THE THEORY OF LEGENDRIAN UNFOLDINGS AND FIRST ORDER DIFFERENTIAL EQUATIONS

Shyuichi Izumiya

Series ♯124. September 1991

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

♯ 97: Y. Okabe, A. Inoue, The theory of KM_2O-Langevin equations and its applications to data analysis (II): Causal analysis (1), 51 pages. 1990.

♯ 98: J. Lawrynowicz, S. Koshi and O. Suzuki, Dualities generated by the generalised Hurwitz problem and variation of the Yang-Mills field, 17 pages. 1991.

♯ 99: R. Agemi, K. Kubota and H. Takamura, On certain integral equations related to nonlinear wave equations, 52 pages. 1991.

♯ 100: S. Izumiya, Geometric singularities for Hamilton-Jacobi equation, 13 pages. 1991.

♯ 101: S. Izumiya, Legendrian singularities and first order differential equations, 16 pages. 1991.

♯ 102: A. Munemasa, Y. Watatani, Orthogonal pairs of $*$-subalgebras and association schemes, 11 pages. 1991.

♯ 103: A. Arai, O. Ogurisu, Meromorphic $N = 2$ Wess-Zumino supersymmetric quantum mechanics, 27 pages. 1991.

♯ 104: H. Takamura, Global existence of classical solutions to nonlinear wave equations with spherical symmetry for small data with noncompact support in three space dimensions, 14 pages. 1991.

♯ 105: R. Agemi, Blow-up of solutions to nonlinear wave equations in two space dimensions, 11 pages. 1991.

♯ 106: T. Nakazi, Extremal problems in H^p, 13 pages. 1991.

♯ 107: T. Nakazi, ρ-dilations and hypo-Dirichlet algebras, 15 pages. 1991.

♯ 108: A. Arai, An abstract sum formula and its applications to special functions, 25 pages. 1991.

♯ 109: Y.-G. Chen, Y. Giga and S. Goto, Analysis toward snow crystal growth, 18 pages. 1991.

♯ 110: T. Hibi, M. Wakayama, A q-analogue of Capelli's identity for $GL(2)$, 7 pages. 1991.

♯ 111: T. Nishimori, A qualitative theory of similarity pseudogroups and an analogy of Sacksteder's theorem, 13 pages. 1991.

♯ 112: K. Matsuda, An analogy of the theorem of Hector and Duminy, 10 pages. 1991.

♯ 113: S. Takahashi, On a regularity criterion uo to the boundary for weak solutions of the Navier-Stokes equations, 23 pages. 1991.

♯ 114: T. Nakazi, Sum of two inner functions and exposed points in H^1, 18 pages. 1991.

♯ 115: A. Arai, De Rham operators, Laplacians, and Dirac operators on topological vector spaces, 27 pages. 1991.

♯ 116: T. Nishimori, A note on the classification of non-singular flows with transverse similarity structures, 17 pages. 1991.

♯ 117: T. Hibi, A lower bound theorem for Ehrhart polynomials of convex polytopes, 6 pages. 1991.

♯ 118: R. Agemi, H. Takamura, The lifespan of classical solutions to nonlinear wave equations in two space dimensions, 30 pages. 1991.

♯ 119: S. Altschuler, S. Angenent and Y. Giga, Generalized motion by mean curvature for surfaces of rotation, 15 pages. 1991.

♯ 120: T. Nakazi, Invariant subspaces in the bidisc and commutators, 20 pages. 1991.

♯ 121: A. Arai, Commutation properties of the partial isometries associated with anticommuting self-adjoint operators, 25 pages. 1991.

♯ 122: Y.-G. Chen, Blow-up solutions to a finite difference analogue of $u_t = \Delta u + u^{1+\alpha}$ in N-dimensional balls, 31 pages. 1991.

♯ 123: A. Arai, Fock-space representations of the relativistic supersymmetry algebra in the two-dimensional space-time, 13 pages. 1991.

THE THEORY OF LEGENDRIAN UNFOLDINGS AND FIRST ORDER DIFFERENTIAL EQUATIONS

SHYUICHI IZUMIYA*

University of Liverpool and Hokkaido University

Abstract. We consider some properties about completely integrable first order differential equations for real-valued functions. In order to study this subject, we introduce the theory of Legendrian unfoldings. One of our theorems asserts that the set of equations with singular solution is an open set in the space of completely integrable equations even though such a set is thin in the space of all equations.

1. Introduction

In this paper we will study some properties about a special class of first order partial differential equations for real-valued functions. In the classical theory, *a first order partial differential equation (or, briefly, an equation)* is written in the form

$$F_k(x_1, \ldots, x_n, y, p_1, \ldots, p_n) = 0$$

for $k = 1, \ldots, 2n+1-r, r \geq n$. *A (classical) solution* of the equation is a smooth function $y = f(x_1, \ldots, x_n)$ and $p_i = \frac{\partial f}{\partial x_i}(x)$. We usually assume that F_k are $(2n+1)$-variable smooth function and $\operatorname{rank}(\frac{\partial F_k}{\partial x_i}, \frac{\partial F_k}{\partial z}, \frac{\partial F_k}{\partial p_j}) = 2n+1-r$.

Define

$$D = \{(x,y) | \text{there exists } p \in \mathbb{R}^n \text{ such that } F_1(x,y,p) = \cdots = F_{2n+1-r}(x,y,p) = 0 \\ \text{and } \operatorname{rank}(\frac{\partial F_k}{\partial p_j})(x,y,p) < \min(n, 2n+1-r)\}.$$

We call D *a discriminant set* of the equation. We also define

$$\Sigma = \{(x,y,p) | F_1(x,y,p) = \cdots = F_{2n+1-r}(x,y,p) = 0 \text{ and} \\ \operatorname{rank}(\frac{\partial F_k}{\partial p_j}(x,y,p) < \min(n, 2n+1-r)\}.$$

We say that Σ is *a singular solution* of the equation if D is a "graph" of a solution. In the history of differential equations, the notion of singular solutions appeared with the notion of complete solutions (cf.[2]). We say that an $(r-n)$-parameter family of (classical) solutions $y = f(t_1, \ldots, t_{r-n}, x_1, \ldots, x_n)$ of the equation is *a (classical) complete solution* if $\operatorname{rank}(\frac{\partial f}{\partial t_j}, \frac{\partial^2 f}{\partial t_i \partial x_j}) = r - n$. In [5] we have studied the case when $r = 2n$ (i.e., the single

---

* The author is a visiting fellow of Anglo-Japanese Scientific Exchange programme (The Royal Society and JSPS).

equation case) and shown that almost all single equations $F = 0$ has no singular solution and Σ consists of singular points of solutions. Hence, the Clairaut's equation (cf. Example 4.3) is not generic in the space of all single equations. But such an equation is well known from old time. We have also shown that, under a certain generic condition, single equations $F = 0$ with singular solution have a (classical) complete solution.

On the other hand, if we try to study the case when $r < 2n$, the involutory condition is very important as in the classical existence theorem (cf. [7]). And involutory equations are not generic in the space of all equations.

According to these facts, we will restrict our attension to the category of equations with (abstract) complete solutions. For details, see §3. The basic result of our theory asserts that the space of equations with (abstract) complete solution is homeomorphic to the space of complete Legendrian unfoldings which will be defined in §2 (cf. Theorem 4.4). By this fact, we can translate generic properties of equations with (abstract) complete solution into corresponding generic properties of complete Legendrian unfoldings. Hence, the notion of Legendrian unfoldings is very important in our theory. In §2 we will develope a general theory of Legendrian unfoldings. Equations with (abstract) complete solution are studied by these tools in §4. One of our main theorems asserts that the set of equations with singular solution is an open set in the space of equations with (abstract) complete solutions. This fact clarifies the reason why the Clairaut's equation is well known from the old time even though it is not generic in the space of all equations.

In [4] we have given another application of the theory of Legendrian unfoldings, in where we have shown that the category of one parameter Legendrian unfoldings supplies a correct class of geometric solutions of the Cauchy problem for Hamilton-Jacobi equation.

All maps, considered here, are class C^∞ unless stated otherwise.

Acknowledgement This work has been completed during the author's stay in the University of Liverpool mainly supported by Anglo-Japanese Scientific Exchange programme (The Royal Society and JSPS). The author would like to thank the Department of Pure Mathematics and especially Professor C.T.C. Wall for his kind encouragement and hospitality.

2. Legendrian unfoldings

The aim of this section is to introduce the theory of Legendrian unfoldings. Let $J^1(\mathbb{R}^n, \mathbb{R})$ be the 1-jet bundle of functions of n-variables. Since we only consider the local situation the 1-jet bundle $J^1(\mathbb{R}^n, \mathbb{R})$ may be considered as $\mathbb{R}^{2n+1}$ with a natural coordinate system

$$(x_1, \dots, x_n, y, p_1, \dots, p_n),$$

where $(x_1, \dots, x_n)$ is a coordinate system of $\mathbb{R}^n$. We have the natural projection

$$\pi : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R} \quad ; \quad \pi(x, y, p) = (x, y).$$

An immersion germ

$$i : (L, q) \to J^1(\mathbb{R}^n, \mathbb{R})$$

is said to be *a Legendrian immersion germ* if

$$\dim L = n \quad \text{and} \quad i^*\theta = 0,$$

where $\theta = dy - \sum_{i=1}^n p_i dx_i$. The image of $\pi \circ i$ is called *a wave front set* of i. We say that $q \in L$ is *a Legendrian singular point* if

$$\operatorname{rank} d(\pi \circ i)_q < n.$$

We now describe the notion of Legendrian unfoldings. Let R be an r-dimensional smooth manifold and

$$\mu : (R, y_0) \to (\mathbb{R}^{r-n}, t_0)$$

be a submersion germ and

$$\ell : (R, y_0) \to J^1(\mathbb{R}^n, \mathbb{R})$$

be a smooth map germ. We say that the pair (μ, ℓ) is *a Legendrian family* if $\ell_t = \ell|\mu^{-1}(t)$ is a Legendrian immersion germ for any $t \in (\mathbb{R}^{r-n}, t_0)$. A Legendrian family (μ, ℓ) is said to be *complete* if ℓ is an immersion germ. Then we have the following simple but a very important lemma.

LEMMA 2.1. *Let (μ, ℓ) be a Legendrian family. Then there exist unique elements*

$$h_1, \dots, h_{r-n} \in C^\infty_{u_0}(R)$$

such that

$$\ell^*\theta = \sum_{i=1}^{r-n} h_i \cdot d\mu_i,$$

where $\mu(u) = (\mu_1(u), \dots, \mu_{r-n}(u))$ and $C^\infty_{u_0}(R)$ is the ring of smooth function germs at u_0.

PROOF: Since ℓ_t is a Legendrian immersion for any $t \in (\mathbb{R}^{r-n}, t_0)$, we have

$$\theta(d\ell(T\mu^{-1}(t))) = 0.$$

This means that

$$\langle d\mu_1, \dots, d\mu_{r-n} \rangle_{C^\infty_{u_0}(R)} \supset \langle \ell^*\theta \rangle_{C^\infty_{u_0}(R)}.$$

The uniqueness follows from the fact that μ is a submersion germ.

We now consider the 1-jet bundle $J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})$ and the canonical 1-form Θ on the space. Let $(t_1, \dots, t_{r-n}, x_1, \dots, x_n)$ be canonical coordinate system on $\mathbb{R}^{r-n} \times \mathbb{R}^n$ and

$$(t_1, \dots, t_{r-n}, x_1, \dots, x_n, y, q_1, \dots, q_{r-n}, p_1, \dots, p_n)$$

be corresponding coordinate system on $J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})$. Then the canonical 1-form is given by

$$\Theta = dy - \sum_{i=1}^{n} p_i \cdot dx_i - \sum_{i=1}^{r-n} q_i \cdot dt_i = \theta - \sum_{i=1}^{r-n} q_i \cdot dt_i.$$

We define the natural projection

$$\Pi : J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R}) \to (\mathbb{R}^{r-n} \times \mathbb{R}^n) \times \mathbb{R}$$

by

$$\Pi(t, x, y, q, p) = (t, x, y)$$

We call the above 1-jet bundle *a unfolded 1-jet bundle.*

Define a map germ

$$\mathcal{L} : (R, u_0) \to J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})$$

by

$$\mathcal{L}(u) = (\mu(u), x \circ \ell(u), y \circ \ell(u), h(u), p \circ \ell(u)).$$

Then we can easily show that $\mathcal{L}$ is a Legendrian immersion germ. If we fix 1-forms Θ and θ, the Legendrian immersion germ $\mathcal{L}$ is uniquely determined by the Legendrian family (μ, ℓ). We call $\mathcal{L}$ *a Legendrian unfolding associated with the Legendrian family* (μ, ℓ). We also say that $\mathcal{L}$ is *complete* if (μ, ℓ) is a complete Legendrian family. We remark that even in the one parameter case the notion of the Legendrian unfoldings is slightly different from the notion of extended Legendrian manifolds in the sense of Zakalyukin[9]. For example, we now consider a Legendrian immersion germ $\mathcal{L} : (\mathbb{R}^2, 0) \to J^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ defined by $\mathcal{L}(u, v) = (u^2 + 3v^2, u, 2v^3, v, -2uv)$, it is the extended Legendrian immersion germ. But it is not a Legendrian unfolding because $u^2 + 3v^2$ is not a submersion.

Since $\mathcal{L}$ is a Legendrian immersion germ, there exists a generating family of $\mathcal{L}$ by the Arnol'd-Zakalyukin's theory ([1],[8],[9]). In this case the generating family is naturally constructed by the $(r - n)$-family of generating families associated with (μ, ℓ).

Let

$$F : ((\mathbb{R}^{r-n} \times \mathbb{R}^n) \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$$

be a function germ such that

$$d_2F|0 \times \mathbb{R}^n \times \mathbb{R}^k$$

is non-singular, where

$$d_2F(t,x,q) = (\frac{\partial F}{\partial q_1}(t,x,q),\ldots,\frac{\partial F}{\partial q_k}(t,x,q)).$$

It follows from the defintion that $C(F) = d_2F^{-1}(0)$ is a smooth r-manifold germ and

$$\pi_F : (C(F),0) \to \mathbb{R}^{r-n}$$

is a submersion germ, where

$$\pi_F(t,x,q) = t.$$

Define map germs

$$\tilde{\Phi}_F : (C(F),0) \to J^1(\mathbb{R}^n,\mathbb{R})$$

by

$$\tilde{\Phi}_F(t,x,q) = (x, F(t,x,q), \frac{\partial F}{\partial x}(t,x,q))$$

and

$$\Phi_F : (C(F),0) \to J^1(\mathbb{R}^{r-n}\times\mathbb{R}^n,\mathbb{R})$$

by

$$\Phi_F(t,x,q) = (t,x,F(t,x,q),\frac{\partial F}{\partial t}(t,x,q),\frac{\partial F}{\partial x}(t,x,q)).$$

Since $\frac{\partial F}{\partial q_i} = 0$ on $C(F)$, we can easily show that

$$(\tilde{\Phi}_F)^*\theta = \sum_{i=1}^{r-n}\frac{\partial F}{\partial t_i}|C(F)\cdot dt_i|C(F).$$

By the definition, Φ_F is a Legendrian unfolding associated with the Legendrian family $(\pi_F, \tilde{\Phi}_F)$. By the same metnod of the theory of Arnol'd-Zakalyukin ([1],[8],[9]), we can show that the following proposition.

PROPOSITION 2.2. *Every Legendrian unfolding germs are constructed by the above method.*

Then F is called *a generalized phase family of* Φ_F. We now consider ambiguity of the choice of generalized phase function germs. Let

$$F,G : ((\mathbb{R}^{r-n}\times\mathbb{R}^n)\times\mathbb{R}^k,0) \to (\mathbb{R},0)$$

be generalized phase families. We say that F and G are *strictly $\mathcal{R}$-equivalent* if there exists a diffeomorphism germ

$$\Phi : ((\mathbb{R}^{r-n}\times\mathbb{R}^n)\times\mathbb{R}^k,0) \to ((\mathbb{R}^{r-n}\times\mathbb{R}^n)\times\mathbb{R}^k,0)$$

of the form $\Phi(t,x,q) = (t,x,\phi(t,x,q))$ such that $F\circ\Phi = G$. If we carefully read proofs of Lemmas 1 and 2 in (page 307 in [1]), we can find the following assertion.

PROPOSITION 2.3. *Let* $F,G : (\mathbb{R}^{r-n}\times\mathbb{R}^n)\times\mathbb{R}^k,0) \to (\mathbb{R},0)$ *be generalized phase function germs such that* $\mathrm{Image}\,\Phi_F = \mathrm{Image}\,\Phi_G$ *and*

$$\operatorname{rank} H(F|0\times\mathbb{R}^k) = \operatorname{rank} H(G|0\times\mathbb{R}^k) = 0,$$

where $H(f)$ is the Hessian matrix of f at the origin. Then F and G are strictly $\mathcal{R}$-equivalent.

3. Geometry of first order differential equations

The aim of this section is to describe the geometric structure connected with first order differential equations. A first order differential equation is most naturally interpreted as being a closed subset of $J^1(\mathbb{R}^n, \mathbb{R})$. Unless the contrary is specifically stated, we use the following definition.

A system of first order differential equations (or, briefly an equation) is an r-dimensional submanifold $E \subset J^1(\mathbb{R}^n, \mathbb{R})$, where $n+1 \leq r \leq 2n$. If $r < 2n$, then E is said to be *overdetermined.* We also say that E is *maximally overdetermined (or holonomic)* if $r = n+1$.

By the philosophy of Lie, we may define the notion of solutions as follows. *An (abstract) solution* of E is a Legendrian immersion

$$i : L \to J^1(\mathbb{R}^n, \mathbb{R})$$

such that

$$i(L) \subset E.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Then $j^1 f : \mathbb{R}^n \to J^1(\mathbb{R}^n, \mathbb{R})$ is a Legendrian embedding. Hence, in our terminology, the (classical) solution of E is a smooth function f such that

$$j^1 f(\mathbb{R}^n) \subset E.$$

On the other hand, we can show that an (abstract) solution $i : L \to J^1(\mathbb{R}^n, \mathbb{R})$ is given by (at least locally) a jet extension $j^1 f$ of a smooth function f if and only if $\pi \circ i$ is a non-singular map. Thus the (abstract) solution has multi-valued near the Legendrian singular point. We also define the notion of singularities of equations. Let $E^r \subset J^1(\mathbb{R}^n, \mathbb{R})$ be an equation. Then $z \in E$ is said to be *a contact singular point* if

$$\theta(T_z E) = 0.$$

We also say that $z \in E$ is *a π-singular point* if

$$\operatorname{rank}(d\pi|E)_z < n+1.$$

We can easily show that if z is a contact singular point of E, then it is a π-singular point of E. Let $\Sigma(\pi|E)$ be the set of π-singular points and $\Sigma_c(E)$ be the set of contact singular points. We say that $D_E = \pi(\Sigma(\pi|E))$ is *a discriminant set* of the equation E. If the π-singular set $\Sigma(\pi|E)$ is a Legendrian submanifold, then we call it *an (abstract) singular solution* of the equation E. In this case, the discriminant set D_E is the graph of the (abstract) singular solution.

Our purpose in this section is to establish the notion of (abstract) complete solutions. Let

$$y = (t_1, \ldots, t_{r-n}, x_1, \ldots, x_n)$$

be the (classical) complete solution of E, then we have a jet extension

$$j^1_* f : \mathbb{R}^{r-n} \times \mathbb{R}^n \to J^1(\mathbb{R}^n, \mathbb{R})$$

which is defined by

$$j^1_* f(t, x) = j^1 f_t(x),$$

where $f_t(x) = f(t, x)$. Then it is easy to show that $j^1_* f$ is an immersion. Since $dim\ E = r$, then $j^1_* f$ gives (at least locally) a parametrization of E and $j^1_* f(t \times \mathbb{R}^n)$ is a (clasical) solution of E for any $t \in \mathbb{R}^{r-n}$. Thus there exists a foliation on E whose leaves are (classical) solutions. Thus we can generalize this notion to an abstract sense. We say that an equation $E \subset J^1(\mathbb{R}^n, \mathbb{R})$ is *completely integrable (or E has an (abstract) complete solution)* if there exists an n-dimensional completely integrable distribution $\mathcal{D}$ on E such that

$$\theta_z(\mathcal{D}_z) = 0$$

for any $z \in E$.

By the Frobenius' theorem, we have the following proposition.

PROPOSITION 3.1. *Let $E^r \subset J^1(\mathbb{R}^n, \mathbb{R})$ be an equation. Then the following conditions are equaivalent.*

(1) *E is completely integrable.*

(2) *For any $q \in E$, there exist a neighbourhood U of q in E and smooth functions*

$$\mu_1, \ldots, \mu_{r-n} \text{ on } U$$

such that

$$d\mu_1 \wedge \cdots \wedge d\mu_{r-n} \neq 0 \text{ on } U$$

and

$$\langle d\mu_1, \ldots, d\mu_{r-n} \rangle_{C^\infty(U)} \supset \langle \theta | U \rangle_{C^\infty(U)}$$

as $C^\infty(U)$-modules, where $C^\infty(U)$ denotes the ring of smooth functions on U.

(3) *For any $q \in E$, there exist a neighbourhood $V \times W$ of 0 in $\mathbb{R}^{r-n} \times \mathbb{R}^n$ and an embedding*

$$f : V \times W \to J^1(\mathbb{R}^n, \mathbb{R})$$

such that

$$f(0) = q,\ f(V \times W) \subset E$$

and

$$f|\{t\} \times W \to J^1(\mathbb{R}^n, \mathbb{R})$$

is Legendrian embedding for any $t \in V$.

4. Completely integrable first order differential equations

In this section we will study some properties of completely integrable equations as an application of the theory of Legendrian unfoldings. Since we will only study local properties, an equation is defined to be an immersion

$$f : U \to J^1(\mathbb{R}^n, \mathbb{R})$$

where U is an open subset of $\mathbb{R}^n$. By Proposition 3.1, we say that f is *completely integrable* if there exists a submersion

$$\mu = (\mu_1, \ldots, \mu_{r-n}) : U \to \mathbb{R}^{r-n}$$

such that

$$\langle d\mu_1, \ldots, \mu_{r-n}\rangle_{C^\infty(U)} \supset \langle f^*\theta\rangle_{C^\infty(U)}.$$

We call $\mu = (\mu_1, \ldots, \mu_{r-n})$ a *complete integral* of f and the pair

$$(\mu, f) : U \to \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R})$$

is called *a first order differential equation with complete integral (or, briefly, an equation with complete integral).*

We can also define the above notions in terms of map germs : *An equation germ* is defined to be an immersion germ

$$f : (\mathbb{R}^r, 0) \to J^1(\mathbb{R}^n, \mathbb{R}).$$

We say that f is *completely integrable* if there exists a submersion germ

$$\mu = (\mu_1, \ldots, \mu_{r-n}) : (\mathbb{R}^r, 0) \to \mathbb{R}^{r-n}$$

such that

$$\langle d\mu_1, \ldots, d\mu_{r-n}\rangle_{\mathcal{E}_u} \supset \langle f^*\theta\rangle_{\mathcal{E}_u},$$

where $u = (u_1, \ldots, u_r)$ is the canonical coordinate of $(\mathbb{R}^r, 0)$ and $\mathcal{E}_u$ is the ring of function germs of u-variables at the origin. Then μ is called *a complete integral* of f and the pair

$$(\mu, f) : (\mathbb{R}^r, 0) \to \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R})$$

is called *an equation germ with complete integral.*

In the terminology in §2, the pair (μ, f) is a complete Legendrian family. Let $\mathcal{L} : U \to J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})$ be a complete Legendrian unfolding associated to the Legendrian family (μ, f) which is defined as in §2. Since $\mathcal{L}$ is uniquely determined by (μ, f), we denote $\ell_{(\mu,f)}$ instead of $\mathcal{L}$.

Conversely, let

$$\mathcal{L} : U \to J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})$$

be a Legendrian immersion such that f is an immersion and μ is a submersion with $\Pi_1 \circ \mathcal{L} = (\mu, f)$, where $\Pi_1 : J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R}) \to J^1(\mathbb{R}^n, \mathbb{R})$ is the canonical projection. Then (μ, f) is an equation with complete integral and $\mathcal{L} = \ell_{(\mu,f)}$.

Some effects of the notion of Legendrian unfoldings on equations with complete integral are given in the following propositions.

PROPOSITION 4.1. *Let* $(\mu, f) : U \to \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R})$ *be an equation with complete integral. Then* $f(\mu^{-1}(t))$ *is a (classical) solution for any* $t \in \mathbb{R}^{r-n}$ *if and only if* $\ell_{(\mu,f)}$ *is Legendrian non-singular.*

PROOF: By the definition, $\ell_{(\mu,f)}$ is Legendrian non-singular if and only if $\Pi \circ \ell_{(\mu,f)}$ is an immersion. Here,

$$\Pi \circ \ell_{(\mu,f)}(u) = (\mu(u), x \circ f(u), y \circ f(u)).$$

We set

$$P_1 : \mathbb{R}^{r-n} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{r-n}$$

the canonical projection. By the definition of the Legendrian family, $P_1 \circ \Pi \circ \ell_{(\mu,f)} = \mu$ is a submersion.

Hence, if $\ell_{(\mu,f)}$ is Legendrian non-singular, then

$$P_1 | \mathrm{Image}\, \Pi \circ \ell_{(\mu,f)}$$

is a submersion. It follows that

$$P_1^{-1}(t) \cap \mathrm{Image}\, \Pi \circ \ell_{(\mu,f)} = \{t\} \times \mathbb{R}^n \times \mathbb{R} \cap \mathrm{Image}\, \Pi \circ \ell_{(\mu,f)}$$

is an n-dimensional submanifold of $\{t\} \times \mathbb{R}^n \times \mathbb{R}$.

We also set

$$P_2 : \mathbb{R}^{r-n} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$$

the canonical projection. Then

$$P_2(\{t\} \times \mathbb{R}^n \times \mathbb{R} \cap \mathrm{Image}\, \Pi \circ \ell_{(\mu,f)})$$

is an n-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}$. It is easy to show that

$$P_2(\{t\} \times \mathbb{R}^n \times \mathbb{R} \cap \mathrm{Image}\ \Pi \circ \ell_{(\mu,f)}) = \pi \circ f(\mu^{-1}(t)).$$

This shows that $f|\mu^{-1}(t)$ is Legendrian non-singular (i.e. the classical solution).

Suppose that $f|\mu^{-1}(t)$ is Legendrian non-singular for any $t \in \mathbb{R}^{r-n}$. We now have a decomposition of the tangent space as follows :

$$T_{u_0} U = T_{u_0} \mu^{-1}(t) \oplus V.$$

Then

$$d(\Pi \circ \ell_{(\mu,f)})| : T_{u_0} \mu^{-1}(t) \to T_{(t,\alpha)}(\{t\} \times \mathbb{R}^n \times \mathbb{R})$$

is a monomorphism and

$$d(\Pi \circ \ell_{(\mu,f)})| : V \to T_t \mathbb{R}^{r-n}$$

is a isomorphism. Thus $\Pi \circ \ell_{(\mu,f)}$ is an immersion at $u_0 \in U$. This completes the proof.

We say that an equation germ with complete integral is *regular* if $\ell_{(\mu,f)}$ is Legendrian non-singular. Let (μ, f) be a regular equation germ with complete integral. By Proposition 4.1, $f(\mu^{-1}(t))$ is a (classical) solution for any $t \in (\mathbb{R}^{r-n}, 0)$ and Image f is foliated by the family $\{f(\mu^{-1}(t))\}_{t\in(\mathbb{R}^{r-n},0)}$. Then we can choose a family of function germ

$$F : (\mathbb{R}^{r-n} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$$

such that Image $j^1 F_t = f(\mu^{-1}(t))$ for any $t \in (\mathbb{R}^{r-n})$ and

$$j_1^1 F : (\mathbb{R}^{r-n} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R}^n, \mathbb{R})$$

is an immersion germ, where $F_t(x) = F(t, x)$ and $j_1^1 F(t, x) = j^1 F_t(x)$. The fact that $j_1^1 F$ is an immersion leads us to the following equality :

$$\operatorname{rank}\left(\frac{\partial F}{\partial t_i}\frac{\partial^2 F}{\partial t_i \partial x_j}\right) = r - n.$$

Thus F is a (classical) complete solution of f. If we consider the 1-jet extension

$$j^1 F : (\mathbb{R}^{r-n} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R}),$$

then it is a Legendrian unfolding associated with $(\pi_{r-n}, j_1^1 F)$.

PROPOSITION 4.2. *Let $(\mu, f) : U \to \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R})$ be an equation with complete integral. For any $u \in U$, we denote*

$$\ell_{(\mu,f)}(u) = (\mu(u), x \circ f(u), y \circ f(u), h(u), p \circ f(u))$$

by the local coordinate of $J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})$. Then f is contact singular at $u_0 \in U$ if and only if

$$h(u_0) = 0.$$

PROOF: Since $\ell_{(\mu,f)}$ is a Legendrian immersion, we have

$$0 = (\ell_{(\mu,f)})^* \Theta = f^* \theta - \sum_{i=1}^{r-n} h_i d\mu_i$$

at any $u \in U$.

By the definition, f is contact singular at u_0 if and only if

$$f^* \theta = 0 \text{ at } u_0.$$

Because μ is a submersion, it is equivalent to the condition

$$h_1(u_0) = \cdots = h_{r-n}(u_0) = 0.$$

W now present a well-known example which is called *the Clairaut's equation.*

EXAMPLE 4.3. The following is the classical example of a partial differential equation with singular solution.

Consider the following partial differential equation on $J^1(\mathbb{R}^n, \mathbb{R})$,

$$y = x_1 \cdot p_1 + \cdots + x_n \cdot p_n + g(x_1, \ldots, p_n).$$

The (classical) complete solution of this equation is given by

$$y = x_1 \cdot t_1 + \cdots + x_n \cdot t_n + g(t_1, \ldots, t_n)$$

for any $(t_1, \ldots, t_n) \in \mathbb{R}^n$.

We now define a submersion

$$\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

by

$$\mu(t, u) = t$$

and an immersion

$$f : \mathbb{R}^n \times \mathbb{R}^n \to J^1(\mathbb{R}^n, \mathbb{R})$$

by

$$f(t, u) = (u, \sum_{i=1}^{n} u_i \cdot t_i + g(t), t).$$

Then we have

$$f^*\theta = (u_1 + \frac{\partial g}{\partial t_1}) \cdot dt_1 + \cdots + (u_n + \frac{\partial g}{\partial t_n}) \cdot dt_n.$$

If we put

$$h(t, u) = (u_1 + \frac{\partial g}{\partial t_1}(t), \ldots, u_n + \frac{\partial g}{\partial t_n}(t)),$$

then the Legendrian unfolding is given by

$$\ell_{(\mu, f)}(t, u) = (t, u, \sum_{i=1}^{n} t_i \cdot u_i + g(t), h(t, u), t).$$

By Proposition 4.2, the contact singular point of this equation is equal to the set which is defined by

$$u_1 + \frac{\partial g}{\partial t_1}(t) = \cdots = u_n + \frac{\partial g}{\partial t_n}(t) = 0.$$

This set is also equal to the π-singular set. Thus it is the singular solution.

We now establish the notion of genericity of equation germs with complete integral. Let $U \subset \mathbb{R}^r$ be an open set. We denote by

$$\mathrm{Int}(U, \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R}))$$

the set of equations with complete integral

$$(\mu, f) : U \to \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R}).$$

We also define

$$L(U, J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R}))$$

to be the set of complete Legendrian unfoldings

$$\ell_{(\mu,f)} : U \to J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R}).$$

These sets are topological spaces equipped with the Whitney C^∞-topology. A subset of

$$\mathrm{Int}(U, \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R})) \text{ (respectively } L(U, J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})))$$

is said to be *generic* if it is an open dense subset in

$$\mathrm{Int}(U, \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R})) \text{ (respectively } L(U, J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R}))).$$

The genericity of a property of germs are defined as follows. Let P be a property of equation germs with complete integral

$$(\mu, f) : (\mathbb{R}^r, 0) \to \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R})$$

(respectively, Legendrian unfoldings

$$\ell_{(\mu,f)} : (\mathbb{R}^r, 0) \to J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})).$$

For an openset $U \subset \mathbb{R}^r$, we define $\mathcal{P}(U)$ to be the set of

$$(\mu, f) \in \mathrm{Int}(U, \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R}))$$

(respectively,

$$\ell_{(\mu,f)} \in L(U, J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})))$$

such that the germ at x whose representative is given by (μ, f) (respectively $\ell_{(\mu,f)}$) has property P for any $x \in U$.

The property P is said to be *generic* if for some neighbourhood U of 0 in $\mathbb{R}^r$, the set $\mathcal{P}(U)$ is a generic subset in $\mathrm{Int}(U, \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R}))$ (respectively $L(U, J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R}))$).

By the construction, we have a well-defined continuous mapping

$$(\Pi_1)_* : L(U, J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})) \to \mathrm{Int}(U, \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R}))$$

defined by

$$(\Pi_1)_*(\ell_{(\mu,f)}) = \Pi_1 \circ \ell_{(\mu,f)} = (\mu, f).$$

The following theorem is fundamental in our theory.

THEOREM 4.4. *The continuous map*

$$(\Pi_1)_* : L(U, J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})) \to \mathrm{Int}(U, \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R}))$$

is a homeomorphism.

PROOF: We now define a mapping

$$\iota : \mathrm{Int}(U, \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R})) \to L(U, J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R}))$$

by

$$\iota((\mu, f)) = \ell_{(\mu,f)}.$$

Then ι is a inverse mapping of $(\Pi_1)_*$. Thus it is enough to show that ι is continuous. The mapping $\ell_{(\mu,f)}$ is given by

$$\ell_{(\mu,f)}(u) = (\mu(u), x \circ f(u), y \circ f(u), h(u), p \circ f(u)),$$

where

$$f^*\theta = \sum_{i=1}^{r-n} h_i \cdot d\mu_i.$$

By the direct calculation, we have

$$f^*\theta = \sum_{j=1}^{r} \left(\frac{\partial y \circ f}{\partial u_j} - \sum_{i-1}^{n} (p_i \circ f) \cdot \frac{\partial x_i \circ f}{\partial u_j}\right) du_j$$

and

$$\sum_{i=1}^{r-n} h_i d\mu_i = \sum_{j=1}^{r} \left(\sum_{\ell=1}^{r-n} h_\ell(u) \cdot \frac{\partial \mu_\ell}{\partial u_j}\right) du_j.$$

Hence, $h(u)$ is determined by the following formula :

$$\begin{pmatrix} \frac{\partial \mu_1}{\partial u_1} & \cdots & \frac{\partial \mu_{r-n}}{\partial u_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mu_1}{\partial u_r} & \cdots & \frac{\partial \mu_{r-n}}{\partial u_r} \end{pmatrix} \begin{pmatrix} h_1(u) \\ \vdots \\ h_{r-n}(u) \end{pmatrix} = \begin{pmatrix} \frac{\partial y \circ f}{\partial u_1} - \sum_{i=1}^{n} (p_i \circ f) \cdot \frac{\partial x_i \circ f}{\partial u_1} \\ \vdots \\ \frac{\partial y \circ f}{\partial u_r} - \sum_{i=1}^{n} (p_1 \circ f) \cdot \frac{\partial x_i \circ f}{\partial u_r} \end{pmatrix}.$$

Since

$$\mathrm{rank}\left(\frac{\partial \mu_i}{\partial u_j}\right) = r - n,$$

there exists a unique solution $h(u)$ of the above linear equation which depends on partial derivatives of μ_i, $x_i \circ f$, $y \circ f$ and $p_i \circ f$ continuously. By the definition of the Whitney C^∞-topology, ι is a continuous map.

This theorem asserts that the genericity of a property of equations with complete integral can be interpreted by the genericity of the corresponding property of Legendrian unfoldings.

On the other hand, by the theory of Legendrian unfoldings, we can study generic properties of Legendrian unfoldings in terms of generating families. In [6] we will classify generic completely integrable holonomic systems of equations by point transformations in the sense of Sophus Lie. The notion of Legendrian unfoldings is the key in the classification. Here, we only consider some generic properties as a consequence of the above theorem.

PROPOSITION 4.5. *For generic equation germ with complete integral*

$$(\mu, f) : (\mathbb{R}^r, 0) \to \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R}),$$

the contact singular set $\Sigma(f_c)$ is empty or an n-dimensional submanifold.

PROOF: By Proposition 2.2 and Theorem 4.4, it is enough to consider a generalized phase family

$$F : ((\mathbb{R}^{r-n} \times \mathbb{R}^n) \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0).$$

We now define a subset Σ of $J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n \times \mathbb{R}^k, \mathbb{R})$ by

$$\{j^1 f(t,x,q) | \frac{\partial f}{\partial t_1}(t,x,q) = \cdots = \frac{\partial f}{\partial t_{r-n}}(t,x,q) = \frac{\partial f}{\partial q_1}(t,x,q) = \cdots = \frac{\partial f}{\partial q_k}(t,x,q) = 0\}.$$

Then Σ is a linear subspace of $J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n \times \mathbb{R}^k, \mathbb{R})$ of $\mathrm{codim}\Sigma = k + r - n$ and it is invariant under the action of the group of strictly $\mathcal{R}$-equivalences. The the transversality of $j^1 F$ to Σ does not depend on the choice of the generalized phase function germs. By the defintion of Φ_F and Proposition 4.2, we have

$$\Sigma(\tilde{\Phi}_c) = j^1 F^{-1}(\Sigma).$$

By the jet transversality theorem, $\Sigma(\tilde{\Phi}_c)$ is an n-dimensional submanifold.

We remark that even for generic equation germs with complete integral (μ, f), we cannot expect that the π-singular set $\Sigma(\pi \circ f)$ is an n-dimensional submanifold. In [3] we classified generic equation germs with complete integral in the case of $n = 1$ (i.e. ordinary differential equations). One of the normal form is given by

$$f(u,v) = (u, v^3 + uv^2, v^2 - 3v - 2u),$$
$$\mu(u,v) = \frac{1}{2}v^2 + u.$$

The π-singular set of this example is given by

$$\Sigma(\pi \circ f) = \{(u,v) | 3v^2 + 2uv\}$$

and it is not a smooth submanifold. We can calculate that $f^*\theta = (3v + 2u)d\mu$, then we have

$$\Sigma(f_c) = \{(u,v) | 3v + 2u = 0\}.$$

Hence $\Sigma(f_c)$ is a smooth submanifold and it is a smooth component of $\Sigma(\pi \circ f)$.

On the other hand, we appreciate another normal form given by

$$f(u,v) = (u, v^2, v),$$
$$\mu(u,v) = v - \frac{1}{2}u.$$

In this case we can easily show that $f^*\theta = 2vd\mu$, then the corresponding Legendrian unfolding is given by

$$\ell_{(\mu,f)}(u,v) = (v - \frac{1}{2}u, u, v^2, 2v, v).$$

It is clear that (μ, f) is a regular equation germ with complete integral. The π-singular set and the contact singular set of this equation is given by $\{(u,v) | v = 0\}$, then it is a smooth submanifold and the singualr solution of f.

In the last part of this paper, we are eventually comming back to the study of equations with (classical) complete solution. By the argument after Proposition 4.1, this class of equations is exactly equal to the class of regular equations with complete integral.

LEMMA 4.6. *For regular equations with complete integral* (μ, f), *we have*

$$\Sigma(\pi \circ f) = \Sigma(f_c).$$

PROOF: Without loss of generality, we may study

$$j^1F : (\mathbb{R}^{r-n} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})$$

instead of $\ell_{(\mu,f)}$. In this case we have

$$h(t,x) = (\frac{\partial F}{\partial t_1}(t,x), \ldots, \frac{\partial F}{\partial t_{r-n}}(t,x)).$$

By the definition, $(t_0, x_0) \in \Sigma(\pi \circ f)$ if and only if

$$\operatorname{rank} \begin{pmatrix} & 0 & & E \\ \frac{\partial F}{\partial t_1} & \cdots & \frac{\partial F}{\partial t_{r-n}} & \frac{\partial F}{\partial x} \end{pmatrix} < n+1$$

at (t_0, x_0). It is equivalent to the fact that (t_0, x_0) satisfies

$$\frac{\partial F}{\partial t_1}(t_0, x_0) = \cdots = \frac{\partial F}{\partial t_{r-n}}(t_0, x_0) = 0.$$

Since $\Sigma(f_c) = h^{-1}(0)$, then we have the required result.

Here, we consider the open subset of $\mathrm{Int}(U, \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R}))$ consisting of regular equations with complete integral. We denote this subset by R-$\mathrm{Int}(U, \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R}))$. If a property P is generic in the space R-$\mathrm{Int}(U, \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R}))$, we say that *generic regular equations with complete integral have property* P. Then the following theorem asserts that generic regular equation germs with complete integral are π-regular or have singular solutions.

THEOREM 4.7. *For generic regular equation germs with complete integral*

$$(\mu, f) : (\mathbb{R}^r, 0) \to \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R}),$$

the π-singular set $\Sigma(\pi \circ f)$ is empty or an n-dimensional submanifold and the discriminant set D_f is an envelope of the family $\{\pi \circ f(\mu^{-1}(t))\}_{t\in(\mathbb{R}^{r-n},0)}$ consisting of graph of a classical complete solution of f.

PROOF: By Proposition 4.5, for generic equation germ with complete integral (μ, f), the contact singular set $\Sigma(f_c)$ is empty or an n-dimensional subspace. It is equal to the π-singular set $\Sigma(\pi \circ f)$ by Lemma 4.6. By the argument after Proposition 4.1, we may consider

$$j^1F : (\mathbb{R}^{r-n} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})$$

instead of (μ, f). In this case the family which is consisting of graph of the complete solution is given by

$$\{(t, x, F(t, x))\}_{t\in(\mathbb{R}^{r-n},0)}.$$

This family is defined by the equation $y - F(t, x)$. Then the set

$$\{(t, x, F(t, x)) | \frac{\partial F}{\partial t_i} = 0 \text{ for } i = 1, \ldots, r - n\}$$

is the envelope of this family by the usual method of the elementary calculus. We can easily show that this set is the criticsal value set of the map germ $\pi \circ j_1^1 F$. This completes the proof.

REMARK. We can explicitly write down the generic condition in the above theorem in terms of $\ell_{(\mu,f)}$. If we assume that $\ell_{(\mu,f)}$ is Legendrian non-singular and h is a submersion, then the same assertion as the above theorem holds.

REFERENCES

1. V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, "Singularities of Differentiable Maps vol. I," Birkhäuser, 1986.
2. C. Carathèodory, "Calculus of Variations and Partial Differential Equations of First Order, Part I, Partial Differential Equations of First Order,," Holden-Day, 1965.
3. A. Hayakawa, G. Ishikawa, S. Izumiya and K. Yamaguchi, *Classification of generic integral diagram and first order ordinary differential equations*, preprint.
4. S. Izumiya, *Geometric singularities for Hamilton-Jacobi equation*, Advanced Studies in Pure Math. (1991) (to appear).
5. S. Izumiya, *Generic properties of first order partial differential equations*, Proceedings of Topology conference in Hawaii, Birkhäuser. (to appear)
6. S. Izumiya, *Classification of generic completely integrable holonomic systems of first order differential equations*, in preparations.
7. V. V. Lychagin, *Local classification of non-linear first order partial differential equations*, Russian Math. Surveys 30 (1975), 105–175.
8. V. M. Zakalyukin, *Lagrangian and Legendrian singularites*, Funct. Anal. Appl. 10 (1976), 23–31.
9. V. M. Zakalyukin, *Reconstructions of fronts and caustics depending on a parameter and verslity of mappings*, Journal of Soviet Math. 27 (1983), 2713–2735.

Keywords. Legendrian unfolding, complete solution, First order differential equation

Department of Pure Mathematics, University of Liverpool, P.O.Box 147, Liverpool L69 3BX, England UK and Department of Mathematics, Faculty of Science, Hokkaido University , Sapporo 060 JAPAN