Fock-Space Representations of the Relativistic Supersymmetry Algebra in the Two-dimensional Space-time

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Fock-Space Representations of the Relativistic Supersymmetry Algebra in the Two-dimensional Space-time

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Abstract. An operator theoretical analysis is made on the representation of the supersymmetry(SUSY) algebra originating from the relativistic supersymmetry in the two-dimensional space-time with a proper treatment of anticommutatvity and commutativity of operators in the SUSY algebra. It is shown that, given on a Hilbert space $\mathfrak{H}$ two self-adjoint supercharges anticommuting *in the strong sense*, one can construct a representation of the SUSY algebra on $\mathfrak{H}$ such that the supersymmetric Hamiltonian and the momentum operator commute *in the strong sense*. As an application, a general class of representations of the SUSY algebra is constructed on an abstract Boson-Fermion Fock space.

AMS subject classification (1990): 81R05, 81T60, 81Q10,17B70, 47A67

1. Introduction

In this paper we consider, from an operator theoretical point of view, the representation of the supersymmetry (SUSY) algebra

$$
\begin{aligned}
&Q_1^2 = H + P, \quad Q_2^2 = H - P,\\
&\{Q_1, Q_2\} = 0,\\
&[Q_j, H] = [Q_j, P] = 0, \quad j = 1, 2,
\end{aligned}
\tag{1.1}
$$

where $\{\,\cdot\,,\,\cdot\,\}$ and $[\,\cdot\,,\,\cdot\,]$ denote anticommutator and commutator, respectively. This SUSY algebra originates from the relativistic supersymmetry in the two-dimensional space-time [16,17]. In the context of supersymmetric quantum field theory (SQFT) the quantities $\{Q_1, Q_2\}$, H, and P are called supercharges, a supersymmetric Hamiltonian, and a momentum operator, respectively. It has been shown in [7–9] that supersymmetric Wess-Zumino models on a two-dimensional cylindrical space-time give realizations of (1.1).

A representation of (1.1) requires a realization of Q_1, Q_2, H, and P as self-adjoint operators satisfying (1.1) on a Hilbert space. However, from an operator theoretical point of view, this is not sufficient to characterize a representation of (1.1) completely, because Q_1, Q_2, H, and P are not necessarily bounded in a realization of them as self-adjoint operators on a Hilbert space and hence the anticommutativity of Q_1 and Q_2 and the commutativity of Q_j with H and P may not be uniquely defined ; we need specify the meaning of them to define a representation of (1.1) unambiguously. In the present paper we use the concept of *strong commutativity* and *strong anticommutativity* given in the following definition.

DEFINITION 1.1. *Let A and B be self-adjoint operators on a Hilbert space.*

(i) *We say that A and B strongly commute if*

$$e^{itA}B \subset Be^{itA}$$

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for all $t \in \mathbf{R}$.

(ii) *We say that* A *and* B *strongly anticommute if*

$$e^{itA}B \subset Be^{-itA}$$

for all $t \in \mathbf{R}$.

Remark. Two self-adjoint operators strongly commute if and only if their spectral projections commute (e.g.,[11, §VIII.5],[13, Part 1, §1.1]). The definition of the strong anticommutativity of two self-adjoint operators is taken from [10,15] (cf. also [13]). Definitions (i) and (ii) are symmetric with respect to A and B.

DEFINITION 1.2. *A representation of (1.1) is a set* $\{Q_1, Q_2, H, P\}$ *of self-adjoint operators on a Hilbert space with the following properties:*

(i)

$$H + P \subset Q_1^2, \quad H - P \subset Q_2^2. \tag{1.2}$$

(ii) Q_1 *and* Q_2 *strongly anticommute.*

(iii) *Each* Q_j $(j = 1, 2)$ *strongly commutes with* H *and* P.

Remark. In our definition of representation of (1.1), we replace the first two equations in (1.1) by (1.2), because $H \pm P$ are not self-adjoint in general.

The outline of the present paper is as follows. In Section 2 we show that, for each pair $\{Q_1, Q_2\}$ of strongly anticommuting self-adjoint operators on a Hilbert space $\mathfrak{H}$, there exists a unique representation $\{Q_1, Q_2, H, P\}$ of (1.1) on $\mathfrak{H}$ such that H and P strongly commute (Theorems 2.4 and 2.5). We also show that, in the representation, the spectral condition for $\{H, P\}$ holds (Corollary 2.6). In view of constructive quantum field theory, it is of particular interest to construct representations of (1.1) on Fock spaces. For this purpose, we define in Section 3 a general class of strongly anticommuting self-adjoint operators on the Boson-Fermion Fock space $\mathfrak{F} = \mathfrak{F}(\mathcal{H}, \mathcal{K})$ based on the pair $\{\mathcal{H}, \mathcal{K}\}$ of two complex Hilbert spaces with $\mathcal{H}$ and $\mathcal{K}$ being the one particle space of bosons and of fermions, respectively. We introduce a set $\mathfrak{A} = \mathfrak{A}(\mathcal{H}, \mathcal{K})$ of pairs $\{A, B\}$ of two densely defined closed linear operators from $\mathcal{H}$ to $\mathcal{K}$. For each $\{A, B\} \in \mathfrak{A}$ and $\lambda \in \mathbf{R}$, we define two strongly anticommuting self-adjoint operators $\{Q_A(\lambda), Q_B\}$ on $\mathfrak{F}$. In the last section, we apply the main result (Theorem 2.4) in Section 2 to construct a Fock-space representation of (1.1) for each $\{A, B\} \in \mathfrak{A}$ and $\lambda \in \mathbf{R}$ with $Q_1 = Q_A(\lambda)$ and $Q_2 = Q_B$. In this way we obtain a general class of Fock-space representations. If $\lambda \neq 0$, then the partial breaking of the supersymmetry occurs, i.e., there exists no state $\Psi \in \mathfrak{F}$ such that $Q_A(\lambda)\Psi = 0$, while there exists a state $\Phi \in \mathfrak{F}$ such that $Q_B\Phi = 0$. We also discuss the unitary equivalence between two of these Fock space representations. Concrete realizations of these Fock-space representations (e.g., $\mathcal{H} = \mathcal{K} = L^2(\mathbf{R})$) corresponds to free SQFT's. But the Fock-space representations constructed here can be used as a starting point for constructing other types of Fock-space representations of (1.1) that correspond to SQFT's with interactions. This subject will be discussed in a forthcoming paper.

2. Representation Based on Two Strongly Anticommuting Self-Adjoint Operators

Let Q_1 and Q_2 be strongly anticommuting self-adjoint operators on a Hilbert space. For each $j = 1, 2$, one can define, via the functional calculus for self-adjoint operators, the 'absolute value' $|Q_j|$ of Q_j, which is the nonnegative self-adjoint operator such that $|Q_j|^2 = Q_j^2$. A fundamental property of $\{Q_1, Q_2\}$ is given by the following lemma [10,15].

LEMMA 2.1. *The self-adjoint operators $|Q_1|$ and $|Q_2|$ strongly commute.*

Using this lemma and the functional calculus for self-adjoint operators, we can prove that the operator

$$H = \frac{1}{2}(Q_1^2 + Q_2^2), \tag{2.1}$$

which is equal to $|Q_1|^2 + |Q_2|^2$, is nonnegative and self-adjoint with domain $D(H) = D(Q_1^2) \cap D(Q_2^2)$ [10,15] and that $Q_1^2 - Q_2^2 (= |Q_1|^2 - |Q_2|^2)$ is essentially self-adjoint. The latter fact allows us to define the self-adjoint operator

$$P = \frac{1}{2}\overline{(Q_1^2 - Q_2^2)}, \tag{2.2}$$

the closure of $(Q_1^2 - Q_2^2)/2$.

LEMMA 2.2.

(i) *Each Q_j ($j = 1, 2$) strongly commutes with H and P.*
(ii) *The operators H and P strongly commute.*

Proof. (i) One can show that Q_1^2 and Q_2 strongly commute (see the proof of [10, Theorem 3.1]). The functional calculus for self-adjoint operators shows that Q_2^2 and Q_2 strongly commute. Hence it follows that H and Q_2 strongly commute. By symmetry H strongly commutes also with Q_1. Similarly we can prove that Q_j strongly commutes with P.

(ii) This follows from the functional calculus for the strongly commuting self-adjoint operators $|Q_j|, j = 1, 2$. ∎

LEMMA 2.3. *The operators H and P defined by (2.1) and (2.2), respectively, satisfy (1.2).*

Proof. It is obvious that

$$D(H) \cap D(P) = D(H) = D(Q_1^2) \cap D(Q_2^2) \subset D(Q_j^2), j = 1, 2.$$

For $f \in D(H) \cap D(P)$, we have

$$Hf = \frac{1}{2}(Q_1^2 + Q_2^2)f, \quad Pf = \frac{1}{2}(Q_1^2 - Q_2^2)f.$$

Hence

$$(H + P)f = Q_1^2 f, \quad (H - P)f = Q_2^2 f \tag{2.3}$$

Thus (1.2) holds. ∎

In summary we have obtained the following result.

THEOREM 2.4. *Let Q_1 and Q_2 be strongly anticommuting self-adjoint operators on a Hilbert space $\mathfrak{H}$. Then $\{Q_1, Q_2, H, P\}$ with H and P given by (2.1) and (2.2), respectively, is a representation of the SUSY algebra (1.1) on $\mathfrak{H}$ such that H and P strongly commute.*

We can prove the uniqueness of the representation given in Theorem 2.4.

THEOREM 2.5. *Let $\{Q_1, Q_2, H, P\}$ be a representation of the SUSY algebra (1.1) such that H and P strongly commute. Then H and P are given by (2.1) and (2.2), respectively.*

Proof. It follows from (1.2) that $D(H) \cap D(P) \subset D(Q_1^2) \cap D(Q_2^2)$ and

$$Hf = \frac{1}{2}(Q_1^2 + Q_2^2)f, \quad Pf = \frac{1}{2}(Q_1^2 - Q_2^2)f, \quad f \in D(H) \cap D(P).$$

Since H and P are strongly commuting self-adjoint operators by the assumption, it follows that $D(H) \cap D(P)$ is a common core for H and P. We already know that $Q_1^2 + Q_2^2$ is self-adjoint and $Q_1^2 - Q_2^2$ is essentially self-adjoint. Thus (2.1) and (2.2) follow. ∎

In concluding this section we show that the 'energy-momentum operator' $\{H, P\}$ in the representation given in Theorem 2.4 satisfies the spectral condition in relativistic quantum field theory in the two-dimensional space-time. For any pair $\{A, B\}$ of strongly commuting self-adjoint operators, there exists a unique resolution of the identity $E_{A,B}$ on $\mathbb{R}^2$ such that $E_{A,B}(S_1 \times S_2) = E_A(S_1)E_B(S_2)$ for all Borel sets $S_1, S_2 \subset \mathbb{R}$, where E_A (resp. E_B) denotes the resolution of the identity of A (resp. B). The joint spectrum $\sigma(A, B)$ of the pair $\{A, B\}$ is defined by

$$\sigma(A, B) = \text{supp } E_{A,B},$$

the suppot of $E_{A,B}$(e.g., [13]).

COROLLARY 2.6 (SPECTRAL CONDITION). *Let $\{Q_1, Q_2, H, P\}$ be the representation of (1.1) constructed in Theorem 2.4. Then the joint spectrum of $\{H, P\}$ is included in the forward light cone in $\mathbb{R}^2$:*

$$\sigma(H, P) \subset \{(p_0, p_1) \in \mathbb{R}^2 | \, p_0^2 - p_1^2 \geq 0, p_0 \geq 0\}.$$

Proof. This follows from the strong commutativity of H and P and the fact that

$$H \pm P \geq 0 \text{ on } D(H) \cap D(P)$$

which follows from (2.3) and the nonnegativity of $Q_j^2, j = 1, 2$. ∎

3. A Class of Strongly Anticommuting Self-Adjoint Operators on Boson-Fermion Fock Spaces

In this section, as a preliminary for the construction of Fock-space representations of the SUSY algebra (1.1), we define a class of strongly anticommuting self-adjoint operators on an abstract Boson-Fermion Fock space. This can be done by employing the theory of infinite dimensional Dirac operators developed in [2,4,5](cf. also [3,6]). In [2,4,5] we used the Q-space representation for the Boson Fock space over an abstract (one particle) Hilbert space (e.g,[12,14]). Here we treat the Boson Fock space in its original form.

Let $\mathcal{H}$ and $\mathcal{K}$ be separable complex Hilbert spaces. Let $\mathfrak{F}_{\mathrm{b}}(\mathcal{H})$ and $\mathfrak{F}_{\mathrm{f}}(\mathcal{K})$ be the Boson Fock space over $\mathcal{H}$ and the Fermion Fock space over $\mathcal{K}$, respectively [11, §II.4]. The Hilbert space in which we shall work is the Boson-Fermion Fock space

$$\mathfrak{F} = \mathfrak{F}(\mathcal{H}, \mathcal{K}) = \mathfrak{F}_{\mathrm{b}}(\mathcal{H}) \otimes \mathfrak{F}_{\mathrm{f}}(\mathcal{K}).$$

Let $a(f), f \in \mathcal{H}$, and $b(u), u \in \mathcal{K}$, be the annihilation operators on $\mathfrak{F}_{\mathrm{b}}(\mathcal{H})$ (*antilinear in* f) and on $\mathfrak{F}_{\mathrm{f}}(\mathcal{K})$ (*antilinear in* u), respectively. We denote by Ω_{b} and Ω_{f} the Fock vacuum in $\mathfrak{F}_{\mathrm{b}}(\mathcal{H})$ and in $\mathfrak{F}_{\mathrm{f}}(\mathcal{K})$, respectively. *The Fock vacuum in* $\mathfrak{F}$ is defined by

$$\Omega = \Omega_{\mathrm{b}} \otimes \Omega_{\mathrm{f}} \in \mathfrak{F}.$$

Every densely defined closed linear operator T on $\mathfrak{F}_{\mathrm{b}}(\mathcal{H})$ (resp. $\mathfrak{F}_{\mathrm{f}}(\mathcal{K})$) extends to a closed linear operator on $\mathfrak{F}$ as $T \otimes I$ (resp. $I \otimes T$). For notational simplicity we denote the extension by the same symbol T.

For a linear operator S on a Hilbert space with domain $D(S)$, we denote by $C^\infty(S)$ the set of C^∞-vectors of S:

$$C^\infty(S) = \cap_{n=1}^\infty D(S^n).$$

Let $A : \mathcal{H} \to \mathcal{K}$ be a densely defined closed linear operator and

$$\begin{aligned} \mathfrak{D} = \mathcal{L}\{a(f_1)^* \cdots a(f_n)^* b(u_1)^* \cdots b(u_m)^* \Omega | f_j \in C^\infty(A^*A), j = 1, \cdots, n, \\ u_k \in C^\infty(AA^*), k = 1, \cdots, m, n, m \geq 0\}, \end{aligned} \tag{3.1}$$

where $\mathcal{L}\{\cdots\}$ means the algebraic linear span of elements in the set $\{\cdots\}$. The subspace $\mathfrak{D}$ is dense in $\mathfrak{F}$. We define an operator d_A with domain $\mathfrak{D}$ by

$$d_A \Psi = (1 - \delta_{n0}) \sum_{j=1}^n a(f_1)^* \cdots \widehat{a(f_j)}^* \cdots a(f_n)^* b(Af_j)^* b(u_1)^* \cdots b(u_m)^* \Omega, \tag{3.2}$$

for $\Psi \in \mathfrak{D}$ of the form

$$\Psi = a(f_1)^* \cdots a(f_n)^* b(u_1)^* \cdots b(u_m)^* \Omega \tag{3.3}$$

and by extending it by linearity to all $\Psi \in \mathfrak{D}$, where $\widehat{a(f_j)}^*$ indicates the omission of $a(f_j)^*$. The operator d_A is well-defined. In fact, for each complete orthonormal system $\{e_n\}_{n=1}^\infty$ of $\mathcal{K}$ with $e_n \in D(A^*)$, we have

$$d_A \Psi = \mathrm{s}-\lim_{N \to \infty} \sum_{n=1}^N a(A^* e_n) b(e_n)^* \Psi, \quad \Psi \in \mathfrak{D}. \tag{3.4}$$

LEMMA 3.1.

(i) *The operator d_A leaves $\mathfrak{D}$ invariant and*

$$d_A^2 = 0. \tag{3.5}$$

(ii) *$\mathfrak{D} \subset D(d_A^*)$ and d_A^* leaves $\mathfrak{D}$ invariant.*
(iii) *d_A is closable.*

Proof. (i) The invariance of $\mathfrak{D}$ under the action of d_A follows from (3.1) and (3.2). For $\Psi \in \mathfrak{D}$ of the form (3.3), we have

$$d_A^2\Psi = (1-\delta_{(n-1)0})(1-\delta_{n0})\sum_{k\neq j}^{n}\sum_{j=1}^{n} a(f_1)^* \cdots \widehat{a(f_j)}^* \cdots \widehat{a(f_k)}^* \cdots a(f_n)^*$$
$$\times\, b(Af_k)^* b(Af_j)^* b(u_1)^* \cdots b(u_m)^*\Omega.$$

Using the anticommutativity of $b(u)^*$ and $b(v)^*, u, v \in \mathcal{K}$, we see that the right hand side is eqaul to $-d_A^2\Psi$. Thus (3.5) follows.

(ii) By (3.4) one can easily check that $\mathfrak{D} \subset D(d_A^*)$ and

$$d_A^*\Psi = (1-\delta_{m0})\sum_{k=1}^{m}(-1)^{k-1}a(A^*u_k)^* a(f_1)^* \cdots a(f_n)^* b(u_1)^* \cdots \widehat{b(u_k)}^* \cdots b(u_m)^*\Omega \quad (3.6)$$

for $\Psi \in \mathfrak{D}$ of the form (3.3). Formula (3.6) shows that d_A^* leaves $\mathfrak{D}$ invariant.

(iii) By part (ii), $D(d_A^*)$ is dense. Hence d_A is closable. ∎

By a limiting argument, we can show that (3.5) with the closure of d_A in place of d_A holds. *We denote the closure of d_A by the same symbol.*

We define *Dirac operators on $\mathcal{F}$* by

$$Q_A = d_A + d_A^*, \quad (3.7)$$
$$\widetilde{Q}_A = i(d_A - d_A^*), \quad (3.8)$$

with $D(Q_A) = D(\widetilde{Q}_A) = D(d_A) \cap D(d_A^*)$.

We denote by $d\Gamma(\cdot)$ (resp. $d\Lambda(\cdot)$) the second quantization on $\mathfrak{F}_b(\mathcal{H})$ (resp. $\mathfrak{F}_f(\mathcal{K})$)[11, §VIII.10]. We introduce the nonnegative self-adjoint operator

$$\Delta_A = d\Gamma(A^*A) + d\Lambda(AA^*) \quad (3.9)$$

on $\mathfrak{F}$.

THEOREM 3.2. *The operators Q_A and $\widetilde{Q}_A$ are self-adjoint and are essentially self-adjoint on every core for Δ_A. Moreover, the following operator equations hold :*

$$\Delta_A = Q_A^2 = \widetilde{Q}_A^2 = d_A^* d_A + d_A d_A^*. \quad (3.10)$$
$$\widetilde{Q}_A = Q_{iA}. \quad (3.11)$$

Proof. The outline of the proof is as follows. We first show that Q_A and $\widetilde{Q}_A$ are closed symmetric (cf.[2, Lemma 4.4(b)]). Then we prove that (3.10) holds on $\mathfrak{D}$. Noting that $\mathfrak{D}$ is a core for Δ_A (apply [11, Theorem VIII.33 and Corollary]), we use a commutator theorem [12, Theorem X.37] to prove the essential self-adjointness of Q_A and $\widetilde{Q}_A$ on $\mathfrak{D}$. See [4] for details. It is easy to see that (3.11) holds on $\mathfrak{D}$. Since $\mathfrak{D}$ is a common core for Q_{iA} and $\widetilde{Q}_A$, the operator equation (3.11) follows. ∎

The Boson-Fock space $\mathfrak{F}$ has the following orthogonal decompsition:

$$\mathfrak{F} = \mathfrak{F}_+ \oplus \mathfrak{F}_-$$

with

$$\mathfrak{F}_+ = \bigoplus_{m=0}^{\infty} \mathfrak{F}_b(\mathcal{H}) \otimes \wedge^{2m}(\mathcal{K}), \quad \mathfrak{F}_- = \bigoplus_{m=0}^{\infty} \mathfrak{F}_b(\mathcal{H}) \otimes \wedge^{2m+1}(\mathcal{K}),$$

where $\wedge^m(\mathcal{K})$ denotes the m-fold antisymmetric tensor product of $\mathcal{K}$ ($\wedge^0(\mathcal{K}) = \mathbf{C}$). Let $P_\pm$ be the orthogonal projection onto $\mathfrak{F}_\pm$ and

$$N_F = P_+ - P_-.$$

Then one can show [4] that

$$N_F : D(Q_A) \to D(Q_A), \tag{3.12}$$

$$\{Q_A, N_F\} = \{\widetilde{Q}_A, N_F\} = 0 \quad \text{on } D(Q_A). \tag{3.13}$$

It follows that the quadruple $\{\mathfrak{F}, \{Q_A, \widetilde{Q}_A\}, \Delta_A, N_F\}$ is a supersytmmetric quantum theory [1,3,17].

Using the boundedness of N_F, (3.12), and (3.13), one can easily see that

$$e^{itN_F} Q_A \subset Q_A e^{-itN_F}, \quad e^{itN_F} \widetilde{Q}_A \subset \widetilde{Q}_A e^{-itN_F}.$$

for all $t \in \mathbf{R}$. Thus we obtain the following result.

PROPOSITION 3.3. *The self-adjoint operator N_F strongly anticommutes with Q_A and $\widetilde{Q}_A$.*

Let $\lambda \in \mathbf{R}$ and

$$Q_A(\lambda) = Q_A + \lambda N_F. \tag{3.14}$$

Then, by Proposition 3.3 and [15, Corollary 2.2], $Q_A(\lambda)$ is self-adjoint and

$$Q_A(\lambda)^2 = Q_A^2 + \lambda^2, \tag{3.15}$$

where we have used the fact $N_F^2 = I$.

Let $C(\mathcal{H}, \mathcal{K})$ be the set of closed linear operators from $\mathcal{H}$ to $\mathcal{K}$. We introduce a set of pairs $\{A, B\}$ with $A, B \in C(\mathcal{H}, \mathcal{K})$.

DEFINITION 3.4. *Let $A, B \in C(\mathcal{H}, \mathcal{K})$. We say that $\{A, B\}$ is in the set $\mathfrak{A} = \mathfrak{A}(\mathcal{H}, \mathcal{K})$ if the following conditions are satisfied:*

(i) *The self-adjoint operators A^*A and B^*B strongly commute.*
(ii) *The self-adjoint operators AA^* and BB^* strongly commute.*
(iii)

$$C^\infty(A^*A) \cap C^\infty(B^*B) \subset D(A^*B) \cap D(B^*A),$$
$$C^\infty(AA^*) \cap C^\infty(BB^*) \subset D(AB^*) \cap D(BA^*),$$

and

$$A^*B + B^*A = 0, \quad AB^* + BA^* = 0.$$

The set $\mathfrak{A}$ is not empty. For example, $\{A, iA\} \in \mathfrak{A}$ for all $A \in C(\mathcal{H}, \mathcal{K})$.

The main result in this section is the following.

THEOREM 3.5. *For all $\{A, B\} \in \mathfrak{A}$ and $\lambda \in \mathbf{R}$, $Q_A(\lambda)$ and Q_B strongly anticommute.*

To prove this theorem, we need some preliminaries. For a self-adjoint operator S on a Hilbert space, we define the subspace

$$C_0^\infty(S) = \cup_{-\infty<a<b<\infty} R(E_S((a, b])),$$

where E_S is the spectral measure of S and $R(T)$ denotes the range of operator T. The subspace $C_0^\infty(S)$ is dense in the Hilbert space on which S is defined. Moreover, each $f \in C_0^\infty(S)$ is an entire analytic vector of S, i.e., for all $t > 0$,

$$\sum_{n=0}^{\infty} \frac{||S^n f||}{n!} t^n < \infty.$$

LEMMA 3.6.

(i) *Let S be a self-adjoint operator on $\mathcal{H}$ and*

$$\mathfrak{D}_S(\mathcal{H}) = \mathcal{L}\{a(f_1)^* \cdots a(f_n)^* \Omega_b | f_j \in C_0^\infty(S), j = 1, \cdots, n, n \geq 0\}.$$

Then $d\Gamma(S)$ leaves $\mathfrak{D}_S(\mathcal{H})$ invariant and each $\Psi \in \mathfrak{D}_S(\mathcal{H})$ is an entire analytic vector of $d\Gamma(S)$.

(ii) *Let T be a self-adjoint operator on $\mathcal{K}$ and*

$$\mathfrak{D}_T(\mathcal{K}) = \mathcal{L}\{b(u_1)^* \cdots b(u_m)^* \Omega_f | u_j \in C_0^\infty(T), j = 1, \cdots, m, m \geq 0\}.$$

Then $d\Lambda(T)$ leaves $\mathfrak{D}_T(\mathcal{K})$ leaves invariant and each $\Psi \in \mathfrak{D}_T(\mathcal{K})$ is an entire analytic vector of $d\Lambda(T)$.

Proof. Let

$$\Psi = a(f_1)^* \cdots a(f_m)^* \Omega_b, \quad f_j \in C_0^\infty(S), j = 1, \cdots, m.$$

Without loss of generality, we can assume that for some $a, b \in \mathbf{R}$, $f_j \in R(E_S((a, b])), j = 1, \cdots, m$. Then $\Psi \in C^\infty(d\Gamma(S))$ and the vector $d\Gamma(S)^n \Psi$ is written as a sum of m^n terms of the form

$$a(S^{j_1} f_1)^* \cdots a(S^{j_m} f_m)^* \Omega_b, \quad 0 \leq j_k \leq n, k = 1, \cdots, m.$$

The standard number operator estimate gives

$$||a(S^{j_1} f_1)^* \cdots a(S^{j_m} f_m)^* \Omega_b|| \leq \sqrt{(m-1)!} \prod_{k=1}^{m} ||S^{j_k} f_k||.$$

Using the spectral representation of S, we can show that

$$||S^k f_j|| \leq c^k ||f_j||$$

with $c = \max\{|a|, |b|\}$. Hence we obtain

$$||a(S^{j_1} f_1)^* \cdots a(S^{j_m} f_m)^* \Omega_b|| \leq \sqrt{(m-1)!} c^{mn} \prod_{j=1}^{m} ||f_j||.$$

Therefore

$$||d\Gamma(S)^n \Psi|| \leq m^n c^{mn} \sqrt{(m-1)!} \prod_{j=1}^{m} ||f_j||.$$

This estimate implies that Ψ is an entire analytic vector of $d\Gamma(S)$. Similarly we can prove part (ii). ∎

LEMMA 3.7. *Let $A \in C(\mathcal{H}, \mathcal{K})$ and*

$$\begin{aligned}\mathfrak{D}_0 = \mathcal{L}\{&a(f_1)^* \cdots a(f_n)^* b(u_1)^* \cdots b(u_m)^* \Omega | f_j \in C_0^\infty(A^*A),\\ &u_k \in C_0^\infty(AA^*), j = 1, \cdots, n, k = 1, \cdots, m, n, m \geq 0\}.\end{aligned}$$

Then Q_A leaves $\mathfrak{D}_0$ invariant and each $\Psi \in \mathfrak{D}_0$ is an entire analytic vector of Q_A.

Proof. By (3.2) and (3.6) one easily sees that $\mathfrak{D}_0 \subset C^\infty(Q_A)$. By Theorem 3.2 we have

$$Q_A^{2n} = \Delta_A^n.$$

It follows from (3.9) and Lemma 3.6 that each $\Psi \in \mathfrak{D}_0$ is an entire analytic vector of Δ_A. Hence we have for all $t > 0$

$$\sum_{n=0}^{\infty} \frac{||Q_A^{2n} \Psi||}{n!} t^n < \infty.$$

Since $Q_A \Psi \in \mathfrak{D}_0$, we have also

$$\sum_{n=0}^{\infty} \frac{||Q_A^{2n+1} \Psi||}{n!} t^n < \infty.$$

These estimates imply that Ψ is an entire analytic vector of Q_A. ∎

Proof of Theorem 3.5: Let $\{A, B\} \in \mathfrak{A}$. The strong commutativity of A^*A and B^*B implies that $C_0^\infty(A^*A) \cap C_0^\infty(B^*B)$ is dense in $\mathcal{H}$. Similarly $C_0^\infty(AA^*) \cap C_0^\infty(BB^*)$ is dense in $\mathcal{K}$. Hence the subspace

$$\begin{aligned}\mathfrak{D}_{A,B} = \mathcal{L}\{&a(f_1)^* \cdots a(f_n)^* b(u_1)^* \cdots b(u_m)^* \Omega | f_j \in C_0^\infty(A^*A) \cap C_0^\infty(B^*B),\\ &u_k \in C_0^\infty(AA^*) \cap C_0^\infty(BB^*), j = 1, \cdots, n, k = 1, \cdots, m, n, m \geq 0\}\end{aligned}$$

is dense in $\mathfrak{F}$. By (3.12), (3.13), and property (iii) in Definition 3.4, we see that

$$\{Q_A(\lambda), Q_B\} = 0 \quad \text{on } \mathfrak{D}_{A,B}.$$

Each $\Psi \in \mathfrak{D}_{A,B} \subset \mathfrak{D}_0$ is an entire analytic vector for Q_B and $Q_A(\lambda)$ leaves $\mathfrak{D}_{A,B}$ invariant. Hence we have for all $t \in \mathbf{R}$

$$\begin{aligned} e^{itQ_B} Q_A(\lambda)\Psi &= \mathrm{s} - \lim_{N\to\infty} \sum_{n=1}^{N} \frac{(it)^n}{n!} Q_B^n Q_A(\lambda)\Psi \\ &= \mathrm{s} - \lim_{N\to\infty} Q_A(\lambda) \sum_{n=1}^{N} \frac{(-it)^n}{n!} Q_B^n \Psi. \end{aligned}$$

Since

$$\mathrm{s} - \lim_{N\to\infty} \sum_{n=1}^{N} \frac{(-it)^n}{n!} Q_B^n \Psi = e^{-itQ_B}\Psi$$

and $Q_A(\lambda)$ is closed, it follows that $e^{-itQ_B}\Psi \in D(Q_A(\lambda))$ and

$$e^{itQ_B} Q_A(\lambda)\Psi = Q_A(\lambda) e^{-itQ_B}\Psi. \tag{3.16}$$

The operator $Q_A(\lambda)$ is essentially self-adjoint on $\mathfrak{D}_{A,B}$, because Q_A is essentially self-adjoint on $\mathfrak{D}_{A,B}$ (the Nelson analytic vector theorem [12, §X.6]) and λN_F is bounded. Hence (3.16) implies that

$$e^{itQ_B} Q_A(\lambda) \subset Q_A(\lambda) e^{-itQ_B}.$$

Thus $Q_A(\lambda)$ and Q_B strongly anticommute. ∎

4. Fock-Space Representations of the SUSY Algebra

We can now apply Theorem 2.4 to construct representations of the SUSY algebra (1.1) on the Boson-Fermion Fock space $\mathfrak{F}$. Let $\{A, B\} \in \mathfrak{A}$ and define

$$H_{A,B}(\lambda) = \frac{1}{2}(Q_A(\lambda)^2 + Q_B^2), \tag{4.1}$$

$$P_{A,B}(\lambda) = \frac{1}{2}\overline{(Q_A(\lambda)^2 - Q_B^2)} \tag{4.2}$$

Theorems 3.5 and 2.4 imply the following result.

THEOREM 4.1. *Let $\{A, B\} \in \mathfrak{A}$ and $\lambda \in \mathbf{R}$. Then $\{Q_A(\lambda), Q_B, H_{A,B}(\lambda), P_{A,B}(\lambda)\}$ is a representation of (1.1) with $Q_1 = Q_A(\lambda), Q_2 = Q_B, H = H_{A,B}(\lambda), P = P_{A,B}(\lambda)$. Moreover we have*

$$H_{A,B}(\lambda) = \frac{1}{2}\{d\Gamma(A^*A + B^*B) + d\Lambda(AA^* + BB^*) + \lambda^2\}, \tag{4.3}$$

$$P_{A,B} = \frac{1}{2}\{d\Gamma(\overline{A^*A - B^*B}) + d\Lambda(\overline{AA^* - BB^*}) + \lambda^2\}. \tag{4.4}$$

Proof. It remains to prove formulae (4.3) and (4.4). They easily follow from (4.1), (4.2), Theorem 3.2 and (3.15). ∎

Let $\lambda \neq 0$. Then formula (4.3) shows that $H_{A,B}(\lambda) \geq \lambda^2/2 > 0$ and hence the supersymmetric Hamiltonian $H_{A,B}(\lambda)$ has no zero-energy states, i.e., the supersymmetry is broken. Similarly fromula (3.15) implies that there exists no vector Ψ such that $Q_A(\lambda)\Psi = 0$. On the other hand, we have $Q_B\Omega = 0$. Thus, in the case $\lambda \neq 0$, the supersymmetry is partially broken in the representation given in Theorem 4.1. One can completely identify the subspace Ker Q_A [4].

We next consider the unitary equivalence of the representations $\{Q_A(\lambda), Q_B, H_{A,B}(\lambda), P_{A,B}(\lambda)\}$, $\{A, B\} \in \mathfrak{A}$.

DEFINITION 4.2. *Let $\{Q_1, Q_2, H, P\}$ and $\{Q_1', Q_2', H', P'\}$ be representations of (1.1) on Hilbert spaces $\mathfrak{H}$ and $\mathfrak{H}'$, respectively. They are said to be unitarily equivalent if there exists a unitary operatror $\Upsilon : \mathfrak{H} \to \mathfrak{H}'$ such that*

$$Q_j' = \Upsilon Q_j \Upsilon^{-1}, j = 1, 2, \ H' = \Upsilon H \Upsilon^{-1}, \ P' = \Upsilon P \Upsilon^{-1}.$$

The following fact is easily proven.

LEMMA 4.3. *Let $\mathcal{H}', \mathcal{K}'$ be Hilbert spaces and $\{A, B\} \in \mathfrak{A}(\mathcal{H}, \mathcal{K})$. Let $U : \mathcal{H} \to \mathcal{H}'$ and $V : \mathcal{K} \to \mathcal{K}'$ be unitary operators and set*

$$A' = VAU^{-1}, \quad B' = VBU^{-1}. \tag{4.5}$$

Then $\{A', B'\} \in \mathfrak{A}(\mathcal{H}', \mathcal{K}')$.

THEOREM 4.4. *Let $\{A, B\} \in \mathfrak{A}(\mathcal{H}, \mathcal{K})$ and $\{A', B'\} \in \mathfrak{A}(\mathcal{H}', \mathcal{K}')$ be as in Lemma 4.3 and $\lambda \in \mathbb{R}$. Then $\{Q_A(\lambda), Q_B, H_{A,B}(\lambda), P_{A,B}(\lambda)\}$ is unitarily equivalent to $\{Q_{A'}(\lambda), Q_{B'}, H_{A',B'}(\lambda), P_{A',B'}(\lambda)\}$.*

Proof. Let (4.5) be satisfied. It is well-known that the operators

$$\Gamma(U) = \bigoplus_{n=0}^{\infty} \underbrace{U \otimes U \otimes \cdots \otimes U}_{n \text{ factors}} : \mathfrak{F}_{\mathrm{b}}(\mathcal{H}) \to \mathfrak{F}_{\mathrm{b}}(\mathcal{H}')$$

and

$$\Lambda(V) = \bigoplus_{m=0}^{\infty} \underbrace{V \otimes V \otimes \cdots \otimes V}_{m \text{ factors}} : \mathfrak{F}_{\mathrm{f}}(\mathcal{K}) \to \mathfrak{F}_{\mathrm{f}}(\mathcal{K}')$$

are unitary [12, §X.7]. Hence

$$\Upsilon = \Gamma(U) \otimes \Lambda(V) : \mathfrak{F}(\mathcal{H}, \mathcal{K}) \to \mathfrak{F}(\mathcal{H}', \mathcal{K}')$$

is unitary. Direct computations using (4.5) show that

$$\Upsilon Q_A(\lambda) \Upsilon^{-1} = Q_{A'}(\lambda), \quad \Upsilon Q_B \Upsilon^{-1} = Q_{B'},$$

on $\mathfrak{D}_{A',B'}$. Since $\mathfrak{D}_{A',B'}$ is a common core for $Q_{A'}(\lambda)$ and $Q_{B'}$, it follows that $\{Q_A(\lambda), Q_B\}$ is unitarily equivalent to $\{Q_{A'}(\lambda), Q_{B'}\}$ via Υ. In the present case, the unitary equivalence of $\{Q_A(\lambda), Q_B\}$ and $\{Q_{A'}(\lambda), Q_{B'}\}$ implies that of $\{H_{A,B}(\lambda), P_{A,B}(\lambda)\}$ and $\{H_{A',B'}(\lambda), P_{A',B'}(\lambda)\}$. ∎

To formulate a sufficient condition for two representations given in Thereom 4.1 not to be unitarily equivalent, we intoroduce a concept related to the spectrum of $\{A, B\} \in \mathfrak{A}(\mathcal{H}, \mathcal{K})$. For $\{A, B\} \in \mathfrak{A}(\mathcal{H}, \mathcal{K})$, we have

$$S_{A,B} \equiv \sigma(A^*A + B^*B) = \{\lambda + \mu | \lambda \in \sigma(A^*A), \mu \in \sigma(B^*B)\}^-,$$

where $\{\ \cdot\ \}^-$ denotes the closure of the set $\{\ \cdot\ \}$. Let

$$\begin{aligned}\Sigma_{A,B} = \cup_{m,n=0,m+n\geq 1}^{\infty} \{&\lambda_1 + \cdots + \lambda_n + \mu_1 + \cdots + \mu_m | \lambda_j \in S_{A,B},\\ &\mu_k \in S_{A^*,B^*}, j = 1, 2 \cdots, n, k = 1, \cdots, m, \}.\end{aligned}$$

THEOREM 4.5. *Let $\{A, B\} \in \mathfrak{A}(\mathcal{H}, \mathcal{K}), \{A', B'\} \in \mathfrak{A}(\mathcal{H}', \mathcal{K}')$. Suppose that $\Sigma_{A,B}^- \neq \Sigma_{A',B'}^-$. Then $\{Q_A(\lambda), Q_B, H_{A,B}(\lambda), P_{A,B}(\lambda)\}$ is not unitarily equivalent to $\{Q_{A'}(\lambda), Q_{B'}, H_{A',B'}(\lambda), P_{A',B'}(\lambda)\}$.*

Proof. By the assumption and the fact

$$\begin{aligned}\sigma(2H_{A,B}(\lambda)) &= \{x + y + \lambda^2 | x \in \sigma(d\Gamma(A^*A + B^*B)), y \in \sigma(d\Lambda(AA^* + BB^*))\}^-\\ &= \{\lambda^2\} \cup \{\Sigma_{A,B}^- + \lambda^2\},\end{aligned}$$

we have $\sigma(2H_{A,B}(\lambda)) \neq \sigma(2H_{A',B'}(\lambda))$, which implies that $H_{A,B}(\lambda)$ cannot be unitarily equivalent to $H_{A',B'}(\lambda)$. Hence the desired result follows. ∎

We give only one example.

EXAMPLE: Let $\mathcal{H} = \mathcal{K} = L^2(\mathbf{R})$ and $\omega(p)$ be a nonnegative measurable function on $\mathbf{R}$ such that

$$|p| \leq \omega(p), \quad p \in \mathbf{R}.$$

Let

$$\nu(p) = (p + \omega(p))^{1/2}$$

and $\theta(p)$ be a real-valued measurable function on $\mathbf{R}$. Define the operators A and B on $L^2(\mathbf{R})$ to be the multiplication operatros by the functions

$$A(p) = i\nu(p)e^{i\theta(p)}$$

and

$$B(p) = \nu(-p)e^{i\theta(p)},$$

respectively. Then we have

$$\begin{aligned}A^*A &= AA^* = p + \omega,\\ B^*B &= BB^* = -p + \omega,\\ A^*B &= -i\nu(p)\nu(-p) = -i\sqrt{\omega^2 - p^2},\\ B^*A &= i\nu(p)\nu(-p) = i\sqrt{\omega^2 - p^2}.\end{aligned}$$

These relations imply that $\{A, B\}$ is in $\mathfrak{A}$ under consideration and

$$H_{A,B}(\lambda) = d\Gamma(\omega) + d\Lambda(\omega) + \frac{\lambda^2}{2},$$

$$P_{A,B}(\lambda) = d\Gamma(p) + d\Lambda(p) + \frac{\lambda^2}{2},$$

Note that $H_{A,B}(\lambda)$ and $P_{A,B}(\lambda)$ are independent of θ. If $\omega(p) = \sqrt{p^2 + m^2}$ with a constant $m \geq 0$, then $H_{A,B}(0)$ and $P_{A,B}(0)$ are the Hamiltonian and the momentum operator of a free relativistic SQFT in the two-dimensional space-time. The present example with $\theta = 0$ is related to the $N = 1$ Wess-Zumino model in the two-dimensional space-time (cf.[8,9]).

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