad \eta^4\lambda(0) \neq 0 \end{cases}$$

Notice that the condition in the right hand side does not depend on the choices of local coordinates and the null vector field η. The subtle difference between these (C^∞-)$\mathcal{A}$-types is expressed by the following *Taylor coefficients condition*: If we write $f = (x, xy + y^5 + \sum_{i+j\geq 6} a_{ij}x^i y^j)$, then

$$f \sim_{\mathcal{A}} (x, xy + y^5 \pm y^7) \iff a_{07} - \tfrac{5}{8}a_{06}^2 \neq 0,$$

otherwise, f is of type elder butterfly. It should be noted that for the butterfly T. Gaffney [6] found the same condition on Taylor coefficients by studying the structure of $\mathcal{A}$-tangent space (Example 1.4. in [6]). Our approach is more direct by extending the method used in [15, 3], and we describe such conditions for all $\mathcal{A}$-types in Rieger's list ($\mathcal{A}$-codimension ≤ 6).

Our second purpose is to demonstrate a systematic use of our criteria for map-germs arising in some specific geometric situation. We develop a method of J. W. Bruce [3] for an application to *extrinsic differential geometry* of surfaces. Look at a generic surface in $\mathbb{R}^3$ from a viewpoint (camera), then we get locally a smooth map from the surface to the plane (screen), that is called the *central projection*. Their singularities have been classified by V. I. Arnold and O. A. Platonova (also O. P. Shcherbak, V. V. Goryunov) [1, 2, 8, 12, 22] based on a different framework. It is shown that some germs of $\mathcal{A}$-codimension 5 *do not appear* generically in central projections, although the reason has not been quite clear from the context of $\mathcal{A}$-classification, as Rieger noted in his paper [15]. Our criteria make the reason very clear – the condition of intrinsic derivatives $\eta^k\lambda$ determines jets of Monge form of the surface, while the condition of Taylor coefficients determines a special position of viewpoints (Remarks 4.11 and 4.12). We present an alternative transparent proof of Arnold-Platonova's theorem within the $\mathcal{A}$-classification theory, moreover, we classify singularities arising in central projections of *moving surfaces* with one-parameter in 3-space (Theorem 4.6).

As a byproduct, in another paper [21] we obtain a generalization of projective classification of jets of Monge forms by Platonova [12]. Our criteria are also useful to determine the bifurcation diagrams of map-germs, especially of corank two. See [24, 25] for the detail.

The rest of this paper is organized as follows. In §2 we briefly introduce the classification of plane-to-plane map-germs. In §3 we give a complete set of criteria for all $\mathcal{A}$-types with $\mathcal{A}$-cod ≤ 6. In §4 we show an application of our criteria to the central projection of smooth surfaces.

2. Preliminary

2.1. **$\mathcal{A}$-classification.** To begin with, we briefly summarize the basics of singularity theory of map-germs. Let $\mathcal{E}_n$ be an $\mathbb{R}$-algebra of smooth map-germs $\mathbb{R}^n, 0 \to \mathbb{R}$ with a unique maximal ideal m_n. The $\mathcal{E}_n$-module consisting of map-germs $\mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ is isomorphic to $m_n\mathcal{E}_n^p$. On this space, the group of diffeomorphism germs defines an equivalence relation: $f, g : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ are *$\mathcal{A}$-equivalent* ($f \sim_{\mathcal{A}} g$), if there exist diffeomorphism germs ϕ and ψ of $\mathbb{R}^n, 0$ and $\mathbb{R}^p, 0$ so that $f = \psi \circ g \circ \phi^{-1}$. We denote by $\mathcal{A}.f$ the $\mathcal{A}$-orbit of f. If an $\mathcal{A}$-orbit has finitely many nearby orbits, then the orbit is called $\mathcal{A}$-*simple*; otherwise, there is some family of $\mathcal{A}$-orbits, called an $\mathcal{A}$-*moduli*.

Let $\xi : \mathbb{R}^n, 0 \to T\mathbb{R}^p$ be a smooth map-germ such that $\pi \circ \xi = f$ (where π is a projection of tangent vector bundle). We call ξ *the vector field along* f or *infinitesimal deformation of* f, and denote the set of all the the vector field along f by $\theta(f)$. In an obvious way, $\theta(f)$ is a $\mathcal{E}_n$-module. For the identity maps $id_n : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$, $id_p : \mathbb{R}^p, 0 \to \mathbb{R}^p, 0$, we write $\theta(n) = \theta(id_n)$, $\theta(p) = \theta(id_p)$, which are the module of vector field-germs. We define $tf : \theta(n) \to \theta(f)$ by the map $\xi \mapsto df \circ \xi$, and $\omega f : \theta(p) \to \theta(f)$ by the map $\eta \mapsto \eta \circ f$. With these notations above, we define the $\mathcal{A}$-*tangent space of* f by

$$T\mathcal{A}(f) := tf(m_n\theta_n) + \omega f(m_p\theta_p) \ \subset m_n\theta(f).$$

In fact, this space consists of all vectors $\frac{d}{dt}(\psi_t \circ f \circ \phi_t^{-1})\big|_{t=0}$ where ψ_t and ϕ_t are deformations of identity maps with $\psi_t(0) = 0$, $\phi_t(0) = 0$. We define the $\mathcal{A}$-*codimension* ($\mathcal{A}$-cod) of f by $\mathrm{cod}(\mathcal{A}, f) := \dim_{\mathbb{R}} m_n\theta(f)/T\mathcal{A}(f)$.

We define *the r-jet space* $J^r(n,p)$ to be the set of r-jets of map-germs at the origin. This is naturally identified with $m_n\mathcal{E}_n^p/m_n^{r+1}\mathcal{E}_n^p$. We say $j^rf(0)$ *is $\mathcal{A}^r$-equivalent to* $j^rg(0)$ ($j^rf(0) \sim_{\mathcal{A}^r} j^rg(0)$) if there exist diffeomorphism germs $\psi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ and $\phi : \mathbb{R}^p, 0 \to \mathbb{R}^p, 0$ such that $j^rf(0) = j^r(\phi \circ g \circ \psi^{-1})(0)$. The r-jets of diffeomorphism germs form a Lie group denoted by $\mathcal{A}^r$, which naturally has an algebraic action on $J^r(n,p)$, hence the orbit $\mathcal{A}^r(j^rf(0))$ is a locally closed semi-algebraic submanifold. For a smooth map-germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, we say f is r-$\mathcal{A}$-*determined*, if $f \sim_{\mathcal{A}} g$ holds for any smooth map-germs $g : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ such that $j^rf(0) = j^rg(0)$. When f is r-determined for some r, we say f *is finitely $\mathcal{A}$-determined*.

From now on, we consider the case $n = p = 2$. We are concerned with the $\mathcal{A}$-classification [15, 16, 17, 18]: In particular, all $\mathcal{A}$-types of $f : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ of corank one with $\mathcal{A}$-cod at most 6 are listed in Table 1. Here we use the notation $\mathcal{A}$-cod for germs with moduli to refer to the codimension of stratum. There are 29 types in the list with additional sign $\pm$, and we use Rieger's notation $1, 2, \cdots, 19$ for the $\mathcal{A}$-types throughout this paper. The type no.14: $(x, xy^2 + y^5)$ is not included in Table 1, since it has $\mathcal{A}$-codimension 7.

Remark 2.1. *Estimating the precise degree of determinacy for a map-germ is a big part of Rieger's classification* [15]*: he studied an algebraic structure of the $\mathcal{A}$-tangent space for each map-germ and got a precise estimate of the determinacy-degree by using results and technics in* [4, 13] *whose basic idea is due to Mather's infinitesimal criterion* [10]. *Thanks to the determinacy results in* [15], *we only have to consider the jet-space of map-germs in our following discussion.*

2.2. **Topological $\mathcal{A}$-classification.** Two germs are *topologically $\mathcal{A}$-equivalent* if they commute via some homeomorphisms of source and target; that is the version where one just replaces diffeomorphisms for $\mathcal{A}$-equivalence by homeomorphisms. By using a theorem of J. Damon [5], several different $\mathcal{A}$-types in Table 1 are combined into a single topological $\mathcal{A}$-type: Those are listed in the following Table 2 [16].

We introduce a coarser classification than topological $\mathcal{A}$-classification for our convenience. We provisionally call the weighted homogeneous part of each normal form in Table 2 the *specified jet* for the corresponding topological $\mathcal{A}$-type, except for 4_k-types; the specified jet of 4_k ($k \geq 3$) is defined to be (x, y^3). Note that

$\mathcal{A}$ -cod	type	normal form
0	1 (regular)	(x, y)
1	2 (fold)	(x, y^2)
2	3 (cusp)	$(x, xy + y^3)$
3	4_2 (beaks and lips)	$(x, y^3 \pm x^2 y)$
	5(swallowtail)	$(x, xy + y^4)$
4	4_3 (goose)	$(x, y^3 + x^3 y)$
	6 (butterfly)	$(x, xy + y^5 \pm y^7)$
	11_5 (gulls)	$(x, xy^2 + y^4 + y^5)$
5	4_4 (ugly goose)	$(x, y^3 \pm x^4 y)$
	7 (elder butterfly)	$(x, xy + y^5)$
	11_7 (ugly gulls)	$(x, xy^2 + y^4 + y^7)$
	12	$(x, xy^2 + y^5 + y^6)$
	16	$(x, x^2 y + y^4 \pm y^5)$
	8 (unimodal)	$(x, xy + y^6 \pm y^8 + \alpha y^9)$
6	4_5	$(x, y^3 + x^5 y)$
	9	$(x, xy + y^6 + y^9)$
	$10^\dagger$ (bimodal)	$(x, xy + y^7 \pm y^9 + \alpha y^{10} + \beta y^{11})$
	11_9	$(x, xy^2 + y^4 + y^9)$
	13	$(x, xy^2 + y^5 \pm y^9)$
	15 (unimodal)	$(x, xy^2 + y^6 + y^7 + \alpha y^9)$
	17	$(x, x^2 y + y^4)$
	$18^\dagger$ (bimodal)	$(x, x^2 y + xy^3 + \alpha y^5 + y^6 + \beta y^7)$
	19 (unimodal)	$(x, x^3 y + \alpha x^2 y^2 + y^4 + x^3 y^2)$

TABLE 1. $\mathcal{A}$-classification up to $\mathcal{A}$-cod ≤ 6 [15]. †: excluding exceptional values of the moduli

topological type	$\mathcal{A}$ -type	normal form
$I_k^\pm$ $(k \geq 2)$	$4_k^\pm$	$(x, y^3 \pm x^k y)$
II_k $(4 \leq k \leq 6)$	$5 - 10$	$(x, xy + y^k)$
III_k $(k \geq 2)$	11_{2k+1}	$(x, xy^2 + y^4 + y^{2k+1})$
IV_5	$12, 13, (14)$	$(x, xy^2 + y^5)$
V_1	$16, 17$	$(x, x^2 y + y^4)$

TABLE 2. Some different $\mathcal{A}$-types (with $\mathcal{A}$-codim ≤ 6) are combined into the same topological $\mathcal{A}$-types [16].

both germs (x, y^3) and $(x, xy^2 + y^4)$ are not finitely $\mathcal{A}$-determined, thus we can not use Damon's theorem [5]; indeed 4_k and 11_{2k+1} for different k may have different topological $\mathcal{A}$-types. However it is useful for our purpose to gather all 4_k of $k \geq 3$ (resp. 11_k) into a group I_* (resp. III_*) of $\mathcal{A}$-types having the same specified jet.

Here we list up all specified jets of germs under consideration in this paper (stable germs are omitted and specified jets of types $15, 18, 19$ are denoted by IV_6, V_2, VI, respectively):

$I_2 : (x, y^3 \pm x^2y)$, $I_* : (x, y^3)$,
$II_4 : (x, xy + y^4)$, $II_5 : (x, xy + y^5)$, $II_6 : (x, xy + y^6)$, $II_7 : (x, xy + y^7)$
$III_* : (x, xy^2 + y^4)$
$IV_5 : (x, xy^2 + y^5)$, $IV_6 : (x, xy^2 + y^6)$,
$V_1 : (x, x^2y + y^4)$, $V_2 : (x, x^2y + xy^3)$ or (x, x^2y)
$VI : (x, y^4 + \alpha x^2y^2 + x^3y)$ or $(x, y^4 + \alpha x^2y^2)$

3. Criteria for map-germs

We state our main result:

Theorem 3.1. *Specified jets of topologically $\mathcal{A}$-equivalent types of plane-to-plane germs with $\mathcal{A}$-codimension up to* 6 *are explicitly characterized by means of geometric terms λ and η as in Table 3: Precisely saying, given a map-germ f of corank one, the jet $j^rf(0)$ is $\mathcal{A}^r$-equivalent to one of the specified r-jets listed in Table 3 if and only if the corresponding condition of λ and η for f in Table 3 is satisfied. A complete set of criteria for detecting $\mathcal{A}$-types of germs with $\mathcal{A}$-codimension up to* 6 *(Table 1) is achieved by adding conditions in coefficients of Taylor expansions, which are precisely described in Proposition 3.3, 3.5, 3.6, and 3.8 below (Table 4 is a brief summary of the criteria).*

Remark 3.2. Our condition in coefficients of Taylor expansions detects $\mathcal{A}$-types among types having the same specified jets, however the geometric meaning is not so clear. For a few cases, T. Gaffney [6] found the same conditions in studying a finer algebraic structure of the corresponding $\mathcal{A}$-tangent space. It would be interesting to compare these two approaches. It would also be reasonable to discuss about the problem in the context of Damon's $\mathcal{K}_D$-theory using the logarithmic vector fields along the A_μ-type discriminant of a stable unfolding. That will be considered somewhere else.

Put $H_\lambda := \text{Hess}\ \lambda$. The proof is divided into the following four cases:

(*case 0*) $d\lambda(0) \neq 0$;
(*case 1*) $d\lambda(0) = 0$ and $\text{rk} H_\lambda(0) = 2$;
(*case 2*) $d\lambda(0) = 0$ and $\text{rk} H_\lambda(0) = 1$;
(*case 3*) $d\lambda(0) = 0$ and $\text{rk} H_\lambda(0) = 0$.

In fact, Table 3 is separated into these four cases by double lines. These cases deal with the same process in recognition trees Fig. 1–5 in [15]: Cases 0, 1, 3 correspond to Fig. 1, 3, 5, respectively, and case 2 corresponds to both Fig. 2 and 4 in [15].

For our simplicity, we omit the case of $\mathcal{A}$-cod ≤ 3, that is the set of characterizations by Whitney and Saji in [19, 20, 23]. In the following proof, we frequently use Rieger's results (e.g., $\mathcal{A}$-determinacy of germs), which should be referred to [15] (see Remark 2.1 and Table 4).

3.1. **Case 0:** $d\lambda \neq 0$ **(**$S(f)$ **is smooth).**

We deal with types $6 - 10$ of $\mathcal{A}$-cod $= 4, 5, 6$.

Proposition 3.3. *For a plane-to-plane map-germ f of corank one,*

specified jet	$\mathcal{A}$-type	condition	type
$II_4 : (x, xy + y^4)$	5	$d\lambda(0) \neq 0$, $\eta\lambda(0) = \eta^2\lambda(0) = 0$, $\eta^3\lambda(0) \neq 0$	A_0 / 4
$II_5 : (x, xy + y^5)$	6, 7	$d\lambda(0) \neq 0$, $\eta\lambda(0) = \eta^2\lambda(0) = \eta^3\lambda(0) = 0$, $\eta^4\lambda(0) \neq 0$	A_0 / 5
$II_6 : (x, xy + y^6)$	8, 9	$d\lambda(0) \neq 0$, $\eta\lambda(0) = \cdots = \eta^4\lambda(0) = 0$ $\eta^5\lambda(0) \neq 0$	A_0 / 6
$II_7 : (x, xy + y^7)$	10	$d\lambda(0) \neq 0$, $\eta\lambda(0) = \cdots = \eta^5\lambda(0) = 0$, $\eta^6\lambda(0) \neq 0$	A_0 / 7
$I_2 : (x, y^3 \pm x^2y)$	$4_2^{\pm}$	$d\lambda(0) = 0$, $\det H_\lambda(0) \neq 0$, $\eta^2\lambda(0) \neq 0$	A_1 / 3
$III_* : (x, xy^2 + y^4)$	11_{odd}	$d\lambda(0) = 0, \det H_\lambda(0) < 0$, $\eta^2\lambda(0) = 0$, $\eta^3\lambda(0) \neq 0$	A_1^- / 4
$IV_5 : (x, xy^2 + y^5)$	12, 13	$d\lambda(0) = 0, \det H_\lambda(0) < 0$, $\eta^2\lambda(0) = \eta^3\lambda(0) = 0$, $\eta^4\lambda(0) \neq 0$	A_1^- / 5
$IV_6 : (x, xy^2 + y^6)$	15	$d\lambda(0) = 0, \det H_\lambda(0) < 0$, $\eta^2\lambda(0) = \eta^3\lambda(0) = \eta^4\lambda(0) = 0$, $\eta^5\lambda(0) \neq 0$	A_1^- / 6
$I_* : (x, y^3)$	4_*	$d\lambda(0) = 0$, $\mathrm{rk}H_\lambda(0) = 1$, $\eta^2\lambda(0) \neq 0$	A_k / 3
$V_1 : (x, x^2y + y^4)$	16, 17	$d\lambda(0) = 0, \mathrm{rk}H_\lambda(0) = 1$, $\eta^2\lambda(0) = 0$, $\eta^3\lambda(0) \neq 0$	A_2 / 4
$V_2 : (x, x^2y + xy^3)$, (x, x^2y)	18	$d\lambda(0) = 0$, $\mathrm{rk}H_\lambda(0) = 1$, $\eta^2\lambda(0) = \eta^3\lambda(0) = 0$	A_k / ≥ 5
$VI : (x, y^4 + \alpha x^2y^2)$, $(x, y^4 + \alpha x^2y^2 + x^3y)$	19	$d\lambda(0) = 0$, $\mathrm{rk}H_\lambda(0) = 0$, $\eta^3\lambda(0) \neq 0$	D_4 / 4

TABLE 3. Geometric criteria for plane-to-plane germs with $\mathcal{A}$-codimension up to 6 (stable germs are omitted). Here H_λ in conditions of the third column denotes Hessian matrix of λ, and the last column means singularity types of λ at 0 and local degree of complexified germs respectively. Refer to Rieger's original list (Table 1 in [15]) for other geometrical invariants.

(1) *For* $r \geq 5$,
$j^r f(0) \sim_{\mathcal{A}^r} (x, xy + y^r) \Longleftrightarrow$

$$d\lambda(0) \neq 0, \;\; \eta^i\lambda(0) = 0 \; (1 \leq i \leq r-2), \;\; \eta^{r-1}\lambda(0) \neq 0.$$

given germs	$\mathcal{A}$-type	r	condition
$II_5 : \left(x, xy + y^5 + \sum_{i+j\geq 6} a_{ij}x^iy^j\right)$	6	7	$a_{07} - \frac{5}{8}a_{06}^2 \neq 0$
	7	7	$a_{07} - \frac{5}{8}a_{06}^2 = 0$
$II_6 : \left(x, xy + y^6 + \sum_{i+j\geq 7} a_{ij}x^iy^j\right)$	8	8	$a_{08} - \frac{3}{5}a_{07}^2 \neq 0$
	9	9	$a_{08} - \frac{3}{5}a_{07}^2 = 0$, $a_{09} - \frac{7}{25}a_{07}^3 \neq 0$
$II_7 : \left(x, xy + y^7 + \sum_{i+j\geq 8} a_{ij}x^iy^j\right)$	$10^\dagger$	11	$a_{09} - \frac{7}{12}a_{08}^2 \neq 0$
$III_* : \left(x, xy^2 + y^4 + \sum_{i+j\geq 5} a_{ij}x^iy^j\right)$	11_5	5	$a_{05} \neq 0$
	11_7	7	$a_{05} = 0$, $a_{07} - 2a_{15} + 4a_{23} \neq 0$
$\left(x, xy^2 + y^4 + \sum_{i+j\geq 8} c_{ij}x^iy^j\right)$	11_9	9	$c_{09} - 2c_{17} \neq 0$
$IV_5 : \left(x, xy^2 + y^5 + \sum_{i+j\geq 6} a_{ij}x^iy^j\right)$	12	6	$a_{06} \neq 0$
	13	9	$a_{06} = 0$, $a_{09} - \frac{5}{2}a_{16} - \frac{5}{6}a_{07}^2 \neq 0$
$IV_6 : \left(x, xy^2 + y^6 + \sum_{i+j\geq 7} a_{ij}x^iy^j\right)$	15	9	$a_{07} \neq 0$
$I_* : \left(x, y^3 + \sum_{i+j\geq 4} a_{ij}x^iy^j\right)$	4_3	4	$a_{31} \neq 0$
	4_4	5	$a_{31} = 0, a_{41} - \frac{1}{3}a_{22}^2 \neq 0$
	4_5	6	$a_{31} = a_{41} - \frac{1}{3}a_{22}^2 = 0$, $a_{51} - \frac{2}{3}a_{32}a_{22} + \frac{1}{3}a_{13}a_{22}^2 \neq 0$
$V_1 : \left(x, x^2y + y^4 + \sum_{i+j\geq 5} a_{ij}x^iy^j\right)$	16	5	$a_{05} \neq 0$
	17	5	$a_{05} = 0$
$V_2 : \left(x, x^2y + xy^3 + \sum_{i+j\geq 5} a_{ij}x^iy^j\right)$	$18^\dagger$	7	$a_{05} \neq \frac{3}{2}, \frac{9}{5}$, $a_{06}(5a_{05} - 9) - 15a_{14}a_{05} \neq 0$
$VI : \left(x, x^3y + \alpha x^2y^2 + y^4 + \sum_{i+j\geq 5} a_{ij}x^iy^j\right)$	19	5	$\Delta \neq 0$

TABLE 4. Complete criteria for plane-to-plane germs with $4 \leq \mathcal{A}$-cod≤ 6. Numbers in the third column mean determinacy-degrees of corresponding $\mathcal{A}$-types, i.e., each $\mathcal{A}$-type is r-$\mathcal{A}$-determined (see Rieger [15]). †: excluding exceptional values of the moduli

(2) *If we write* $f = (x, xy + y^5 + \sum_{i+j\geq 6} a_{ij}x^iy^j)$,

$$a_{07} - \tfrac{5}{8}a_{06}^2 \neq 0 \iff f \sim_{\mathcal{A}} (x, xy + y^5 \pm y^7) \cdots \boxed{6},$$
$$a_{07} - \tfrac{5}{8}a_{06}^2 = 0 \iff f \sim_{\mathcal{A}} (x, xy + y^5) \cdots \boxed{7}.$$

(3) *If we write* $f = (x, xy + y^6 + \sum_{i+j\geq 7} a_{ij}x^iy^j)$,

$$a_{08} - \tfrac{3}{5}a_{07}^2 \neq 0 \quad \Longleftrightarrow \quad f \sim_{\mathcal{A}} (x, xy + y^6 \pm y^8 + \alpha y^9) \cdots \boxed{8},$$

$$\begin{cases} a_{08} - \tfrac{3}{5}a_{07}^2 = 0 \\ a_{09} - \tfrac{7}{25}a_{07}^3 \neq 0 \end{cases} \quad \Longleftrightarrow \quad f \sim_{\mathcal{A}} (x, xy + y^6 + y^9) \cdots \boxed{9}.$$

(4) *If we write* $f = (x, xy + y^7 + \sum_{i+j\geq 8} a_{ij}x^iy^j)$,

$$a_{09} - \tfrac{7}{12}a_{08}^2 \neq 0 \quad \Longleftrightarrow \quad j^{11}f(0) \sim_{\mathcal{A}^{11}} (x, xy + y^7 \pm y^9 + \alpha y^{10} + \beta y^{11}),$$

and excluding exceptional values of α *and* β

$$f \sim_{\mathcal{A}} (x, xy + y^7 \pm y^9 + \alpha y^{10} + \beta y^{11}) \cdots \boxed{10}.$$

In order to prove 1 in Proposition 3.3, we need the next lemma based on Lemma 2.6 in [19]. Notice that λ is changed by multiplying a non-zero function when we take another coordinates, and also that there is an ambiguity to choose the null vector field η.

Lemma 3.4. *The conditions on the right hand side of 1 in Proposition 3.3 are independent from the choice of coordinates of the source and target and the choice of* η.

Proof : The proof is similar to that of Lemma 2.6 in [19]. □

Proof of 1 in Proposition 3.3 : It is easily checked that for the r-jet $(x, xy + y^r)$, the condition in the right hand side holds. Thus the "only if" part of 1 follows from Lemma 3.4.

The "If" part is shown by finding a suitable coordinate change. Assume that the condition on the right hand side of 1 holds for f. Since f is of corank 1 at 0, we may write

$$f(x, y) = (x, \textstyle\sum_{i+j\geq 2} a_{ij}x^iy^j).$$

Take $\eta = \frac{\partial}{\partial y}$ and

$$\lambda(x, y) = \textstyle\sum_{i+j\geq 2} j \cdot a_{ij}x^iy^{j-1}.$$

For this choice of coordinates and η, by Lemma 3.4, we have

$$d\lambda(0) \neq 0, \ \ \eta\lambda(0) = \eta^2\lambda(0) = \cdots = \eta^{r-2}\lambda(0) = 0, \ \ \eta^{r-1}\lambda(0) \neq 0.$$

Then $a_{11} \neq 0$, $a_{02} = a_{03} = \cdots = a_{0\,r-1} = 0$, $a_{0r} \neq 0$. By some coordinate change, we have $j^rf(0) = (x, xy + y^r)$. □

The following proof of the claim 2 uses a simple trick for eliminating a certain term in the normal form. This trick is standard for the classification of map-germs as seen in Bruce's work [3], and will implicitly appear several times in other cases.

Proof of 2 in Proposition 3.3 : The basic fact is that determinacy-degrees of map-germs $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ of corank one with $\mathcal{A}$-cod≤ 6 are given by Rieger [15] as in Table 4. Here both 6-type and 7-type are 7-determined, hence our task is to show

$$(x, xy + y^5 + cy^6 + dy^7) \sim_{\mathcal{A}^7} (x, xy + y^5 + (d - \tfrac{5}{8}c^2)y^7).$$

Write $xy + y^5 + cy^6 + dy^7 = xy + y^5(1 + \alpha y) + \beta y^6 + dy^7$ with $\alpha + \beta = c$. By the coordinate change so that

$$\bar{x} = x, \quad \bar{y}^5 = y^5(1 + \alpha y),$$

the 7-jet has the form

$$(\bar{x}, \bar{x}(\bar{y} - \tfrac{1}{5}\alpha\bar{y}^2 + \tfrac{4}{25}\alpha^2\bar{y}^3 + h.o.t.) + \bar{y}^5 + \beta(\bar{y} - \tfrac{1}{5}\alpha\bar{y}^2 + h.o.t.)^6 + d\bar{y}^7).$$

By the coordinate change

$$\begin{aligned} x &= \bar{x}(1 - \tfrac{1}{5}\alpha\bar{y} + \tfrac{4}{25}\alpha^2\bar{y}^2 + h.o.t.) \\ y &= \bar{y} \end{aligned}$$

the jet is written by

$$(x(1 + \tfrac{1}{5}\alpha y - \tfrac{3}{25}\alpha^2 y^2 + h.o.t.),\ xy + y^5 + \beta y^6 + (d - \tfrac{6}{5}\alpha\beta)y^7).$$

Then, by the coordinate change of target

$$(X, Y) \to (X - \tfrac{1}{5}\alpha Y, Y),$$

we eliminate the term $\frac{1}{5}\alpha xy$ in the first component; Hence the jet becomes

$$\begin{pmatrix} x(1 - \frac{3}{25}\alpha^2 y^2 + h.o.t.) - \frac{1}{5}\alpha(y^5 + \beta y^6 + (d - \frac{6}{5}\alpha\beta)y^7) \\ xy + y^5 + \beta y^6 + (d - \frac{6}{5}\alpha\beta)y^7 \end{pmatrix}.$$

Take $\tilde{x}$ to be the first component and $\tilde{y} = y$, then the jet is written by (after rewriting variables)

$$(x, xy(1 + \tfrac{3}{25}\alpha^2 y^2 + h.o.t.) + y^5 + (\tfrac{1}{5}\alpha + \beta)y^6 + (d - \alpha\beta)y^7).$$

Now we choose $\alpha = \frac{5}{4}c$ and $\beta = -\frac{1}{4}c$ to kill the term y^6 in the second component. Finally by $\tilde{x} = x$ and $\tilde{y} = y(1 + \frac{3}{25}\alpha^2 y^2 + h.o.t.)$, we obtain the form

$$(x, xy + y^5 + (d - \tfrac{5}{8}c^2)y^7).$$

This completes the proof. □

Proof of 3 in Proposition 3.3 : The proof is similar to that of the claim 2 just described above: First we eliminate the terms including x of order ≥ 7, and then we directly show that

$$(x, xy + y^6 + cy^7 + dy^8 + ey^9) \sim_{\mathcal{A}^9} (x, xy + y^6 + (d - \tfrac{3}{5}c^2)y^8 + (e - \tfrac{7}{5}cd + \tfrac{14}{25}c^3)y^9).$$

In fact, rewriting variables as $\tilde{x}, \tilde{y}$ of the germ in the left hand side, substitute

$$\begin{aligned} \tilde{x} &= x + \tfrac{c}{5}xy + \tfrac{c}{5}y^6 - \tfrac{3c^3}{25}y^8 + \tfrac{cd}{5}y^8 + \tfrac{14c^4}{125}y^9 - \tfrac{7c^2d}{25}y^9 + \tfrac{ce}{5}y^9, \\ \tilde{y} &= y - \tfrac{c}{5}y^2 + \tfrac{c^2}{25}y^3 - \tfrac{c^3}{125}y^4 + \tfrac{c^4}{625}y^5 - \tfrac{c^5}{3125}y^6 \\ &\quad + \tfrac{c^6}{15625}y^7 - \tfrac{c^7}{78125}y^8 + \tfrac{2c^8}{390625}y^9, \end{aligned}$$

and take the coordinate change of the target

$$(X, Y) \mapsto \left(X - \tfrac{c}{5}Y, Y\right),$$

then we get the equivalence. Here $c = a_{07}$, $d = a_{08}$, $e = a_{09}$, and both 8-type and 9-type are 9-determined, thus we have the claim 3. □

Proof of 4 in Proposition 3.3 : Also in a similar way as above we see

$$(x, xy + y^7 + cy^8 + dy^9) \sim_{\mathcal{A}^9} (x, xy + y^7 + (d - \tfrac{7}{12}c^2)y^9).$$

In fact, it is achieved by

$$\begin{aligned}\tilde{x} &= x + \tfrac{c}{6}xy + \tfrac{c}{6}y^7 - \tfrac{7c^3}{72}y^9 + \tfrac{cd}{6}y^9,\\ \tilde{y} &= y - \tfrac{c}{6}y^2 + \tfrac{c^2}{36}y^3 - \tfrac{c^3}{216}y^4 + \tfrac{c^4}{1296}y^5 - \tfrac{c^5}{7776}y^6 + \tfrac{c^6}{46656}y^7\\ &\quad - \tfrac{c^7}{279936}y^8 - \tfrac{5c^8}{93312}y^9\end{aligned}$$

and

$$(X, Y) \mapsto (X - \tfrac{c}{6}Y, Y).$$

In addtion, we easily get

$$(x, xy + y^7 \pm y^9 + O(10)) \sim_{\mathcal{A}} (x, xy + y^7 \pm y^9 + \alpha y^{10} + \beta y^{11} + O(12))$$

for some α, $\beta \in \mathbb{R}$. In Rieger [15], it is shown that the 10-type is 11-determined for generic α and β excluding some values explicitly given in [15, p.359]. This implies the claim 4. $\square$

3.2. **Case 1:** $d\lambda(0) = 0$, $\mathrm{rk}H_\lambda(0) = 2$. We deal with types 11_{2k+1} $(k = 2, 3, 4)$, $12, 13, 15$ of $\mathcal{A}$-cod $= 4, 5, 6$.

Proposition 3.5. *For a plane-to-plane map-germ f of corank one,*

(1) *For $r \geq 4$,*
$j^r f(0) \sim_{\mathcal{A}^r} (x, xy^2 + y^r) \iff$

$$d\lambda(0) = 0,\ \det H_\lambda(0) < 0,\ \eta^i\lambda(0) = 0\ (2 \leq i \leq r-2),\ \eta^{r-1}\lambda(0) \neq 0.$$

(2) *If we write $f = (x, xy^2 + y^4 + \sum_{i+j\geq 5} a_{ij}x^iy^j)$,*

$$\begin{aligned} a_{05} \neq 0 &\iff f \sim_{\mathcal{A}} (x, xy^2 + y^4 + y^5) \cdots \boxed{11_5},\\ \begin{cases} a_{05} = 0\\ a_{07} - 2a_{15} + 4a_{23} \neq 0\end{cases} &\iff f \sim_{\mathcal{A}} (x, xy^2 + y^4 + y^7) \cdots \boxed{11_7},\\ a_{05} = a_{07} - 2a_{15} + 4a_{23} = 0 &\iff j^7 f(0) \sim_{\mathcal{A}^7} (x, xy^2 + y^4).\end{aligned}$$

Furthermore, if we write $f = (x, xy^2 + y^4 + \sum_{i+j\geq 8} c_{ij}x^iy^j)$,

$$c_{09} - 2c_{17} \neq 0 \iff f \sim_{\mathcal{A}} (x, xy^2 + y^4 + y^9) \cdots \boxed{11_9}.$$

(3) *If we write $f = (x, xy^2 + y^5 + \sum_{i+j\geq 6} a_{ij}x^iy^j)$,*

$$\begin{aligned} a_{06} \neq 0 &\iff f \sim_{\mathcal{A}} (x, xy^2 + y^5 + y^6) \cdots \boxed{12},\\ \begin{cases} a_{06} = 0\\ a_{09} - \tfrac{5}{2}a_{16} - \tfrac{5}{6}a_{07}^2 \neq 0\end{cases} &\iff f \sim_{\mathcal{A}} (x, xy^2 + y^5 + y^9) \cdots \boxed{13}.\end{aligned}$$

(4) *If we write $f = (x, xy^2 + y^6 + \sum_{i+j\geq 7} a_{ij}x^iy^j)$,*

$$a_{07} \neq 0 \iff f \sim_{\mathcal{A}} (x, xy^2 + y^6 + y^7 + \alpha y^9) \cdots \boxed{15}.$$

Note that in claim 1 of Proposition 3.5, $d\lambda(0) = 0$ implies $\eta\lambda(0) = 0$.

Proof of 1 in Proposition 3.5 : The proof is similar to that of 1 in Proposition 3.3. $\square$

Proof of 2 in Proposition 3.5 : Let $f = (x, xy^2 + y^4 + \sum_{i+j\geq 5} a_{ij}x^iy^j)$. By routine coordinate changes, f is equivalent to

$$(x, xy^2 + y^4 + a_{05}y^5 + \sum_{i+j\geq 6} a'_{ij}x^iy^j)$$

for some a'_{ij} ($a'_{06} = a_{06} - 2a_{14}$ etc). Since $(x, xy^2 + y^4 + y^5)$ is 5-determined, $a_{05} \neq 0$ leads to

$$f \sim_{\mathcal{A}} (x, xy^2 + y^4 + y^5).$$

Next we suppose $a_{05} = 0$. A similar coordinate change as above shows that f is equivalent to

$$(x, xy^2 + y^4 + (a_{06} - 2a_{14})y^6 + (a_{07} - 2a_{15} + 4a_{23})y^7 + \textstyle\sum_{i+j\geq 8} b_{ij}x^iy^j)$$

for some b_{ij}. Since the germ $(x, xy^2 + y^4 + y^7)$ is 7-determined, we want to eliminate the term y^6 from the second component of the right-hand side. By a similar argument as in the proof of 2 in Proposition 3.3, we have

$$(x, xy^2 + y^4 + cy^6 + dy^7) \sim_{\mathcal{A}^7} (x, xy^2 + y^4 + dy^7).$$

In fact this is achieved by an explicit coordinate change of source (writing variables as $\tilde{x}, \tilde{y}$ of the germ in the left hand side)

$$\begin{aligned}\tilde{x} &= x + cxy^2 + cy^4 + cdy^7\\ \tilde{y} &= y - \tfrac{c}{2}y^3 + \tfrac{3c^2}{8}y^5 - \tfrac{9c^3}{16}y^7\end{aligned}$$

and $(X, Y) \mapsto (X - cY, Y)$ of target. With this coordinate change and some adding coordinate change of source, f is equivalent to

$$(x, xy^2 + y^4 + (a_{07} - 2a_{15} + 4a_{23})y^7 + \textstyle\sum_{i+j\geq 8} b'_{ij}x^iy^j).$$

for some b'_{ij}. Hence $a_{07} - 2a_{15} + 4a_{23} \neq 0$ leads to

$$f \sim_{\mathcal{A}} (x, xy^2 + y^4 + y^7).$$

Finally suppose $a_{07} - 2a_{15} + 4a_{23} = 0$; Put $f = (x, xy^2 + y^4 + \sum_{i+j\geq 8} c_{ij}x^iy^j)$ for some c_{ij}. We see

$$(x, xy^2 + y^4 + cy^8 + dy^9) \sim_{\mathcal{A}^9} (x, xy^2 + y^4 + dy^9)$$

by the change of source (writing variables as $\tilde{x}, \tilde{y}$ of the germ in the left hand side)

$$\begin{aligned}\tilde{x} &= x - \tfrac{c}{2}x^2y^2 - \tfrac{c}{2}xy^4 + \tfrac{c^2}{4}x^3y^4 + \tfrac{c^2}{2}x^2y^6 + \tfrac{c^2}{4}xy^8\\ \tilde{y} &= y + \tfrac{c}{4}xy^3 - \tfrac{c}{4}y^5 - \tfrac{c^2}{32}x^2y^5 - \tfrac{c^2}{16}xy^7 - \tfrac{5c^2}{8}y^9\end{aligned}$$

and the change of target

$$(X, Y) \mapsto (X + \tfrac{c}{2}XY, Y).$$

Then it turns out that f is $\mathcal{A}$-equivalent to

$$(x, xy^2 + y^4 + (c_{09} - 2c_{17})y^9 + O(10)).$$

Since $(x, xy^2 + y^4 + y^9)$ is 9-determined, $c_{09} - 2c_{17} \neq 0$ leads to

$$f \sim_{\mathcal{A}} (x, xy^2 + y^4 + y^9).$$

This completes the proof. □

Proof of 3 and 4 in Proposition 3.5 : The proof is similar to that of 2 in Proposition 3.5. We can directly show that

$$(x, xy^2 + y^5 + by^7 + cy^8 + dy^9) \sim_{\mathcal{A}^9} (x, xy^2 + y^5 + (d - \tfrac{5}{6}b^2)y^9)$$

by suitable coordinate changes.

On the other hand

$$(x, xy^2 + y^6 + y^7 + O(8)) \sim_{\mathcal{A}} (x, xy^2 + y^6 + y^7 + \alpha y^9 + O(10))$$

for $\alpha \in \mathbb{R}$, is shown by Rieger in the proof of Lemma 3.2.1:3 in [15]. This completes the proof. □

3.3. **Case 2:** $d\lambda(0) = 0$**,** $\mathrm{rk}H_\lambda(0) = 1$**.** We deal with types 4_k $(k = 3, 4, 5), 16 - 18$ of $\mathcal{A}$-cod $= 4, 5, 6$.

Proposition 3.6. *For a plane-to-plane map-germ f of corank one,*

(1) $j^3 f(0) \sim_{\mathcal{A}^3} (x, y^3) \Longleftrightarrow$

$$d\lambda(0) = 0, \ \ \mathrm{rk}H_\lambda(0) = 1, \ \ \eta^2\lambda(0) \neq 0.$$

(2) *If we write* $f = (x, y^3 + \sum_{i+j\geq 4} a_{ij}x^iy^j)$,

$$a_{31} \neq 0 \quad \Longleftrightarrow \quad f \sim_{\mathcal{A}} (x, y^3 + x^3y) \cdots \boxed{4_3},$$

$$\begin{cases} a_{31} = 0 \\ a_{41} - \frac{1}{3}a_{22}^2 \neq 0 \end{cases} \quad \Longleftrightarrow \quad f \sim_{\mathcal{A}} (x, y^3 \pm x^4y) \cdots \boxed{4_4},$$

$$\begin{cases} a_{31} = a_{41} - \frac{1}{3}a_{22}^2 = 0 \\ a_{51} - \frac{2}{3}a_{32}a_{22} + \frac{1}{3}a_{13}a_{22}^2 \neq 0 \end{cases} \quad \Longleftrightarrow \quad f \sim_{\mathcal{A}} (x, y^3 \pm x^5y) \cdots \boxed{4_5}.$$

(3) $j^4 f(0) \sim_{\mathcal{A}^4} (x, x^2y + y^4) \Longleftrightarrow$

$$d\lambda(0) = 0, \ \ \mathrm{rk}H_\lambda(0) = 1, \ \ \eta^2\lambda(0) = 0, \ \ \eta^3\lambda(0) \neq 0.$$

(4) *If we write* $f = (x, x^2y + y^4 + \sum_{i+j\geq 5} a_{ij}x^iy^j)$,

$$a_{05} \neq 0 \quad \Longleftrightarrow \quad f \sim_{\mathcal{A}} (x, x^2y + y^4 \pm y^5) \cdots \boxed{16},$$

$$a_{05} = 0 \quad \Longleftrightarrow \quad f \sim_{\mathcal{A}} (x, x^2y + y^4) \qquad \cdots \boxed{17}.$$

(5) $j^4 f(0) \sim_{\mathcal{A}^4} (x, x^2y + xy^3)$ *or* $(x, x^2y) \Longleftrightarrow$

$$d\lambda(0) = 0, \ \ \mathrm{rk}H_\lambda(0) = 1, \ \ \eta^2\lambda(0) = \eta^3\lambda(0) = 0.$$

(6) *If we write* $f = (x, x^2y + xy^3 + \sum_{i+j\geq 5} a_{ij}x^iy^j)$,

$$\begin{cases} a_{05} \neq \frac{3}{2}, \frac{9}{5} \\ a_{06}(5a_{05} - 9) - 15a_{14}a_{05} \neq 0 \end{cases}$$

$$\Longleftrightarrow j^7 f(0) \sim_{\mathcal{A}^7} (x, x^2y + xy^3 + \alpha y^5 + y^6 + \beta y^7),$$

and excluding exceptional values of α and β,

$$f \sim_{\mathcal{A}} (x, x^2y + xy^3 + \alpha y^5 + y^6 + \beta y^7) \quad \cdots \boxed{18}.$$

Note that we exclude the type (x, x^2y), because it has codimension 7, while the type 18 has codimension 6.

Proof of 1, 3 and 5 in Proposition 3.6. We can prove these statements by similar way to proof of 1 in Proposition 3.3. □

Proof of 2 in Proposition 3.6. Let

$$f(x, y) = (x, y^3 + \textstyle\sum_{i+j\geq 4} a_{ij}x^iy^j).$$

By a coordinate change of the source plane f is equivalent to

$$(x, y^3 + a_{31}x^3y + \textstyle\sum_{i+j\geq 5} a'_{ij}x^iy^j)$$

for some a'_{ij} where $a'_{41} = a_{41} - \frac{1}{3}a_{22}^2 - \frac{1}{3}a_{13}a_{31}$. Since 4_3-type is 4-determined, $a_{31} \neq 0$ leads to

$$f \sim_{\mathcal{A}} (x, y^3 + x^3y).$$

Suppose $a_{31} = 0$. In entirely the same way as above, f is equivalent to

$$(x, y^3 + (a_{41} - \tfrac{1}{3}a_{22}^2)x^4y + \textstyle\sum_{i+j\geq 6} b_{ij}x^iy^j)$$

for some b_{ij}. Since 4_4-type is 5-determined, $a_{41} - \frac{1}{3}a_{22}^2 \neq 0$ leads to

$$f \sim_{\mathcal{A}} (x, y^3 \pm x^4y).$$

If $a_{41} - \frac{1}{3}a_{22}^2 = 0$, then f is equivalent to

$$(x, y^3 + (a_{51} - \tfrac{2}{3}a_{32}a_{22} + \tfrac{1}{3}a_{13}a_{22}^2)x^5y + O(7)).$$

Since 4_5-type is 6-determined, the claim 2 follows. □

Proof of 4 in Proposition 3.6. Let

$$f(x, y) = (x, x^2y + y^4 + \textstyle\sum_{i+j\geq 5} a_{ij}x^iy^j).$$

Rewrite variables, and substitute

$$\tilde{x} = x$$
$$\tilde{y} = y - \textstyle\sum_{i+j=5,\ i\geq 2} a_{ij}x^{i-2}y^j,$$

then we see that f is equivalent to

$$(x, x^2y + y^4 + a_{14}xy^4 + a_{05}y^5 + O(6)).$$

Now we show

$$(x, x^2y + y^4 + cxy^4 + dy^5) \sim_{\mathcal{A}^5} (x, x^2y + y^4 + dy^5).$$

This is explicitly given by $\tilde{x} = x$ and $\tilde{y} = y - \frac{c}{3}xy$ and the coordinate change of the target:

$$(X, Y) \mapsto \left(X, Y + \tfrac{c}{3}XY + \tfrac{c^2}{9}X^2Y + \tfrac{c^3}{27}X^3Y\right).$$

Since 16 and 17-types are 5-determined, the claim is proved. □

Proof of 6 in Proposition 3.6. At first, for $d \neq \frac{9}{5}$

$$\begin{aligned}&(x, x^2y + xy^3 + cxy^4 + dy^5 + exy^5 + gy^6)\\ &\qquad \sim_{\mathcal{A}^6} (x, x^2y + xy^3 + dy^5 + Pxy^5 + (g - \tfrac{15cd}{5d-9})y^6)\end{aligned}$$

holds where P is a constant. This is also given by some complicated coordinate changes.

Next we see

$$(x, x^2y + xy^3 + dy^5 + exy^5 + gy^6) \sim_{\mathcal{A}^6} (x, x^2y + xy^3 + dy^5 + gy^6)$$

for $d \neq \frac{3}{2}$. This follows from

$$\tilde{x} = x$$
$$\tilde{y} = y + \tfrac{e}{2(3-2d)}xy + \tfrac{e^2}{4(3-2d)^2}x^2y + \tfrac{e^3}{8(3-2d)^3}x^3y - \tfrac{e}{3-2d}y^3 - \tfrac{5e^2}{4(3-2d)^2}xy^3$$

and $(X, Y) \mapsto \left(X, Y + \frac{e}{2(2d-3)}XY\right)$. Now let

$$f(x, y) = (x, x^2y + xy^3 + \textstyle\sum_{i+j\geq 5} a_{ij}x^iy^j).$$

A simple coordinate change shows that f is equivalent to

$$(x, x^2y + xy^3 + a_{14}xy^4 + a_{05}y^5 + Qxy^5 + a_{06}y^6 + O(7))$$

where Q is a constant, and hence by the above argument, f is equivalent to

$$(x, x^2y + xy^3 + a_{05}y^5 + (a_{06} - \tfrac{15a_{14}a_{05}}{5a_{05}-9})y^6 + O(7))$$

for $a_{05} \neq \frac{3}{2}, \frac{9}{5}$. Finally by a similar coordinate change as above again, f is equivalent to

$$(x, x^2y + xy^3 + a_{05}y^5 + (a_{06} - \tfrac{15a_{14}a_{05}}{5a_{05}-9})y^6 + b_{16}xy^6 + b_{07}y^7 + O(8))$$

for some b_{ij}; Then xy^6 in the second component is killed, and moreover, if the coefficient of y^6 is not zero, then f is $\mathcal{A}$-equivalent to

$$(x, x^2y + xy^3 + \alpha y^5 + y^6 + \beta y^7 + O(8))$$

that follows from the same argument as in the proof of Proposition 3.2.2:2 in [15]. For generic values α and β, it implies the claim 6, by the determinacy result. This completes the proof. $\square$

Remark 3.7. 4_k-type can be characterized as follows: For a plane-to-plane map-germ f of corank one,

$$f \sim_{\mathcal{A}} (x, y^3 \pm x^k y) \iff \lambda \text{ is } A_{k-1}\text{-type and } \eta^2\lambda(0) \neq 0.$$

This special feature of 4_k-type would be explained with the augmentation-theory (see [11]). This will be studied somewhere else.

3.4. **Case 3:** $d\lambda(0) = 0$**,** $\mathrm{rk} H_\lambda(0) = 0$**.** Finally we deal with type 19 of $\mathcal{A}$-cod $= 6$.

Proposition 3.8. *For a plane-to-plane map-germ f of corank one,*

(1) $j^4 f(0) \sim_{\mathcal{A}^4} (x, x^3y + \alpha x^2y^2 + y^4)$ *or* $(x, \alpha x^2y^2 + y^4) \iff$

$$d\lambda(0) = 0, \quad \mathrm{rk}\, H_\lambda(0) = 0, \quad \eta^3\lambda(0) \neq 0.$$

(2) (Rieger [15]) *If we write* $f = (x, x^3y + \alpha x^2y^2 + y^4 + \sum_{i+j\geq 5} a_{ij}x^iy^j)$,

$$\Delta = 8\alpha a_{41} - 12a_{32} - 4\alpha^2 a_{23} + 4\alpha a_{14} + (3 + 2\alpha^3)a_{05} \neq 0$$

$$\iff f \sim_{\mathcal{A}} (x, x^3y + \alpha x^2y^2 + y^4 + x^3y^2) \cdots \boxed{19}$$

Note that the $\mathcal{A}$-codimension of $(x, \alpha x^2y^2 + y^4)$ is greater than 6, so we exclude it, while the type 19 has codimension 6. The claim 2 is due to Rieger, that can be seen in the proof of Prop. 3.2.3.1 in [15], and we can show the first claim in the same manner with the previous sections.

4. Application to projection of surface in 3-space

This section is devoted to an application of our criteria to singularities arising in parallel/central projection of a surface in 3-space. Our standing point is to look at this problem as a typical one of *$\mathcal{A}$-recognition problem of plane-to-plane map-germs arising in a concrete geometric setting.*

4.1. **Parallel and central projections.** Let $\iota : M \hookrightarrow \mathbb{R}^3$ be an embedding of smooth surface.

Definition 4.1. *A parallel projection of a smooth surface M to the plane is the restriction to M of a linear orthogonal projection $pr : \mathbb{R}^3 \to \mathbb{R}^2$.*

The direction of orthogonal projection has two dimensional freedom; the space of directions is just the 2-dimensional sphere S^2. There is naturally produced a 2-parameter family of parallel projections, $M \times U \to \mathbb{R}^2$, where U is any small open subset of S^2. Hence a naÏve guess is that any plane-to-plane germs of $\mathcal{A}$-cod ≤ 4 might appear generically in parallel projection of surface M at some points. In fact it is true.

Theorem 4.2 (Arnold [1], Gaffney-Ruas [7, 6], Bruce [3]). *For a generic surface M, singularities arising in parallel projections of M are $\mathcal{A}$-equivalent to the germs of $\mathcal{A}$-cod ≤ 4 in Table 1.*

Remark 4.3. We should remark about what the word "generic" means. A precise statement is as follows: there exists a residual subset of the space of all embeddings of M into $\mathbb{R}^3$ (equipped with C^∞-topology) so that for each element $\iota : M \hookrightarrow \mathbb{R}^3$ of this subset, any parallel projection $pr|_M : M \to \mathbb{R}^2$ admits only singularities of $\mathcal{A}$-cod ≤ 4 listed in Table 1. Below we abuse the word "generic" in the same manner for several similar situations; Perhaps that would not cause any confusion.

Definition 4.4. *A central projection of M from a viewpoint $p = (a, b, c) \in \mathbb{R}^3 - M$ is defined by the restriction*

$$\varphi_p := \pi_p|_M : M \to \mathbb{R}P^2$$

of the canonical surjection on the projective plane

$$\pi_p : \mathbb{R}^3 - \{p\} \to \mathbb{R}P^2, \quad x \mapsto \textit{line generated by } x - p.$$

There is 3-dimensional freedom of the choice of viewpoint p; there is naturally produced a 3-parameter family of central projection, $M \times U \to \mathbb{R}P^2$, where U is any small open subset of the complement $\mathbb{R}^3 - M$. Therefore we might have expected that any plane-to-plane germs of $\mathcal{A}$-cod ≤ 5 would appear in central projection generically. However it is not the case. Arnold and Platonova proved the following remarkable theorem [1, 12]:

Theorem 4.5 (Arnold [1], Platonova [12]). *For a generic surface M, and for any $p \in \mathbb{R}^3$ not lying on M, the germ $\varphi_p : M, x \to \mathbb{R}P^2, \varphi_p(x)$ at any point $x \in M$ is $\mathcal{A}$-equivalent to one of the list of germs with $\mathcal{A}$-cod ≤ 5 in Table 1 except for* $12, 16$ *and unimodal type* 8.

So the three types $12, 16$ and 8 are excluded in the list of singularities arising in central projection of a generic surface, in other words, this geometric setting makes a strong restriction on the appearance of singularities of plane-to-plane germs of $\mathcal{A}$-cod $= 5$. Our criteria are applied to detecting $\mathcal{A}$-types of map-germs arising in this special geometric setting. Then we give not only a new transparent proof of Theorem 4.5 in the context of Rieger's classification but also some extension as stated in the following theorem:

Theorem 4.6. *For a generic one-parameter family of embeddings $M \times I \to \mathbb{R}^3$, $(x, t) \mapsto \iota_t(x)$, the central projection $\pi_p \circ \iota_t : M \to \mathbb{R}P^2$ for any t and any viewpoint*

p admit only $\mathcal{A}$-types with $\mathcal{A}$-cod ≤ 5 *and types* $12, 16, 8, 4_5, 9, 11_9, 13, 17, 19$ *with* $\mathcal{A}$-cod 6. *Namely, each type of* $10, 15, 18$ *with* $\mathcal{A}$-cod 6 *does not appear generically.*

In Rieger [17], parallel projection of moving surfaces with one-parameter has been considered. Theorem 4.6 generalizes it in a much more general form.

4.2. **Proofs of Theorems 4.5 and 4.6.** First we explain the main idea of the proof: This is a slightly modified version of the method which J. W. Bruce used for the proof of Theorem 4.2 in [3]. Now our setting is for central projection. Let (x, y, z) be the coordinates in $\mathbb{R}^3$. Assume that $x_0 = (1, 0, 0) \in M$ and $T_{x_0}M$ is xy-plane, and take viewpoints $p = (a, b, c) \in \mathbb{R}^3$ such that $a \neq 1$.

Then M is locally expressed by its *Monge form* $z = f(x, y)$ centered at $x_0 \in M$, i.e.,

$$M = \{(1 + x, y, f(x, y)) \in \mathbb{R}^3\}$$

as a set-germ at x_0 and $df(0, 0) = 0$. Note that infinitesimal information of M at x_0 can be deduced from the Taylor expansion of f. In particular, the germ at x_0 of the central projection from the viewpoint p,

$$\varphi_{p,f} : M, x_0 \to \mathbb{R}P^2, y_0,$$

$(y_0 = \varphi_{p,f}(y_0))$ is explicitly written by

$$\varphi_{p,f}(x, y) := \left(\frac{y - b}{1 + x - a}, \frac{f(x, y) - c}{1 + x - a}\right)$$

using local coordinates (x, y) of M and $[1 : X : Y]$ of $\mathbb{R}P^2$.

Let V_ℓ be the ℓ-jet space of the Monge form $z = f(x, y)$ at 0 (that is the space of polynomials of degree greater than 1 and less than or equal to ℓ). Also denote by $J^\ell(2, 2)$ the jet space of $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$. We then define $\Phi(p, j^\ell f(0)) \in J^\ell(2, 2)$ to be the ℓ-jet of $\varphi_{p,f}(x, y) - \varphi_{p,f}(0, 0)$ at the origin, and consider the following diagram:

$$\begin{array}{ccc} \mathbb{R}^3 \times V_\ell & \xrightarrow{\Phi} & J^\ell(2,2), \\ {\scriptstyle pr}\downarrow & & \\ V_\ell & & \end{array} \qquad \begin{array}{ccc} (p, j^\ell f(0)) & \xrightarrow{\Phi} & j^\ell \varphi_{p,f}(0). \\ {\scriptstyle pr}\downarrow & & \\ j^\ell f(0) & & \end{array}$$

Note that $J^\ell(2, 2)$ is stratified by $\mathcal{A}^\ell$-orbits (those strata of low codimension are given in Rieger's list). Therefore Φ induces a stratification of $\mathbb{R}^3 \times V_\ell$.

Definition 4.7. *For an* $\mathcal{A}^\ell$*-orbit* $W \subset J^\ell(2, 2)$*, define*

$$G_W := pr(\Phi^{-1}(W)) \quad \subset \quad V_\ell.$$

Since any $\mathcal{A}^\ell$-orbit W is a semi-algebraic subset of $J^\ell(2, 2)$, $\Phi^{-1}(W)$ and hence G_W turns out to be semi-algebraic.

Next we discuss a certain variant of Thom's transversality theorem. First we fix an Euclidean metric of $\mathbb{R}^3$ and its orientation. Suppose that we are given an embedding $M \subset \mathbb{R}^3$ with the unit normal vector field $\boldsymbol{n} : M \to \mathbb{R}^3$. Let U be an open subset of M with unit tangent vector fields $\boldsymbol{v}$ and $\boldsymbol{w}$ so that $\{\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{n}\}$ is an orthonormal frame with respect to the fixed orientation. At each point $q \in U$, linear coordinates x_q, y_q, z_q of $\mathbb{R}^3$ centered at q are chosen to coincide with the oriented lines defined by vectors $\boldsymbol{v}(q), \boldsymbol{w}(q), \boldsymbol{n}(q)$, respectively. Writing U near q as

cod G_W	$\mathcal{A}$-cod	type W
0	0	1
	1	2
	2	3
1	3	$4_2, 5$
	4	4_3
2	4	$6, 11_5$
	5	$4_4, 7, 11_7$
3	5	$12, 16, 8$
	6	$4_5, 9, 11_9, 13, 17, 19$
4	6	$10, 15, 18$

Table 5. Codimension of G_W and $\mathcal{A}$-codimension of W

the Monge form $z_q = f_q(x_q, y_q)$, we associate to each point q the Taylor expansion of f_q truncated to degree ℓ: Then

$$\Theta : U \to V_\ell, \quad \Theta(q) := j^\ell f_q(0)$$

is defined after rewriting the variables such as $x_q = x$, $y_q = y$, $z_q = z$.

Globally, we take an open cover $\{U_i\}$ of M so that for each subscript i, we have $\Theta_i : U_i \to V_\ell$ in the same way just as mentioned. Note that there is the right linear action on V_ℓ of the rotation group $SO(2)$: For $q \in U_i \cap U_j$, the difference between Θ_i and Θ_j at q is only caused by this action. Then, in [3, Thm.1], the following version of transversality theorem is proved:

Proposition 4.8 (Bruce [3])**.** *Let $X \subset V_\ell$ be an $SO(2)$ invariant submanifold. For generic surface M in $\mathbb{R}^3$, $\Theta_i : U_i \to V_\ell$ is transverse to X.*

Obviously, G_W is an $SO(2)$-invariant subset. It immediately implies the following assertion by a standard argument of transversality theorem:

Corollary 4.9. *(1) If codim $G_W \geq 3$, then for a generic embedded surface M, the central projection $\varphi_p : M \to \mathbb{R}P^2$ from any viewpoint p does not admit W-type singularity at any point of M. (2) For a generic s-parameter family of embeddings of M into $\mathbb{R}^3$, any central projection admits only singularities of type W with codim $G_W \leq 2 + s$.*

From Corollary 4.9, our main task for proving Theorem 4.6 is to determine codim G_W for all W in consideration. To do this, we describe explicitly the defining equations of G_W. We obtain the following result:

Proposition 4.10. *Table 5 is the list of* codim G_W *for all the map-germs of $\mathcal{A}$-codim ≤ 6, with ℓ large enough. In addition,* codim $G_W \geq 4$ *holds for all the map-germs of $\mathcal{A}$-*cod ≥ 7.

Theorems 4.5 and 4.6 immediately follow from Proposition 4.10 and Corollary 4.9.

To show Proposition 4.10, we consider the same cases as in the previous section: cases $0, 1, 2, 3$. The proof will be done as follows. From now on we write

$$f(x, y) = \sum_{i+j \geq 2} c_{ij} x^i y^j.$$

For each $\mathcal{A}$-type in Table 1, we will apply our criteria in Chapter 2 to the plane-to-plane germ of the following form

$$\varphi_{p,f}(x,y) = \left(\frac{y-b}{1+x-a}, \frac{\sum c_{ij}x^i y^j - c}{1+x-a}\right).$$

Then we obtain a certain condition in variables

$$a, b, c, c_{20}, c_{11}, c_{02}, c_{30}, c_{21}, \cdots$$

so that $\varphi_{p,f}$ is $\mathcal{A}$-equivalent to the $\mathcal{A}$-type. That is nothing but the condition defining the semi-algebraic subset $\Phi^{-1}(W)$ in $\mathbb{R}^3 \times V_\ell$ for the corresponding $\mathcal{A}^\ell$-orbit $W \subset J^\ell(2,2)$ (with ℓ larger than the determinacy order). The condition consists of polynomial equations and inequalities. Simply we call the (system of) equations the *defining equation of* $\Phi^{-1}(W)$. By eliminating the variables a, b, c from the equation, we obtain the defining equation of G_W. The inequalities do not affect the codimension.

In general the codimension of $\Phi^{-1}(W)$ is equal to that of W, therefore the main task is to check how the projection pr affects the defining equation of G_W.

4.3. **Case 0.** Here we think of the case $d\lambda \neq 0$. This case automatically implies $\eta\lambda(0) = 0$ in common other than 2-type. By this condition, it follows that

$$(1-a)^2 c_{20} + (1-a)(-b)c_{11} + (-b)^2 c_{02} = 0$$

and $\det \mathrm{Hess} f(0) < 0$, that is, f is hyperbolic at the origin. Hence from now on we assume that

$$c_{20} = c_{02} = 0, \quad \text{and} \quad c_{11} \neq 0$$

by taking a suitable coordinate x, y via $SO(2)$-action. With this condition, our calculations become much easier.

Let us determine the defining equation of $\Phi^{-1}(W)$, G_W and codim G_W for each $\mathcal{A}^\ell$-orbit W.

$\boxed{2}$ $\Phi^{-1}(W)$ is given by $c = 0$. Therefore there is no defining equation for G_W. Hence codim $G_W = 0$.

$\boxed{3}$ $\Phi^{-1}(W)$ is given by $b = c = 0$. Therefore there is no defining equation for G_W. Hence codim $G_W = 0$.

$\boxed{5}$ $\Phi^{-1}(W)$ is given by $b = c = 0$ and $c_{30} = 0$. Then G_W is given by $c_{30} = 0$. Hence codim $G_W = 1$.

$\boxed{6}$ $\Phi^{-1}(W)$ is given by $b = c = 0$ and $c_{30} = c_{40} = 0$. Then G_W is given by $c_{30} = c_{40} = 0$. Hence codim $G_W = 2$.

$\boxed{7}$ $\Phi^{-1}(W)$ is given by $b = c = 0$, $c_{30} = c_{40} = 0$ and $A(1-a)^2 + B(1-a) + C = 0$ where A, B, C are some polynomials in c_{ij}. We get this equation from the additional conditions of Taylor expansions: $a_{07} - \frac{5}{8}a_{06}^2 = 0$ in our criterion for 7-type (see Proposition 3.3): Under the condition

$$d\lambda(0) \neq 0, \eta\lambda(0) = \eta^2\lambda(0) = \eta^3\lambda(0) = 0, \eta^4\lambda(0) \neq 0,$$

$\varphi_{p,f}$ is $\mathcal{A}$-equivalent to $(x, xy + y^5 + \sum_{i+j\geq 6} a_{ij}x^iy^j)$ where

$$a_{06} = \frac{1}{c_{11}c_{50}(1-a)}\left\{(-5c_{21}c_{50} + 6c_{11}c_{60})(1-a) - c_{11}c_{50}\right\},$$

and

$$a_{07} = \frac{1}{c_{11}^2c_{50}(1-a)^2}\left\{\begin{array}{l}(20c_{21}^2c_{50} - 5c_{11}c_{31}c_{50} - 6c_{11}c_{21}c_{60} + c_{11}^2c_{70})(1-a)^2 \\ +(6c_{11}c_{21}c_{50} - c_{11}^2c_{60})(1-a) + c_{11}^2c_{50}\end{array}\right\}.$$

(See Proof of 1 in Proposition 3.3.) Here $a_{07} - \frac{5}{8}a_{06}^2 = 0$ gives an equation in the variable a with A, B, C depending only on c_{ij}:

$$A(1-a)^2 + B(1-a) + C = 0.$$

In addition, A (also B, C) is independent from the other two equations $c_{30} = c_{40} = 0$; for instance, we see that there is a monomial $c_{11}c_{31}c_{50}^2$ in A. The variable a is solved in c_{ij} generically; the locus in V_ℓ where a is not solved is defined by $A = B = C = 0$, but it has high codimension, so this quadratic equation does not affect codim G_W. Thus, G_W is given by $c_{30} = c_{40} = 0$. Hence codim $G_W = 2$.

Remark 4.11. For a generic surface, hyperbolic points where the Monge form satisfies that $c_{30} = c_{40} = 0$ are isolated, since G_W has codimension 2 in V_ℓ. Look at such a point of the surface from a viewpoint lying on the a-axis ($b = c = 0$), then the central projection produces the butterfly singularity (6-type). However, there is an exception: from at most two points on the a-axis which are given by the solution $a = a(c_{ij})$ of the quadric equation, the central projection admits the elder-butterfly singularity (7-type). These exceptional points are called *h-focal points* ("h" for "hyperbolic") by Platonova [12].

$\boxed{8}$ $\Phi^{-1}(W)$ is given by $b = c = 0$ and $c_{30} = c_{40} = c_{50} = 0$. Then G_W is given by $c_{30} = c_{40} = c_{50} = 0$. Hence codim $G_W = 3$. The difference between 7-type and 8-type (although they have the same $\mathcal{A}$-codimension) is the difference between closed conditions $a_{07} - \frac{5}{8}a_{06}^2 = 0$ and $\eta^4\lambda(0) = 0$. As mentioned in Remark 4.11, the former condition on coefficients determines the position of viewpoint, while the geometric condition $\eta^4\lambda(0) = 0$ is that of the Monge form. Also in the following other calculations, this kind of difference makes the difference of codim G_W.

$\boxed{9}$ $\Phi^{-1}(W)$ is given by $b = c = 0$, $c_{30} = c_{40} = c_{50} = 0$, $A(1-a)^2 + B(1-a) + C = 0$ for some polynomials A, B, C in c_{ij}. The last equation comes from the condition $a_{08} - \frac{3}{5}a_{07}^2 = 0$ in our criterion (Proposition 3.3) in the similar way as the case of no. 7 (e.g., A contains the monomial $c_{11}c_{31}c_{60}^2$, so it is independent from other three equations). Then G_W is given by $c_{30} = c_{40} = c_{50} = 0$. Hence codim $G_W = 3$.

$\boxed{10}$ $\Phi^{-1}(W)$ is given by $b = c = 0$, $c_{30} = c_{40} = c_{50} = c_{60} = 0$. Then G_W is given by $c_{30} = c_{40} = c_{50} = c_{60} = 0$. Hence codim $G_W = 4$.

4.4. **Case 1.** Here we think of the case that $d\lambda(0) = 0$ and $H_\lambda(0)$ is non-degenerate. Note that we can always assume that $c_{11} = 0$ by taking a suitable rotation of xy-plane. Then the condition $\frac{\partial}{\partial x}\lambda(0) = \frac{\partial}{\partial x}\lambda(0) = 0$ leads to

$$(1-a)c_{20} = 0, \quad b\, c_{02} = 0,$$

i.e., $c_{20} = b = 0$ or $c_{20} = c_{02} = 0$, hence f is parabolic or umbilic (flat) at the origin, while $\mathrm{rk}\, H_\lambda(0) = 2$ leads to an open condition. Note that the locus consisting of umbilic Monge forms is defined by vanishing the 2-jet, so it has codimension 3 in V_ℓ. Below, for the simplicity, we treat only with the case that $c_{20} = b = 0$ (i.e., parabolic forms); In fact, taking $c_{02} = 0$ instead of $b = 0$ (i.e., considering umbilic forms) does not affect the codimension of G_W as seen in 11_5-type.

$\boxed{4_2}$ $\Phi^{-1}(W)$ is given by $b = c = 0$ and $c_{20} = 0$. Then G_W is given by $c_{20} = 0$. Hence codim $G_W = 1$. That is, the locus of parabolic Monge forms has codimension 1 in V_ℓ.

$\boxed{11_5}$ $\Phi^{-1}(W)$ is given by $b = c = 0$ and $c_{20} = c_{30} = 0$. Then G_W is given by $c_{20} = c_{30} = 0$. Hence codim $G_W = 2$.

Precisely saying, the component of G_W having parabolic Monge forms has codimension 2 in V_ℓ, where $\eta^2\lambda(0) = 0$ leads to the extra equation $c_{30} = 0$. If we take $c_{02} = 0$ instead of $b = 0$, i.e., we consider the umbilic Monge form, then $\eta^2\lambda(0) = 0$ implies an equation of a, b and c_{ij} $(i + j = 3)$, instead of $c_{30} = 0$. So we can generically solve a or b in c_{ij}. Thus the component of G_W having umbilic Monge forms remains to be of codimension 3 in V_ℓ and it does not affect the codimension of G_W. This argument is also valid in the following other types, so we will not repeat it below.

$\boxed{11_7}$ $\Phi^{-1}(W)$ is given by $b = c = 0$, $c_{20} = c_{30} = 0$, $A(1 - a) + B = 0$ for some polynomials A, B in c_{ij}. Then G_W is given by $c_{20} = c_{30} = 0$. Hence codim $G_W = 2$.

Remark that the equation $A(1-a)+B = 0$ arises from the condition $a_{05} = 0$ in our criterion (see Proposition 3.5) after rewriting $\varphi_{p,f}$ to be $(x, xy^2+\sum_{i+j\geq 5} a_{ij}x^iy^j)$ by an explicit coordinate change.

Remark 4.12. As seen in Remark 4.11, we also have an exceptional point here. We look at parabolic points on the surface where $c_{20} = c_{30} = 0$ from a viewpoint lying on the a-axis ($b = c = 0$), that is the unique asymptotic line. Then the gulls singularity (11_5-type) appears on the line except for the point $(a, 0, 0)$ where a is given by $A(1 - a) + B = 0$. This exceptional point is called *p-focal* point ("p" for parabolic) by Platonova [12], and at this point the ugly-gulls singularity (11_7-type) appears.

$\boxed{11_9}$ $\Phi^{-1}(W)$ is given by $b = c = 0$, $c_{20} = c_{30} = 0$, $A(1 - a) + B = 0$, $a_{07} - 2a_{15} + 4a_{23} = 0$ where a_{ij} are functions in c_{ij}'s and a obtained in entirely the same way as the above case 11_7. Solve the variable a by $A(1 - a) + B = 0$, and then the last equation yields a non-trivial equation, say $C(c_{ij}) = 0$, which is independent from other equations. Therefore G_W is given by $c_{20} = c_{30} = 0$ and $C(c_{ij}) = 0$. Hence codim $G_W = 3$.

$\boxed{12}$ $\Phi^{-1}(W)$ is given by $b = c = 0$, $c_{20} = c_{30} = c_{40} = 0$. Then G_W is given by $c_{20} = c_{30} = c_{40} = 0$. Hence codim $G_W = 3$.

$\boxed{13}$ $\Phi^{-1}(W)$ is given by $b = c = 0$, $c_{20} = c_{30} = c_{40} = 0$, $A(1-a)+B = 0$ for some polynomials A, B in c_{ij}. Here the last equation comes from the condition $a_{06} = 0$

in our criterion (see Proposition 3.5). Then G_W is given by $c_{20} = c_{30} = c_{40} = 0$. Hence codim $G_W = 3$.

$\boxed{15}$ $\Phi^{-1}(W)$ is given by $b = c = 0$, $c_{20} = c_{30} = c_{40} = c_{50} = 0$. Then G_W is given by $c_{20} = c_{30} = c_{40} = c_{50} = 0$. Hence codim $G_W = 4$.

4.5. **Case 2.** Here we think of the case $d\lambda(0) = 0$ and $H_\lambda(0)$ is degenerate (rank $=$ 1). Since $d\lambda(0) = 0$, as seen in case 1, we can assume $c_{11} = 0$. Remark again that in this condition, $d\lambda(0) = 0$ leads to $c_{20} = b = 0$ and

$$H_\lambda(0) = -\frac{1}{(1-a)^3}\begin{pmatrix} 3(1-a)c_{30} & (1-a)c_{21} \\ (1-a)c_{21} & (1-a)c_{12} + c_{02} \end{pmatrix}.$$

Then $\det H_\lambda(0) = 0$ leads to

$$(3c_{30}c_{12} - c_{21}^2)(1-a) + 3c_{02}c_{30} = 0.$$

Write it by $C(1-a) + D = 0$.

$\boxed{4_3}$ $\Phi^{-1}(W)$ is given by $b = c = 0$, $c_{20} = 0$, $C(1-a) + D = 0$. Then G_W is given by $c_{20} = 0$. Hence $\operatorname{codim} G_W = 1$. Here $C(1-a) + D = 0$ does not affect codim G_W because the condition $C = 0$ increases the codimension.

Remark 4.13. The lips and beaks singularities arise on the unique asymptotic line at a parabolic point; there is one exceptional point on the line, given by $C(1-a) + D = 0$, where the goose singularity appears. This point divides the line into two half lines, each of which corresponds to viewpoints for either the lips singularity or the beaks singularity of the projection.

$\boxed{4_4}$ $\Phi^{-1}(W)$ is given by $b = c = 0$, $c_{20} = 0$, $C(1-a) + D = 0$, $a_{31} = 0$. Here a_{31} is a function in c_{ij} obtained by rewriting $\varphi_{p.f}$ to be the suitable form as in Proposition 3.6. Solve a by $C(1-a) + D = 0$ and substitute it into a_{31}, then denote the resulting equation by $A(c_{ij}) = 0$, that is independent from others. Then G_W is given by $c_{20} = 0$ and $A(a_{ij}) = 0$. Hence codim $G_W = 2$.

$\boxed{4_5}$ $\Phi^{-1}(W)$ is given by $b = c = 0$, $c_{20} = 0$, $C(1-a)+D = 0$, $a_{31} = a_{41} - \frac{1}{3}a_{22}^2 = 0$. Here a_{ij} are obtained by Proposition 3.6. Then G_W is given by $c_{20} = 0$ and $A(c_{ij}) = B(c_{ij}) = 0$. Hence codim $G_W = 3$. $A(c_{ij})$ is the equation $a_{31} = 0$, and $B(c_{ij}) = 0$ is the equation $a_{41} - \frac{1}{3}a_{22}^2 = 0$ after the substitution of a.

$\boxed{16}$ $\Phi^{-1}(W)$ is given by $b = c = 0$, $c_{20} = c_{30} = c_{21} = 0$. Then G_W is given by $c_{20} = c_{30} = c_{21} = 0$. Hence codim $G_W = 3$. $c_{30} = c_{21} = 0$ come from $\eta^2\lambda(0) = 0$ and $\det H_\lambda(0) = 0$.

$\boxed{17}$ $\Phi^{-1}(W)$ is given by $b = c = 0$, $c_{20} = c_{30} = c_{21} = 0$, $A(1-a)^2 + B(1-a) + C = 0$. Then G_W is given by $c_{20} = c_{30} = c_{21} = 0$. Hence codim $G_W = 3$. Here A, B, C come from the condition $a_{05} = 0$ (see Proposition 3.6). Hence this quadratic equation does not affect codim G_W.

$\boxed{18}$ $\Phi^{-1}(W)$ is given by $b = c = 0$, $c_{20} = c_{30} = c_{21} = c_{40} = 0$. Then G_W is given by $c_{20} = c_{30} = c_{21} = c_{40} = 0$. Hence codim $G_W = 4$.

4.6. **Case 3.** Here we think of the case $d\lambda(0) = 0$ and $H_\lambda(0) = O$. In fact, the germ of $\mathcal{A}$-codim= 6 in (case 3)-singularities is just 19-type. As seen in the previous sections, we can also suppose $c_{11} = 0$.

$\boxed{19}$ $\Phi^{-1}(W)$ is given by $b = c = 0$, $c_{20} = c_{30} = c_{21} = 0$ and $(1-a)c_{12} + c_{02} = 0$. Then G_W is given by $c_{20} = c_{30} = c_{21} = 0$. Hence codim $G_W = 3$.

4.7. **Types of $\mathcal{A}$-codimension ≥ 7.** We prove the second claim in Proposition 4.10. In Rieger's recognition trees, terminating lines lead to $\mathcal{A}$-orbits of $\mathcal{A}$-cod ≥ 7. All jets in $J^\ell(2,2)$ indicated by terminating lines satisfy closed conditions obtained by replacing some inequalities in our criteria by equalities. For instance, let us look at an orbit W with $\mathcal{A}$-cod ≥ 7 over 5-jets of $\mathcal{A}$-orbit of (x, y^3). Any jet $(x, y^3 + \sum_{\ell \geq i+j \geq 4} a_{ij} x^i y^j)$ belonging to W satisfies the following closed conditions obtained from the criterion (2) in Proposition 3.6 :

$$d\lambda(0) = 0, \ \ \mathrm{rk} H_\lambda(0) = 1,$$
$$a_{31} = a_{41} - \tfrac{1}{3}a_{22}^2 = a_{51} - \tfrac{2}{3}a_{32}a_{22} + \tfrac{1}{3}a_{13}a_{22}^2 = 0.$$

Namely, the last equation $a_{51} - \cdots = 0$ is added to the condition for 4_5-type just described above. Then, by calculation, one can get codim $G_W \geq 4$. It is similar for other cases. □

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(Y. Kabata) Department of Mathematics, Graduate School of Science, Hokkaido University, Sapporo 060-0810, Japan

E-mail address: kabata@math.sci.hokudai.ac.jp