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**A lower bound theorem for Ehrhart polynomials
of convex polytopes**

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Let $\mathcal{P} \subset \mathbb{R}^N$ be an *integral* convex polytope, i.e., a convex polytope any of whose vertices has integer coordinates, of dimension d and $\partial\mathcal{P}$ the boundary of \mathcal{P} . Given a positive integer n we write $i(\mathcal{P}, n)$ for the number of those rational points $(\alpha_1, \alpha_2, \dots, \alpha_N)$ in \mathcal{P} such that each $n\alpha_i$ is an integer. In other words,

$$i(\mathcal{P}, n) = \#(n\mathcal{P} \cap \mathbb{Z}^N).$$

Here $n\mathcal{P} := \{n\alpha; \alpha \in \mathcal{P}\}$ and $\#(X)$ is the cardinality of a finite set X . The systematic study of $i(\mathcal{P}, n)$ originated in the work of Ehrhart (cf. [Ehr]), who established that the function $i(\mathcal{P}, n)$ possesses the following fundamental properties :

(0.1) $i(\mathcal{P}, n)$ is a polynomial in n of degree d .

(Thus $i(\mathcal{P}, n)$ can be defined for *every* integer n .)

(0.2) $i(\mathcal{P}, 0) = 1$.

(0.3) ("loi de réciprocité") $(-1)^d i(\mathcal{P}, -n) = \#(n(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N)$
for every integer $n > 0$.

We say that $i(\mathcal{P}, n)$ is the *Ehrhart polynomial* of \mathcal{P} . See, e.g., [Sta₃, pp.235-241] for an introduction to Ehrhart polynomials.

We define the sequence $\delta_0, \delta_1, \delta_2, \dots$ of integers by the formula

$$(1 - \lambda)^{d+1} \left[1 + \sum_{n=1}^{\infty} i(\mathcal{P}, n) \lambda^n \right] = \sum_{i=0}^{\infty} \delta_i \lambda^i. \quad (1)$$

Then, the basic facts (0.1) and (0.2) on $i(\mathcal{P}, n)$ together with a fundamental result on generating functions, e.g., [Sta₃, Corollary (4.3.1)] guarantee that $\delta_i = 0$ for every $i > d$. We say that the sequence $\delta(\mathcal{P}) := (\delta_0, \delta_1, \dots, \delta_d)$ which appears in Eq. (1) is the δ -vector of \mathcal{P} . Thus $\delta_0 = 1$ and $\delta_1 = \#(\mathcal{P} \cap \mathbb{Z}^N) - (d + 1)$. On the other hand, it follows easily from (0.3) that $\delta_d = \#((\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N)$. Moreover, each δ_i is non-negative [Sta₁].

Now, our result in this paper is

(1.1) THEOREM. Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d with the δ -vector $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ and suppose that $(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N$ is non-empty, i.e., $\delta_d \neq 0$. Then we have the inequality $\delta_1 \leq \delta_i$ for every $1 \leq i < d$.

The proof of Theorem (1.1) relies on the following two well-known facts. See, e.g., [Sta₂] for fundamental definitions and results concerning f -vectors and h -vectors of triangulations of balls and spheres.

(1.2) LEMMA (cf. [B-M]). Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d with the δ -vector $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$. Also, let Δ be a triangulation of \mathcal{P} with the vertex set $\mathcal{P} \cap \mathbb{Z}^N$ whose h -vector is $h(\Delta) = (h_0, h_1, \dots, h_d, h_{d+1})$. (Thus $h_1 = \delta_1$, $h_d = \delta_d$ [Sta₂, pp.80-81] and $h_{d+1} = 0$ [Sta₂, p.67].) Then $h_i \leq \delta_i$ for every $0 \leq i \leq d$.

(1.3) LEMMA ([Bar₁], [Bar₂]). Let $h(\Delta) = (h_0, h_1, \dots, h_d)$ be the h -vector of a triangulation of the boundary $\partial\mathcal{P}$ of a convex polytope \mathcal{P} of dimension d . (Thus, in particular, $h_i = h_{d-i}$ for each $0 \leq i \leq d$ [Sta₂, p.77].) Then $h_1 \leq h_i$ for every $1 \leq i < d$.

We are now in the position to give a proof of Theorem (1.1). Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d and suppose that $(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N$ is non-empty, say $(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N = \{v_1, v_2, \dots, v_\ell\}$. First, we take any triangulation $\Delta(0)$ of the boundary $\partial\mathcal{P}$ of \mathcal{P} with the vertex set $\partial\mathcal{P} \cap \mathbb{Z}^N$, and then we construct a triangulation $\Delta(j)$ of \mathcal{P} with the vertex set $(\partial\mathcal{P} \cap \mathbb{Z}^N) \cup \{v_1, \dots, v_j\}$ for each $1 \leq j \leq \ell$ in the following way:

(i) Define $\Delta(1)$ to be the triangulation of \mathcal{P} which consists of those simplices $\sigma \subset \mathcal{P}$ such that σ is the convex hull of $\tau \cup \{v_1\}$ in \mathbb{R}^N for some $\tau \in \Delta(0)$, i.e., $\Delta(1)$ is the cone over $\Delta(0)$ with apex v_1 .

(ii) If $\Delta(j)$ is constructed and $1 \leq j < \ell$, then let $\tau(j) \in \Delta(j)$ be the smallest face which contains v_{j+1} and write τ' for the subdivision of $\tau(j)$ which is the cone over the boundary $\partial\tau(j)$ of $\tau(j)$ with apex v_{j+1} . Also, let $\text{link}_{\Delta(j)}(\tau(j))$ (resp. $\text{star}_{\Delta(j)}(\tau(j))$) be the link [Sta₂, p.70] (resp. star [Sta₂, p.72]) of $\tau(j)$ in $\Delta(j)$. We then define $\Delta(j+1)$ to be the triangulation of \mathcal{P} which consists of those simplices $\sigma \subset \mathcal{P}$ such that σ is either (a) the convex hull of $\zeta \cup \xi$ in \mathbb{R}^N for some $\zeta \in \tau'$ and $\xi \in \text{link}_{\Delta(j)}(\tau(j))$ or (b) $\sigma \in \Delta(j) - \text{star}_{\Delta(j)}(\tau(j))$.

We now investigate the relation between the h -vector of $\Delta(j)$ and that of $\Delta(j+1)$ for each $0 \leq j < \ell$. We write $(h_0^{(j)}, h_1^{(j)}, h_2^{(j)}, \dots)$ for the h -vector $h(\Delta(j))$ of $\Delta(j)$. Since $\Delta(1)$ is the cone over $\Delta(0)$ with apex v_1 , we know that $h_i(0) = h_i(1)$ for each $0 \leq i \leq d$. Thus, by virtue of Lemma (1.3), we have $h_1^{(1)} \leq h_i^{(1)}$ for every $1 \leq i < d$. Now, let $j \geq 1$ and suppose that $h_1^{(j)} \leq h_i^{(j)}$ for every $1 \leq i < d$. Let v be a vertex of $\tau(j)$ and set $\Delta^* = \text{star}_{\Delta(j+1)}(\{v_{j+1}\})$. Then we easily see

$$f_i(\Delta(j+1)) = f_i(\Delta(j)) + f_{i-1}(\text{link}_{\Delta^*}(\{v\})) \quad (2)$$

for every $1 \leq i \leq d$. Here, e.g., $(f_0(\Delta(j)), f_1(\Delta(j)), f_2(\Delta(j)), \dots)$ is the f -vector of Δ_j , i.e., $f_i(\Delta(j))$ is the number of i -dimensional simplices of $\Delta(j)$. On the other hand, we set

$$\rho := \text{link}_{\Delta(j+1)}(\{v\}) \cap \text{link}_{\Delta(j+1)}(\{v_{j+1}\}).$$

Then ρ is a triangulation of a $(d-2)$ -sphere and $\text{link}_{\Delta^*}(\{v\})$ is the cone over ρ with apex v_{j+1} . Let $h(\rho) = (h_0, h_1, \dots, h_{d-1})$ be the h -vector of ρ . It follows from Eq. (2) that

$$h_i(j+1) = h_i(j) + h_{i-1}$$

for each $1 \leq i \leq d$. Hence $h_1(j+1) \leq h_1(j+1)$ for every $1 \leq i < d$ because of each $h_i \geq h_0 (= 1)$.

Thanks to Lemma (1.2), the δ -vector $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ of \mathcal{P} satisfies $h_i(\ell) \leq \delta_i$ for every $0 \leq i \leq d$. On the other hand, we know $h_i(\ell) \geq h_1(\ell) (= \delta_1)$ for each $1 \leq i < d$, thus we have the inequality $\delta_1 \leq \delta_i$ for every $1 \leq i < d$ as required. Q.E.D.

(1.4) REMARK. (a) When $\tau(j) \in \Delta(j)$ is a facet (maximal face) of $\Delta(j)$, then $h(\rho) = (1, 1, \dots, 1)$. This fact immediately shows that if $(h_0, h_1, \dots, h_d, 0)$ is the h -vector of a triangulation of a d -ball, then $(h_0, h_1+1, h_2+1, \dots, h_d+1, 0)$ is also the h -vector of a triangulation of a d -ball. Thus, in particular, given positive integers d and n , there exists a triangulation Δ of a d -ball with the h -vector $h(\Delta) = (1, n, 1, \dots, 1, 0) \in \mathbb{Z}^{d+2}$.

(b) Let Δ be a triangulation of a d -ball and suppose that each facet possesses a vertex contained in the interior $\Delta - \partial\Delta$ of Δ . Then the h -vector $h(\Delta) = (h_0, h_1, \dots, h_d, 0)$ of Δ satisfies the linear inequality

$$h_0 + h_1 + \dots + h_i \leq h_d + h_{d-1} + \dots + h_{d-i}$$

for every $0 \leq i \leq [d/2]$.

(c) A technique similar to what was done in the proof of Theorem (1.1) enables us to obtain the linear inequalities

$$\delta_{d-1} + \delta_{d-2} + \dots + \delta_{d-i} \leq \delta_2 + \delta_3 + \dots + \delta_i + \delta_{i+1},$$

$0 \leq i \leq [(d-1)/2]$, for the δ -vector $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ of an arbitrary integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$ of dimension d . In particular, $\delta_{d-1} \leq \delta_2$, however, [Sta5, Example (3.4)] shows that unfortunately we cannot expect $\delta_{d+1-i} \leq \delta_i$ when $2 < i$.

It would, of course, be of great interest to find a combinatorial characterization of the δ -vectors of integral convex polytopes.

We refer the reader to [Sta₄, (3.4)], [Sta₄, (4.1)], [Sta₅, (3.3)] for further information about δ -vectors (= h^* -vectors). Also, see [H₁], [H₂], and [H₃].

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