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And
Commutators**

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Invariant Subspaces In The Bidisc
And
Commutators

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Abstract. Let M be an invariant subspace of $L^2(\mathbb{T}^2)$ on the bidisc. V_1 and V_2 denote the multiplication operators on M by coordinate functions z and w , respectively. In this paper we study the relation between M and the commutator of V_1 and V_2^* . For example, M is studied when the commutator is self-adjoint or finite rank.

§1. Introduction

We let T^2 be the torus that is the cartesian product of 2 unit circles in \mathbb{C} . The usual Lebesgue spaces, with respect to the Haar measure m of T^2 , are denoted by $L^p = L^p(T^2)$, and $H^p = H^p(T^2)$ is the space of all f in L^p whose Fourier coefficients

$$\hat{f}(j, \ell) = \int_{T^2} f(z, w) \bar{z}^j \bar{w}^\ell dm(z, w)$$

are 0 as soon as at least one component of (j, ℓ) is negative.

A closed subspace M of L^2 is said to be invariant if

$$zM \subset M \text{ and } wM \subset M.$$

One can ask for a classification or an explicit description (in some sense) of all invariant subspaces of L^2 , but this seems out of reach. In the previous paper [8], the author studied the relation between M and the structure of $M \ominus wM$. An important role was played by invariant subspaces of the form FN ,

where F is unimodular and $H^2 \subseteq N \subseteq$ the closure of $\bigcup_{n=0}^{\infty} \bar{z}^n H^2$.

Such invariant subspaces relate with invariant subspaces which were studied previously in [1], [4], [3] and [2]. However the condition on $M \ominus wM$ ^{in [8]} is a little unnatural. In this paper we will find natural conditions on M which imply that M is of the form FN .

Given an invariant subspace M of L^2 , V_1 and V_2 denote the restriction of multiplications by z and w on M , respectively. Put

$$A_n = V_1^n V_2^* - V_2^* V_1^n \quad (n \geq 1)$$

and write $A = A_1$. In this paper we will describe invariant subspaces of L^2 when $A = 0$ or $\bigcap_{n=1}^{\infty} \text{Ker } A_n \neq \emptyset$. Mandrekar [6] described M when $A = 0$ and M is in H^2 . In fact, Theorem 2 in [6] is a corollary of (2) of Theorem 5 in [8] that was independently proved. In Section 2 we study invariant subspaces under several hypotheses on the restriction of V_1 to the kernel of V_2^* . These are previously known results [8] except for one of them. In Section 3 A_n ($n \geq 1$) is studied using results in Section 2. In Section 4 invariant subspaces are studied when A is finite rank. In Section 5 we try to prove that if A is selfadjoint then $A = 0$. In Section 6 we give several examples that results in the previous sections can be applied to.

We define several subspaces of L^2 which will be used later. Let $C(T^2)$ be the space of complex-valued continuous functions.

(1) H_1 or H_2 is the set of f (in L^2) with Fourier series

$$\sum_{\substack{j>0 \\ k=0}} a_{jk} z^j w^k \quad \text{or} \quad \sum_{\substack{k>0 \\ j=0}} a_{jk} z^j w^k,$$

respectively. Put $A_j = H_j \cap C(T^2)$ for $j = 1, 2$.

(2) L_1 or L_2 is the set of f (in L^2) with Fourier series

$$\sum_{k=0}^{\infty} a_{jk} z^j w^k \quad (\text{no restriction on } j)$$

or

$$\sum_{j=0}^{\infty} a_{jk} z^j w^k \quad (\text{no restriction on } k) ,$$

respectively. Put $C_j = L_j \cap C(T^2)$ for $j = 1, 2$.

(iii) H_1 or H_2 is the set of f (in L^2) with Fourier series :

$$\sum_{k \geq 0} a_{jk} z^j w^k \quad (\text{no restriction on } j)$$

or

$$\sum_{j \geq 0} a_{jk} z^j w^k \quad (\text{no restriction on } k)$$

respectively. Put $B_j = H_j \cap C(T^2)$ for $j = 1, 2$.

§2. The restriction of V_1 to $\text{Ker } V_2^*$

Let M be an invariant subspace of L^2 . Put

$$S_1 = M \ominus wM \quad \text{and} \quad S_2 = M \ominus zM.$$

$S_1 = \text{Ker } V_2^*$ and $S_2 = \text{Ker } V_1^*$. In this section we give a new result and results in the previous paper [8], which will be used in this paper.

Proposition 1. Let M be an invariant subspace of L^2 . V_2^* is an one-to-one operator if and only if $M = \chi_{E_1} F H_2 + \chi_{E_2} L^2$ where F is unimodular, and χ_{E_j} is a characteristic function of Borel set E_j on T^2 , $\chi_{E_1} \in L_2$ and $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e..

Proof is in [3] and [7, pl64 ~ pl65] because V_2^* is one-to-one if and only if $S_1 = \{0\}$.

Proposition 2. Let M be an invariant subspace of L^2 in which V_2^* has a nontrivial kernel.

(1) $V_1(\text{Ker } V_2^*) = \text{Ker } V_2^*$ if and only if $M = \chi_{E_1} F H_1 + \chi_{E_2} L^2$ where F is unimodular and, χ_{E_1} is a nonzero function in L_1 and $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e..

(2) $V_1(\text{Ker } V_2^*) \subsetneq \text{Ker } V_2^*$ if and only if $M = F H^2$ for some unimodular function F .

Proof. Theorem 5 in [8] shows that $zS_1 = S_1$ (or $zS_1 \subsetneq S_1$) if and only if M has the form : $M = \chi_{E_1} F H_1 + \chi_{E_2} L^2$ in (1) (or the form : $M = F H^2$ in (2), respectively). This implies the proposition because $\text{Ker } V_2^* = S_1$.

By Propositions 1 and 2, we are interested in an invariant subspace such that $\text{Ker } V_2^*$ is not invariant under V_1 .

Proposition 3. Let M be an invariant subspace of L^2 in which V_2^* has a nontrivial kernel.

(1) There exists a nonzero function f in $\text{Ker } V_2^*$ such that $V_1^n f$ belongs to $\text{Ker } V_2^*$ for any integer n if and only if $M = \chi_{E_1} F H_1 + \chi_{E_2} L^2$ where F is unimodular, and χ_{E_1} is a non-zero function in L_1 and $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e..
($V_1^n = V_1^{*(-n)}$ when $n < 0$)

(2) There exists a function f in $\text{Ker } V_2^*$ such that $V_1^n f$ belongs to $\text{Ker } V_2^*$ for any $n \geq 0$ and $V_1^\ell f$ is not in $\text{Ker } V_2^*$ for some $\ell < 0$ if and only if $M = FN$ where N is an invariant subspace which contains H^2 and is contained properly in H_1 , and F is unimodular.

Proof. (1) (the hypothesis) under that $|f| > 0$ a.e., and (2) were proved in [8, Theorem 6]. We will prove (1) in general. Put $M_1 = \bigcap_{n \geq 0} w^n M$ then $M = (\sum_{n \geq 0} w^n S_1) \oplus M_1$. Let D be the largest closed subspace of S_1 with $zD \subseteq D$. If we let $D_3 = S_1 \ominus D$, $D_2 = \bigcap_{n \geq 0} z^n D$ and $D_1 = D \ominus D_2$ then

$$M = \left(\sum_{n \geq 0} w^n D_1 \right) \oplus \left(\sum_{n \geq 0} w^n D_2 \right) \oplus \left(\sum_{n \geq 0} w^n D_3 \right) \oplus M_1.$$

Since $zD_2 = D_2$, by [3] and [7, p164 - p165]

$$\left(\sum_{n \geq 0} \oplus D_2 w^n \right) \oplus M_1 = \chi_{E_1} F_1 H_1 \oplus \chi_{E_2} F_2 H_2 \oplus \chi_{E_3} L^2$$

where $\chi_{E_j} \in L_j$ ($j = 1, 2$), $\chi_{E_1} + \chi_{E_3} \leq 1$ a.e. and $\chi_{E_2} + \chi_{E_3} \leq 1$ a.e.. If there exists a nonzero function f in $\text{Ker } V_2^*$ such that $z^n f$ belongs to $\text{Ker } V_2^*$ for any integer n , then χ_{E_1}

is nonzero. Since $\chi_{E_1} (\overline{F}_1 M \ominus H_1)$ is invariant under multiplication of w , $\chi_{E_1} (\overline{F}_1 M \ominus H_1) = \{0\}$ and hence $\chi_{E_1} M = \chi_{E_1} F_1 H_1$.

Since $\chi_{E_1} M \subset M$, $(1 - \chi_{E_1})M \subset M$ and $M = \chi_{E_1} M \oplus (1 - \chi_{E_1})M$. We can prove $z(1 - \chi_{E_1})M = (1 - \chi_{E_1})M$. For $zA_1(1 - \chi_{E_1})M \subset (1 - \chi_{E_1})M$ and

$$[zA_1(1 - \chi_{E_1})] = (1 - \chi_{E_1})L_1 \ni \bar{z}(1 - \chi_{E_1}).$$

Therefore $\bar{z}(1 - \chi_{E_1})M \subset (1 - \chi_{E_1})M$ and hence $z(1 - \chi_{E_1})M = (1 - \chi_{E_1})M$. Hence

$$(1 - \chi_{E_1})M = \chi_{E_4} F_4 H_1 + \chi_{E_3} L^2$$

where $\chi_{E_4} \in L_1$, $\chi_{E_4} + \chi_{E_3} \leq 1$ a.e. and F_4 is unimodular.

Thus M has the form $\chi_{E'} F H_1 + \chi_{E''} L^2$ for some unimodular F where $\chi_{E'} = \chi_{E_1} + \chi_{E_4}$ and $\chi_{E''} = \chi_{E_3}$

§3. Commutator of V_1^n and V_2^*

Put $A_n = V_1^n V_2^* - V_2^* V_1^n$ ($n \geq 1$) and write $A = A_1$. The following trivial lemmas are important.

Lemma 1. $A_n V_2 = 0$ for any $n \geq 1$ and hence $\bigcap_{n=1}^{\infty} \text{Ker } A_n \supseteq \text{WM}$.

Lemma 2. For any f in $\text{Ker } V_2^*$, $f \in \text{Ker } A_n$ if and only if $z^n f \in \text{Ker } V_2^*$.

Proof. When $f \in \text{Ker } V_2^*$, if $f \in \text{Ker } A_n$ then $V_2^* V_1^n f = 0$ and hence $z^n f \in \text{Ker } V_2^*$. Conversely if $z^n f \in \text{Ker } V_2^*$ then $A_n f = V_1^n V_2^* f = 0$.

In general A_n is a nonzero operator.

The structure of M is simple $\left(\text{---} \right)$ when $A = 0$ or $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supseteq \text{WM}$.

WM. The following theorems are corollaries of [8, Theorem 5],
(, which make this precise,)

Theorem 4. Let M be an invariant subspace of L^2 with $A = 0$. Then one and only one of the following occurs and vice versa.

(1) $M = \chi_{E_1} F H_1 + \chi_{E_2} L^2$ where χ_{E_1} is in L_1 , $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e. and F is unimodular.

(2) $M = \chi_{E_1} F H_2 + \chi_{E_2} L^2$ where χ_{E_1} is in L_2 , $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e. and F is unimodular.

(3) $M = F H^2$ for some unimodular function.

Conversely, if (1), (2) or (3) holds for an invariant subspace M of L^2 , then $A = 0$.

Proof. If M has the form (1) then $S_2 = \{0\}$ and hence $A^* = 0$ because $A^*V_1 = 0$. Therefore $A = 0$. If M has the form (2) then $S_1 = \{0\}$ and hence $A = 0$ because $AV_2 = 0$ by Lemma 1. If M has the form (3) then $zS_1 \subseteq S_1$ and hence $V_1 \text{Ker } V_2^* \subseteq \text{ker } V_2^*$. Therefore $A = 0$ on $\text{Ker } V_2^*$ and $A = 0$ because $AV_2 = 0$. Conversely suppose $A = 0$. Then $zS_1 \subseteq S_1$. If $S_1 = 0$, then by Proposition 1, M has the form (2). If $S_1 \neq 0$, then (since $zS_1 \subseteq S_1$) Proposition 2 implies that M has either the form (1) or the form (3).

Mandrekar [6] considered Theorem 4 when M is in H^2 . Then since $\bigcap_{n=1}^{\infty} z^n H^2 = \bigcap_{n=1}^{\infty} w^n H^2 = \{0\}$, M has the form (3). Now we wish to consider invariant subspaces with $A \neq 0$.

Theorem 5. Let M be an invariant subspace of L^2 such that $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supseteq wM$. Then one and only one of the following holds.

(1) $M = \chi_{E_1} F \mathbb{H}_1 + \chi_{E_2} L^2$ where χ_{E_1} is a nonzero function in L_1 , $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e. and F is unimodular.

(2) $M = FN$ where N is an invariant subspace which contains H^2 and is contained properly in \mathbb{H}_1 , and F is unimodular.

Conversely, if (1), (2) or (3) holds for an invariant subspace M of L^2 , then $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supseteq wM$.

Proof. Suppose $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supseteq wM$. Then there exists $f \neq 0$ in $(\bigcap_{n=1}^{\infty} \text{Ker } A_n) \ominus wM$. In particular, $f \in \text{Ker } V_2^*$. By Lemma 2, $z^n f \in \text{Ker } V_2^*$ for $n \geq 0$, so by Proposition 3, M is of either the form (1) or the form (2). Conversely if M has the form (1), then $A = 0$ by Theorem 4. Then V_2^* commutes with V_1 , so it commutes with every power of V_1 , hence $A_n = 0$ for $n \geq 1$. Thus $\bigcap_{n=1}^{\infty} \text{Ker } A_n = M$. Since χ_{E_1} is a nonzero function, $M \neq wM$ because $M \ominus wM = \chi_{E_1} FL_1$. Therefore $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supseteq wM$. If M has the form (2), (by (2)) of Proposition 3 there exists a function f in $\text{Ker } V_2^*$ such that $z^n f \in \text{Ker } V_2^*$ for any $n \geq 0$. By Lemma 2, $f \in \text{Ker } A_n$ while f is orthogonal to wM because $f \in \text{Ker } V_2^*$. Therefore $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supseteq wM$.

§4. Finite rank commutator

Theorem 4 describes those invariant subspaces with $A = 0$. Now we are interested in invariant subspaces in which A has finite rank. The following lemma was pointed out to the author by Professor K. Takahashi. It implies if $A_n = 0$, then $A = 0$.

Lemma 3. $V_1^* A_n = A_{n-1}$ for $n > 1$ and hence $\text{Ker } A_n \subset \text{Ker } A_{n-1}$.

Proof is clear.

Proposition 6. Let M be an invariant subspace of L^2 .

(1) If $\dim \text{Ker } V_2^*$ is finite then A_n is finite rank r_n , $\sup_n r_n < \infty$, and $\bigcap_{n=1}^{\infty} \text{Ker } A_n = \text{wM}$.

(2) Suppose $\dim \text{Ker } V_2^*$ is infinite. If A_n is finite rank r_n and $\sup_n r_n < \infty$ then $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supset \text{wM}$.

Proof. (1) By Lemma 1 A_n is finite rank r_n and $r_n \leq \dim \text{Ker } V_2^*$. By Lemma 2 if $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supset \text{wM}$, there exists a nonzero function f such that $f \in M \ominus \text{wM}$ and $z^n f \in M \ominus \text{wM}$ for any $n \geq 0$, and hence this implies that $\dim \text{Ker } V_2^*$ is infinite. (2) By Lemma 3 and hypothesis, $r_n \leq r_{n+1}$, so ultimately, r_n is constant. Also $r_n = \dim (\text{Ker } A_n)^\perp$, while $(\text{Ker } A_{n+1})^\perp$ contains $(\text{Ker } A_n)^\perp$, so ultimately $(\text{Ker } A_n)^\perp$ does not change with n . But then neither does $\text{Ker } A_n$. In other words, $\text{Ker } A_n =$ $\text{Ker } A_{n_0}$ if $n \geq n_0$. Hence $\bigcap_{n=1}^{\infty} \text{Ker } A_n = \text{Ker } A_{n_0}$ for some $n_0 \geq 1$. Therefore

if $\bigcap_{n=1}^{\infty} \text{Ker } A_n = wM$ then $\text{Ker } A_{n_0} = wM$ and hence $(\text{Ker } A_{n_0})^{\perp} = \text{Ker } V_2^*$. Since $\dim \text{Ker } V_2^*$ is infinite, this contradicts the hypothesis that A_{n_0} is finite rank and hence $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supseteq wM$.

Corollary 1. Let M be an invariant subspace of H^2 . If A_n is finite rank r_n and $\sup_n r_n < \infty$ then $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supseteq wM$.

Proof. If $\dim \text{Ker } V_2^*$ is finite, by [8, Theorem 3] there exists a nonzero function g in L^{∞} such that $gM \subset M$ and $g \notin H^{\infty}$. By [2, Proposition 3] this implies that $M \not\subset H^2$, so $\dim \text{Ker } V_2^*$ is infinite. (2) of Proposition 6 implies the corollary.

Corollary 2. Let M be an invariant subspace with $\bigcap_{n=1}^{\infty} z^n M = \{0\}$ then the following (1) and (2) are equivalent.

(1) $\dim \text{Ker } V_2^* = \infty$, A_n is finite rank r_n and $\sup_n r_n = r < \infty$.

(2) $M = FN$ for some unimodular F and some invariant subspace with $N = K \oplus H^2 \subsetneq H_1$. Moreover $N \ominus wN = (K \ominus wK) \oplus H_1$ and S is the largest closed subspace of finite codimension r of $N \ominus wN$ such that $zS \subset S$ then $S \supseteq H_1$.

Proof. (1) \Rightarrow (2). By (2) of Proposition 6 $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supseteq wM$ and hence by hypothesis and Theorem 5 M has the form FN . If we

put $S' = [\bigcap_{n=1}^{\infty} \text{Ker } A_n] \ominus wM$, then by Lemma 2 S' is the largest closed subspace of $S_1 = M \ominus wM$ with $zS' \subset S'$, and by hypothesis $\dim(S_1 \ominus S') = r < \infty$. Put $S = \overline{FS'}$ then S is the desired subspace and (2) follows. (2) \Rightarrow (1). By hypothesis of (2), FS is the largest closed subspace of S_1 with $zFS \subset FS$ and hence $FS = [\bigcap_{n=1}^{\infty} \text{Ker } A_n] \ominus wM$. This implies (1).

§5. Selfadjoint commutator

In this section we will study the following conjecture : If $A = A^*$ then $A = 0$. Unfortunately we could not do it. However we will give a few partial answers.

Lemma 4. $A_n = \sum_{j=0}^{n-1} V_1^{n-1-j} A V_1^j$. If $A = A^*$ then $A_n = V_1^{n-1} A$ and hence $\text{Ker } A_n = \text{Ker } A$.

Proof. For any $n > 1$

$$\begin{aligned} A_n &= V_1^{n-1} (V_1 V_2^*) - (V_2^* V_1^{n-1}) V_1 \\ &= V_1^{n-1} (V_1 V_2^*) - (V_1^{n-1} V_2^* - A_{n-1}) V_1 \\ &= V_1^{n-1} A + A_{n-1} V_1 \end{aligned}$$

and hence $A_n = \sum_{j=0}^{n-1} V_1^{n-1-j} A V_1^j$. If $A = A^*$ then $A V_1 = 0$ and hence $A_n = V_1^{n-1} A$.

Proposition 7. Suppose M is an invariant subspace with $A = A^*$.

- (1) $A_n^2 = 0$ for any $n > 1$.
- (2) If A is finite rank r then A_n is also finite rank r for any $n > 1$.
- (3) If $\text{Ker } V_1^* \cap \text{Ker } V_2^* = \{0\}$ then $A = 0$.
- (4) If $\text{Ker } V_1^* \cap \text{Ker } V_2^* \neq \{0\}$ then $\bigcap_{n=1}^{\infty} \text{Ker } A_n \supseteq \text{wM}$.

Proof. (1) If $A = A^*$ then $A_n^2 = V_1^{n-1} A V_1^{n-1} = 0$ for $n > 1$ by Lemma 4 because $A^* V_1 = 0$. (2) is clear by Lemma 4. Since $A V_1 = A V_2 = 0$, $A = 0$ on $[zM + wM]$, that is, the closed linear span of $zM + wM$.

If $\text{Ker } V_1^* \cap \text{Ker } V_2^* = \{0\}$ then $M = [zM + wM]$ and hence $A = 0$. Suppose $\text{Ker } V_1^* \cap \text{Ker } V_2^* \neq \{0\}$. If $zM \not\subseteq wM$ then $[zM + wM] \not\supseteq wM$ and hence $\text{Ker } A \not\supseteq wM$. By Lemma 4 $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supseteq wM$. It does not happen that $zM \subset wM$ (pointed out me privately by Professor K.Takahashi). For if $f \in \text{Ker } V_1^* \cap \text{Ker } V_2^*$ then $A = A^*$ implies

$$V_2^* V_1 f = V_1^* V_2 f.$$

If $zM \subseteq wM$ then $V_2^* V_1 f = \bar{w} z f$, and hence

$$\|f\| = \|V_2^* V_1 f\| = \|V_1^* V_2 f\|.$$

Thus $V_1^* V_2 f = \bar{z} w f$, and so $\bar{w} z f = \bar{z} w f$, hence $f = 0$. (Otherwise, $\underbrace{z^2}_{= w^2}$ in a set of positive measure).

We don't know whether if $A = A^*$ then $A = 0$. However there exist a lot of invariant subspaces such that A is unitarily equivalent to A^* and $A \neq 0$ (see Example 2). Put $Uf(z, w) = f(w, z)$ for any f in L^2 , then U is a unitary operator on L^2 and U^2 is the identity operator I on L^2 . Let M be an invariant subspace which is invariant under U . Then U is isometric on M and $U^2 = I$ on M . Hence U can be assumed to be a unitary operator on M .

Proposition 8. Let M be an invariant subspace of L^2 which is invariant under U . Then $V_2 U = U V_1$ and $U A^* U = A$.

§6. Examples

In the previous sections, invariant subspaces M , which satisfy $\bigcap_{n=0}^{\infty} \text{Ker } A_n \supsetneq wM$, were important. In this section, we will give several examples of such invariant subspaces.

Example 1. Suppose M is a non trivial invariant subspace in H^2 . Let R be the orthogonal projection in L^2 with range $H^2 \ominus M$, and let the operator J_z on $H^2 \ominus M$ be defined by $J_z f = R(zf)$. If J_z is of finite rank n then there exists an analytic polynomial p of z of degree n such that $p(S_z) = 0$ and hence $p(z)H^2 \subset M$. The inner part of $p(z)$ is a finite Blaschke product $F = F(z)$ of degree m and $m \neq 0$ (because $M \neq H^2$.) Since $\bar{F}H^2$ is in H_1 , $N = \bar{F}M$ lies between H^2 and H_1 . Then $M = FN$, $N = K \oplus H^2$ and $\dim K \ominus wK \leq m$. For

$$K \subset \sum_{j=0}^{\infty} \oplus (\bar{F}H_1 \ominus H_1)w^j$$

and hence $\dim K \ominus wK \leq \dim (\bar{F}H_1 \ominus H_1)$. By Theorem 5, $\bigcap_{n=1}^{\infty} \text{Ker } A_n \supsetneq wM$ and by Corollary 2, A_n is finite rank r_n and $\sup_n r_n \leq m$.

Since $\dim (H^2 \ominus M) < \infty$ and $\dim (H^2 \ominus \bar{F}H^2) = \infty$, $M \neq \bar{F}H^2$ and hence $\dim K \ominus wK \neq 0$. By Corollary 2, $0 < \sup_n r_n \leq m$. For if $\sup_n r_n = 0$ then $(K \ominus wK) \oplus H_1$ is an invariant subspace under the multiplication of z and $(K \ominus wK) \oplus H_1 \subseteq \bar{F}H_1$. By Beurling's theorem (cf. [4, p4])

$$(K \ominus wK) \oplus H_1 = \bar{q}H_1$$

and q is nonconstant finite Blaschke product of degree $\leq m$ because $K \ominus wK \neq \{0\}$. Therefore $N = \bar{q}H^2$. If $q \neq F$ then $M = GH^2$ and $m > l = \text{degree of } G \text{ where } G = F\bar{q}$. Hence J_z is of finite rank l . This contradiction implies that $0 \neq \sup_n r_n$. By Theorem 4, M does not have the form qH^2 for any unimodular q . This is known in [2, Corollary 2].

Example 2. If M is an invariant subspace in H^2 , of finite codimension n , then by Example 1 $M = F_1N_1 = F_2N_2$ where F_1 and F_2 are finite Blaschke products of z and w , respectively, and $H^2 \subset N_j \subset H_j$ ($j = 1, 2$). Then both A and A^* are finite rank of degree $m \leq n$ and $m \neq 0$. By [8, Theorem 3], $\dim \text{Ker } V_2^* = \dim \text{Ker } V_1^* = \infty$. By Example 1, M does not have the form $\bar{q}H^2$ for any unimodular q . Put $M = [zH^2 + wH^2]$ then M is of finite codimension 1. Moreover M is invariant under U . Hence A is rank one and $UA^*U = A$.

Example 3. Let M be an invariant subspace of L^2 . Invariant subspaces M which satisfy $w^n M \supset zM$ for any $n \geq 1$ or $z^n M \supset wM$ for any $n \geq 1$, were studied in [3], [4] and [7]. In general, if $wM \supset zM$ $AV_1 = 0$. For $AV_2 = 0$ by Lemma 1 and hence $AV_1 = 0$ because $V_2M \supset V_1M$. Hence by the first part of Lemma 4 $A_n = V_1^{n-1}A$. Thus $\text{Ker } A_n = \text{Ker } A$ for any $n \geq 1$. If $w^n M \supset zM$ for any $n \geq 1$ it is known (see [7]) that

$$M = q(H_2 \oplus z.H_2)$$

or

$$M = \chi_{E_1} H_2 \oplus \chi_{E_2} L^2$$

where q is unimodular, $\chi_{E_1} \in L_2$ and $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e..

Hence if $wM \neq M$ then $M \ominus wM = \{q\}$, $\dim \text{Ker } V_2^* = 1$, $\text{Ker } A_n = wM$ and A_n is rank 1 for any $n \geq 1$. If $z^n M \supset wM$ for any $n \geq 1$, by Proposition 6 then $\bigcap_{n=1}^{\infty} \text{Ker } A_n \supsetneq wM$ and $M \ominus wM = q(H_1 \oplus \bar{z}H_1)$

for some

unimodular q . In [3], the authors considered generalizations of the above invariant subspaces. That is, for any fixed $\ell \geq 1$ M satisfies $w^n M \supset z^\ell M$ for any $n \geq 1$ or $z^n M \supset w^\ell M$ for any $n \geq 1$. They described completely such invariant subspaces and showed that if $zM \neq M$ or $wM \neq M$ then $M = FN$, where F is unimodular, and $H_1 \supset N \supset z^\ell H_1$ or $H_2 \supset N \supset w^\ell H_2$. Hence if $zM \neq M$ and $z^n M \supset w^\ell M$ then $\bigcap_{n=1}^{\infty} \text{Ker } A_n \supsetneq wM$.

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