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Vector fields near a generic submanifold

Goo Ishikawa, Shyūichi Izumiya and Kazuo Watanabe

Abstruct

It is given a classification of generic vector fields near a generic submanifold. The normal forms are linear vector fields near the local model of the submanifold. Similar results are obtained for vector fields near a hypersurface with boundary and near a piecewise-smooth hypersurface.

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0. Introduction

In this paper we shall study generic local normal forms of vector fields near a generic submanifold. This classification problem arises naturally in setting up a "directional derivative problem" (cf. V.I. Arnold [1]).

Let M be a smooth n-manifold, Q a smooth $(n-k)$-manifold and $\mathrm{Emb}(Q, M)$ the space of smooth embeddings of Q in M endowed with the Whitney topology. We also consider vector fields on M. By the Thom's transversality theorem, a generic vector field has only isolated singularities. Thus we may assume that a vector field X on M has only isolated singularities. One of our results is as follows:

Theorem A. *There exists a residual subset $\mathcal{O} \subset \mathrm{Emb}(Q, M)$ with the following property: For any $i \in \mathcal{O}$ and $q \in Q$, there exist a coordinate neighborhood $(U, (x_1, \dots, x_n))$ around $i(q)$ in M and an integer $t = t(q)$, $(0 \le t \le \frac{n}{k} - 1)$, such that*

$$(1) \quad X|U = \sum_{j=1}^{t}\left(\sum_{i=1}^{k} x_{jk+i}\frac{\partial}{\partial x_{(j-1)k+i}}\right) + \frac{\partial}{\partial x_{tk+1}},$$

$$(2) \quad i(Q) \cap U = \{(x_1, \dots, x_n) | x_1 = \dots = x_k = 0\}.$$

We remark that the same normal forms have been obtained by S.M.Vishik [6] in the case when $k = 1$. Our assertion and the method of the proof are, however, slightly different from those of him.

We observe that if $n < 2k$ then the generic normal form of vector fields near a generic submanifold is the trivial vector field $\partial/\partial x_1$.

In §1 we shall prepare a kind of transversality theorem which describes some generic properties of embeddings. The local properties of embeddings will be studied in §2.

In §3 we will prove Theorem A. The main idea of the proof is as follows: Since X has only isolated singularities, we may assume that X is non-singular near the image of a generic embedding. Then X is transformed to $\partial/\partial x_1$ by a local diffeomorphism on M. Moreover, the image is the zero set of a local submersion. In the first place, we shall classify zero sets of local submersions by diffeomorphisms which preserve the vector field $\partial/\partial x_1$. After this procedure, we shall transform local submersions and vector fields to corresponding normal forms in Theorem A by appropriate diffeomorphisms.

In §4, we shall state other results. We treat the same problem in the case of a generic submanifold with boundary in Theorem B. We also study the case of a generic piecewise smooth hypersurface in Theorem C. Since we can prove Theorems B and C by the same method as that used to show Theorem A, we shall omit details of the proofs of Theorems B and C.

The multi-germ version of the method we used here can be applied to study shadows of submanifolds in a Euclidian space (K.Watanabe [7]).

All map germs considered here, are differentiable of class C^∞, unless stated otherwise.

1. A transversality theorem

Since the vector field X has only isolated singularities, there exists a residual set $\mathcal{O}_1 \subset \mathrm{Emb}(Q, M)$ such that X is non-singular at every points of $i(Q)$ for any $i \in \mathcal{O}_1$.

Let $i : Q \to M$ be an embedding. For any $q \in Q$, we may suppose locally Q is the zero set of a submersion germ $F : (M, i(q)) \to (\mathbf{R}^k, 0)$. We call F *a local equation of* $i(Q)$ *at* $i(q)$. Of course, the choice of F is not unique. It depends only on the $\mathcal{C}$-equivalence in the sense of J.Mather [5]. Thus we use notions and results of the theory on the $\mathcal{K}$-equivalence (cf. J.Mather [5] and J.Martinet [4]).

Let $\mathcal{S}$ be a $\mathcal{K}$-invariant Whitney stratification of the fiber $J^r(1, k)$ of the r-jet space $J^r(\mathbf{R}, \mathbf{R}^k)$. We define a stratification $\tilde{\mathcal{S}}$ of the r-jet space by

$$\{\mathbf{R} \times \{\mathbf{R}^k - \{0\}\} \times J^r(1, k), \mathbf{R} \times \{0\} \times S \mid S \in \mathcal{S}\},$$

where $J^r(\mathbf{R}, \mathbf{R}^k) = \mathbf{R} \times \mathbf{R}^k \times J^r(1, k)$. Our transversality theorem is as follows:

THEOREM 1.1. *There exists a residual set* $\mathcal{O} \subset \mathit{Emb}(Q, M)$ *with the following properties: For any* $i \in \mathcal{O}$ *and* $q \in Q$, *there exists a local coordinate* $(x_1, \ldots, x_n)$ *of* M *near* $i(q)$ *such* *that $X = \partial/\partial x_1$ and the germ $j_1^r F : (M, i(q)) \to J^1(\mathbb{R}, \mathbb{R}^k)$ is transverse to $\tilde{\mathcal{S}}$. Here F is a local equation of $i(Q)$ at $i(q)$, $j_1^r F(x_1, \ldots, x_n) = j^r F_{(x_2,\ldots,x_n)}(x_1)$ and $F_{(x_2,\ldots,x_n)}(x_1) = F(x_1, \ldots, x_n)$.*

PROOF: Let Σ_X be the singular set of X. For any $p \in M - \Sigma_X$, there exists a coordinate neighborhood $(U, (x_1, \ldots, x_n))$ around p with $X = \partial/\partial x_1$. We fix countably many such coordinate neighborhoods $U_\ell, (\ell \in \mathbb{N})$, of $M - \Sigma_X$.

Let $J^r E(Q, M)$ be the open subset of $J^r(Q, M)$ consisting of r-jets of local embeddings. For each $\mathbf{d} = (d_1, \ldots, d_k)$, with $1 < d_1 < \cdots < d_k \leq n$, and for each $\ell \in \mathbb{N}$, define an open subset $\mathcal{R}_{\mathbf{d},\ell}$ of $J^r E(Q, M)$ as follows: $j^r i(q) \in \mathcal{R}_{\mathbf{d},\ell}$ if and only if $i(q) \in U_\ell$ and $\det(\partial x_j \circ i)/(\partial u_s)(q) \neq 0$ for $j \neq d_1, \ldots, d_k$, $1 \leq s \leq n - k$, where $(u_1, \ldots, u_{n-k})$ is a local coordinate of Q around q.

For any $j^r i(q) \in \mathcal{R}_{\mathbf{d},\ell}$, we may choose

$$x^{\mathbf{d}} = (x_1, \ldots, x_{d_1-1}, x_{d_1+1}, \ldots, x_{d_k-1}, x_{d_k+1}, \ldots, x_n),$$

as a local coordinate of Q around q. With respect to this local coordinate, a representative of $j^r i(q)$ is given by

$$\tilde{i}(x^{\mathbf{d}}) = (x_1, \ldots, x_{d_1-1}, g_{d_1}(x^{\mathbf{d}}), x_{d_1+1}, \ldots, x_{d_k-1}, g_{d_k}(x^{\mathbf{d}}), x_{d_k+1}, \ldots, x_n).$$

We now define a map-germ

$$F^{\mathbf{d},\ell} : (\mathbb{R}^n, i(q)) \to (\mathbb{R}^k, 0)$$

by

$$F^{\mathbf{d},\ell} = (x_{d_1} - g_{d_1}(x^{\mathbf{d}}), \ldots, x_{d_k} - g_{d_k}(x^{\mathbf{d}})).$$

Then we have

$$(F^{\mathbf{d},\ell})^{-1}(0) = \operatorname{Im} \tilde{i}.$$

We also define a surjective submersion

$$L_{\mathbf{d},\ell} : \mathcal{R}_{\mathbf{d},\ell} \to J^r(\mathbb{R}, \mathbb{R}^k)$$

by

$$L_{\mathbf{d},\ell}(j^r i(q)) = j_1^r F^{\mathbf{d},\ell}(i(q)).$$

Then $\mathcal{S}_{\mathbf{d},\ell} = L_{\mathbf{d},\ell}^{-1}(\mathcal{S})$ is a Whitney stratification of $\mathcal{R}_{\mathbf{d},\ell}$.

Let $i : Q \to M$ be an embedding with $i(Q) \cap \Sigma_X = \emptyset$. It is easy to show that $j^r i$ is transverse to $\mathcal{S}_{\mathbf{d},\ell}$ on $\mathcal{R}_{\mathbf{d},\ell}$ near $q \in Q$ if and only if $j_1^r F^{\mathbf{d},\ell}$ is transverse to $\mathcal{S}$ near $i(q)$.

Define $T_{\mathbf{d},\ell}$ as the set of embeddings $i : Q \to M$ such that $j^r i$ is transverse to $\mathcal{S}_{\mathbf{d},\ell}$ on $\mathcal{R}_{\mathbf{d},\ell}$. Since $\mathcal{S}_{\mathbf{d},\ell}$ is a Whitney stratification, $T_{\mathbf{d},\ell}$ is an open dense subset of $\operatorname{Emb}(Q, M)$.

Because the choice of local equations depends only on the $\mathcal{C}$-equivalence, the set

$$\mathcal{O} = \mathcal{O}_1 \cap (\bigcap_{\mathbf{d},\ell} T_{\mathbf{d},\ell})$$

has required properties.

2. Local properties of submanifolds

In order to detect normal forms of vector fields near a generic submanifold, we shall study some properties of local equations. Denote by $\mathcal{E}_n$ the ring of function germs of n-variables at 0 and by $\mathfrak{m}_n$ the unique maximal ideal.

Let $F, G : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^k, 0)$ be map germs. We say that F and G are *p-$\mathcal{C}^+$-equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R} \times \mathbb{R}^\ell, 0)$ of the form $\Phi(x, u) = (x + \alpha(u), \phi(u))$ such that $\Phi^* I(F) = I(G)$, where $I(F) = F^*(\mathfrak{m}_k)\mathcal{E}_{1+\ell}$.

We remark that diffeomorphism germs of the above form preserve the vector field $\partial/\partial x$. The purpose of this section is to classify local equations by the p-$\mathcal{C}^+$-equivalence. For this purpose, we use some notations and results in [3], [4] and [5].

For each map-germ $f : (\mathbb{R}, 0) \to (\mathbb{R}^k, 0)$, we set

$$T_e\mathcal{K}(f) = \langle \frac{\partial f}{\partial x} \rangle_{\mathcal{E}_1} + I(f)\mathcal{E}_{1,k},$$

and

$$\mathcal{K}\text{-cod}(f) = \dim_{\mathbb{R}} \mathcal{E}_{1,k}/T_e\mathcal{K}(f),$$

where $\mathcal{E}_{1,k}$ is the $\mathcal{E}_1$-module of map germs $(\mathbb{R}, 0) \to \mathbb{R}^k$. Then we have the following well-known classification (cf.[3]).

LEMMA 2.1. *Let $f : (\mathbb{R}, 0) \to (\mathbb{R}^k, 0)$ be a map-germ with $\mathcal{K}$-cod$(f) < +\infty$. Then f is $\mathcal{C}$-equivalent to the map-germ $(x^{t+1}, 0, \ldots, 0)$ for some non-negative integer t.*

By the direct calculation, we have

$$\mathcal{K}\text{-cod}(x^{t+1}, 0, \ldots, 0) = (t+1)k - 1.$$

The following notion is rather an important chain of ideas: A map germ $F : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^k, 0)$ is an infinitesimally $\mathcal{C}^+$-versal deformation of $f = F|\mathbb{R} \times 0$ if

$$\mathcal{E}_{1,k} = \langle \frac{df}{dx} \rangle_{\mathbb{R}} + I(f)\mathcal{E}_{1,k} + \langle \frac{\partial F}{\partial u_1}|\mathbb{R} \times 0, \ldots, \frac{\partial F}{\partial u_\ell}|\mathbb{R} \times 0 \rangle_{\mathbb{R}}.$$

Then we have the following theorem, which is a corollary of Damon's general versality theorem in [2].

THEOREM 2.2. *Let* $F, G : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^k, 0)$ *be infinitesimally* $\mathcal{C}^+$*-versal deformations of* f. *Then* F *and* G *are p-*$\mathcal{C}^+$*-equivalent.*

Since f is a map-germ of one-variable, we have

$$\langle \frac{df}{dx} \rangle_{\mathbb{R}} + I(f)\mathcal{E}_{1,k} = \langle \frac{df}{dx} \rangle_{\mathcal{E}_1} + I(f)\mathcal{E}_{1,k}.$$

Thus it follows that F is an infinitesimally $\mathcal{C}^+$-versal deformation if and only if it is infinitesimally $\mathcal{K}$-versal. Hence the following classification theorem holds as a corollary of Theorem 2.2.

COROLLARY 2.3. *Let* $F, G : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^k, 0)$ *be infinitesimally* $\mathcal{K}$*-versal deformations of* $f, g : (\mathbb{R}, 0) \to (\mathbb{R}^k, 0)$ *respectively. Then* F *and* G *are p-*$\mathcal{C}^+$*-equivalent if and only if* f *and* g *are* $\mathcal{C}$*-equivalent.*

By Lemma 2.1 and Corollary 2.3, we can detect normal forms of local equations.

PROPOSITION 2.4. *Let* $F : (\mathbb{R} \times \mathbb{R}^{n-1}, 0) \to (\mathbb{R}^k, 0)$ *be an infinitesimally* $\mathcal{K}$*-versal deformation of* $f : (\mathbb{R}, 0) \to (\mathbb{R}^k, 0)$. *Then* F *is p-*$\mathcal{C}^+$*-equivalent to*

$$(*) \quad (x^{t+1} + \sum_{i=1}^{t} u_{1,i} x^{t-i}, \sum_{i=1}^{t+1} u_{2,i} x^{t+1-i}, \dots, \sum_{i=1}^{t+1} u_{k,i} x^{t+1-i}),$$

for some integer $t, (0 \leq t \leq \frac{n}{k} - 1)$. *Here,*

$$(x, u) = (x, u_{1,1}, \dots, u_{1,t}, u_{2,1}, \dots, u_{2,t+1}, \dots, u_{k,1}, \dots, u_{k,t+1}, u_{(t+1)k+1}, \dots, u_n).$$

PROOF: By the assumption and Lemma 2.1, f is $\mathcal{C}$-equivalent to $(x^{t+1}, 0, \dots, 0)$ for some integer t. Then $\mathcal{K}$-cod$(f) = (t+1)k - 1$ and $0 \leq t \leq \frac{n}{k} - 1$. By a simple calculation, the $(n+1)$-parameter infinitesimally $\mathcal{K}$-versal deformation of $(x^{t+1}, 0, \dots, 0)$ is given by (*). It follows from Corollary 2.3 that F is p-$\mathcal{C}^+$-equivalent to (*).

3. Proof of Theorem A

We now define a Whitney stratification $\mathcal{S}$ of $J^r(1,k)$ as follows:

$$\mathcal{S} = \{\mathrm{o}, \mathcal{K}(j^r(x^{t+1}, 0, \ldots, 0)) \mid 0 \le t \le r-1\},$$

where $\mathrm{o} = j^r(0,\ldots,0)$ and $\mathcal{K}z$ denotes the $\mathcal{K}$-orbit through a jet z. Let $F : (\mathbf{R}\times\mathbf{R}^{n-1}, 0) \to (\mathbf{R}^k, 0)$ be a map germ such that $j_1^r F$ is transverse to $\mathbf{R} \times \{0\} \times \mathcal{K}(j^r(x^{t+1}, 0, \ldots, 0))$ for $r > \frac{n}{k} - 1$. By the usual calculation, F is an infinitesimally $\mathcal{K}$-versal deformation of $F|\mathbf{R}\times 0$. Then F is p-$\mathcal{C}^+$-equivalent to a map germ (*) in Proposition 2.4.

We adopt a coordinate transformation of $(\mathbf{R}\times\mathbf{R}^{n-1}, 0)$ as follows:

$$\begin{aligned}
x_{\ell k+1} &= \frac{1}{(t+1-\ell)!}x^{t+1-\ell} + \sum_{i=1}^{t-\ell} \frac{(t-i)!}{(t+1)!(t-i+\ell)!} u_{1,i} x^{t-i-\ell}, \quad (0 \le \ell \le t),\\
x_{\ell k+j} &= \sum_{i=1}^{t+1-\ell} \frac{(t-i+1)!}{(t-i+1-\ell)!} u_{j,i} x^{t-i+1-\ell}, \quad (0 \le \ell \le t, 2 \le j \le k),\\
x_m &= u_m, \quad ((t+1)k+1 \le m \le n).
\end{aligned}$$

By this coordinate transformation, the zero set of the germ (*) in Proposition 2.4 is diffeomorphic to the set $\{x_1 = \cdots = x_k = 0\}$, and the vector field $\partial/\partial x$ is transformed to

$$\sum_{j=1}^{t}\Big(\sum_{i=1}^{k} x_{jk+i}\frac{\partial}{\partial x_{(j-1)k+i}}\Big) + \frac{\partial}{\partial x_{tk+1}}.$$

By Theorem 1.1, the set of embeddings $Q \to M$ satisfying the property in Theorem A is residual. This completes the proof.

4. Some general cases

In this section we consider more general situations which include two cases. One of these cases is that $i(Q)$ is a generic hypersurface with boundary in M and another case is that $i(Q)$ is a generic piecewise smooth hypersurface. We now divide these two cases.

i) Hypersurface with boundary: In this case the local equation of $i(Q)$ at a boundary point $i(q)$ is given by a local submersion $(F_1, F_2) : (M, i(q)) \to (\mathbb{R}^2, 0)$ such that $F_1^{-1}(0) \cap \{F_2 \geq 0\} = (i(Q), i(q))$ and $F_1^{-1}(0) \cap F_2^{-1}(0) = (i(\partial Q), i(q))$. Then we use the following equivalence relation: Let $(F_1, F_2), (G_1, G_2) : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^2, 0)$ be map germs. We say that (F_1, F_2) and (G_1, G_2) are *p-$(\mathcal{B}, \mathcal{C}^+)$-equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R} \times \mathbb{R}^\ell, 0)$ of the form $\Phi(x, u) = (x + \alpha(u), \phi(u))$ such that

$$A \cdot \begin{pmatrix} \Phi^*(F_1) \\ \Phi^*(F_2) \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$$

for some $A \in \Delta_+(2, \mathcal{E}_{1+\ell})$. Here

$$\Delta_+(2, \mathcal{E}_{1+\ell}) = \left\{ \begin{pmatrix} \xi_{11} & 0 \\ \xi_{21} & \xi_{22} \end{pmatrix} \middle| \, \xi_{ij} \in \mathcal{E}_{1+\ell}, \xi_{11} \cdot \xi_{22} \neq 0 \text{ and } \xi_{22} > 0 \right\}.$$

We can classify generic local equations by the p-$(\mathcal{B}, \mathcal{C}^+)$-equivalence almost the same way as that of Proposition 2.4. After that we also transform local equations and vector field $\partial/\partial x$ to normal forms in the following theorem:

THEOREM B. *Local normal forms of vector fields at a boundary point of a generic hypersurface with boundary are given by*

$$\sum_{j=1}^{r} \left(x_{2j+1} \frac{\partial}{\partial x_{2j-1}} + x_{2j+2} \frac{\partial}{\partial x_{2j}} \right) + \frac{\partial}{\partial x_{2r+1}} + \sum_{i=0}^{s-r-1} x_{2(r+1)+i+1} \frac{\partial}{\partial x_{2(r+1)+i}},$$

for some pairs of integers (r, s) with $0 \leq r \leq \frac{n-4}{2}$ and $r+1 \leq s \leq \frac{n}{2}$, where the hypersurface is locally given by

$$\{(x_1, \ldots, x_n) | x_1 = 0 \text{ and } x_2 \geq 0\}.$$

ii) Piecewise-smooth hypersurface: In what follows, we shall take as a local model of piecewise-smooth hypersurface the germ at the origin of sets

$$Q_k = \bigcup_{i=1}^{k} \{(x_1, \ldots, x_n) | x_1 \geq 0, x_2 \geq 0, \ldots, x_{i-1} \geq 0, x_i = 0, x_{i+1} \geq 0, \ldots, x_k \geq 0\},$$

$(k = 1, \ldots, n)$.

Any piecewise-smooth hypersurface is locally the image of Q_k by a diffeomorphism germ $i : (\mathbf{R}^n, 0) \to (M, q)$ for some k. In this case the local equation is given by a local submersion $F = (F_1, \ldots, F_k) : (M, q) \to (\mathbf{R}^k, 0)$ such that

$$\bigcup_{i=1}^{k} (F_i^{-1}(0) \cup \{F_j \geq 0 | 1 \leq j \leq k, j \neq i\}) = (i(Q_k), q).$$

Then we use the following equivalence relation: Let $F, G : (\mathbf{R} \times \mathbf{R}^\ell, 0) \to (\mathbf{R}^k, 0)$ be map-germs. We say that F and G are *p-$(Q_k, \mathcal{C}^+)$-equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbf{R} \times \mathbf{R}^\ell, 0) \to (\mathbf{R} \times \mathbf{R}^\ell, 0)$ of the form $\Phi(x, u) = (x + \alpha(u), \phi(u))$ such that

$$A \cdot \begin{pmatrix} \Phi^*(F_1) \\ \vdots \\ \Phi^*(F_k) \end{pmatrix} = \begin{pmatrix} G_1 \\ \vdots \\ G_k \end{pmatrix}$$

for some $A \in D_+(k, \mathcal{E}_{1+\ell})$. Here,

$$D_+(k, \mathcal{E}_{1+\ell}) = \left\{ \begin{pmatrix} \xi_1 & & 0 \\ & \ddots & \\ 0 & & \xi_k \end{pmatrix} \middle| \xi_i \in \mathcal{E}_{1+\ell}, \xi_i > 0 \right\}.$$

Similarly to Theorems A and B, we have the following classification:

THEOREM C. *Local normal forms of generic vector fields at a point of a generic piecewise-smooth hypersurface are labelled by a positive integer k and integers $t_i, 1 \leq i \leq k$, with*

$$\sum_{i=1}^{k} t_i \leq n \quad \text{and} \quad t_1 \geq t_2 \geq \cdots \geq t_k \geq 1,$$

and given by

$$\sum_{i=1}^{m} \Big(x_{t_1 + \cdots + t_{i-1} + k + 2 - i} \partial/\partial x_i + \sum_{j=1}^{t_i - 2} x_{t_1 + \cdots + t_{i-1} + k + 2 - i + j} \partial/\partial x_{t_1 + \cdots + t_{i-1} + k + 1 - i + j}$$

$$+ \partial/\partial x_{t_1 + \cdots + t_i + k - i} \Big) + \sum_{i=m+1}^{k} (\pm \partial/\partial x_i),$$

and the piecewise-smooth hypersurface is locally given by Q_k, where m is an integer with $t_m \geq 2, t_{m+1} = 1$.

EXAMPLE 4.1: Applying Theorem C to the case $n = 3$ and $k \geq 2$, we have the local normal forms of generic vector fields at a non-smooth point of a generic piecewise-smooth surface in $\mathbf{R}^3$:

$$\pm \frac{\partial}{\partial x_1} \pm \frac{\partial}{\partial x_2} \pm \frac{\partial}{\partial x_3}, \qquad \text{at the summit of } Q_3,$$

$$x_3 \frac{\partial}{\partial x_1} \pm \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, \quad \pm \frac{\partial}{\partial x_1} \pm \frac{\partial}{\partial x_2}, \qquad \text{at the edge of } Q_2.$$

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