Invariant Subspaces In The Bidisc
And
Commutators

Takahiko Nakazi

Series ♯120. August 1991

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

♯ 93: A. Arai, A theorem on essential self-adjointness with application to Hamiltonians in non-relativistic quantum field theory, 23 pages. 1990.

♯ 94: Y. Okabe, A. Inoue, On the exponential decay of the correlation functions for KMO-Langevin equations, 13 pages. 1990.

♯ 95: T. Sano, Y. Watatani, Angles between two subfactors, 62 pages. 1990.

♯ 96: S. Ninomiya, The Fourier-Sato transformation of pure sheaves, 22 pages. 1990.

♯ 97: Y. Okabe, A. Inoue, The theory of KM_2O-Langevin equations and its applications to data analysis (II): Causal analysis (1), 51 pages. 1990.

♯ 98: J. Ławrynowicz, S. Koshi and O. Suzuki, Dualities generated by the generalised Hurwitz problem and variation of the Yang-Mills field, 17 pages. 1991.

♯ 99: R. Agemi, K. Kubota and H. Takamura, On certain integral equations related to nonlinear wave equations, 52 pages. 1991.

♯ 100: S. Izumiya, Geometric singularities for Hamilton-Jacobi equation, 13 pages. 1991.

♯ 101: S. Izumiya, Legendrian singularities and first order differential equations, 16 pages. 1991.

♯ 102: A. Munemasa, Y. Watatani, Orthogonal pairs of $*$-subalgebras and association schemes, 11 pages. 1991.

♯ 103: A. Arai, O. Ogurisu, Meromorphic $N = 2$ Wess-Zumino supersymmetric quantum mechanics, 27 pages. 1991.

♯ 104: H. Takamura, Global existence of classical solutions to nonlinear wave equations with spherical symmetry for small data with noncompact support in three space dimensions, 14 pages. 1991.

♯ 105: R. Agemi, Blow-up of solutions to nonlinear wave equations in two space dimensions, 11 pages. 1991.

♯ 106: T. Nakazi, Extremal problems in H^p, 13 pages. 1991.

♯ 107: T. Nakazi, ρ-dilations and hypo-Dirichlet algebras, 15 pages. 1991.

♯ 108: A. Arai, An abstract sum formula and its applications to special functions, 25 pages. 1991.

♯ 109: Y.-G. Chen, Y. Giga and S. Goto, Analysis toward snow crystal growth, 18 pages. 1991.

♯ 110: T. Hibi, M. Wakayama, A q-analogue of Capelli's identity for $GL(2)$, 7 pages. 1991.

♯ 111: T. Nishimori, A qualitative theory of similarity pseudogroups and an analogy of Sacksteder's theorem, 13 pages. 1991.

♯ 112: K. Matsuda, An analogy of the theorem of Hector and Duminy, 10 pages. 1991.

♯ 113: S. Takahashi, On a regularity criterion uo to the boundary for weak solutions of the Navier-Stokes equations, 23 pages. 1991.

♯ 114: T. Nakazi, Sum of two inner functions and exposed points in H^1, 18 pages. 1991.

♯ 115: A. Arai, De Rham operators, Laplacians, and Dirac operators on topological vector spaces, 27 pages. 1991.

♯ 116: T. Nishimori, A note on the classification of non-singular flows with transverse similarity structures, 17 pages. 1991.

♯ 117: T. Hibi, A lower bound theorem for Ehrhart polynomials of convex polytopes, 6 pages. 1991.

♯ 118: R. Agemi, H. Takamura, The lifespan of classical solutions to nonlinear wave equations in two space dimensions, 30 pages. 1991.

♯ 119: S. Altschuler, S. Angenent and Y. Giga, Generalized motion by mean curvature for surfaces of rotation, 15 pages. 1991.

Invariant Subspaces In The Bidisc And Commutators

Takahiko Nakazi*

Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060, Japan

*This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education.

1980 Mathematics Subject Classification. Primary 47 A 15, 47 B 47, 47 B 20, 46 J 15, 32 A 35.

Abstract. Let M be an invariant subspace of $L^2(T^2)$ on the bidisc. V_1 and V_2 denote the multiplication operators on M by coordinate functions z and w, respectively. In this paper we study the relation between M and the commutator of V_1 and V_2^*. For example, M is studied when the commutator is self-adjoint or finite rank.

§1. Introduction

We let T^2 be the torus that is the cartesian product of 2 unit circles in $\mathbb{C}$. The usual Lebesgue spaces, with respect to the Haar measure m of T^2, are denoted by $L^p = L^p(T^2)$, and $H^p = H^p(T^2)$ is the space of all f in L^p whose Fourier coefficients

$$\hat{f}(j,\ell) = \int_{T^2} f(z, w)\bar{z}^j\bar{w}^\ell dm(z, w)$$

are 0 as soon as at least one component of (j, ℓ) is negative.

A closed subspace M of L^2 is said to be invariant if

$$zM \subset M \quad \text{and} \quad wM \subset M.$$

One can ask for a classification or an explicit description (in some sense) of all invariant subspaces of L^2, but this seems out of reach. In the previous paper [8], the author studied the relation between M and the structure of $M \ominus wM$. An important role was played by invariant subspaces of the form FN, where F is unimodular and $H^2 \subsetneq N \subsetneq$ the closure of $\bigcup_{n=0}^{\infty} \bar{z}^n H^2$. Such invariant subspaces relate with invariant subspaces which were studied previously in [1], [4], [3] and [2]. However the condition on $M \ominus wM$ in [8] is a little unnatural. In this paper we will find natural conditions on M which imply that M is of the form FN.

Given an invariant subspace M of L^2, V_1 and V_2 denote the restriction of multiplications by z and w on M, respectively. Put

$$A_n = V_1^n V_2^* - V_2^* V_1^n \quad (n \geq 1)$$

and write $A = A_1$. In this paper we will describe invariant subspaces of L^2 when $A = 0$ or $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$. Mandrekar [6] described M when $A = 0$ and M is in H^2. In fact, Theorem 2 in [6] is a corollary of (2) of Theorem 5 in [8] that was independently proved. In Section 2 we study invariant subspaces under several hypotheses on the restriction of V_1 to the kernel of V_2^*. These are previously known results [8] except for one of them. In Section 3 A_n $(n \geq 1)$ is studied using results in Section 2. In Section 4 invariant subspaces are studied when A is finite rank. In Section 5 we try to prove that if A is selfadjoint then $A = 0$. In Section 6 we give several examples that results in the previous sections can be applied to.

We define several subspaces of L^2 which will be used later. Let $C(T^2)$ be the space of complex-valued continuous functions.

(i) H_1 or H_2 is the set of f (in L^2) with Fourier series

$$\sum_{\substack{j \geq 0 \\ k = 0}} a_{jk} z^j w^k \quad \text{or} \quad \sum_{\substack{k > 0 \\ j = 0}} a_{jk} z^j w^k,$$

respectively. Put $A_j = H_j \cap C(T^2)$ for $j = 1. 2$.

(ii) L_1 or L_2 is the set of f (in L^2) with Fourier series

$$\sum_{k=0} a_{jk} z^j w^k \quad \text{(no restriction on } j\text{)}$$

or

$$\sum_{j=0} a_{jk} z^j w^k \quad \text{(no restriction on } k) ,$$

respectively. Put $C_j = L_j \cap C(T^2)$ for $j = 1, 2$.

(iii) $\mathbb{H}_1$ or $\mathbb{H}_2$ is the set of f (in L^2) with Fourier series :

$$\sum_{k \geq 0} a_{jk} z^j w^k \quad \text{(no restriction on } j)$$

or

$$\sum_{j \geq 0} a_{jk} z^j w^k \quad \text{(no restriction on } k)$$

respectively. Put $B_j = \mathbb{H}_j \cap C(T^2)$ for $j = 1, 2$.

§2. The restriction of V_1 to Ker V_2^*

Let M be an invariant subspace of L^2. Put

$$S_1 = M \ominus wM \quad \text{and} \quad S_2 = M \ominus zM.$$

$S_1 = \text{Ker } V_2^*$ and $S_2 = \text{Ker } V_1^*$. In this section we give a new result and results in the previous paper [8], which will be used in this paper.

Proposition 1. Let M be an invariant subspace of L^2. V_2^* is an one-to-one operator if and only if $M = \chi_{E_1} F H_2 + \chi_{E_2} L^2$ where F is unimodular, and χ_{E_j} is a characteristic function of Borel set E_j on T^2, $\chi_{E_1} \in \mathcal{L}_2$ and $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e..

Proof is in [3] and [7, p164 ~ p165] because V_2^* is one-to-one if and only if $S_1 = \{0\}$.

Proposition 2. Let M be an invariant subspace of L^2 in which V_2^* has a nontrivial kernel.

(1) $V_1(\text{Ker } V_2^*) = \text{Ker } V_2^*$ if and only if $M = \chi_{E_1} F H_1 + \chi_{E_2} L^2$ where F is unimodular and, χ_{E_1} is a nonzero function in $\mathcal{L}_1$ and $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e..

(2) $V_1(\text{Ker } V_2^*) \subsetneq \text{Ker } V_2^*$ if and only if $M = FH^2$ for some unimodular function F.

Proof. Theorem 5 in [8] shows that $zS_1 = S_1$ (or $zS_1 \subsetneq S_1$) if and only if M has the form : $M = \chi_{E_1} F H_1 + \chi_{E_2} L^2$ in (1) (or the form : $M = FH^2$ in (2), respectively). This implies the proposition because $\text{Ker } V_2^* = S_1$.

By Propositions 1 and 2, we are interested in an invariant subspace such that $\operatorname{Ker} V_2^*$ is not invariant under V_1.

Proposition 3. Let M be an invariant subspace of L^2 in which V_2^* has a nontrivial kernel.

(1) There exists a nonzero function f in $\operatorname{Ker} V_2^*$ such that $V_1^n f$ belongs to $\operatorname{Ker} V_2^*$ for any integer n ($V_1^n = V_1^{*(-n)}$ when $n < 0$) if and only if $M = \chi_{E_1} F \mathbb{H}_1 + \chi_{E_2} L^2$ where F is unimodular, and χ_{E_1} is a nonzero function in $\mathcal{L}_1$ and $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e..

(2) There exists a function f in $\operatorname{Ker} V_2^*$ such that $V_1^n f$ belongs to $\operatorname{Ker} V_2^*$ for any $n \geq 0$ and $V_1^{\ell} f$ is not in $\operatorname{Ker} V_2^*$ for some $\ell < 0$ if and only if $M = FN$ where N is an invariant subspace which contains H^2 and is contained properly in $\mathbb{H}_1$, and F is unimodular.

Proof. (1) under the hypothesis that $|f| > 0$ a.e., and (2) were proved in [8, Theorem 6]. We will prove (1) in general. Put $M_1 = \bigcap_{n \geq 0} w^n M$ then $M = (\sum_{n \geq 0} \oplus\, w^n S_1) \oplus M_1$. Let D be the largest closed subspace of S_1 with $zD \subseteq D$. If we let $D_3 = S_1 \ominus D$, $D_2 = \bigcap_{n \geq 0} z^n D$ and $D_1 = D \ominus D_2$ then

$$M = (\sum_{n \geq 0} \oplus\, w^n D_1) \oplus (\sum_{n \geq 0} \oplus\, w^n D_2) \oplus$$

$$(\sum_{n \geq 0} \oplus\, w^n D_3) \oplus M_1.$$

Since $zD_2 = D_2$, by [3] and [7, p164 - p165]

$$(\sum_{n\geq 0} \oplus D_2 w^n) \oplus M_1 = \chi_{E_1} F_1 \mathbb{H}_1 \oplus \chi_{E_2} F_2 \mathbb{H}_2 \oplus \chi_{E_3} L^2$$

where $\chi_{E_j} \in \mathcal{L}_j$ $(j = 1, 2)$, $\chi_{E_1} + \chi_{E_3} \leq 1$ a.e. and $\chi_{E_2} + \chi_{E_3} \leq 1$ a.e.. If there exists a nonzero function f in $\mathrm{Ker}\, V_2^*$ such that $z^n f$ belongs to $\mathrm{Ker}\, V_2^*$ for any integer n, then χ_{E_1} is nonzero. Since $\chi_{E_1}(\bar{F}_1 M \ominus \mathbb{H}_1)$ is invariant under multiplication of w, $\chi_{E_1}(\bar{F}_1 M \ominus \mathbb{H}_1) = \{0\}$ and hence $\chi_{E_1} M = \chi_{E_1} F_1 \mathbb{H}_1$.

Since $\chi_{E_1} M \subset M$, $(1 - \chi_{E_1})M \subset M$ and $M = \chi_{E_1} M \oplus (1 - \chi_{E_1})M$. We can prove $z(1 - \chi_{E_1})M = (1 - \chi_{E_1})M$. For $zA_1(1 - \chi_{E_1})M \subset (1 - \chi_{E_1})M$ and

$$[zA_1(1 - \chi_{E_1})] = (1 - \chi_{E_1})\mathcal{L}_1 \ni \bar{z}(1 - \chi_{E_1}).$$

Therefore $\bar{z}(1 - \chi_{E_1})M \subset (1 - \chi_{E_1})M$ and hence $z(1 - \chi_{E_1})M = (1 - \chi_{E_1})M$. Hence

$$(1 - \chi_{E_1})M = \chi_{E_4} F_4 \mathbb{H}_1 + \chi_{E_3} L^2$$

where $\chi_{E_4} \in \mathcal{L}_1$, $\chi_{E_4} + \chi_{E_3} \leq 1$ a.e. and F_4 is unimodular.

Thus M has the form $\chi_E' F \mathbb{H}_1 + \chi_E'' L^2$ for some unimodular F where $\chi_E' = \chi_{E_1} + \chi_{E_4}$ and $\chi_E'' = \chi_{E_3}$

§3. Commutator of V_1^n and V_2^*

Put $A_n = V_1^n V_2^* - V_2^* V_1^n$ $(n \geq 1)$ and write $A = A_1$. The following trivial lemmas are important.

Lemma 1. $A_n V_2 = 0$ for any $n \geq 1$ and hence $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supseteq wM$.

Lemma 2. For any f in $\operatorname{Ker} V_2^*$, $f \in \operatorname{Ker} A_n$ if and only if $z^n f \in \operatorname{Ker} V_2^*$.

Proof. When $f \in \operatorname{Ker} V_2^*$, if $f \in \operatorname{Ker} A_n$ then $V_2^* V_1^n f = 0$ and hence $z^n f \in \operatorname{Ker} V_2^*$. Conversely if $z^n f \in \operatorname{Ker} V_2^*$ then $A_n f = V_1^n V_2^* f = 0$.

In general A_n is a nonzero operator. The structure of M is simple when $A = 0$ or $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$. The following theorems, which make this precise, are corollaries of [8, Theorem 5].

Theorem 4. Let M be an invariant subspace of L^2 with $A = 0$. Then one and only one of the following occurs and vice versa.

(1) $M = \chi_{E_1} F H_1 + \chi_{E_2} L^2$ where χ_{E_1} is in $\mathcal{L}_1$, $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e. and F is unimodular.

(2) $M = \chi_{E_1} F H_2 + \chi_{E_2} L^2$ where χ_{E_1} is in $\mathcal{L}_2$, $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e. and F is unimodular.

(3) $M = FH^2$ for some unimodular function.

Conversely, if (1), (2) or (3) holds for an invariant subspace M of L^2, then $A = 0$.

Proof. If M has the form (1) then $S_2 = \{0\}$ and hence $A^* = 0$ because $A^*V_1 = 0$. Therefore $A = 0$. If M has the form (2) then $S_1 = \{0\}$ and hence $A = 0$ because $AV_2 = 0$ by Lemma 1. If M has the form (3) then $zS_1 \subseteq S_1$ and hence $V_1 \operatorname{Ker} V_2^* \subseteq \ker V_2^*$. Therefore $A = 0$ on $\operatorname{Ker} V_2^*$ and $A = 0$ because $AV_2 = 0$. Conversely suppose $A = 0$. Then $zS_1 \subseteq S_1$. If $S_1 = 0$, then by Proposition 1, M has the form (2). If $S_1 \neq 0$, then (since $zS_1 \subseteq S_1$) Proposition 2 implies that M has either the form (1) or the form (3).

Mandrekar [6] considered Theorem 4 when M is in H^2. Then since $\bigcap_{n=1}^{\infty} z^n H^2 = \bigcap_{n=1}^{\infty} w^n H^2 = \{0\}$, M has the form (3). Now we wish to consider invariant subspaces with $A \neq 0$.

Theorem 5. Let M be an invariant subspace of L^2 such that $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$. Then one and only one of the following holds.

(1) $M = \chi_{E_1} F \mathbb{H}_1 + \chi_{E_2} L^2$ where χ_{E_1} is a nonzero function in L_1, $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e. and F is unimodular.

(2) $M = FN$ where N is an invariant subspace which contains H^2 and is contained properly in $\mathbb{H}_1$, and F is unimodular.

Conversely, if (1),(2) or (3) holds for an invariant subspace M of L^2, then $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$.

Proof. Suppose $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$. Then there exists $f \neq 0$ in $(\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n) \ominus wM$. In particular, $f \in \operatorname{Ker} V_2^*$. By Lemma 2, $z^n f \in \operatorname{Ker} V_2^*$ for $n \geq 0$, so by Proposition 3, M is of either the form (1) or the form (2). Conversely if M has the form (1), then $A = 0$ by Theorem 4. Then V_2^* commutes with V_1, so it commutes with every power of V_1, hence $A_n = 0$ for $n \geq 1$. Thus $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n = M$. Since χ_{E_1} is a nonzero function, $M \neq wM$ because $M \ominus wM = \chi_{E_1} F L_1$. Therefore $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$. If M has the form (2), (by (2)) of Proposition 3 there exists a function f in $\operatorname{Ker} V_2^*$ such that $z^n f \in \operatorname{Ker} V_2^*$ for any $n \geq 0$. By Lemma 2, $f \in \operatorname{Ker} A_n$ while f is orthogonal to wM because $f \in \operatorname{Ker} V_2^*$. Therefore $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$.

§4. Finite rank commutator

Theorem 4 describes those invariant subspaces with $A = 0$. Now we are interested in invariant subspaces in which A has finite rank. The following lemma was pointed out to the author by Professor K.Takahashi. It implies if $A_n = 0$, then $A = 0$.

Lemma 3. $V_1^* A_n = A_{n-1}$ for $n > 1$ and hence $\operatorname{Ker} A_n \subset \operatorname{Ker} A_{n-1}$.

Proof is clear.

Proposition 6. Let M be an invariant subspace of L^2.

(1) If $\dim \operatorname{Ker} V_2^*$ is finite then A_n is finite rank r_n, $\sup_n r_n < \infty$, and $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n = wM$.

(2) Suppose $\dim \operatorname{Ker} V_2^*$ is infinite. If A_n is finite rank r_n and $\sup_n r_n < \infty$ then $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$.

Proof. (1) By Lemma 1 A_n is finite rank r_n and $r_n \le \dim \operatorname{Ker} V_2^*$. By Lemma 2 if $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$, there exists a nonzero function f such that $f \in M \ominus wM$ and $z^n f \in M \ominus wM$ for any $n \ge 0$, and hence this implies that $\dim \operatorname{Ker} V_2^*$ is infinite. (2) By Lemma 3 and hypothesis, $r_n \le r_{n+1}$, so ultimately, r_n is constant. Also $r_n = \dim (\operatorname{Ker} A_n)^{\perp}$, while $(\operatorname{Ker} A_{n+1})^{\perp}$ contains $(\operatorname{Ker} A_n)^{\perp}$, so ultimately $(\operatorname{Ker} A_n)^{\perp}$ does not change with n. But then neither does $\operatorname{Ker} A_n$. In other words, $\operatorname{Ker} A_n = \operatorname{Ker} A_{n_0}$ if $n \ge n_0$. Hence $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n = \operatorname{Ker} A_{n_0}$ for some $n_0 \ge 1$. Therefore if $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n = wM$ then $\operatorname{Ker} A_{n_0} = wM$ and hence $(\operatorname{Ker} A_{n_0})^{\perp} = \operatorname{Ker} V_2^*$. Since $\dim \operatorname{Ker} V_2^*$ is infinite, this contradicts the hypothesis that A_{n_0} is finite rank and hence $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneqq wM$.

Corollary 1. Let M be an invariant subspace of H^2. If A_n is finite rank r_n and $\sup_n r_n < \infty$ then $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneqq wM$.

Proof. If $\dim \operatorname{Ker} V_2^*$ is finite, by [8, Therem 3] there exists a nonzero function g in L^∞ such that $gM \subset M$ and $g \notin H^\infty$. By [2, Proposition 3] this implies that $M \not\subset H^2$, so $\dim \operatorname{Ker} V_2^*$ is infinite. (2) of Proposition 6 implies the corollary.

Corollary 2. Let M be an invariant subspace with $\bigcap_{n=1}^{\infty} z^n M = \{0\}$ then the following (1) and (2) are equivalent.

(1) $\dim \operatorname{Ker} V_2^* = \infty$, A_n is finite rank r_n and $\sup_n r_n = r < \infty$.

(2) $M = FN$ for some unimodular F and some invariant subspace with $N = K \oplus H^2 \subsetneqq \mathbb{H}_1$. Moreover $N \ominus wN = (K \ominus wK) \oplus \mathcal{H}_1$ and S is the largest closed subspace of finite codimension r of $N \ominus wN$ such that $zS \subset S$ then $S \supseteq \mathcal{H}_1$.

Proof. (1) $\Rightarrow$ (2). By (2) of Proposition 6 $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneqq wM$ and hence by hypothesis and Theorem 5 M has the form FN. If we put $S' = [\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n] \ominus wM$, then by Lemma 2 S' is the largest closed subspace of $S_1 = M \ominus wM$ with $zS' \subset S'$, and by hypothesis $\dim (S_1 \ominus S') = r < \infty$. Put $S = \bar{F}S'$ then S is the desired subspace and (2) follows. (2) $\Rightarrow$ (1). By hypothesis of (2), FS is the largest closed subspace of S_1 with $zFS \subset FS$ and hence $FS = [\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n] \ominus wM$. This implies (1).

§5. Selfadjoint commutator

In this section we will study the following conjecture : If $A = A^*$ then $A = 0$. Unfortunately we could not do it. However we will give a few partial answers.

Lemma 4. $A_n = \sum_{j=0}^{n-1} V_1^{n-1-j} A V_1^j$. If $A = A^*$ then $A_n = V^{n-1}A$ and hence $\operatorname{Ker} A_n = \operatorname{Ker} A$.

Proof. For any $n > 1$

$$\begin{aligned} A_n &= V_1^{n-1}(V_1V_2^*) - (V_2^*V_1^{n-1})V_1 \\ &= V_1^{n-1}(V_1V_2^*) - (V_1^{n-1}V_2^* - A_{n-1})V_1 \\ &= V_1^{n-1}A + A_{n-1}V_1 \end{aligned}$$

and hence $A_n = \sum_{j=0}^{n-1} V_1^{n-1-j} A V_1^j$. If $A = A^*$ then $AV_1 = 0$ and hence $A_n = V_1^{n-1}A$.

Proposition 7. Suppose M is an invariant subspace with $A = A^*$.

(1) $A_n^2 = 0$ for any $n > 1$.

(2) If A is finite rank r then A_n is also finite rank r for any $n > 1$.

(3) If $\operatorname{Ker} V_1^* \cap \operatorname{Ker} V_2^* = \{0\}$ then $A = 0$.

(4) If $\operatorname{Ker} V_1^* \cap \operatorname{Ker} V_2^* \neq \{0\}$ then $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$.

Proof. (1) If $A = A^*$ then $A_n^2 = V_1^{*n-1}AV_1^{n-1}A = 0$ for $n \geq 1$ by Lemma 4 because $A^*V_1 = 0$. (2) is clear by Lemma 4. Since $AV_1 = AV_2 = 0$, $A = 0$ on $[zM + wM]$, that is, the closed linear span of $zM + wM$.

If $\operatorname{Ker} V_1^* \cap \operatorname{Ker} V_2^* = \{0\}$ then $M = [zM + wM]$ and hence $A = 0$. Suppose $\operatorname{Ker} V_1^* \cap \operatorname{Ker} V_2^* \neq \{0\}$. If $zM \not\subset wM$ then $[zM + wM] \supsetneqq wM$ and hence $\operatorname{Ker} A \supsetneqq wM$. By Lemma 4 $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneqq wM$. It does not happen that $zM \subset wM$ (pointed out me privately by Professor K.Takahashi). For if $f \in \operatorname{Ker} V_1^* \cap \operatorname{Ker} V_2^*$ then $A = A^*$ implies

$$V_2^*V_1f = V_1^*V_2f.$$

If $zM \subseteq wM$ then $V_2^*V_1f = \bar{w}zf$, and hence

$$\| f \| = \| V_2^*V_1f \| = \| V_1^*V_2f \|.$$

Thus $V_1^*V_2f = \bar{z}wf$, and so $\bar{w}zf = \bar{z}wf$, hence $f = 0$. (Otherwise, $z^2 = w^2$ in a set of positive measure).

We don't know whether if $A = A^*$ then $A = 0$. However there exist a lot of invariant subspaces such that A is unitarily equivalent to A^* and $A \neq 0$ (see Example 2). Put $Uf(z, w) = f(w, z)$ for any f in L^2, then U is a unitary operator on L^2 and U^2 is the identity operator I on L^2. Let M be an invariant subspace which is invariant under U. Then U is isometric on M and $U^2 = I$ on M. Hence U can be assumed to be a unitary operator on M.

Proposition 8. Let M be an invariant subspace of L^2 which is invariant under U. Then $V_2U = UV_1$ and $UA^*U = A$.

§6. Examples

In the previous sections, invariant subspaces M, which satisfy $\bigcap_{n=0}^{\infty} \operatorname{Ker} A_n \supsetneqq wM$, were important. In this section, we will give several examples of such invariant subspaces.

Example 1. Suppose M is a non trivial invariant subspace in H^2. Let R be the orthogonal projection in L^2 with range $H^2 \ominus M$, and let the operator J_z on $H^2 \ominus M$ be defined by $J_z f = R(zf)$. If J_z is of finite rank n then there exists an analytic polynomial p of z of degree n such that $p(S_z) = 0$ and hence $p(z)H^2 \subset M$. The inner part of $p(z)$ is a finite Blashke product $F = F(z)$ of degree m and $m \neq 0$ because $M \neq H^2$. Since $\bar{F}H^2$ is in $\mathbb{H}_1$, $N = \bar{F}M$ lies between H^2 and $\mathbb{H}_1$. Then $M = FN$, $N = K \oplus H^2$ and $\dim K \ominus wK \leq m$. For

$$K \subset \dot{\sum_{j=0}^{\infty}} \oplus (\bar{F}H_1 \ominus H_1)w^j$$

and hence $\dim K \ominus wK \leq \dim (\bar{F}H_1 \ominus H_1)$. By Theorem 5, $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneqq wM$ and by Corollary 2, A_n is finite rank r_n and $\sup_n r_n \leq m$.

Since $\dim (H^2 \ominus M) < \infty$ and $\dim (H^2 \ominus FH^2) = \infty$, $M \neq FH^2$ and hence $\dim K \ominus wK \neq 0$. By Corollary 2, $0 < \sup_n r_n \leq m$. For if $\sup_n r_n = 0$ then $(K \ominus wK) \oplus H_1$ is an invariant subspace under the multiplication of z and $(K \ominus wK) \oplus H_1 \subseteq \bar{F}H_1$. By Beurling's theorem (cf. [4, p4])

$$(K \ominus wK) \oplus H_1 = \bar{q}H_1$$

and q is nonconstant finite Blaschke product of degree $\leq m$ because $K \ominus wK \neq \{0\}$. Therefore $N = \bar{q}H^2$. If $q \neq F$ then $M = GH^2$ and $m > \ell$ = degree of G where $G = F\bar{q}$. Hence J_z is of finite rank ℓ. This contradiction implies that $0 \neq \sup_n r_n$. By Theorem 4, M does not have the form qH^2 for any unimodular q. This is known in [2, Corollary 2].

Example 2. If M is an invariant subspace in H^2, of finite codinension n, then by Example 1 $M = F_1N_1 = F_2N_2$ where F_1 and F_2 are finite Blashke products of z and w, respectively, and $H^2 \subset N_j \subset \mathbb{H}_j$ $(j = 1, 2)$. Then both A and A^* are finite rank of degree $m \leq n$ and $m \neq 0$. By [8, Theorem 3], $\dim \operatorname{Ker} V_2^* = \dim \operatorname{Ker} V_1^* = \infty$. By Example 1, M does not have the form $\bar{q}H^2$ for any unimodular q. Put $M = [zH^2 + wH^2]$ then M is of finite codimension 1. Moreover M is invariant under U. Hence A is rank one and $UA^*U = A$.

Example 3. Let M be an invariant subspace of L^2. Invariant subspaces M which satisfy $w^nM \supset zM$ for any $n \geq 1$ or $z^nM \supset wM$ for any $n \geq 1$, were studied in [3], [4] and [7]. In general, if $wM \supset zM$ $AV_1 = 0$. For $AV_2 = 0$ by Lemma 1 and hence $AV_1 = 0$ because $V_2M \supset V_1M$. Hence by the first part of Lemma 4 $A_n = V_1^{n-1}A$. Thus $\operatorname{Ker} A_n = \operatorname{Ker} A$ for any $n \geq 1$. If $w^nM \supset zM$ for any $n \geq 1$ it is known (see [7]) that

$$M = q(H_2 \oplus z\,\mathbb{H}_2)$$

or

$$M = \chi_{E_1}\mathbb{H}_2 \oplus \chi_{E_2}L^2$$

where q is unimodular, $\chi_{E_1} \in L_2$ and $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e.. Hence if $wM \neq M$ then $M \ominus wM = \{q\}$, $\dim \operatorname{Ker} V_2^* = 1$, $\operatorname{Ker} A_n = wM$ and An is rank 1 for any $n \geq 1$. If $z^n M \supset wM$ for any $n \geq 1$, by Proposition 6 then $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$ and $M \ominus wM = q(H_1 \oplus \bar{z}H_1)$ for some

unimodular q. In [3], the authors considered generalizations of the above invariant subspaces. That is, for any fixed $\ell \geq 1$ M satisfies $w^n M \supset z^\ell M$ for any $n \geq 1$ or $z^n M \supset w^\ell M$ for any $n \geq 1$. They described completely such invariant subspaces and showed that if $zM \neq M$ or $wM \neq M$ then $M = FN$, where F is unimodular, and $\mathbb{H}_1 \supset N \supset z^\ell \mathbb{H}_1$ or $\mathbb{H}_2 \supset N \supset w^\ell \mathbb{H}_2$. Hence if $zM \neq M$ and $z^n M \supset w^\ell M$ then $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$.

References

1. P.R.Ahern and D.N.Clark, Invariant subspaces and analytic continuation in several variables, J. Math. Mech, 19(1970), 963-969.
2. O.P.Agrawal, D.N.Clark and R.G.Douglas, Invariant subspaces in the polydisk, Pacific J. Math. 121(1986), 1-11.
3. R.E.Curto, P.S.Muhly, T.Nakazi and T.Yamamoto, On superalgebras of the polydisc algebra, Acta Sci. Math., 51(1987), 413-421.
4. H.Helson, Analyticity on compact abelian groups, in Algebras in Analysis (Proc. Instructional Conference and NATO Advanced Study Inst., Birmingham, 1973), Academic Press, London, 1975, 1-62.
5. K.Izuchi, Unitarily equivalence of invariant subspaces in the polydisk, Pacific J. Math. 130 (1987), 351 - 358.
6. V.Mandrekar, The validity of Beurling theorems in polydiscs, Proc. Amer. Math. Soc. 103(1988), 145-148.
7. T.Nakazi, Invariant subspaces of weak-* Dirichlet algebras, Pacific J. Math. 69(1977), 151-167.
8. T.Nakazi, Certain invariant subspaces of H^2 and L^2 on a bidisc, Can. J. Math. XL(1988), 1272-1280.
9. W.Rudin, Invariant subspaces of H^2 on a torus, J. Funct. Anal. 61(1985), 378-384.