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R. Agemi and H. Takamura

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The lifespan of classical solutions to nonlinear wave equations in two space dimensions

Rentaro Agemi and Hiroyuki Takamura

We consider, in two space dimensions, the initial value problems for $\partial_t^2 u - \Delta u = A|u|^p$ with small initial data of compact support. It is known that a global C^2- solution exists if $p > p_0(2) = (3+\sqrt{17})/2$ and the lifespan of a C^2- solution is finite if $1 < p \le p_0(2)$. In this paper we look for upper and lower bounds for the lifespan of C^2- solution. We also study the lifespan in the case where the support of initial data is non compact.

§1 Introduction

In the present paper we study the lifespan of solutions to initial value problems for nonlinear wave equations of the form

$$
(1.1) \qquad \begin{aligned} &\partial_t^2 u(x,t) - \Delta u(x,t) = A|u(x,t)|^p, \quad (x,t) \in \mathbf{R}^n \times [0,\infty), \\ &u(x,0) = f(x), \ \partial_t u(x,0) = g(x), \quad x \in \mathbf{R}^n, \end{aligned}
$$

where p and A are positive constants and $n = 2, 3$.

F. John [6] has proved the following remarkable results in three space dimensions. The global classical solution to (1.1) exist for small initial data with compact support provided $p > p_0(3) = 1+\sqrt{2}$, and the lifespan of classical solution to (1.1) is finite provided $1 < p < p_0(3)$, $f = 0$, $g \ge 0$ (also see F. John [7], p.32). Here $p_0(n)$ stands for the positive root of the quadratic $q(n,p) = 0$ where

$$
(1.2) \qquad q(n,p) = \frac{n-1}{2}p^2 - \frac{n+1}{2}p - 1
$$

and the lifespan of a solution u to (1.1) means the largest T such that $u(x,t) \in C^2(\mathbb{R}^n \times [0,T))$. He also proved in [6] that the lifespan T_ε of solutions to (1.1) with $f(x) = \varepsilon\varphi(x)$ and $g(x) = \varepsilon\psi(x)$ is equivalent to ε^{-2} for $p = 2$. Recently H. Lindblad [9] has refined this result by showing that the following limit exists for $1 < p < p_0(3)$:

$$\lim_{\varepsilon \to +0} \varepsilon^{-p(p-1)/q(3,p)} T_\varepsilon.$$

R. T. Glassey [3] has proved in two space dimensions that the global solution to (1.1) exists for small initial data with compact support provided $p > p_0(2)$. R. T. Glassey [4] proved that if $1 < p < p_0(2)$ then the lifespan of a solution to (1.1) is finite. Moreover, J. Schaeffer [10] proved that the lifespan is finite for critical values $p = p_0(2)$ and $p_0(3)$.

The main aim of this paper is to look for the upper and lower bounds for the lifespan in two space dimensions.

THEOREM 1. *Let u_ε be a C^2- solution to (1.1) with initial data $f(x) = \varepsilon\varphi(x)$ and $g(x) = \varepsilon\psi(x)$, where $\varepsilon > 0$, $\varphi \in C_0^3(\mathbb{R}^2)$ and $\psi \in C_0^2(\mathbb{R}^2)$. Then there exist positive constants ε_0 and C depending only on p, A, φ and ψ such that the lifespan T_ε of u_ε satisfies the following inquality for $0 < \varepsilon < \varepsilon_0$:*

$$\begin{aligned}
&T_\varepsilon \geq \exp(C\varepsilon^{-(p-1)}) && \textit{for} \quad p = p_0(2),\\
&\varepsilon^{p-1} T_\varepsilon^{-q(2,p)/p} \log T_\varepsilon \geq C && \textit{for} \quad 2 \leq p < p_0(2),\ p \neq 3,\\
&\varepsilon^2\, T_\varepsilon^{1/3} (\log T_\varepsilon)^2 \geq C && \textit{for} \quad p = 3.
\end{aligned}$$

THEOREM 2. *Let u_ε be a C^2- solution to (1.1) with initial data $f(x) = 0$ and $g(x) = \varepsilon\psi(x)$, where $\varepsilon > 0, \psi(x) \geq 0$ and $\psi \in C^2(\mathbb{R}^2)$. Then there exists a positive constant C depending only on p, A and ψ such that the lifespan T_ε of u_ε has an upper bound:*

$$T_\varepsilon \leq C\varepsilon^{p(p-1)/q(2,p)} \qquad \textit{for} \quad 1 < p < p_0(2).$$

We give here some remarks. Firstly, H. Lindblad [9] has proved for $p = 2$ that if $\int_{\mathbf{R}^2} \psi(x)dx = 0$ then $\lim_{\varepsilon \to 0} \varepsilon T_\varepsilon$ exists and if $\int_{\mathbf{R}^2} \psi(x)dx \neq 0$ then $\lim_{\varepsilon \to 0} a(\varepsilon)^{-1} T_\varepsilon$ exists, where $a(\varepsilon)$ is defined by $a(\varepsilon)^2 \varepsilon^2 \log(a(\varepsilon) + 1) = 0$. His results are much sharper than ours. However, his method is not applicable to the case where $3 \leq p \leq p_0$ (2) (See [9], Lemma 8.8). Secondly, making use of the method proving Theorem 1 or the one in K. Kubota [8], we can prove simply the existence of global solutions in R. T. Glassey [3] (see Appendix). Finally the assumption of Theorem 2 does not require that the support of initial data is compact.

When the supports of initial data is non compact and $p > p_0(3)$, F. Asakura [2] has proved in three space dimensions the following results. Let $D^\alpha f(x)$, $D^\beta g(x)$ $= O(|x|^{-1-\kappa})$ as $|x| \to \infty$ $(|\alpha| \leq 3, |\beta| \leq 2)$. Then the global solution to (1.1) exists for small initial data provided $\kappa > 2/(p-1)$. Moreover, he also proved that the lifespan of a solution to (1.1) is finite if $0 < \kappa < 2/(p-1)$ and initial data satisfy

$$f(x) = 0 \quad \text{and} \quad g(x) \geq \varepsilon(1 + |x|)^{-1-\kappa}. \tag{1.3}$$

The next aim of this paper is to show that, in two space dimensions, the lifespan is finite under the same assumption above. For global existence of solutions, see K. Kubota [8].

THEOREM 3. *Let u_ε be a C^2- solution to (1.1) with initial data satisfying (1.3). Then there exists a positive constant C depending only on A, p and κ such that the lifespan T_ε of u_ε has an upper bound:*

$$T_\varepsilon \leq C\varepsilon^{(\kappa - \frac{2}{p-1})^{-1}} \quad \textit{for} \quad p > 1 \quad \textit{and} \quad 0 < \kappa < \frac{2}{p-1}.$$

For the relations between the power of ε in Theorem 2 and the one in Theorem 3, see K. Kubota [8].

In § 2, we define the norm to be used and formulate a priori estimate which plays an important role in the proof of the existence theorem. Making use of a priori estimate, we prove Theorem 1, employing the iteration method in F. John [6]. In §3, we prove a priori estimate mentioned above. Theorem 2 will be proved in §4 by making use of the methods in F. John [6] and R. Agemi [1]. Theorem 3 will be also proved in §5 by the same method as in §4. In Appendix, we give a simple proof of the global existence theorem for $p > p_0(2)$.

We close the introduction, pointing out that the results similar to Theorem 3 are proved in the different way from us by K. Tsutaya's preprint "Global existence theorem for semilinear wave equations with non compact data in two space dimensions."

§2 Proof of Theorem 1

Let initial data $f(x) = \varepsilon\varphi(x)$ and $g(x) = \varepsilon\psi(x)$ be supported in $|x| < k$. Then we find from [7], Appendix (also see [1]) that a C^2- solution u to (1.1) with $p \geq 1$ is unique and

$$u(x,t) = 0 \qquad \text{for} \qquad |x| > t + k. \tag{2.1}$$

Considering this fact, we define the norm for a continuous function $u(x,t)$ in $Q_T = \mathbb{R}^2 \times [0,T)$ satisfying (2.1):

$$||u|| = \sup_{(x,t)\in Q_T} N(|x|,t)|u(x,t)|, \tag{2.2}$$

where

$$N(r,t) = k^{-(p+2)/2p}(t+r+2k)^{1/2}(t-r+2k)^{1/p}, \ 2 \leq p \leq p_0(2). \tag{2.3}$$

As is well known, a solution to (1.1) has to satisfy the integral equation of the form

$$u(x,t) = u_0(x,t) + AL(|u|^p)(x,t), \tag{2.4}$$

where

$$L(|u|^p)(x,t) = \frac{1}{2\pi}\int_0^t d\tau \int_0^{t-\tau} \frac{\rho d\rho}{\sqrt{(t-\tau)^2-\rho^2}} \int_{|\omega|=1} |u|^p(x+\rho\omega,\tau)dS_\omega \tag{2.5}$$

and u_0 is a solution to a linear wave equation

$$
\begin{aligned}
&\partial_t^2 u_0(x,t) - \Delta u_0(x,t) = 0, \\
&u_0(x,0) = \varepsilon\varphi(x), \ \partial_t u_0(x,0) = \varepsilon\psi(x).
\end{aligned} \tag{2.6}
$$

By definition of the operator L, we see that $L(|u|^p)$ satisfies (2.1) for u with (2.1).

Now we can formulate a priori estimate which is a core in the proof of Theorem 1.

LEMMA 2.1. *There exists a positive constant C depending only on p such that, for any continuous function $u(x,t)$ satisfying (2.1),*

$$
||L(|u|^p)|| \leq Ck^2 M(k,p,T)||u||^p \quad \textit{for} \quad 2 \leq p \leq p_0(2), \tag{2.7}
$$

where

$$
M(k,p,t) = \begin{cases} \left(\dfrac{T+2k}{k}\right)^{-q(2,p)/p} \log \dfrac{T+2k}{k} & \textit{for} \quad p \neq 3, \\ \left(\dfrac{T+2k}{k}\right)^{1/3} \left(\log \dfrac{T+2k}{k}\right)^2 & \textit{for} \quad p = 3. \end{cases} \tag{2.8}
$$

Let X denotes the Banach space of functions $u(x,t)$ for which the $D^\alpha u$ are continuous in Q_T for $|\alpha| \leq 2$ and satisfy (2.1) and $||D^\alpha u|| < \infty$. Here D^α stands for the space differentiation $D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2}$.

We shall look for a solution u to (1.1) in X, employing the classical iteration method:

$$
u_{n+1} = u_0 + AL(|u_n|^p), \quad n \in \mathbb{N} \tag{2.9}
$$

It follows from Lemma 1 in [3], p.236 that a solution u_0 to (2.6) satisfies

$$
|D^\alpha u_0(x,t)| \leq C(\varphi,\psi)\varepsilon(t+r+2k)^{-1/2}(t-r+2k)^{-1/2} \tag{2.10}
$$

for $|\alpha| \leq 2$ and $t \geq 0$, where a positive constant $C(\varphi,\psi)$ depends on $D^\alpha\psi$ and $D^\beta\varphi$ for $|\beta| \leq 3$. Hence we find from (2.2), (2.3) and (2.10) that

$$
||D^\alpha u_0|| \leq C(\varphi,\psi)k^{-1}\varepsilon, \tag{2.11}
$$

which implies $u_0 \in X$. Here we have used the fact that $p \geq 2$ and $t - r + 2k > k$.

Assume a priori estimate (2.7). Then F. John [6] proved that the sequence $\{u_n\}$ defined by (2.9) converges in X if u_0 satisfies the inequality (56e) in [6], p.24. Applying the inequality to our case, we know that $\{u_n\}$ converges in X if

$$ACk^2 M(k,p,T)\|u_0\|^{p-1} \leq \frac{1}{p2^p}, \quad \|u_0\| < \frac{1}{2} \tag{2.12}$$

for $2 \leq p \leq p_0(2)$. Therefore we find from (2.8), (2.11) that (2.12) holds if

$$ACk^2\left(\frac{T+2k}{k}\right)^{-q(2,p)/p} \log \frac{T+2k}{k} \varepsilon^{p-1}(k^{-1}C(\varphi,\psi))^{p-1} \leq \frac{1}{p2^p} \quad \text{for} \quad p \neq 3,$$
$$ACk^2\left(\frac{T+2k}{k}\right)^{1/3}\left(\log \frac{T+2k}{k}\right)^2 \varepsilon^2(k^{-1}C(\varphi,\psi))^2 \leq \frac{1}{24} \quad \text{for} \quad p = 3$$

and

$$C(\varphi,\psi)k^{-1}\varepsilon < \frac{1}{2}.$$

Thus Theorem 1 is proved by taking ε_0 small.

§3 Proof of a priori estimate

Let u be a continuous function in $Q_T = \mathbb{R}^2 \times [0,T)$ satisfying (2.1). Then $L(|u|^p)$ satisfies (2.1) and hence we can assume hereafter that

$$r < t + k, \qquad r = |x|. \tag{3.1}$$

It follows from (2.2) and (2.5) that

$$\begin{aligned} &|L(|u|^p)(x,t)| \\ &\leq \frac{\|u\|^p}{2\pi} \int_0^t d\tau \int_0^{t-\tau} \frac{\rho d\rho}{\sqrt{(t-\tau)^2 - \rho^2}} \int_{|\omega|=1} N(|x + \rho\omega|, \tau)^{-p} dS_\omega \end{aligned} \tag{3.2}$$

The integral over the unit circle in (3.2) is equal to

$$\begin{aligned} &2\int_0^\pi N((|x|^2 + \rho^2 + 2\rho|x|\cos\theta)^{1/2}, \tau)^{-p} d\theta \\ &= 2\int_{-1}^1 (1-\eta^2)^{-1/2} N(r^2 + \rho^2 + 2\rho r\eta)^{1/2}, \tau)^{-p} d\eta. \end{aligned}$$

Changing variables

$$\lambda = (r^2 + \rho^2 + 2\rho r\eta)^{1/2},$$

we find that the integral becomes

$$4\int_{|\rho-r|}^{\rho+r} \lambda N(\lambda,\tau)^{-p} h(\lambda,\rho;r)^{-1/2},$$

where

$$h(\lambda,\rho;r) = (\rho^2 - (\lambda - r)^2)((\lambda + r)^2 - \rho^2). \tag{3.3}$$

Therefore we get from (3.2)

$$\begin{aligned} &|L(|u|^p)(x,t)| \\ &\le \frac{2\|u\|^p}{\pi}\int_0^t d\tau \int_0^{t-\tau} \frac{\rho d\rho}{\sqrt{(t-\tau)^2-\rho^2}} \int_{|\rho-r|}^{\rho+r} \frac{\lambda N(\lambda,\tau)^{-p}}{\sqrt{h(\lambda,\rho;r)}} d\lambda. \end{aligned} \tag{3.4}$$

In what follows, we can assume that

$$\lambda < \tau + k, \tag{3.5}$$

because of (2.1).

Inverting the order of the (ρ,λ)- integral, we find that the integral in (3.4) is equal to

$$\begin{aligned} &\int_0^t d\tau \int_{|r-t+\tau|}^{r+t-\tau} \lambda N(\lambda,\tau)^{-p} d\lambda \int_{|\lambda-r|}^{t-\tau} \frac{\rho d\rho}{\sqrt{((t-\tau)^2-\rho^2)h(\lambda,\rho;r)}} \\ &+ \int_0^{t-r} d\tau \int_0^{t-\tau-r} \lambda N(\lambda,\tau)^{-p} d\lambda \int_{|\lambda-r|}^{\lambda+r} \frac{\rho d\rho}{\sqrt{((t-\tau)^2-\rho^2)h(\lambda,\rho;r)}} \\ &= I_1(r,t) + I_2(r,t), \end{aligned} \tag{3.6}$$

where the last equality gives the difinitions of I_1 and I_2, and we regard I_2 as zero for $t < r$.

It follows from (2.2), (3.4) and (3.6) that a priori estimate (2.7) in Lemma 2.1 is valid provided

$$I_j(r,t) \le C(p)k^2 M(k,p,T) N(r,t)^{-1} \quad \text{for} \quad j = 1.2, \tag{3.7}$$

where M is defined in (2.8) and we denote here and hereafter by $C(p)$ various positive constants depending only on p.

Firstly, we investigate the integral I_1. In the interval of the ρ-integral, we have

$$\begin{aligned} &\lambda + r - \rho \geq \lambda + r - t + \tau, \\ &\lambda + r + \rho \geq \lambda + r + |\lambda - r| \geq 2\lambda \quad \text{or} \quad 2r. \end{aligned} \tag{3.8}$$

Since

$$\int_a^b \frac{\rho d\rho}{\sqrt{\rho^2 - a^2}\sqrt{b^2 - \rho^2}} = \frac{1}{2}B(\frac{1}{2}, \frac{1}{2}) = \frac{\pi}{2} \quad \text{for} \quad 0 \leq a < b, \tag{3.9}$$

it follows from (3.3), (3.6), (3.8) and (3.9) with $a = |\lambda - r|$, $b = t - \tau$ that

$$\int_{|\lambda - r|}^{t-\tau} \frac{\rho d\rho}{\sqrt{((t-\tau)^2 - \rho^2)h(\lambda, \rho; r)}} \leq \frac{\pi}{2\sqrt{2}(\sqrt{r} \text{ or } \sqrt{\lambda})\sqrt{\tau + \lambda - t + r}}$$

Thus we get

$$I_1(r,t) \leq \frac{\pi}{2\sqrt{2}} \int_0^t d\tau \int_{|r-t+\tau|}^{r+t-\tau} \frac{\lambda N(\lambda, \tau)^{-p}}{(\sqrt{r} \text{ or } \sqrt{\lambda})\sqrt{\tau + \lambda - t + r}} d\lambda \tag{3.10}$$

Introducing new variables of integration

$$\alpha = \tau + \lambda, \quad \beta = \tau - \lambda$$

and extending the domain of (α, β)-integration, we find from (3.5) and (3.10) that

$$I_1(r,t) \leq \frac{\pi}{4\sqrt{2}} \int_{t-r}^{t+r} d\alpha \int_{-k}^{t-r} \frac{\lambda N(\lambda, \tau)^{-p}}{(\sqrt{r} \text{ or } \sqrt{\lambda})\sqrt{\alpha - (t - r)}} d\beta. \tag{3.11}$$

We shall prove (3.7) for I_1 deviding into two cases.

CASE 1: $4r > t + r + 2k$.

Since $2\lambda = \alpha - \beta < \alpha + k$ in the domain of integration and, by definition,

$$N(\lambda, \tau) = k^{-(p+2)/2p}(\alpha + 2k)^{1/2}(\beta + 2k)^{1/p}, \tag{3.12}$$

we get from (3.11)

$$I_1(r,t) \leq \frac{\pi k^{(p+2)/2}}{8\sqrt{2r}} \int_{t-r}^{t+r} \frac{(\alpha + 2k)^{(2-p)/2}}{\sqrt{\alpha - (t - r)}} d\alpha \int_{-k}^{t-r} (\beta + 2k)^{-1} d\beta. \tag{3.13}$$

Note that $\alpha - (t-r) \le \alpha + k$, because of (3.1). Then the integration by parts yields

$$\begin{aligned}
&\int_{t-r}^{t+r} \frac{(\alpha+2k)^{(2-p)/2}}{\sqrt{\alpha-(t-r)}}d\alpha \\
(3.14) \qquad &\le 2\sqrt{2r}(t+r+2k)^{(2-p)/2} + (p-2)\int_{t-r}^{t+r}(\alpha+2k)^{(1-p)/2}d\alpha.
\end{aligned}$$

Let $3 < p \le p_0(2)$. Then it follows from (3.14) that the α-integral in (3.13) is dominated by

$$\begin{aligned}
&2\sqrt{2}(t+r+2k)^{(3-p)/2} + \frac{2(p-2)}{p-3}(t-r+2k)^{(3-p)/2} \\
&\le 2(\sqrt{2}+\frac{p-2}{p-3})(t-r+2k)^{(3-p)/2}.
\end{aligned}$$

Therefore, it follows from this, (1.2), (2.3) and (3.13) that

$$\begin{aligned}
I_1(r,t) &\le C(p)k^{(p+2)/2}(t+r+2k)^{-1/2}(t-r+2k)^{(3-p)/2} \log \frac{t-r+2k}{k} \\
(3.15) \qquad &= C(p)N(r,t)^{-1}k^{(p+2)(p-1)/2p}(t-r+2k)^{-q(2,p)/p} \log \frac{t-r+2k}{k} \\
&\le C(p)k^2N(r,t)^{-1}(\frac{T+2k}{k})^{-q(2,p)/p} \log \frac{T+2k}{k},
\end{aligned}$$

which is (3.7) for I_1.

Let $2 \le p < 3$. Then it follows from (3.14) that the α-integral in (3.13) is dominated by

$$2(\sqrt{2}+\frac{p-2}{3-p})(t+r+2k)^{(3-p)/2} \le 2(\sqrt{2}+\frac{p-2}{3-p})(2t+3k)^{(3-p)/2}.$$

Therefore, we get

$$\begin{aligned}
I_1(r,t) &\le C(p)k^{(p+2)/2}(t+r+2k)^{-1/2}(T+2k)^{(3-p)/2} \log \frac{T+2k}{k} \\
&\le C(p)k^2N(r,t)^{-1}(\frac{T+2k}{k})^{-q(2,p)/p} \log \frac{T+2k}{k}.
\end{aligned}$$

Let $p=3$. Then, in the same way as in the above, the α-integral in (3.13) is dominated by

$$2\sqrt{2r}(t+r+2k)^{-1/2} + \log\frac{t+r+2k}{t-r+2k} \le 2\sqrt{2} + \log\frac{2t+3k}{k}.$$

therefore, we get

$$I_1(r,t) \le C(3)k^{5/2}(t+r+2k)^{-1/2}(\log\frac{T+2k}{k})^2$$
$$\le C(3)k^2N(r,t)^{-1}(\frac{T+2k}{k})^{1/3}(\log\frac{T+2k}{k})^2.$$

Thus we have proved (3.7) for I_1 in Case 1.

CASE 2: $4r < t+r+2k$, i.e., $t+r+2k < 2(t-r+2k)$.

In this case, we choose $\sqrt{\lambda}$ in a dedominator of the integrand in (3.11). Then we have

$$I_1(r,t) \le \frac{\pi}{8}k^{(p+2)/2}\int_{t-r}^{t+r}\frac{(\alpha+2k)^{(1-p)/2}}{\sqrt{\alpha-(t-r)}}d\alpha\int_{-k}^{t-r}(\beta+2k)^{-1}d\beta. \tag{3.16}$$

The α-integral in (3.16) is dominated, in the same way as in Case 1, by

$$2\sqrt{2r}(t+r+2k)^{(1-p)/2}+(p-1)\int_{t-r}^{t+r}(\alpha+2k)^{-p/2}d\alpha. \tag{3.17}$$

Let $2 < p \le p_0(2)$. Since $(t+r+2k)$ is equivalent to $(t-r+2k)$ in Case 2, we find from (3.17) that the α-integral in (3.16) is dominated by

$$C(p)(t+r+2k)^{-1/2}(t-r+2k)^{(3-p)/2}.$$

Therefore we have

$$I_1(r,t) \le C(p)k^{(2+p)/2}(t+r+2k)^{-1/2}(t-r+2k)^{(3-p)/2}\log\frac{t-r+2k}{k}$$
$$\le C(p)k^2N(r,t)^{-1}(\frac{T+2k}{k})^{-q(2,p)/p}\log\frac{T+2k}{k}.$$

Let $p=2$. Then the α-integral in (3.16) is dominated by

$$2\sqrt{2r}(t+r+2k)^{-1/2}+\log\frac{t+r+2k}{t-r+2k} \le 2\sqrt{2}+\log 2.$$

Therefore, we have

$$I_1(r,t) \le C(2)k^2\log\frac{T+2k}{k}$$
$$\le C(2)k^2N(r,t)^{-1}(\frac{T+2k}{k})\log\frac{T+2k}{k}.$$

Thus the proof of (3.7) for I_1 is finished.

Next we investigate the integral I_2 defined for $t > r$. In the domain of the integration, we have

$$t-\tau-\rho \geq t-\tau-(\lambda+r) = t-r-\alpha$$
$$t-\tau+\rho \geq t-\tau+|\lambda-r| \geq 2r \quad \text{or} \quad t-r-\beta.$$

It follows from this, (3.3) and (3.9) with $a = |\lambda - r|$, $b = \lambda + r$ that

$$\int_{|\lambda-r|}^{\lambda+r} \frac{\rho d\rho}{\sqrt{((t-\tau)^2-\rho^2)h(\lambda,\rho;r)}} \leq \frac{\pi}{2(\sqrt{2r} \quad \text{or} \quad \sqrt{t-r-\beta})\sqrt{t-r-\alpha}}.$$

Hence we have from this and (3.6)

$$\text{(3.18)} \qquad I_2(r,t) \leq \frac{\pi}{2}\int_0^{t-r} d\tau \int_0^{t-\tau-r} \frac{\lambda N(\lambda,\tau)^{-p} d\lambda}{(\sqrt{2r} \quad \text{or} \quad \sqrt{t-r-\beta})\sqrt{t-r-\alpha}}.$$

Extending the domain of (α,β)-integral, we find from (3.5) and (3.18) that

$$I_2(r,t) \leq \frac{\pi}{4}\int_0^{t-r} d\alpha \int_{-k}^{t-r} \frac{\lambda N(\lambda,\tau)^{-p} d\beta}{(\sqrt{2r} \quad \text{or} \quad \sqrt{t-r-\beta})\sqrt{t-r-\alpha}}.$$

Since $2\lambda = \alpha - \beta < \alpha + k$, we get from this and (3.12)

$$\text{(3.19)} \qquad I_2(r,t) \leq \frac{\pi k^{(2+p)/2}}{8\sqrt{2r}}\int_0^{t-r} \frac{(\alpha+2k)^{(2-p)/2}}{\sqrt{t-r-\alpha}} d\alpha \int_{-k}^{t-r} (\beta+2k)^{-1} d\beta$$

or

$$\text{(3.20)} \qquad I_2(r,t) \leq \frac{\pi k^{(2+p)/2}}{8}\int_0^{t-r} \frac{(\alpha+2k)^{(2-p)/2}}{\sqrt{t-r-\alpha}} d\alpha \int_{-k}^{t-r} \frac{(\beta+2k)^{-1}}{\sqrt{t-r-\beta}} d\beta$$

We first show that, for $2 \leq p \leq p_0(2)$,

$$\text{(3.21)} \qquad \int_0^{t-r} \frac{(\alpha+2k)^{(2-p)/2}}{\sqrt{t-r-\alpha}} d\alpha \leq C(p)(t-r+2k)^{(3-p)/2}.$$

In fact, let $0 < t-r < k$, i.e., $3k > t-r+2k$. Then, by integration by parts, the α-integral is dominated by

$$2\sqrt{t-r}(2k)^{(2-p)/2} \leq 2\left(\frac{2}{3}\right)^{(2-p)/2}(t-r+2k)^{(3-p)/2},$$

where we have used the fact that $p \geq 2$. Let $t - r > k$ which implies $t - r > (t - r + 2k)/4$. Then, breaking the integral up into two pieces, we get

$$\int_0^{(t-r)/2} \frac{(\alpha + 2k)^{(2-p)/2}}{\sqrt{t - r - \alpha}} d\alpha \leq \sqrt{2}(t - r)^{-1/2} \int_0^{(t-r)/2} (\alpha + 2k)^{(2-p)/2} d\alpha$$
$$\leq \frac{4\sqrt{2}}{4 - p}(t - r + 2k)^{(3-p)/2},$$

because $p_0(2) < 4$. On the complement,

$$\int_{(t-r)/2}^{t-r} \frac{(\alpha + 2k)^{(2-p)/2}}{\sqrt{t - r - \alpha}} d\alpha \leq (\frac{t - r}{2} + 2k)^{(2-p)/2} \int_{(t-r)/2}^{t-r} \frac{d\alpha}{\sqrt{t - r - \alpha}}$$
$$\leq \sqrt{2}\ 2^{(p-2)/2}(t - r + 2k)^{(3-p)/2}.$$

Next we show in a similar way to the above that

$$\int_{-k}^{t-r} \frac{(\beta + 2k)^{-1}}{\sqrt{t - r - \beta}} d\beta \leq C(t - r + 2k)^{-1/2} \log \frac{t - r + 2k}{k}. \tag{3.22}$$

In fact, let $0 < t - r < k$. Then, by integration by parts, the β-integral is dominated by

$$2\sqrt{t - r + k}\ k^{-1} \leq 6(t - r + 2k)^{-1/2}.$$

Let $k < t - r$. Then, breaking the integral up into two pieces, we get

$$\int_{-k}^{(t-r)/2} \frac{(\beta + 2k)^{-1}}{\sqrt{t - r - \beta}} d\beta \leq \sqrt{2}(t - r)^{-1/2} \int_{-k}^{(t-r)/2} (\beta + 2k)^{-1} d\beta$$
$$\leq 2\sqrt{2}(t - r + 2k)^{-1/2} \log \frac{t - r + 4k}{2k}.$$

On the complement,

$$\int_{(t-r)/2}^{t-r} \frac{(\beta + 2k)^{-1}}{\sqrt{t - r - \beta}} d\beta \leq (\frac{t - r}{2} + 2k)^{-1} \int_{(t-r)/2}^{t-r} \frac{1}{\sqrt{t - r - \beta}} d\beta$$
$$\leq 2\sqrt{2}(t - r + 2k)^{-1/2}.$$

When $4r > t + r + 2k$, it follows from (3.19) and (3.21) that

$$I_2(r, t) \leq C(p)k^{(2+p)/2}(t + r + 2k)^{-1/2}(t - r + 2k)^{(3-p)/2} \log \frac{T + 2k}{k}. \tag{3.23}$$

When $4r < t+r+2k$, i.e., $t+r+2k < 2(t-r+2k)$, we obtain (3.23) from (3.20), (3.21) and (3.22). Thus the estimate (3.7) for I_2 follows from (3.23), by the same way as in (3.15). Therefore, the proof of Lemma 2.1 is completed.

§4 Proof of Theorem 2

Let $u(x,t)$ be a C^2-solution to (1.1) in $\mathbf{R}^2 \times [0,T)$ with the initial data defined in the statement of Theorem 2. In this section we shall show that if T exceeds a certain constant we get a contradiction. This implies that the lifespan of u has an upper bound which is expressed by such constant.

By the assumption of Theorem 2 on ψ, we assume that

$$\psi(x_0) > 0 \quad \text{for some} \quad x_0 \in \mathbf{R}^2. \tag{4.1}$$

Throughout this section we use the following notation. Let $\overline{v}$ be the spherical mean of $v \in C^0(\mathbf{R}^2 \times [0,\infty))$ about a point x_0;

$$\overline{v}(r,t) = \frac{1}{2\pi}\int_{|\omega|=1} v(x_0 + r\omega, t)dS\omega. \tag{4.2}$$

Then (4.1) yields

$$\overline{\psi}(0) > 0. \tag{4.3}$$

Hence one can find $\delta > 0$ so small that

$$\overline{\psi}(2\delta) > 0. \tag{4.4}$$

As in section 2, we know that u satisfies the integral equation (2.4). Here we employ the following lemmas.

LEMMA 4.1. *Let u_0 be a solution to (2.6) with $\varphi(x) = 0$. Then*

$$\overline{u_0}(r,t) \geq \frac{\varepsilon}{2\pi\sqrt{r}}\int_{|t-r|}^{t+r} \sqrt{\lambda}\,\overline{\psi}(\lambda)d\lambda \quad \textit{for} \quad 0 < t \leq 2r. \tag{4.5}$$

Lemma 4.2. *Let u be the solution to (1.1) with $f(x) = 0$, $g(x) \geq 0$ and $p > 1$. Then*

$$\overline{u}(r,t) \geq \frac{A}{2\pi\sqrt{r}} \iint_{T_{r,t}} \sqrt{\lambda}|\overline{u}(\lambda,\tau)|^p d\lambda d\tau \tag{4.6}$$

for $0 < t - r \leq r$, where

$$T_{r,t} = \{(\lambda,\tau) \in \mathbb{R}^2; t - r \leq \tau + \lambda \leq t + r,\ \tau - \lambda \leq t - r,\ \tau \geq 0\}. \tag{4.7}$$

These lemmas are due to R. Agemi [1].

Lemma 2 follows from the proof of (2.11) of [1]. Applying its argument to u_0, we readily get Lemma 1. For the sake of completeness, we shall review proofs of these lemmas.

Proof of Lemma 4.2: It follows from (2.4) that

$$\overline{u}(r,t) = \overline{u_0}(r,t) + \overline{AL(|u|^p)}(r,t). \tag{4.8}$$

Since the assumption that $f(x) = 0$, $g(x) \geq 0$ yields $\overline{u_0} \geq 0$, we have

$$\overline{u}(r,t) \geq \overline{AL(|u|^p)}(r,t). \tag{4.9}$$

We now employ the following fundamental identity for iterated spherical means by F. John [5], p.81;

$$\begin{aligned}\frac{1}{(2\pi)^2} \int_{|\eta|=1} \int_{|\omega|=1} & v(r\eta + \rho\omega) dS\omega dS\eta \\ &= \frac{2}{\pi} \int_{|\rho - r|}^{\rho + r} \frac{\lambda \overline{v}(\lambda)}{\sqrt{h(\lambda,\rho;r)}} d\lambda,\end{aligned} \tag{4.10}$$

where $h(\lambda,\rho;r)$ is defined by (3.3). Applying (4.10) with $v = |u|^p$ to the right hand side of (4.9), we get

$$\overline{u}(r,t) \geq \frac{2A}{\pi} \int_0^t d\tau \int_0^{t-\tau} \frac{\rho d\rho}{\sqrt{(t-\tau)^2 - \rho^2}} \int_{|\rho - r|}^{\rho + r} \frac{\lambda |\overline{u}|^p(\lambda,\tau)}{\sqrt{h(\lambda,\rho;r)}} d\lambda. \tag{4.11}$$

Here we have used the Jensen's inequality

$$\overline{|u|^p} \geq |\overline{u}|^p \quad \text{for} \quad p > 1.$$

Note that

$$t - \tau - r \leq r \quad \text{for} \quad 0 < t - r \leq r \quad \text{and} \quad \tau \geq 0.$$

Inverting the other of (λ, ρ)-integral, we find that the right hand side of (4.11) equals to

$$\begin{aligned}
&\frac{2A}{\pi} \int_0^t d\tau \int_{|t-\tau-r|}^{t-\tau-r} \lambda |\overline{u}|^p(\lambda, \tau) d\lambda \int_{|\lambda - r|}^{t-\tau} \frac{\rho d\rho}{\sqrt{h(\lambda, \rho; r)((t-\tau)^2 - \rho^2)}} \\
&+ \frac{2A}{\pi} \int_0^{t-r} d\tau \int_0^{t-\tau-r} \lambda |\overline{u}|^p(\lambda, \tau) d\lambda \int_{|\lambda - r|}^{\lambda + r} \frac{\rho d\rho}{\sqrt{h(\lambda, \rho; r)((t-\tau)^2 - \rho^2)}}.
\end{aligned}$$

Hence it follows from this and (4.11) that

$$\begin{aligned}
&\overline{u}(r,t) \\
(4.12) \qquad &\geq \frac{2A}{\pi} \int_0^t d\tau \int_{|t-\tau-r|}^{t-\tau+r} \lambda |\overline{u}|^p(\lambda, \tau) d\lambda \int_{|\lambda - r|}^{t-\tau} \frac{\rho d\rho}{\sqrt{h(\lambda, \rho; r)((t-\tau)^2 - \rho^2)}}.
\end{aligned}$$

In the domain of (ρ, λ)-integral of (4.12), we know that

$$(4.13) \qquad h(\lambda, \rho; r) \leq 12 r \lambda ((t-\tau)^2 - (\lambda - r)^2) \quad \text{for} \quad 0 < t - r \leq r.$$

In fact, $\rho \leq t - \tau$ implies

$$\rho^2 - (\lambda - r)^2 \leq (t-\tau)^2 - (\lambda - r)^2.$$

Since $\lambda \leq t + r$, $\rho \leq t - \tau$ and $t \leq 2r$, we have

$$\lambda + r + \rho \leq t + r + r + t - \tau \leq 2t + 2r \leq 6r.$$

Moreover, $\rho \geq |\lambda - r|$ yields

$$\lambda + r - \rho \leq \lambda + r - |\lambda - r| \leq \lambda + r + \lambda - r = 2\lambda.$$

Thus we get (4.13). Using the fact that

$$\int_{|\lambda-r|}^{t-r} \frac{\rho d\rho}{\sqrt{(t-r)^2-\rho^2}} = \sqrt{(t-r)^2-(\lambda-r)^2},$$

we obtain (4.6) by (4.12) and (4.13).

PROOF OF LEMMA 4.1: As is well known, a solution to (2.6) with $\varphi(x) = 0$ is expressed in the form

$$u_0(x,t) = \frac{1}{2\pi}\int_0^t \frac{\rho d\rho}{\sqrt{t^2-\rho^2}} \int_{|\omega|=1} \varepsilon\psi(x+\rho\omega)dS\omega. \tag{4.14}$$

Applying (4.10) with $v = \psi$ to the right hand side of (4.14), we get

$$\overline{u_0}(r,t) = \frac{2\varepsilon}{\pi}\int_0^t \frac{\rho d\rho}{\sqrt{t^2-\rho^2}} \int_{|\rho-r|}^{\rho+r} \frac{\lambda\overline{\psi}(\lambda)}{\sqrt{h(\lambda,\rho;r)}}d\lambda. \tag{4.15}$$

As in the proof of Lemma 4.2, inverting the order of (λ,ρ)-integral yields for $0 < t-r \le r$

$$\overline{u_0}(r,t) = \frac{2\varepsilon}{\pi}\int_{|t-r|}^{t+r} \lambda\overline{\psi}(\lambda)d\lambda \int_{|\lambda-r|}^{t} \frac{\rho d\rho}{\sqrt{h(\lambda,\rho;r)(t^2-\rho^2)}}. \tag{4.16}$$

Replacing $t-r$ by t in (4.13), we obtain (4.5) by same way as in the proof of Lemma 4.2.

Now, define the region

$$S = \{(r,t) \in \mathbb{R}^2;\ 3\delta \le t+r,\ \delta \le t-r \le 2\delta\}, \tag{4.17}$$

where δ is the one in (4.4). Then it follows from (4.4), (4.5) and (4.17) that

$$\overline{u_0}(r,t) \ge \frac{C'\varepsilon}{\sqrt{r}} \quad \text{for} \quad (r,t) \in S \quad \text{and} \quad t \le 2r, \tag{4.18}$$

where C' is a constant defined by

$$C' = \frac{1}{2\pi}\int_{2\delta}^{3\delta} \sqrt{\lambda}\overline{\psi}(\lambda)d\lambda. \tag{4.19}$$

Since (2.4) yields

$$\overline{u}(r,t) \geq \overline{u_0}(r,t), \tag{4.20}$$

we have

$$\overline{u}(r,t) \geq \frac{C'\varepsilon}{\sqrt{r}} \quad \text{for} \quad (r,t) \in S \quad \text{and} \quad t \leq 2r. \tag{4.21}$$

Let Σ denote the set

$$\Sigma = \{(r,t) \in \mathbf{R}^2;\ 3\delta \leq t - r \leq r\}. \tag{4.22}$$

For $(r,t) \in \Sigma$ we introduce the sets

$$\begin{aligned} S_{r,t} &= \{(\lambda,\tau);\ t - r \leq \lambda,\ \tau + \lambda \leq t + r,\ \delta \leq \tau - \lambda \leq 2\delta\}, \\ R_{r,t} &= \{(\lambda,\tau);\ t - r \leq \lambda,\ \tau + \lambda \leq t + r,\ 3\delta \leq \tau - \lambda \leq t - r\}. \end{aligned} \tag{4.23}$$

We note that for $(r,t) \in \Sigma$

$$S_{r,t},\ R_{r,t}\ \subset T_{r,t},\ S_{r,t}\ \subset S,\ R_{r,t}\ \subset \Sigma. \tag{4.24}$$

Hence it follows from (4.6), (4.21) and (4.24) that

$$\overline{u}(r,t) \geq \frac{A}{2\pi\sqrt{r}} \iint_{S_{r,t}} \sqrt{\lambda}\left(\frac{C'\varepsilon}{\sqrt{\lambda}}\right)^p d\lambda d\tau \quad \text{for} \quad (r,t) \in \Sigma.$$

Changing the variables by

$$\alpha = \tau + \lambda, \quad \beta = \tau - \lambda, \tag{4.25}$$

we have for $(r,t) \in \Sigma$

$$\overline{u}(r,t) \geq \frac{A(C')^p}{4\pi\sqrt{r}}\varepsilon^p \int_\delta^{2\delta} d\beta \int_{2(t-r)+\beta}^{t+r} \left(\frac{\alpha-\beta}{2}\right)^{(1-p)/2} d\alpha.$$

Since $p > 1$ and $\alpha - \beta \leq \alpha \leq t + r \leq 3r$,

$$\overline{u}(r,t) \geq \frac{A(C')^p}{4\pi}\left(\frac{2}{3}\right)^{(p-1)/2} \varepsilon^p r^{-p/2} \int_\delta^{2\delta} (3r - t - \beta) d\beta.$$

Here we find that for $(r,t) \in \Sigma$

$$3r - t - \beta \geq r - \beta \geq r - 2\delta \geq r/3.$$

Hence

$$\overline{u}(r,t) \geq \frac{\delta A(C')^p}{12\pi} (\frac{2}{3})^{(p-1)/2} \varepsilon^p r^{(2-p)/2}.$$

Therefore we obtain for $(r,t) \in \Sigma$

$$\overline{u}(r,t) \geq \begin{cases} C_0 \ \varepsilon^p \ r^{-1/2}(t-r-s)^{(3-p)/2} & \text{if} \quad 1 < p < 3, \\ C_0 \ \varepsilon^p \ r^{-(p-2)/2} & \text{if} \quad 3 \leq p < p_0(2), \end{cases} \tag{4.26}$$

where we set $s = 3\delta$ and

$$C_0 = \frac{\delta A(C')^p}{12\pi} (\frac{2}{3})^{(p-1)/2}. \tag{4.27}$$

Now, assume that $\overline{u}$ has more general estimate

$$\overline{u}(r,t) \geq C r^{-q}(t-r-s)^a (t-r)^{-b} \quad \text{for} \quad (r,t) \in \Sigma \tag{4.28}$$

with $C > 0$, $q \geq 1/2$, $a \geq 0$, $b \geq 0$. Then it follows from (4.6), (4.24), (4.25) and (4.28) that for $(r,t) \in \Sigma$

$$\begin{aligned} \overline{u}(r,t) &\geq \frac{A}{2\pi\sqrt{r}} \iint_{R_{r,t}} \sqrt{\lambda} |\overline{u}(\lambda, \tau)|^p d\lambda d\tau \\ &\geq \frac{AC^p}{2\pi\sqrt{r}} \iint_{R_{r,t}} \lambda^{1/2-pq} (\tau - \lambda - s)^{pa} (\tau - \lambda)^{-pb} d\lambda d\tau \\ &\geq \frac{AC^p}{4\pi\sqrt{r}} \int_s^{t-r} d\beta \int_{2(t-r)+\beta}^{t+r} (\frac{\alpha-\beta}{2})^{1/2-pq} (\beta - s)^{pa} \beta^{-pb} d\alpha. \end{aligned}$$

We divide the estimate for $\overline{u}$ into following two cases.

CASE 1: $t - r \geq r/2$.

Since $pq > q \geq 1/2$ and

$$\alpha - \beta \leq \alpha \leq t + r \leq 3r \leq 6(t-r)$$

for $(r,t)\in\Sigma$ in this case, we have

$$\overline{u}(r,t)\geq\frac{3^{1/2-pq}AC^p}{4\pi\sqrt{r}(t-r)^{pb+pq-1/2}}\int_s^{t-r}(\beta-s)^{pa}(3r-t-\beta)d\beta.$$

By $3r-t\geq r\geq t-r$, β-integral equals to

$$\int_s^{t-r}(\beta-s)^{pa}(t-r-\beta)d\beta=\frac{(t-r-s)^{pa+2}}{(pa+1)(pa+2)}.$$

Hence

$$\overline{u}(r,t)\geq\frac{3^{1/2-pq}AC^p(t-r-s)^{pa+2}}{4\pi(pa+2)^2\sqrt{r}(t-r)^{pb+pq-1/2}}.$$

CASE 2: $t-r\leq r/2$.

Since $t+r\geq 2r\geq 4(t-r)$ and $2(t-r)+\beta\leq 3(t-r)$ for $(r,t)\in\Sigma$ in this case, we have

$$\begin{aligned}\overline{u}(r,t)&\geq\frac{AC^p}{4\pi\sqrt{r}}\int_s^{t-r}(\beta-s)^{pa}\beta^{-pb}d\beta\int_{3(t-r)}^{4(t-r)}(\frac{\alpha}{2})^{1/2-pq}d\alpha\\&\geq\frac{2^{1/2-pq}AC^p}{4\pi\sqrt{r}(t-r)^{pb+pq-3/2}}\int_s^{t-r}(\beta-s)^{pa}d\beta\\&\geq\frac{2^{1/2-pq}AC^p(t-r-s)^{pa+1}}{4\pi(pa+1)\sqrt{r}(t-r)p^{pb+pq-3/2}}\\&\geq\frac{3^{1/2-pq}AC^p(t-r-s)^{pa+2}}{4\pi(pa+2)^2\sqrt{r}(t-r)^{pb+pq-1/2}}.\end{aligned}$$

Combining these two cases, we obtain

$$\overline{u}(r,t)\geq C^*r^{-1/2}(t-r-s)^{a^*}(t-r)^{-b^*}\quad\text{for}\quad(r,t)\in\Sigma,\tag{4.29}$$

where

$$\begin{aligned}&a^*=pa+2,\qquad b^*=pb+pq-\frac{1}{2},\\&C^*=\frac{D_qC^p}{(pa+2)^2},\ D_q=\frac{3^{1/2-pq}A}{4\pi}\end{aligned}\tag{4.30}$$

Using the values

$$\begin{aligned}&q=\frac{1}{2},\ a=\frac{3-p}{2},\ b=0,\ C=C_0\varepsilon^p\quad\text{if}\quad 1<p<3,\\&q=\frac{p-2}{2},\ a=b=0,\ C=C_0\varepsilon^p\quad\text{if}\quad 3\leq p<p_0(2)\end{aligned}\tag{4.31}$$

Corresponding to (4.26), we have (4.29) with

$$
(4.32) \qquad \begin{aligned}
a^* &= \frac{p(3-p)}{2} + 2,\ b^* = \frac{p-1}{2},\ C^* = \frac{D_{1/2}(C_0\varepsilon^p)^p}{(p(3-p)/2+2)^2} \quad \text{if} \quad 1 < p < 3, \\
a^* &= 2,\ b^* = \frac{p(p-2)-1}{2},\ C^* = \frac{D_{(p-2)/2}(C_0\varepsilon^p)^p}{4} \quad \text{if} \quad 3 \le p < p_0(2)
\end{aligned}
$$

Define the sequences $\{a_\ell\}, \{b_\ell\}, \{C_\ell\}$ $(\ell \in \mathbb{N})$ by

$$
(4.33) \qquad \begin{aligned}
& a_{\ell+1} = pa_\ell + 2,\ b_{\ell+1} = pb_\ell + \frac{p-1}{2},\ C_{\ell+1} = \frac{D_{1/2}C_\ell^p}{(pa+2)^2}, \\
& a_1 = a^*,\ b_1 = b^*,\ C_1 = C^* \quad \text{as given in (4.32).}
\end{aligned}
$$

Then (4.29) will hold with $q = 1/2$, $a^* = a_\ell$, $b^* = b_\ell$, $C^* = C_\ell$ for $\ell \in \mathbb{N}$. Solving the above sequences we have

$$
(4.34) \qquad \begin{aligned}
a_\ell &= \left(\frac{3-p}{2} + \frac{2}{p-1}\right)p^\ell - \frac{2}{p-1},\quad b_\ell = \frac{1}{2}p^\ell - \frac{1}{2} \quad \text{if} \quad 1 < p < 3, \\
a_\ell &= \frac{2}{p-1}p^\ell - \frac{2}{p-1},\quad b_\ell = \frac{p-2}{2}p^\ell - \frac{1}{2} \quad \text{if} \quad 3 \le p < p_0(2).
\end{aligned}
$$

If $1 < p < 3$, then

$$
\begin{aligned}
pa_\ell + 2 &= \left(\frac{3-p}{2} + \frac{2}{p-1}\right)p^{\ell+1} - \frac{2}{p-1}. \\
&\le \frac{2}{p-1}(p^{\ell+2} - 1) \le 2(\ell+2)p^{\ell+1}.
\end{aligned}
$$

If $3 \le p < p_0(2)$, then

$$
pa_\ell + 2 = \frac{2}{p-1}(p^{\ell+1} - 1) \le 2(\ell+2)p^{\ell+1}.
$$

Hence it follows from (4.32) that

$$
C_{\ell+1} \ge \frac{D_{1/2}\, C_\ell^p}{4(\ell+2)^2\, p^{2\ell+2}} \quad \text{for} \quad 1 < p < p_0(2),
$$

which implies that

$$
C_\ell \ge \exp\Big[p^\ell\Big(\frac{1}{p}\log C_1 - \sum_{j=1}^{\ell-1}\frac{2\log(j+2) + 2(j+1)\log\ p + \log 4D_{1/2}^{-1}}{p^{j+1}}\Big)\Big]
$$

For sufficiently large ℓ we have

$$(4.35) \qquad C_\ell \geq \exp\left[p^\ell (p^{-1}\log C_1 - S(p))\right] \quad \text{for} \quad 1 < p < p_0(2),$$

where

$$(4.36) \qquad S(p) = \sum_{j=1}^{\infty} \frac{2\log(j+2) + 2(j+1)\ \log\ p + \log 4D_{1/2}^{-1}}{p^{j+1}}.$$

We note that $S(p)$ is finite because $p > 1$ and each term is positive for sufficiently large ℓ. Therefore it follows from (4.29), (4.33), (4.34) and (4.35) that for $(r,t) \in \Sigma$

$$(4.37) \qquad \overline{u}(r,t) \geq \frac{\sqrt{t-r}}{\sqrt{r}(t-r-s)^{2/(p-1)}} \exp[p^\ell J(r,t)],$$

where

$$(4.38) \qquad J(r,t) = p^{-1}\log\ C_1 - S(p) + \left\{ \begin{array}{ll} (\frac{3-p}{2} + \frac{2}{p-1})\log(t-r-s) - \frac{1}{2}\log(t-r) & \text{if} \quad 1 < p < 3, \\ \frac{2}{p-1}\log(t-r-s) - \frac{p-2}{2}\log(t-r) & \text{if} \quad 3 \leq p < p_0(2). \end{array} \right.$$

If there exists a point $(r,t) \in \Sigma$ such that

$$(4.39) \qquad J(r,t) > 0,$$

then we have $\overline{u}(r,t) = \infty$ letting $\ell \to \infty$, which implies that u cannot be a C^2-solution to (1.1). This contradicts the assumption that u is a C^2-solution to (1.1), so that we shall look for $(r,t) \in \Sigma$ which satisfies (4.39). In view of (4.38), if $t - r \geq 2s$, (4.39) follows from

$$\left(\frac{2}{p-1} - \frac{p-2}{2}\right)\log(t-r) > -\frac{1}{p}\log C_1 + S(p) + \frac{2p}{p-1}\log 2 \quad \text{for} \quad 1 < p < p_0(2).$$

Hence the definition of C_1, (4.32), gives the following sufficient condition to (4.39);

$$\frac{q(2,p)}{1-p}\log(t-r) > \log C''\varepsilon^{-p} \quad \text{for} \quad 1 < p < p_0(2),$$

where $q(2,p)$ is defined by (1.2) and we set

$$C'' = (5^2 D_1^{-1})^{1/p} C_0^{-1} e^{S(p)+2p/(p-1)}. \tag{4.40}$$

C_0 is defined by (4.27). Here we have used the following fact which follows from the definition of D_q, (4.30).

$$D_{1/2} \geq D_q \quad \text{for} \quad q \geq 1/2. \tag{4.41}$$

Recall that $q(2,p) < 0$ for $1 < p < p_0(2)$. Therefore, setting $t = 2r$, we find that (4.39) follows from

$$t > C\varepsilon^{p(p-1)/q(2,p)} \quad \text{for} \quad 1 < p < p_0(2),$$

where

$$C = 2(C'')^{(1-p)/q(2,p)}.$$

Thus we conclude that T has to satisfy

$$T \leq C\varepsilon^{p(p-1)/q(2,p)}.$$

This completes the proof of Theorem 2.

§5. Proof of Theorem 3

In this section we shall prove Theorem 3 by using the same argument as in the proof of Theorem 2.

Let $u(x,t)$ be a C^2-solution to (1.1) in $\mathbf{R}^2 \times [0,T)$ with initial data satisfying (1.8). In the definition of the spherical mean, (4.2), we set $x_0 = 0$. Define the set

$$\Sigma' = \{(r,t) \in \mathbf{R}^2;\ s < t - r \leq r\} \tag{5.1}$$

for some fixed constant $s > 0$. By virtue of Lemma 4.1, (4.5), we have for $(r,t) \in \Sigma'$

$$\begin{aligned}\overline{u_0}(r,t) &\geq \frac{\varepsilon}{2\pi\sqrt{r}}\int_{t-r}^{t+r}\sqrt{\lambda}\overline{\psi}(\lambda)d\lambda \\ &\geq \frac{\varepsilon}{2\pi\sqrt{r}}\int_{t-r}^{t+r}\sqrt{\lambda}(1+\lambda)^{-\kappa-1}d\lambda \\ &\geq \frac{1}{2\pi}\left(\frac{s}{1+s}\right)^{\kappa+1}\frac{\varepsilon}{\sqrt{r}}\int_{t-r}^{t+r}\lambda^{-\kappa-1/2}d\lambda.\end{aligned}$$

If $0 < \kappa \leq 1/2$, then

$$\begin{aligned}\overline{u_0}(r,t) &\geq \frac{1}{2\pi}\left(\frac{s}{1+s}\right)^{\kappa+1}\frac{\varepsilon}{\sqrt{r}(t+r)^{\kappa+1/2}}\int_{t-r}^{t+r}d\lambda \\ &\geq \frac{1}{3\pi}\left(\frac{s}{1+s}\right)^{\kappa+1}\frac{\varepsilon}{r^{\kappa}}.\end{aligned}$$

If $\kappa > 1/2$, then

$$\begin{aligned}\overline{u_0}(r,t) &\geq \frac{1}{\pi(2\kappa-1)}\left(\frac{s}{1+s}\right)^{\kappa+1}\frac{\varepsilon}{\sqrt{r}(t-r)^{\kappa-1/2}}\left[1-\left(\frac{t-r}{t+r}\right)^{\kappa-1/2}\right] \\ &\geq \frac{1}{\pi(2\kappa-1)}\left(\frac{s}{1+s}\right)^{\kappa+1}\frac{\varepsilon}{\sqrt{r}(t-r)^{\kappa-1/2}}\min\{1,\kappa-\frac{1}{2}\}\left(1-\frac{t-r}{t+r}\right) \\ &\geq \frac{1}{3\pi}\min\{\frac{2}{2\kappa-1},1\}\left(\frac{s}{1+s}\right)^{\kappa+1}\frac{\varepsilon}{\sqrt{r}(t-r)^{\kappa-1/2}}.\end{aligned}$$

Hence we obtain for $(r,t) \in \Sigma'$

$$\overline{u_0}(r,t) \geq \begin{cases} \dfrac{C_0\varepsilon}{r^{\kappa}} & \text{if } 0 < \kappa \leq \frac{1}{2}, \\ \dfrac{C_0\varepsilon}{\sqrt{r}(t-r)^{\kappa-1/2}} & \text{if } \kappa > \frac{1}{2}, \end{cases} \tag{5.2}$$

where C_0 is a constant defined by

$$C_0 = \frac{1}{3\pi}\min\{\left|\frac{2}{2\kappa-1}\right|,1\}\left(\frac{s}{1+s}\right)^{\kappa+1}. \tag{5.3}$$

For $(r,t) \in \Sigma'$ we find that

$$r^{\kappa} \leq r^{\kappa+1/2}(t-r-s)^{-1/2}.$$

Therefore it follows from (4.20) and (5.2) that

$$\overline{u}(r,t) \geq \begin{cases} \dfrac{C_0\varepsilon(t-r-s)^{1/2}}{r^{\kappa+1/2}} & \text{if } 0<\kappa\leq \frac{1}{2}, \\ \dfrac{C_0\varepsilon}{\sqrt{r}(t-r)^{\kappa-1/2}} & \text{if } \kappa > \frac{1}{2}. \end{cases} \tag{5.4}$$

Now, as in the proof of Theorem 2, assume that $\overline{u}$ has more general estimate (4.28). Hence we have (4.29), (4.30) for $(r,t)\in\Sigma'$. Using the values

$$\begin{aligned} &q=\kappa+\frac{1}{2},\ a=\frac{1}{2},\ b=0,\ C=C_0\varepsilon \quad \text{if} \quad 0<\kappa\leq\frac{1}{2}, \\ &q=\frac{1}{2},\ a=0,\ b=\kappa-\frac{1}{2},\ C=C_0\varepsilon \quad \text{if} \quad \kappa>\frac{1}{2} \end{aligned} \tag{5.5}$$

corresponding to (5.4), we obtain (4.29) with

$$\begin{aligned} &a^*=\frac{p}{2}+2,\ b^*=p\kappa+\frac{p-1}{2},\ C^*=\frac{D_{\kappa+1/2}(C_0\varepsilon)^p}{(p/2+2)^2} \quad \text{if} \quad 0<\kappa\leq\frac{1}{2}, \\ &a^*=2,\ b^*=p\kappa-\frac{1}{2},\ C^*=\frac{D_{1/2}(C_0\varepsilon)^p}{4} \quad \text{if} \quad \kappa>\frac{1}{2}. \end{aligned} \tag{5.6}$$

Define the sequence $\{a_\ell\}$, $\{b_\ell\}$, $\{C_\ell\}$ $(\ell\in\mathbb{N})$ by (4.33) in which (4.32) is replaced by (5.6). Solving these sequences we get

$$\begin{aligned} &a_\ell=\left(\frac{2}{p-1}+\frac{1}{2}\right)p^\ell-\frac{2}{p-1},\ b_\ell=\left(\kappa+\frac{1}{2}\right)p^\ell-\frac{1}{2} \quad \text{if} \quad 0<\kappa\leq\frac{1}{2}, \\ &a_\ell=\frac{2}{p-1}p^\ell-\frac{2}{p-1},\ b_\ell=\kappa p^\ell-\frac{1}{2} \quad \text{if} \quad \kappa>\frac{1}{2}. \end{aligned} \tag{5.7}$$

If $0<\kappa\leq 1/2$, then

$$pa_\ell+2=\left(\frac{2}{p-1}+\frac{1}{2}\right)p^{\ell+1}-\frac{2}{p-1} \quad \leq 2(\ell+2)p^{\ell+1}.$$

If $\kappa>1/2$, then

$$pa_\ell+2=\frac{2}{p-1}(p^{\ell+1}-1) \ \leq 2(\ell+2)p^{\ell+1}.$$

Thus, in the same manner as the proof of Theorem 2, the estimate for $\overline{u}$, (4.37), in which J is replaced by J', holds for $(r,t)\in\Sigma'$. $J'(r,t)$ is defined by

$$J'(r,t)=p^{-1}\log C_1-S(p)+ \tag{5.8}$$

$$+\begin{cases} (\frac{2}{p-1}+\frac{1}{2})\log(t-r-s)-(\kappa+\frac{1}{2})\log(t-r) & \text{if} \quad 0<\kappa\le\frac{1}{2}, \\ \frac{2}{p-1}\log(t-r-s)-\kappa\log(t-r) & \text{if} \quad \kappa>\frac{1}{2}, \end{cases}$$

where $S(p)$ is the one in (4.36). $J'(r,t)>0$ which lead to the contradiction as in section 4 follows from

$$(\frac{2}{p-1}-\kappa)\log(t-r)>\log C'\varepsilon^{-1} \quad \text{for} \quad (r,t)\in\Sigma' \quad \text{and} \quad t-r\ge 2s,$$

where

$$C'=((p/2+2)^2D_1^{-1})^{1/p}C_0^{-1}e^{S(p)+2p/(p-1)}$$

because (4.41) is still valid in this section though C_0 is defined by (5.3). Therefore, setting $t=2r$, we find that $J'(r,t)>0$ for $(r,t)\in\Sigma'$ follows from

$$t>C\,\varepsilon^{(\kappa-\frac{2}{p-1})^{-1}} \quad \text{for} \quad p>1 \quad \text{and} \quad 0<\kappa<\frac{2}{p-1},$$

where

$$C=2(C')^{(\kappa-\frac{2}{p-1})^{-1}}.$$

Thus we conclude that T has to satisfy

$$T\le C\,\varepsilon^{(\kappa-\frac{2}{p-1})^{-1}}.$$

This completes the proof of Theorem 3.

Appendix

In Appendix, we give a simple proof of the global existence of solutions to (1.1) with $p>p_0(2)$. Following R. T. Glassey [3], we define the norm for a continuous function $u(x,t)$ in $\mathbb{R}^2\times[0,\infty)$ satisfying (2.1):

$$\|u\|=\sup_{(x,t)\in\mathbb{R}^2\times[0,\infty)} N(|x|,t)|u(x,t)|, \tag{A.1}$$

where

$$N(r,t)= \begin{cases} k^{-q-1/2}(t+r+2k)^{1/2}(t-r+2k)^q & \text{for} \quad p>p_0(2),\ p\neq 4 \\ k^{-1}(t+r+2k)^{1/2}(t-r+2k)^{1/2}(\log\frac{t-r+3k}{k})^{-1} & \text{for} \quad p=4 \end{cases} \tag{A.2}$$

and

$$
\text{(A.3)} \qquad \begin{aligned} q &= \frac{p-3}{2} \quad \text{for} \quad p_0(2) < p < 4, \\ q &= \frac{1}{2} \qquad \text{for} \quad p > 4. \end{aligned}
$$

We notice that the norm is slightly modified the one in [3].

We shall show the following lemma which assures, by the method in §2, the global in time existence of solutions to (1.1) with small initial data.

LEMMA A.1. *There exist a positive constant C depending only on p such that*

$$
\text{(A.4)} \qquad \|L(|u|^p)\| \le Ck^2\|u\|^p, \quad p > p_0(2),
$$

for any continuous function u in $\mathbf{R}^2 \times [0, \infty)$ satisfying (2.1).

PROOF: The proof will be done by same method as in §3. In order to show a priori estimate (A.4), it is enough to prove, instead of (3.7),

$$
\text{(A.5)} \qquad I_j(r,t) \le Ck^2 N(r,t)^{-1} \quad \text{for} \quad j = 1, 2,
$$

where the I_j are defined in (3.6).

We first treat the integral I_1. When $4r > t + r + 2k$, we get, instead of (3.13),

$$
\text{(A.6)} \qquad I_1(r,t) \le \frac{\pi k^{pq+p/2}}{8\sqrt{2r}} \int_{t-r}^{t+r} \frac{(\alpha + 2k)^{(2-p)/2}}{\sqrt{\alpha - (t-r)}} d\alpha \int_{-k}^{t-r} (\beta + 2k)^{-pq} d\beta
$$
$$
\text{for} \quad p \neq 4
$$

The β-integral for $p = 4$ is

$$
\text{(A.7)} \qquad \int_{-k}^{t-r} (\beta + 2k)^{-2} (\log \frac{\beta + 3k}{k})^4 d\beta.
$$

Since $p > p_0(2) > 3$, it follows from (3.14) that the α-integral in (A.6) is dominated by

$$
2(\sqrt{2} + \frac{p-2}{p-3})(t - r + 2k)^{(3-p)/2}.
$$

Note that $pq > 1$ for $p > p_0$ (2). Then the β-integral in (A.6) is dominated by

$$\int_{-k}^{\infty} (\beta + 2k)^{-pq} d\beta = \frac{k^{1-pq}}{pq-1}.$$

The β-integral (A.7) is also dominated by

$$Ck^{-\delta} \int_{-k}^{\infty} (\beta + 2k)^{-2+\delta} d\beta = \frac{Ck^{-1}}{1-\delta}$$

for some small $\delta > 0$ and for some $C > 0$. Hence we get

$$\text{(A.8)} \qquad I_1(r,t) \le C(p) k^{1+p/2} (t + r + 2k)^{-1/2} (t - r + 2k)^{(3-p)/2}.$$

Since

$$(t - r + 2k)^{(4-p)/2} \le k^{(4-p)/2} \qquad \text{for} \quad p > 4,$$
$$\log \frac{t - r + 3k}{k} > \log 2 \qquad \text{for} \quad p = 4,$$

we conclude from (A.8) that (A.5) for I_1 is valid.

When $4r < t + r + 2k$, we get, instead of (3.16)

$$\text{(A.9)} \qquad I_1(r,t) \le \frac{\pi}{8} k^{pq+p/2} \int_{t-r}^{t+r} \frac{(\alpha + 2k)^{(1-p)/2}}{\sqrt{\alpha - (t-r)}} d\alpha \int_{-k}^{t-r} (\beta + 2k)^{-pq} d\beta$$
$$\text{for} \quad p \ne 4$$

The β-integral becomes (A.7) for $p = 4$. Since $p > p_0(2) > 2$ and $(t + r + 2k)$ is equivalent to $(t - r + 2k)$, it follows from (3.17) that the α-integral in (A.9) is dominated by

$$C(p)(t + r + 2k)^{-1/2} (t - r + 2k)^{(3-p)/2}.$$

Hence we also conclude in the same way as in the above that (A.5) for I_1 is valid.

Next we treat the integral I_2. Making use of the method deriving (3.21), we find that the α-integral in (3.19) or (3.20) is dominated by

$$\text{(A.10)} \qquad \begin{array}{ll} C(p)(t - r + 2k)^{(3-p)/2} & \text{for} \quad p_0(2) < p < 4, \\ C(p)(t - r + 2k)^{-1/2} \log \dfrac{t - r + 3k}{k} & \text{for} \quad p = 4, \\ C(p) k^{(4-p)/2} (t - r + 2k)^{-1/2} & \text{for} \quad p > 4. \end{array}$$

When $4r > t + r + 2k$, we get, instead of (3.19),

$$\text{(A.11)} \qquad I_2(r,t) \leq \frac{\pi k^{pq+p/2}}{8\sqrt{r}} \int_0^{t-r} \frac{(\alpha+2k)^{(2-p)/2}}{\sqrt{t-r-\alpha}} d\alpha \int_{-k}^{t-r} (\beta+2k)^{-pq} d\beta$$

$$\text{for} \quad p \neq 4.$$

The β-integral becomes (A.7) for $p = 4$. Since the β-integral in (A.11) is dominated by $C(p)k^{1-pq}$, we find from (A.10) and (A.11) that (A.5) for I_2 is valid.

When $4r < t+r+2k$, i.e., $t+r+2k < 2(t-r+2k)$, we get, instead of (3.20),

$$\text{(A.12)} \qquad I_2(r,t) \leq \frac{\pi}{8} k^{pq+p/2} \int_0^{t-r} \frac{(\alpha+2k)^{(2-p)/2}}{\sqrt{t-r-\alpha}} d\alpha \int_{-k}^{t-r} \frac{(\beta+2k)^{-pq}}{\sqrt{t-r-\beta}} d\beta$$

$$\text{for} \quad p \neq 4$$

The β-integral for $\quad p = 4 \quad$ is

$$\text{(A.13)} \qquad \int_{-k}^{t-r} \frac{(\beta+2k)^{-2}}{\sqrt{t-r-\beta}} (\log \frac{\beta+3k}{k})^4 d\beta.$$

Making use of the method deriving (3.22), we find that the β-integrals are dominated by

$$\text{(A.14)} \qquad C(p)k^{1-pq}(t+r+2k)^{-1/2}.$$

We give here the proof of (A.14) for the critical case where $p = 4$ and $t - r > k$, which implies $t-r > (t-r+2k)/4$. We find that, for some $C > 0$ and some small $\delta > 0$, the integral (A.13) dominated by

$$Ck^{-\delta} \int_{-k}^{t-r} \frac{(\beta+2k)^{-2+\delta}}{\sqrt{t-r-\beta}} d\beta.$$

Then, breaking the integral up into two pieces, we get

$$\begin{aligned} Ck^{-\delta} \int_{-k}^{(t-r)/2} \frac{(\beta+2k)^{-2+\delta}}{\sqrt{t-r-\beta}} d\beta &\leq \sqrt{2}Ck^{-\delta}(t-r)^{-1/2} \int_{-k}^{\infty} (\beta+2k)^{-2+\delta} d\beta \\ &\leq \frac{2\sqrt{2}Ck^{-1}}{1-\delta}(t-r+2k)^{-1/2} \\ &\leq \frac{4Ck^{-1}}{1-\delta}(t+r+2k)^{-1/2}. \end{aligned}$$

On the complement,

$$Ck^{-\delta}\int_{(t-r)/2}^{t-r}\frac{(\beta+2k)^{-2+\delta}}{\sqrt{t-r-\beta}}d\beta \le Ck^{-\delta}\left(\frac{t-r}{2}+2k\right)^{-2+\delta}\int_{(t-r)/2}^{t-r}\frac{d\beta}{\sqrt{t-r-\beta}}$$
$$\le Ck^{-\delta}2^{2-\delta}(t-r+2k)^{\delta-3/2}.$$
$$\le C\sqrt{2}2^{2-\delta}k^{-1}(t+r+2k)^{-1/2}.$$

Here we have used that $(t-r+2k)^{\delta-1} \le k^{\delta-1}$. Therefore, we conclude from (A.10), (A.12) and (A.14) that (A.5) for I_2 is valid. Thus the proof of Lemma A.1 is completed.

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Department of Mathematics
Hokkaido University
060 Sapporo, Japan