Commutation Properties of the Partial Isometries Associated with Anticommuting Self-adjoint Operators

Asao Arai

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Commutation Properties of the Partial Isometries Associated with Anticommuting Self-adjoint Operators

Asao Arai

Department of Mathematics, Hokkaido University, Sapporo 060, Japan

Abstract. It is proven that, for every pair $\{A, B\}$ of anticommuting self-adjoint operators, iAB is essntially self-adjoint on a suitable domain and its closure $C(A, B)$ anticommutes with A and B. For every self-adjoint opeartor S, a partial isometry U_S is defined by the polar decomposition $S = U_S|S|$. Let P_S be the orthogonal projection onto (Ker $S)^\perp$. The commutation properties of the operators $U_A, U_B, U_{C(A,B)}, P_A, P_B$, and $P_A P_B$ are investigated. These operators multiplied by some constants satisfy a set of commutation relations, which may be regarded as an extension of that satisfied by the standard basis of the Lie algebra $su(2, \mathbb{C})$ of the special unitary group $SU(2)$. It is shown that there exists a Lie algebra $\mathfrak{M}$ associated with those operators and that, if A and B are injective, then $\mathfrak{M}$ gives a completely reducible representation of $su(2, \mathbb{C})$ with the heighest weight of each irreducible component being $1/2$. Moreover, the "diagonalization" of $A + B$ is given.

I. Introduction

Recently S. Pedersen [2] has presented several equivalent definitions of anticommutativity of (unbounded) self-adjoint operators and discussed some fundamental properties of anticommuting self-adjoint operators (cf. also [4]).

In this paper we further develop theory of anticommuting self-adjoint operators. The outline is as follows. In Section II, after summarizing some fundamental facts on anticommuting self-adjoint operators, we consider the product iAB of two anticommuting self-adjoint operators A and B in a Hilbert space $\mathcal{H}$. We show that it is essentially self-adjoint on a suitable domain and its closure $C(A, B)$ anticommutes with A and B (Theorems 2.3, 2.5, and 2.6). This means that for every pair $\{A, B\}$ of anticommuting self-adjoint operators, we have a family $\{A, B, C(A, B)\}$ of mutually anticommuting self-adjoint operators.

Section III is concerned with the partial isometries appearing in the polar decompositions of A, B, and $C(A, B)$. It is known [1, p.358] that the polar decomposition of a self-adjoint operator A with spectral family $\{E_A(\lambda)\}$ is of the form

$$A = U_A|A|, \tag{1.1}$$

with

$$U_A = 1 - E_A(0) - E_A(-0), \tag{1.2}$$

which is a partial isometry with initial space, $(\text{Ker } A)^{\perp}$, and final space, $\overline{\text{Ran } A}$. We call U_A the *partial isometry associated with* A. Let P_A be the orthogonal projection onto $(\text{Ker } A)^{\perp}$. We prove that the family $\mathfrak{A} = \{U_A, U_B, U_{C(A,B)}, P_A, P_B, P_A P_B\}$ of bounded self-adjoint operators satisfies some interesting commutation relations (Theorem 3.5), which may be regarded as an extension of those satisfied by the standard basis of the Lie algebra $su(2, \mathbb{C})$ of the special unitary group $SU(2)$. In deriving these commutation relations, a formula for U_A plays an important role (Lemma 3.1). We see that there exists a Lie algebra $\mathfrak{M}$ associated with $\mathfrak{A}$ (Theorem 3.6). In particular, if A and B are injective, then $\mathfrak{M}$ gives a completely reducible representation of $su(2, \mathbb{C})$ with the heighest weight of each irreducible component being $1/2$ (Theorems 3.7 and 3.8).

In the last section, we consider the sum $A + B$ in the case where B is injective. The Hilbert space $\mathcal{H}$ has an orthogonal decomposition into two closed subspaces which diagonalizes B (Thereom 4.1). We prove by an explicit construction that there exists a unitary transformation diagonalizing $A + B$ with respect to the decomposition (Theorem 4.4).

II. Products of Anticommuting Self-adjoint Operators

We first recall a definition of the anticommutativity of self-adjoint operators [2](cf. also [4]): Two self-adjoint operators A and B in a Hibert space $\mathcal{H}$ are said to *anticommute* if $\exp(itA)B \subset B\exp(-itA)$ for all $t \in \mathbf{R}$. This definition of the anticommutativity is symmetric in A and B (see [2]). For the reader's convenience, we summarize as a lemma some known facts on anticommuting self-adjoint operators.

LEMMA 2.1 (VASILESCU [4], PEDERSEN [2]). *Let A and B be anticommuting self-adjoint operators. Then the following (i)-(vii) hold:*

(i) $U_B A \subset -AU_B$ *and* $U_A B \subset -BU_A$.

(ii) $U_B|A| \subset |A|U_B$ *and* $U_A|B| \subset |B|U_A$.

(iii) $|A|$ *and* $|B|$ *commute.*

(iv) $U_A U_B = -U_B U_A$.

(v) A *and* $|B|$ *commute and* B *and* $|A|$ *commute.*

(vi) $D(A) \cap D(B) \cap D(AB) = D(A) \cap D(B) \cap D(BA)$ *and*

$$(AB + BA)f = 0, \quad f \in D(A) \cap D(B) \cap D(AB).$$

(vii) $A + B$ *is self-adjoint.*

Let A and B be anticommuting self-adjoint operators. Then, by Lemma 2.1(iii), we can define a two dimensional spectral measure E by

$$E = E_{|A|} \otimes E_{|B|}. \tag{2.1}$$

Let

$$\mathcal{D} = \cup_{a,b>0} \operatorname{Ran} E([-a,a] \times [-b,b]). \tag{2.2}$$

Then $\mathcal{D}$ is dense.

Lemma 2.2. *The spectral measure E commutes with U_A and U_B, respectively. In particular, U_A and U_B leave $\mathcal{D}$ invariant.*

Proof: It follows from (1.2) that U_A commutes with $|A|$ and hence with $E_{|A|}$. Lemma 2.1(ii) implies that U_A commutes with $E_{|B|}$. Therefore U_A commutes with E. Similarly one can prove that U_B commutes with E. ∎

For two anticommuting self-adjoint operators A and B, we define

$$C_0(A,B) = iAB \tag{2.3}$$

with $D(C_0(A,B)) = D(A) \cap D(B) \cap D(AB)$. It follows from Lemma 2.1(vi) that

$$D(C_0(A,B)) = D(C_0(B,A)), \tag{2.4}$$

and

$$[C_0(A,B) + C_0(B,A)]f = 0, \quad f \in D(C_0(A,B)). \tag{2.5}$$

Theorem 2.3.

(i) *For all $k \in \mathbb{N}$, $C_0(A,B)^k$ is essentially self-adjoint on $\mathcal{D}$.*

(ii) *Let $C(A,B)$ be the closure of $C_0(A,B)$. Then*

$$C(A,B) = -C(B,A). \tag{2.6}$$

Proof: (i) The Hermiteness of $C_0(A,B)^k$ follows from (2.4) and (2.5). We show that $\mathcal{D}$ is an invariant set of analytic vectors for $C_0(A,B)^k$. Then the assertion follows from a variant of Nelson's analytic vector theorem [3, X.6, Corollary 2]. Let

$$L = C_0(A,B)^k.$$

We have by (1.1)

$$AB = U_A|A|U_B|B|,$$

which, together with Lemma 2.2, implies that $\mathcal{D} \subset D(AB)$ and AB leaves $\mathcal{D}$ invariant. Hence L leaves $\mathcal{D}$ invariant and

$$L^n f = (U_A U_B)^{n+k}(|A||B|)^{n+k} f$$

for all $f \in \mathcal{D}$. Since $\|U_A\|, \|U_B\| \leq 1$, we have

$$\|L^n f\| \leq a^{n+k} b^{n+k} \|f\|$$

for $f \in \text{Ran } E([-a,a] \times [-b,b]) \in \mathcal{D}$. This implies that f is an analytic vector for L.

(ii) This follows from part(i) with $k = 1$ and (2.5). ∎

It is interesting to find other cores for the self-adjoint operator $C(A,B)$. For this purpose, we need a lemma.

LEMMA 2.4. *Let A and B be anticommuting self-adjoint operators. Then, for all $m, n \in \mathbb{N} \cup \{0\}$, $A^{2m} + B^{2n}$ is self-adjoint and nonnegative. Moreover, the subspace $\mathcal{D}$ given by (2.2) is a core for $A^{2m} + B^{2n}$.*

PROOF: The self-adjointness of $A^2 + B^2$ was proved in [4]. Let

$$T_{m,n} = A^{2m} + B^{2n}.$$

Then we have

$$T_{m,n} = |A|^{2m} + |B|^{2n}.$$

It follows from Lemma 2.1(iii) and the operator calculus for commuting self-adjoint operators that $|A|^{2m} + |B|^{2n}$ is self-adjoint. Note that $D(|A|^{2m}) \cap D(|B|^{2n}) = D(A^{2m}) \cap D(B^{2n})$. Thus $T_{m,n}$ is self-adjoint. The nonnegativity of $T_{m,n}$ is obvious.

We have $\mathcal{D} \subset D(A^{2m} + B^{2n})$. Let $f \in D(A^{2m} + B^{2n})$ and define

$$f_k = E([-k,k]) \times [-k,k])f, \quad k \in \mathbb{N}.$$

Then, using the operator calculus with respect to the spectral measure E given by (2.3), one can easily show that, for all k, $f_k \in \mathcal{D}$, and

$$f_k \to f, \quad T_{m,n} f_k \to T_{m,n} f,$$

as $k \to \infty$. Hence $\mathcal{D}$ is a core for $T_{m,n}$. ∎

THEOREM 2.5. *Let A and B be anticommuting self-adjoint operators. Then $C(A,B)$ is essentially self-adjoint on every core of $A^2 + B^2$.*

PROOF: By Lemma 2.4, the operator

$$N = A^2 + B^2 + 1$$

is self-adjoint with $N \geq 1$ and is essentially self-adjoint on $\mathcal{D}$. Since A^2 and B^2 are commuting nonnegative self-adjoint operators, it follows that

$$||A^2 f||^2 + ||B^2 f||^2 \leq ||Nf||^2, \quad f \in D(N) = D(A^2) \cap D(B^2).$$

We denote by $< \cdot , \cdot >$ the inner product of the Hilbert space $\mathcal{H}$. For all $f \in \mathcal{D}$,

$$\begin{aligned} ||C(A,B)f||^2 &= < A^2 f, B^2 f > \\ &\leq ||A^2 f|| \, ||B^2 f|| \\ &\leq \frac{1}{2}(||A^2 f||^2 + ||B^2 f||^2) \\ &\leq \frac{1}{2}||Nf||^2. \end{aligned}$$

Moreover, one can see that

$$< C(A,B)f, Ng > - < Nf, C(A,B)g > = 0, \quad f, g \in \mathcal{D}.$$

Therefore we can apply a variant of Nelson's commutator theorem [3, Theorem X.37] to obtain the desired result. ∎

Theorem 2.6. *The operator $C(A,B)$ anticommutes with A, B, and $A+B$.*

Proof: In the same way as in the proof of Theorem 2.3, we can show that $\mathcal{D}$ is a set of analytic vectors for A. We already know that $C(A,B)\mathcal{D} \subset \mathcal{D}$. Hence, for all $f \in \mathcal{D}$ and all $t \in \mathbb{R}$,

$$e^{itA}C(A,B)f = \lim_{M\to\infty} g_M$$

with

$$g_M = \sum_{m=0}^{M} \frac{(it)^m}{m!} A^m C(A,B)f.$$

It follows from (2.6) that

$$A^m C(A,B)f = (-1)^m C(A,B)A^m f.$$

Hence we can write

$$g_M = C(A,B)h_M$$

with

$$h_M = \sum_{m=0}^{M} \frac{(-it)^m}{m!} A^m f.$$

We have

$$h_M \to e^{-itA}f \quad (M \to \infty)$$

and

$$C(A,B)h_M \to e^{itA}C(A,B)f \quad (M \to \infty).$$

Since $C(A,B)$ is closed, we conclude that $\exp(-itA)f \in D(C(A,B))$ and

$$C(A,B)e^{-itA}f = e^{itA}C(A,B)f. \tag{2.7}$$

By Theorem 2.3(i), $\mathcal{D}$ is a core for $C(A,B)$. Hence (2.7) implies that

$$e^{itA}C(A,B) \subset C(A,B)e^{-itA},$$

i.e., $C(A,B)$ and A anticommute. Similarly we can prove the anticommutativity of $C(A,B)$ and B. The anticommutativity of $C(A,B)$ with $A+B$ follows from that of $C(A,B)$ with A and B. ∎

III. Commutation Relations of the Partial Isometries Associated with Anticommuting Self-adjoint Operators

Let A and B be anticommuting self-adjoint operators in a Hilbert space $\mathcal{H}$. In this section we compute commutation relations of the partial isometries U_A, U_B, and $U_{C(A,B)}$ and show that there exists a Lie algebra containing these partial isometries. If A and B are injective, then this algebra gives a completely reducible (infinite dimensional) representation of $su(2, \mathbb{C})$ with the heighest weight of each irreducible component being $1/2$.

A key tool to compute products of the partial isometries associated with self-adjoint operators is the following formula.

Lemma 3.1. *Let A be a self-adjoint operator. Then*

$$U_A = s - \lim_{\epsilon \to +0} A(A^2 + \epsilon)^{-1/2} \tag{3.1}$$

Proof: For $f \in \text{Ker } A$, (3.1) is obvious, since $\text{Ker } A = \text{Ker } U_A$. Let $f \in (\text{Ker } A)^\perp$ and $\epsilon > 0$. Then

$$\| \, |A|(A^2 + \epsilon)^{-1/2} f - f \|^2 = \int u_\epsilon(\lambda) d\|E_A(\lambda) f\|^2,$$

where

$$u_\epsilon(\lambda) = \left| \frac{|\lambda|}{\sqrt{|\lambda|^2 + \epsilon}} - 1 \right|^2 .$$

Obviously we have

$$0 \leq u_\epsilon(\lambda) \leq 4$$

and

$$\lim_{\epsilon \to +0} u_\epsilon(\lambda) = 0, \quad \lambda \in \mathbb{R} \setminus \{0\}.$$

In the present case, the spectral measure $\|E_A(\cdot) f\|^2$ has no mass at $\lambda = 0$. Hence, by the dominated convergence theorem, we obtain

$$\lim_{\epsilon \to +0} |A|(A^2 + \epsilon)^{-1/2} f = f.$$

Since U_A is bounded, it follows that

$$\lim_{\epsilon\to+0} U_A|A|(A^2+\epsilon)^{-1/2}f = U_A f,$$

which, together with (1.1), implies that

$$\lim_{\epsilon\to+0} A(A^2+\epsilon)^{-1/2}f = U_A f.$$

Thus (3.1) follows. ∎

Remark. The partial isometry U_A has another representation given by

$$U_A = \mathrm{sgn}(A)P_A,$$

where $\mathrm{sgn}(\lambda) = \lambda/|\lambda|$. This follows from operator calculus as above.

Let A and B be anticommuting self-adjoint operators and $\epsilon > 0$. We define

$$T_\epsilon(A,B) = \int (\lambda^2+\epsilon)^{1/2}(\mu^2+\epsilon)^{1/2}dE(\lambda,\mu), \tag{3.2}$$

where E is the spectral measure given by (2.1). The operator $T_\epsilon(A,B)$ is self-adjoint with

$$T_\epsilon(A,B) \geq \epsilon.$$

The operator calculus with respect to E gives

$$(A^2+\epsilon)^{1/2}(B^2+\epsilon)^{1/2} \subset T_\epsilon(A,B)$$

and

$$T_\epsilon(A,B)^{-1} = (A^2+\epsilon)^{-1/2}(B^2+\epsilon)^{-1/2}. \tag{3.3}$$

It follows that $|A||B|T_\epsilon(A,B)^{-1}$ is bounded.

We have

$$C(A,B)^2 \restriction \mathcal{D} = (A^2B^2) \restriction \mathcal{D} = (B^2A^2) \restriction \mathcal{D} \subset \int \lambda^2\mu^2 dE(\lambda,\mu).$$

By Theorem 2.3, the operator on the left hand side is essentially self-adjoint. Hence we obtain

$$C(A,B)^2 = \int \lambda^2\mu^2 dE(\lambda,\mu), \tag{3.4}$$

which implies that $|A||B|(C(A,B)^2+\epsilon)^{-1/2}$ is bounded.

LEMMA 3.2. *We have*

$$s-\lim_{\epsilon\to+0}\left\{|A||B|T_\epsilon(A,B)^{-1}-|A||B|(C(A,B)^2+\epsilon)^{-1/2}\right\}=0. \tag{3.5}$$

PROOF: We have for all $f\in\mathcal{H}$

$$\|\{|A||B|T_\epsilon(A,B)^{-1}-|A||B|(C(A,B)^2+\epsilon)^{-1/2}\}f\|^2=\int_{\lambda,\mu\geq 0} w_\epsilon(\lambda,\mu)d\|E(\lambda,\mu)f\|^2,$$

where

$$w_\epsilon(\lambda,\mu)\doteq\left|\frac{\lambda\mu}{\sqrt{(\lambda^2+\epsilon)(\mu^2+\epsilon)}}-\frac{\lambda\mu}{\sqrt{\lambda^2\mu^2+\epsilon}}\right|^2.$$

It is easy to see that

$$|w_\epsilon(\lambda,\mu)|\leq 4$$

and

$$\lim_{\epsilon\to+0} w_\epsilon(\lambda,\mu)=0,\quad \lambda,\mu\geq 0.$$

Hence, by the dominated convergence theorem, we obtain (3.5). ∎

For a self-adjoint operator A, we denote by P_A the orthogonal projection onto $(\text{Ker } A)^\perp$.

LEMMA 3.3. *Let A and B be anticommuting self-adjoint operators. Then :*

(i) *P_A and P_B commute.*

(ii) *P_A and U_B commute and P_B and U_A commute.*

PROOF: Since Ker $A=$ Ker $|A|$, it follows that

$$P_A=P_{|A|},$$

Note that $P_{|A|}=E_{|A|}((0,\infty))$. Since $E_{|A|}$ and $E_{|B|}$ commute, part(i) follows.

The proof of Lemma 2.2 gives that U_B commutes with $E_{|A|}$ and hence with $P_{|A|}$. Thus U_B commutes with P_A. It follows from symmetry in A and B that P_B and U_A commute. ∎

Theorem 3.4. *The following formulae hold:*

$$U_A U_B = -iU_{C(A,B)}. \tag{3.6}$$

$$U_{C(A,B)} U_A = -iP_A U_B = -iU_B P_A \tag{3.7}$$

$$U_{C(A,B)} U_B = iP_B U_A = iU_A P_B. \tag{3.8}$$

Proof: It follows from Lemma 2.1(v) that

$$(A^2 + \epsilon)^{-1/2} B \subset B(A^2 + \epsilon)^{-1/2}.$$

Using this fact and formula (3.1), we have

$$\begin{aligned} U_A U_B &= \mathrm{s} - \lim_{\epsilon \to +0} A(A^2 + \epsilon)^{-1/2} B(B^2 + \epsilon)^{-1/2} \\ &= \mathrm{s} - \lim_{\epsilon} ABT_\epsilon(A, B)^{-1} \\ &= \mathrm{s} - \lim_{\epsilon} U_A U_B |A||B| T_\epsilon(A, B)^{-1}. \end{aligned}$$

It follows from (3.5) and the boundedness of $U_A U_B$ that

$$\mathrm{s} - \lim_{\epsilon \to +0} \left\{ U_A U_B |A||B| T_\epsilon(A, B)^{-1} - U_A U_B |A||B| (C(A, B)^2 + \epsilon)^{-1/2} \right\} = 0.$$

Hence

$$U_A U_B = \mathrm{s} - \lim_{\epsilon \to +0} U_A U_B |A||B| (C(A, B)^2 + \epsilon)^{-1/2},$$

which implies that for all $f \in \mathcal{D}$ and $g \in \mathcal{H}$,

$$\begin{aligned} < f, U_A U_B g > &= \mathrm{s} - \lim_{\epsilon \to +0} < |B||A| U_B U_A f, (C(A, B)^2 + \epsilon)^{-1/2} g > \\ &= \mathrm{s} - \lim_{\epsilon \to +0} < iC(A, B) f, (C(A, B)^2 + \epsilon)^{-1/2} g > \\ &= \mathrm{s} - \lim_{\epsilon \to +0} < f, (-i) C(A, B)(C(A, B)^2 + \epsilon)^{-1/2} g > \\ &= < f, -iU_{C(A,B)} g > . \end{aligned}$$

Thus (3.6) follows.

Let $f \in \mathcal{D}$. Then $A(A^2+\epsilon)^{-1/2}f \in \mathcal{D}$ and hence

$$
\begin{aligned}
C(A,B)A(A^2+\epsilon)^{-1/2}f &= iABA(A^2+\epsilon)^{-1/2}f \\
&= -iBA^2(A^2+\epsilon)^{-1/2}f \\
&= -iU_B|B|A^2(A^2+\epsilon)^{-1/2}f.
\end{aligned}
$$

Using the commutativity of U_B with $|C(A,B)|$, which follows from Theorem 2.6 and Lemma 2.1(ii), we have

$$
\begin{aligned}
U_{C(A,B)}U_A f &= \lim_{\epsilon\to+0} C(A,B)(C(A,B)^2+\epsilon)^{-1/2}A(A^2+\epsilon)^{-1/2}f \\
&= \lim_{\epsilon\to+0}(C(A,B)^2+\epsilon)^{-1/2}C(A,B)A(A^2+\epsilon)^{-1/2}f \\
&= -iU_B \lim_{\epsilon\to+0} f_\epsilon,
\end{aligned}
$$

where

$$f_\epsilon = (C(A,B)^2+\epsilon)^{-1/2}|B|A^2(A^2+\epsilon)^{-1/2}f.$$

The operator calculus with respect to the spectral measure E gives

$$f_\epsilon = \int_{\lambda,\mu\geq 0} u_\epsilon(\lambda,\mu)dE(\lambda,\mu)f,$$

where

$$u_\epsilon(\lambda,\mu) = \frac{\lambda^2\mu}{\sqrt{(\lambda^2\mu^2+\epsilon)(\lambda^2+\epsilon)}}.$$

We have

$$|u_\epsilon(\lambda,\mu)| \leq 1, \quad \lambda,\mu \geq 0,$$

and

$$\lim_{\epsilon\to+0} u_\epsilon(\lambda,\mu) = 1, \quad \lambda,\mu > 0.$$

Let $f \in (\mathrm{Ker}\ A)^\perp \cap (\mathrm{Ker}\ B)^\perp$. Then, for all Borel sets $M \subset \mathbb{R}$,

$$\|E(\{0\}\times M)f\|^2 = \|E(M\times\{0\})f\|^2 = 0.$$

Theorefore, by the dominated convergence theorem, we obtain

$$\lim_{\epsilon \to +0} f_\epsilon = f.$$

Thus

$$U_{C(A,B)} U_A f = -i U_B f.$$

It is obvious that, for $f \in \text{Ker } A \cup \text{Ker } B$, $f_\epsilon = 0$ and hence $U_{C(A,B)} U_A f = 0$. For all $f \in \mathcal{D}$, we have

$$f = P_A P_B f + P_A(1 - P_B) f + (1 - P_A) P_B f + (1 - P_A)(1 - P_B) f.$$

Note that $P_A P_B f \in (\text{Ker } A)^\perp \cap (\text{Ker } B)^\perp$ and

$$P_A(1 - P_B) \in \text{Ker } B, (1 - P_A) P_B \in \text{Ker } A, (1 - P_A)(1 - P_B) \in \text{Ker } A \cap \text{Ker } B.$$

Hence it follows that, for all $f \in \mathcal{D}$,

$$U_{C(A,B)} U_A f = -i U_B P_A P_B f = -i P_A U_B P_B f.$$

Moreover, we have $U_B P_B f = U_B f$, because $\text{Ker } B = \text{Ker } U_B$. Hence we obtain

$$U_{C(A,B)} U_A f = -i P_A U_B f = -i U_B P_A f.$$

Since $\mathcal{D}$ is dense, this equation extends to all $f \in \mathcal{H}$. Thus (3.7) follows.

Formula (3.8) follows from (3.7) with A (resp. B) replaced by B (resp. A) and the fact $U_{C(A,B)} = -U_{C(B,A)}$. ∎

Let A and B be anticommuting self-adjoint operators and define

$$X_1 = i\frac{U_A}{2},\ X_2 = i\frac{U_B}{2},\ X_3 = i\frac{U_{C(A,B)}}{2}, \tag{3.9}$$

$$Y_1 = 1,\ Y_2 = P_B,\ Y_3 = P_A,\ Y_4 = P_A P_B. \tag{3.10}$$

For bounded linear operators X, Y on $\mathcal{H}$, we define

$$[X, Y] = XY - YX. \tag{3.11}$$

THEOREM 3.5. *The following commutation relations hold:*

$$[X_j, X_k] = \sum_{\ell=1}^{3} \epsilon_{jk\ell} X_\ell Y_j, \quad , j, k = 1, 2, 3, \tag{3.12}$$

$$[X_j, Y_m] = [Y_m, Y_n] = 0, \quad j = 1, 2, 3, \ m, n = 1, 2, 3, 4, \tag{3.13}$$

where $\epsilon_{jk\ell}$ is the Levi-Civita symbol with $\epsilon_{123} = 1$.

PROOF: This follows from Lemma 3.3 , Theorem 3.4 and the fact that $U_A U_B = -U_B U_A$ for any two anticommuting self-adjoint operators A and B(Lemma 2.1(iv)). ∎

The vector space of all bounded linear operators on $\mathcal{H}$ is a Lie algebra with the Lie bracket $[\,\cdot\,,\,\cdot\,]$ given by (3.11). We denote it by $\mathfrak{L}(\mathcal{H})$.

THEOREM 3.6. *Let $\mathfrak{M} \subset \mathfrak{L}(\mathcal{H})$ be a subspace spanned by $X_k Y_m, k = 1, 2, 3, m = 1, 2, 3, 4$. Then $\mathfrak{M}$ is a Lie subalgebra of $\mathfrak{L}(\mathcal{H})$.*

PROOF: We need only to show that $[\mathfrak{M}, \mathfrak{M}] \subset \mathfrak{M}$. By Theorem 3.5, we have for $j, k = 1, 2, 3$, and $m, n = 1, 2, 3, 4$

$$[X_j Y_m, X_k Y_n] = [X_j, X_k] Y_m Y_n = \sum_{\ell=1}^{3} \epsilon_{jk\ell} X_\ell Y_j Y_m Y_n.$$

It follows from (3.10) that there exists a number $\alpha(j, m, n) \in \{1, 2, 3, 4\}$ such that

$$Y_j Y_m Y_n = Y_{\alpha(j,m,n)}.$$

Hence $[X_j Y_m, X_k Y_n] \in \mathfrak{M}$. Thus $\mathfrak{M}$ is a Lie algebra. ∎

As is well-known, the Lie algebra $su(2, \mathbb{C})$ of the special unitary group $SU(2)$ is the set of 2×2 complex skew-Hermitian matrices of trace zero and has a basis $\{e_j\}_{j=1}^{3}$ which satisfy the commutation relations

$$[e_j, e_k] = \sum_{\ell=1}^{3} \epsilon_{jk\ell} e_\ell, \quad j, k = 1, 2, 3. \tag{3.14}$$

We define a linear map $\rho : su(2, \mathbb{C}) \to \mathfrak{L}(\mathcal{H})$ by

$$\rho(\sum_{j=1}^{3} \alpha_j e_j) = \sum_{j=1}^{3} \alpha_j X_j, \quad \alpha_j \in \mathbb{C}, j = 1, 2, 3. \tag{3.15}$$

THEOREM 3.7. *Suppose that A and B are injective. Then ρ is a faithful representation of the $su(2, \mathbb{C})$ with Ran $\rho = \mathfrak{M}$.*

PROOF: Under the present assumption, we have $P_A = P_B = 1$. Hence $Y_m = 1$, $m = 1, 2, 3, 4$. Thus (3.12) and (3.13) reduce to

$$[X_j, X_k] = \sum_{\ell=1}^{3} \epsilon_{jk\ell} X_\ell, \quad , j, k = 1, 2, 3. \tag{3.16}$$

Therefore ρ is a representation of $su(2, \mathbb{C})$. The map ρ is injective, since $X_j, j = 1, 2, 3$ are linear independent. Hence ρ is faithful. ∎

If the Hilbert space $\mathcal{H}$ is infinite dimensional, then ρ is an infinite dimensional representation of $su(2, \mathbb{C})$. We analyze the structure of this representation in detail. We prove the following theorem.

THEOREM 3.8. *The representation ρ of $su(2, \mathbb{C})$ is completely reducible and the heighest weight of each irreducible component of it is $1/2$.*

To prove this theorem, we prepare some lemmas. We denote by $\sigma(S)$ the spectrum of operator S.

LEMMA 3.9. *Let A and B be anticommuting self-adjoint operators and injective. Then :*

(i) *U_A, U_B and $U_{C(A,B)}$ are unitary operators.*

(ii)

$$\sigma\ (U_A) = \sigma\ (U_B) = \sigma\ (U_{C(A,B)}) = \{-1, 1\}. \tag{3.17}$$

PROOF: (i) In general, if a self-adjoint operator is injective, then the associated partial isometry is unitary. This easily follows from the definition of the polar decomposition. Hence the assertion about U_A and U_B follows. If $f \in \text{Ker } U_{C(A,B)} = \text{Ker } C(A,B)$, then we have from (3.16)

$$(U_A U_B - U_B U_A) f = 0.$$

Since $U_A U_B = -U_B U_A$, we obtain

$$U_A U_B f = 0,$$

which means that $U_B f \in \text{Ker } U_A = \{0\}$, i.e., $U_B f = 0$. Hence $f = 0$. Thus $U_{C(A,B)}$ is injective and hence unitary.

(ii) Since U_A is self-adjoint and unitary by part (i), either 1 or -1 is contained in $\sigma\,(U_A)$. Let $1 \in \sigma\,(U_A)$ and $U_A f = f$ $(f \neq 0)$. Then

$$U_B U_A f = U_B f.$$

The left hand side is equal to $-U_A U_B f$. Since $U_B f \neq 0$, it follow that $U_B f$ is an eigenvector of U_A with eigenvalue -1. Hence $-1 \in \sigma\,(U_A)$. Similarly, one can show that, if $-1 \in \sigma\,(U_A)$, then $1 \in \sigma\,(U_A)$. Thus (3.17) follows. ∎

Let A and B be anticommuting, injective self-adjoint operators. The Weyl canonical basis of the representation ρ is given by the triple $\{H, E_+, E_-\}$ defined by

$$\begin{aligned}
E_+ &= \frac{1}{2}(iX_1 - X_2) = -\frac{1}{4}(U_A + iU_B),\\
E_- &= iX_1 + X_2 = \frac{1}{2}(-U_A + iU_B)\\
H &= iX_3 = -\frac{1}{2}U_{C(A,B)}.
\end{aligned}$$

These operators satisfy the following commutation relations:

$$[H, E_+] = E_+,\ [H, E_-] = -E_-,\ [E_+, E_-] = H. \tag{3.18}$$

Note that

$$E_{\pm}^2 = 0, H^2 = \frac{1}{4},$$
$$E_{+}^{*} = \frac{1}{2}E_{-}. \tag{3.19}$$

Moreover, we have from (3.17)

$$\sigma(H) = \left\{\pm\frac{1}{2}\right\}. \tag{3.20}$$

Let

$$\begin{aligned}\mathcal{K}_{\pm} &= \{f \in \mathcal{H} | U_{C(A,B)}f = \mp f\}\\ &= \{f \in \mathcal{H} | Hf = \pm\frac{1}{2}f\}\end{aligned} \tag{3.21}$$

Then we have the orthogonal decomposition

$$\mathcal{H} = \mathcal{K}_{+} \oplus \mathcal{K}_{-}. \tag{3.22}$$

Lemma 3.10. *The operator E_{-} (resp. $2E_{+}$) is a unitary operator from $\mathcal{K}_{+}$ (resp.$\mathcal{K}_{-}$) onto $\mathcal{K}_{-}$ (resp. $\mathcal{K}_{+}$).*

Proof: Let $f, g \in \mathcal{K}_{+}$. Then we have from (3.19) and the last commutation relation in (3.18)

$$\begin{aligned}< E_{-}f, E_{-}g > = 2 < f, E_{+}E_{-}g >= 2 < f, Hg > +2 < f, E_{-}E_{+}g >\\ =< f, g > +2 < f, E_{-}E_{+}g > .\end{aligned}$$

The first commutation relation in (3.18) implies that $HE_{+}g = (3/2)E_{+}g$. But, by (3.20), $3/2$ cannot be an eigenvalue of H. Hence $E_{+}g$ must be zero. Thus we obtain

$$< E_{-}f, E_{-}g >=< f, g >,$$

i.e., E_{-} is an isometry from $\mathcal{K}_{+}$ to $\mathcal{K}_{-}$. Similarly we can show that $2E_{+}$ is an isometry from $\mathcal{K}_{-}$ to $\mathcal{K}_{+}$. To prove the surjectivity of E_{-}, let $h \in \mathcal{K}_{-}$ and $f = 2E_{+}h$. As in the case of E_+g, we can show that $E_-h = 0$. Using this fact and (3.18), we see that $f \in \mathcal{K}_+$ and

$$E_-f = h$$

Hecne E_- is surjective. Thus E_- is unitary from $\mathcal{K}_+$ to $\mathcal{K}_-$. Similarly we can prove that $2E_+$ is unitary from $\mathcal{K}_-$ to $\mathcal{K}_+$. ∎

PROOF OF THEOREM 3.8: Let $\{f_n\}_{n=1}^{\infty}$ be the complete orthonmormal system (C.O.N.S) of $\mathcal{K}_+$. Then, by Lemma 3.10, $\{E_-f_n\}_{n=1}^{\infty}$ is a C.N.O.S of $\mathcal{K}_-$. Therefore $\{f_n, E_-f_n\}_{n=1}^{\infty}$ forms a C.N.O.S of $\mathcal{H}$. Thus we obtain the orthogonal decomposition

$$\mathcal{H} = \oplus_{n=1}^{\infty}\mathcal{H}_n$$

with

$$\mathcal{H}_n = \{\alpha f_n + \beta E_-f_n | \alpha, \beta \in \mathrm{C}\}.$$

Note that $H, E_\pm$ leave $\mathcal{H}_n$ invariant and ρ is irreducible in $\mathcal{H}_n$ with the heighest weight 1/2. Thus the desired result follows. ∎

IV. Diagonalization of the Sum of Two Anticommuting Self-adjoint Operators

In [2, Corollary 3.3] it was shown that, for all injective anticommuting self-adjoint operators A and B in a Hilbert space $\mathcal{H}$, there exists a decomposition

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- = \begin{pmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{pmatrix} \tag{4.1}$$

into orthogonal closed subspaces $\mathcal{H}_\pm$ such that

$$A = \begin{pmatrix} 0 & L_A a^* \\ a L_A & 0 \end{pmatrix},$$

and

$$B = \begin{pmatrix} B_+ & 0 \\ 0 & -aB_+a^* \end{pmatrix},$$

where a is an isometric isomorphism mapping from $\mathcal{H}_+$ onto $\mathcal{H}_-$ and B_+ and L_A are injective nonnegative self-adjoint commuting operators on $\mathcal{H}_+$ (Remark: our A (resp.B) is B (resp. A) in [2,Corollary 3.3]). This theorem can be generalized to the case where A is not necessarily injective.

Theorem 4.1. *Let A and B be anticommuting self-adjoint operators and B be injective. Then there exists a decomposition (4.1) such that*

$$A = \begin{pmatrix} 0 & a^* L_A^{(-)} \\ a L_A^{(+)} & 0 \end{pmatrix}, \tag{4.2}$$

and

$$B = \begin{pmatrix} B_+ & 0 \\ 0 & -B_- \end{pmatrix}, \tag{4.3}$$

where a is a partial isometry from $\mathcal{H}_+$ to $\mathcal{H}_-$, B_+ (resp. B_-) and $L_A^{(+)}$ (resp. $L_A^{(-)}$) are nonnegative self-adjoint commuting operators in $\mathcal{H}_+$ (resp. $\mathcal{H}_-$), and B_- is a nonnegative self-adjoint operator in $\mathcal{H}_-$ with $aB_+ \subset -B_-a$.

The proof of this theorem is similar to that of [2, Corollary 3.3]. Hence we omit it.

We consider the sum

$$T = A + B \tag{4.4}$$

of anticommuting self-adjoint operators A and B. The purpose of the rest of this section is to show that T can be diagonalized with respect to the decomposition given in Theorem 4.1. Let A and B be anticommuting self-adjoint operators and B be injective. Then the operator

$$\Lambda = \arctan(|A||B|^{-1}), \tag{4.5}$$

defined by the operator calculus with respect to the spectral measure E given by (2.1), is a bounded self-adjoint operator.

LEMMA 4.2. *Let Λ be given by (4.5), Then*

$$\sin\Lambda = |A|(A^2 + B^2)^{-1/2}, \tag{4.6}$$

$$\cos\Lambda = |B|(A^2 + B^2)^{-1/2}. \tag{4.7}$$

Moreover,

$$[X_j, \Lambda] = [Y_m, \Lambda] = 0, \quad j = 1, 2, 3, m = 1, 2, 3, 4, \tag{4.8}$$

where X_j and Y_m are defined by (3.9) and (3.10).

PROOF: Since $\sigma(\Lambda) \subset [0, \pi/2]$, we have

$$\sin\Lambda = \tan\Lambda(1 + (\tan\Lambda)^2)^{-1/2},$$

$$\cos\Lambda = (1 + (\tan\Lambda)^2)^{-1/2}.$$

It follows from operator calculus that

$$|A|(A^2 + B^2)^{-1/2} \subset \tan\Lambda(1 + (\tan\Lambda)^2)^{-1/2}.$$

It is easy to see that the operator on the left hand side is a bounded self-adjoint. Hence (4.6) follows. Similarly we can prove (4.7).

The commutation relations (4.8) follow from the commutativity of E with U_A, U_B, $U_{C(A,B)}, P_A$,and P_B $(= 1)$. ∎

Since $-iX_3$ and Λ are commuting bounded self-adjoint operators, $-iX_3\Lambda$ is bounded and self-adjoint. Hence the operator

$$V = e^{X_3\Lambda} \tag{4.9}$$

is unitary.

LEMMA 4.3. *Let Λ be given by (4.5). Then*

$$VX_1V^{-1} = (1 - P_A)X_1 + P_A(X_1 \cos\Lambda + X_2 \sin\Lambda), \tag{4.10}$$

$$VX_2V^{-1} = (1 - P_A)X_2 + P_A(-X_1 \sin\Lambda + X_2 \cos\Lambda), \tag{4.11}$$

PROOF: Since $X_3\Lambda$ and X_1 are bounded, we have

$$VX_1V^{-1} = \sum_{n=0}^{\infty} \frac{1}{n!}(\mathrm{ad}X_3\Lambda)^n X_1,$$

where

$$(\mathrm{ad}X)^0 Y = Y, \quad (\mathrm{ad}X)^n Y = [X, (\mathrm{ad}X)^{n-1}Y], \ n \geq 1.$$

It is easy to check that

$$(\mathrm{ad}X_3\Lambda)^{2n-1}X_1 = (-1)^{n-1}\Lambda^{2n-1}X_2P_A,$$
$$(\mathrm{ad}X_3\Lambda)^{2n}X_1 = (-1)^n\Lambda^{2n}X_1P_A, \quad n \geq 1.$$

Thus (4.10) follows. Similarly, using the formulae

$$(\mathrm{ad}X_3\Lambda)^{2n-1}X_2 = (-1)^n\Lambda^{2n-1}X_1P_A,$$
$$(\mathrm{ad}X_3\Lambda)^{2n}X_2 = (-1)^n\Lambda^{2n}X_2P_A,$$

we can prove (4.11). ∎

Theorem 4.4. *Let A and B be anticommuting self-adjoint operators and B be injective. Then*

$$VTV^{-1} = P_A U_B (A^2 + B^2)^{1/2} + (1 - P_A)B. \tag{4.12}$$

In particular, if A is also injective, then

$$VTV^{-1} = U_B (A^2 + B^2)^{1/2}. \tag{4.13}$$

Proof: we can write

$$T = -2i(X_1|A| + X_2|B|).$$

Since V commutes with $|A|$ and $|B|$, it follows that

$$V|A|V^{-1} = |A|, \quad V|B|V^{-1} = |B|.$$

Hence, for all $f \in D(A) \cap D(B)$, we have

$$VTV^{-1}f = -2i(VX_1V^{-1}|A|f + VX_2V^{-1}|B|f).$$

Putting (4.10) and (4.11) into the right hand side and using the fact $(1 - P_A)|A| = 0$, we obtain (4.12). If A is injective, then $P_A = 1$. Hence (4.12) gives (4.13). ∎

One can easily see that the right hand sides of (4.12) and (4.13) are diagonal with respect to the decomposition given in Theorem 4.1.

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