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Comparison and Nuclearity of Spaces of Differential Forms on Topological Vector Spaces

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Abstract

Two types of fundamental spaces of differential forms on infinite dimensional topological vector spaces are considered; one is a fundamental space of Hida's type and the other is one of Malliavin's. It is proven that the former space is smaller than the latter. Moreover, it is shown that, under some conditions, the fundamental space of Hida's type is nuclear as a complete countably normed space, while that of Malliavin's in the L^2 sense is not.

1. Introduction

Recently differential forms on infinite dimensional spaces have been studied [1,3,4,5,16]. In [1,3] a class of infinite dimensional Dirac type operators was defined and, for each of them, an index theorem was established in terms of a functional integral representation. In [4] we introduced a general class of de Rham complexies based on a family of operators of de Rham type and defined the Lapalcians associated with them. Moreover, we proved de Rham-Hodge-Kodaira decomposition theorems on some fundamental spaces of differential forms, which include as a special case the decomposition theorem proven in [16]. Operator theoretical aspects related to the de Rham operators defined in [4] have been discussed in [5]. The framework for infinite dimensional analysis presented in [1,3,4,5] may be a natural one as a generalization of analysis on finite dimensional manifolds. Moreover, one of the present authors has shown in [3] that it gives a unified mathematical point of view for models in supersymmetric quantum field theory [2,6,10].

In this paper we concentrate our attention on two of the fundamental spaces of differental forms introduced in [4]; one is a fundamental space of Hida's type [9] and the other is one of Malliavin's [12,18]. We first prove the following comparison theorem : the former space is smaller than the latter. We then show that, under certain conditions, the fundamental space of Hida's type is nuclear as a complete countably normed space, while that of Malliavin's in the L^2 sense is not.

The outline of the present paper is as follows. In Section 2 we review the framework of the theory developed in [4] and summarize some results obtained there. Section 3 is devoted to the proof of the comparison theorem stated above. The main ingredients of the proof are Khinchine's inequalities and estimates for second quantization operators. In the last section, we prove the result on nuclearity mentioned above. This is done by applying Theorem A in Appendix that gives a necessary and sufficient condition for a complete countably Hilbert space to be nuclear.

2. Framework

In this section we review some results in [4]. We shall use the following general notations. For a real locally convex topological vector space E, we denote by E^* the topological dual space of E and by $< x, \xi >$ ($x \in E, \xi \in E^*$) the canonical bilinear form on $E \times E^*$. If E is a Hilbert space, we denote by $\| \cdot \|_E$ and $(\cdot , \cdot)_E$ the norm and the inner product of E, respectively. For a linear operator S, we denote by $\mathfrak{D}(S)$ its domain and define

$$C^\infty(S) = \cap_{n=1}^\infty \mathfrak{D}(S^n),$$

the set of C^∞-vectors of S.

Let B be a real locally convex topological vector space and H a separable real Hilbert space densely and continuously embedded in B. Identifying H^* with H, we have a Gelfand triplet $\{B^*, H, B\}$:

$$B^* \subset H \subset B.$$

Let μ be the Gaussian measure on B given by

$$\int_B e^{i<x,\xi>} d\mu(x) = e^{-\frac{\|\xi\|_H^2}{2}}, \quad \xi \in B^*.$$

The bilinear form $< x, \xi >$ can be continuously extended to all $\xi \in H$ as an element of $\cap_{1 \le p < \infty} L^p(B, d\mu)$. We denote the extension by the same symbol. For $\xi = \xi_1 + i\xi_2$ ($i = \sqrt{-1}$), we define $< x, \xi > = < x, \xi_1 > + i < x, \xi_2 >$.

Let K be a separable complex Hilbert space and $\bigwedge^n(K)$ the n-fold antisymmetric tensor product of K. For $f_k \in K, k = 1, \cdots, n$, we define their exterior product $f_1 \wedge \cdots \wedge f_n \in \bigwedge^n(K)$ by

$$f_1 \wedge \cdots \wedge f_n = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (\mathrm{sign}\ \sigma) f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes \cdots \otimes f_{\sigma(n)},$$

where $\mathfrak{S}_n$ is the symmetric group of degree n.

A basic Hilbert space in the theory developed in [4] is

$$\bigwedge^n(B,K) = L^2(B, d\mu) \otimes \bigwedge^n(K),$$

which is identified with

$$\bigwedge^n(B,K) = L^2(B, d\mu; \bigwedge^n(K)),$$

the Hilbert space of $\bigwedge^n(K)$-valued square integrable functions on (B, μ).

Let H_c be the complexification of H. We denote by $C(H_c, K)$ the set of densely defined closed linear operators from H_c to K. Let $A \in C(H_c, K)$ and $\mathcal{P}(\bigwedge^n(K))$ the subspace (in $\bigwedge^n(B,K)$) spanned by $\bigwedge^n(K)$-valued polynomials of the form

$$\omega(x) = P_m(< x, \xi_1 >, \cdots, < x, \xi_m >) f_1 \wedge \cdots \wedge f_n, \; m = 0, 1, 2, \cdots, \quad (2.1)$$

with

$$\xi_j \in C^\infty(A^*A), \quad j = 1, 2, \cdots, m,$$
$$f_k \in C^\infty(AA^*), \quad k = 1, 2, \cdots, n,$$

where P_m is a polynomial of order m. For each $n = 0, 1, 2 \cdots$, we define an operator d_n from $\bigwedge^n(B,K)$ to $\bigwedge^{n+1}(B,K)$ with domain $\mathcal{P}(\bigwedge^n(K))$ by

$$(d_n\omega)(x) = \sqrt{n+1} \sum_{j=1}^{m} \partial_j P_m(< x, \xi_1 >, \cdots, < x, \xi_m >) A\xi_j \wedge f_1 \wedge \cdots \wedge f_n$$

for ω of the form (2.1) and by extending it by linearity to all $\omega \in \mathcal{P}(\bigwedge^n(K))$. It is shown that d_n is well-defined and closable [3,4,5]; we denote the closure by the same symbol. One can prove [3,4,5] that

$$d_{n+1}d_n = 0,$$

which means that the sequence $\{d_n\}_{n=0}^{\infty}$ of operators forms a complex of de Rham type. We set $d_{-1} = 0$.

The Laplacians associated with $\{d_n\}_{n=0}^{\infty}$ are defined by

$$\Delta_n = d_n^* d_n + d_{n-1}d_{n-1}^*, n = 0, 1, 2, \cdots,$$

which are nonnegative and self-adjoint in $\bigwedge^n(B, K)$, respectively [4] (see (3.6) in Section 3 in the present paper).

The space of differential n-forms $W^\infty(\bigwedge^n(K))$ of Malliavin's type is defined to be the completion of $\mathcal{P}(\bigwedge^n(K))$ in all the norms $\|\omega\|_{p,s}, 1 < p < \infty, s = 0, 1, 2, \cdots$, given by

$$\begin{aligned}\|\omega\|_{p,s} &= \left[\int_B \|(I + \Delta_n)^s \omega(x)\|^p_{\bigwedge^n(K)} d\mu(x)\right]^{1/p} \\ &= \|(I + \Delta_n)^s \omega\|_{L^p(B, d\mu; \bigwedge^n(K))}.\end{aligned}$$

Let $W_{p,s}(\bigwedge^n(K))$ be the completion of $\mathcal{P}(\bigwedge^n(K))$ in the norm $\|\cdot\|_{p,s}$. Then we have

$$W^\infty(\bigwedge^n(K)) = \bigcap_{1<p<\infty} \bigcap_{s=0}^{\infty} W_{p,s}(\bigwedge^n(K)).$$

We define the fundamental space $W_2^\infty(\bigwedge^n(K))$ of Malliavin's type in the L^2 sense by

$$W_2^\infty(\bigwedge^n(K)) = \bigcap_{s=0}^{\infty} W_{2,s}(\bigwedge^n(K)).$$

REMARK: The standard Malliavin's fundamental space is given by $W^{\infty}(\bigwedge^0(K))$ (i.e., 0-forms) with $A^*A = I$ (see [4]).

A fundamental space of Hida'type is defined in terms of the Γ operator in the mathematical theory of second quantization [15,17]. It is well-known [15,17] that for each self-adjoint operator S in H_c, there exists a unique self-adjoint operator $\Gamma(S)$ in $L^2(B, d\mu)$ such that

$$\Gamma(S)1 = 1$$

and

$$\begin{aligned}&\Gamma(S) :< x, \xi_1 >< x, \xi_2 > \cdots < x, \xi_n >: \\ &=:< x, S\xi_1 >< x, S\xi_2 > \cdots < x, S\xi_n >:, \\ &\xi_j \in \mathfrak{D}(S), \quad j = 1, 2, \cdots, n, \; n \geq 1,\end{aligned}$$

where : $\cdot$: denotes the Wick product [17]. If $S \geq 0$, then $\Gamma(S) \geq 0$.

We define the operator $\Gamma_n(AA^*)$ in $\bigwedge^n(K)$ by

$$\Gamma_n(AA^*) = \underbrace{AA^* \otimes AA^* \otimes \cdots \otimes AA^*}_{n \text{ times}},$$

which is non-negative and self-adjoint.

Let

$$\Gamma_n = \Gamma(A^*A) \otimes \Gamma_n(AA^*).$$

For $A \in C(H_c, K)$ such that $A^*A \geq 1 + \epsilon$ with some $\epsilon > 0$, Γ_n is unbounded with $\Gamma_n \geq 1$. We define a fundamental space $H_n(B)$ of Hida's type to be the completion of $\mathcal{P}(\bigwedge^n(K))$ in all the norms $|||\omega|||_{p,s}, 1 < p < \infty, s = 0, 1, 2, \cdots$, given by

$$|||\omega|||_{p,s} = \|(I + \Gamma_n)^s \omega\|_{L^p(B,d\mu;\bigwedge^n(K))}.$$

Let $V_s^p(\bigwedge^n(K))$ be the completion of $\mathcal{P}(\bigwedge^n(K))$ in the norm $|||\cdot|||_{p,s}$. Then we have

$$H_n(B) = \bigcap_{1<p<\infty} \bigcap_{s=0}^{\infty} V_s^p(\bigwedge^n(K)).$$

We also define

$$H_{n,2}(B) = \bigcap_{s=0}^{\infty} V_s^p(\bigwedge^n(K)).$$

We remark that Hida's space considered in [9] is $H_0(B)$ (i.e., 0-forms) with $H = L^2(\mathrm{R}), B = \mathcal{S}(\mathrm{R})^*$, and $A = (d/dt) - t$ (see [4]).

3. A Comparison Theorem

In this section we prove the following comparison theorem.

THEOREM 3.1. *Let*

$$A^*A \geq 1 + \epsilon \tag{3.1}$$

with a constant $\epsilon > 0$. *Then*

$$H_n(B) \subset W^\infty(\bigwedge^n(K)).$$

This theorem is obtained as a corollary of the following one.

THEOREM 3.2. *Assume that* A *satisfies (3.1). Then the following hold:*

(1)

$$H_{n,2}(B) \subset W^\infty(\bigwedge^n(K)).$$

(2)

$$H_{n,2}(B) = H_n(B).$$

We shall prove Theorem 3.2. For this purpose we need some preliminaries.

LEMMA 3.3 (KHINCHINE'S INEQUALITIES [18]). *Let* $\{\gamma_i(\omega)\}$ *be a coin tossing sequence, i.e., a sequence of i.i.d. random variables with*

$$P(\gamma_i(\omega) = 1) = P(\gamma_i(\omega) = -1) = 1/2,$$

where P denotes the probability measure controlling $\gamma_i(\omega), i = 1, 2, \cdots$. Then the following hold:

(1) *Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of real numbers and $1 < p < \infty$. Then there exist positive constants $c_j(p) < \infty, j = 1, 2,$ such that*

$$\begin{aligned}(\sum_{i=1}^{\infty} | a_i |^2)^{p/2} &\le c_1(p) E[| \sum_{i=1}^{\infty} \gamma_i(\omega) a_i |^p] \\ &\le c_2(p) (\sum_{i=1}^{\infty} | a_i |^2)^{p/2}. \qquad (3.2)\end{aligned}$$

(2) *Let $\{a_{ij}\}$ be a double sequence of complex numbers such that for all $n = 1, 2, \cdots$, the matrix $(a_{m_i m_j})_{1 \le i,j \le n}$ is positive semi-definite and let $1 < p < \infty$. Then there exists a positive constant $C(p) < \infty$ such that*

$$E[(\sum_{i,j=1}^{\infty} \gamma_i(\omega)\gamma_j(\omega) a_{ij})^{p/2}] \le C(p) (\sum_{i=1}^{\infty} a_{ii})^{p/2}. \qquad (3.3)$$

Let $\{T_t\}_{t \ge 0}$ be the Ornstein-Uhlenbeck semigroup on $L^2(B, d\mu)$:

$$T_t = \Gamma(e^{-t}).$$

Then T_t^{-1} exists and is given by

$$T_t^{-1} = \Gamma(e^t).$$

Since T_t^{-1} is self-adjoint, it has a natural extension to $\bigwedge^n(B, K)$ as

$$\hat{T}_t^{-1} = T_t^{-1} \otimes I = \Gamma(e^t) \otimes I, \qquad (3.4)$$

where I denotes identity.

LEMMA 3.4. *Suppose that (3.1) holds. Then, for each $t > 0$ and $s = 0, 1, 2, \cdots$, there exist positive constants β and C such that for all $\omega \in \mathfrak{D}(\Gamma_n^\beta)$,*

$$||\hat{T}_t^{-1}(1+\Delta_n)^s\omega||_{\bigwedge^n(B,K)} \leq C||(1+\Gamma_n)^\beta\omega||_{\bigwedge^n(B,K)}. \tag{3.5}$$

Before proceeding to the proof of this lemma, we recall some facts obtained in [4]. We have the orthogonal decomposition (the Fock-Wiener-Itô decomposition):

$$\bigwedge^n(B,K) = \oplus_{m=0}^\infty \mathfrak{L}_m(H) \otimes \bigwedge^n(K),$$

where $\mathfrak{L}_m(H)$ is the closed subspace generated by $:< x, \xi_1 > \cdots < x, \xi_m >:$ $, \xi_j \in H, j = 1, 2 \cdots, m$. In [4](cf. also [1,3]) we proved that

$$\Delta_n = d\Gamma(A^*A) \otimes I + I \otimes d\Gamma_n(AA^*), \tag{3.6}$$

where $d\Gamma(\,\cdot\,)$ and $d\Gamma_n(\,\cdot\,)$ denote the second quantizations in $L^2(B,\mu)$ and $\bigwedge^n(K)$, respectively [14,15,17]. Let

$$\mathcal{F}_{m,n} = \mathfrak{L}_m(H) \otimes \bigwedge^n(K).$$

The nonnegative self-adjoint operator Δ_n is reduced by each $\mathcal{F}_{m,n}$; we denote the reduction by $\Delta_{n,m}$. The operator $\Delta_{n,m}$ has a spectral representation as is shown in the following [4]. Let

$$A_j = I \otimes \cdots \otimes I \otimes \overset{j}{\check{A^*A}} \otimes I \otimes \cdots \otimes I, \quad j = 1, 2, \cdots, m,$$

and

$$B_k = I \otimes \cdots \otimes I \otimes \overset{k}{\check{AA^*}} \otimes I \otimes \cdots \otimes I, \quad k = 1, 2, \cdots, n.$$

Then, $\{A_j \otimes I,\ \ I \otimes B_k \mid\ \ j = 1, 2, \cdots, m,\ \ k = 1, 2, \cdots, n\}$ is a family of commuting self-adjoint operators in $\mathcal{F}_{m,n}$. Hence, by the spectral theorem for self-adjoint operators, there exists a unique family $\{E(\lambda, \nu) \mid \lambda \in \mathbf{R}^m,\ \nu \in \mathbf{R}^n\}$ of projection-valued measures on $\mathbf{R}^m \times \mathbf{R}^n$ such that

$$\Delta_{n,m} = \int (\sum_{j=1}^{m} \lambda_j + \sum_{k=1}^{n} \nu_k) dE(\lambda, \nu).$$

Also the operator Γ_n is reduced by $\mathcal{F}_{m,n}$; we denote the reduction by $\Gamma_{n,m}$. Then we have

$$\Gamma_{n,m} = \int (\prod_{j=1}^{m} \lambda_j)(\prod_{k=1}^{n} \nu_k) dE(\lambda, \nu).$$

Now we are ready to prove Lemma 3.4.

Proof of Lemma 3.4: We need only to prove (3.5) for all $\omega \in \mathcal{P}(\bigwedge^n(K))$, because $\mathcal{P}(\bigwedge^n(K))$ is a core for $(1 + \Gamma_n)^s, s = 1, 2, \cdots$. Let $\omega \in \mathcal{P}(\bigwedge^n(K))$. We denote by Q_m the orthogonal projection from $\bigwedge^n(B, K)$ to $\mathcal{F}_{m,n}$. Then we have

$$\|\hat{T}_t^{-1}(I + \Delta_n)^s \omega\|^2_{\bigwedge^n(B,K)} = \sum_{m=0}^{\infty} \|Q_m \hat{T}_t^{-1}(I + \Delta_n)^s \omega\|^2_{\bigwedge^n(B,K)}.$$

Since $Q_m, \hat{T}_t^{-1}$, and Δ_n are mutually commutative, we have

$$Q_m \hat{T}_t^{-1}(I + \Delta_n)^s \omega = e^{mt}(I + \Delta_{n,m})^s \omega_m,$$

with $\omega_m = Q_m \omega$. Hence we obtain

$$\|\hat{T}_t^{-1}(I + \Delta_n)^s \omega\|^2_{\bigwedge^n(B,K)} = \sum_{m=0}^{\infty} e^{2mt} \|(I + \Delta_{n,m})^s \omega_m\|^2_{\mathcal{F}_{m,n}}. \qquad (3.7)$$

Let $\sigma(T)$ denote the spectrum of operator T. It is well-known [7] that for any $T \in C(H_c, K)$,

$$\sigma(T^*T) \setminus \{0\} = \sigma(TT^*) \setminus \{0\}. \tag{3.8}$$

Applying this fact to $T = A$, we have for all $(\lambda_1, \cdots, \lambda_n, \nu_1, \cdots, \nu_m) \in$ supp E

$$\lambda_j, \nu_k \geq 1 + \epsilon, \; j = 1, \cdots, m, \; k = 1, \cdots, n.$$

Hence

$$\begin{aligned} \sum_{j=1}^{m} \lambda_j + \sum_{k=1}^{n} \nu_k &\leq \frac{m+n}{(1+\epsilon)^{m+n-1}} (\Pi_{j=1}^{m} \lambda_j)(\Pi_{k=1}^{n} \nu_k) \\ &\leq C(\Pi_{j=1}^{m} \lambda_j)(\Pi_{k=1}^{n} \nu_k), \end{aligned}$$

where

$$C = (1+\epsilon) \sup_{x>0} x e^{-x \log(1+\epsilon)} < \infty.$$

Therefore, putting

$$C_s = (\max\{1, C\})^{2s},$$

we have

$$\begin{aligned} &\text{the right hand side (RHS) of (3.7)} \\ &= \sum_{m=0}^{\infty} e^{2mt} \int (1 + \sum_{j=1}^{m} \lambda_j + \sum_{k=1}^{n} \nu_k)^{2s} d\|E(\lambda, \nu)\omega_m\|^2_{\mathcal{F}_{m,n}} \\ &\leq \sum_{m=0}^{\infty} e^{2mt} C_s \int (1 + (\Pi_{j=1}^{m} \lambda_j)(\Pi_{k=1}^{n} \nu_k))^{2s} d\|E(\lambda, \nu)\omega_m\|^2_{\mathcal{F}_{m,n}} \\ &= C_s \sum_{m=0}^{\infty} e^{2mt} \|(I + \Gamma_{n,m})^s \omega_m\|^2_{\mathcal{F}_{m,n}} \\ &= C_s \|\hat{T}_t^{-1} (I + \Gamma_n)^s \omega\|^2_{\bigwedge^n (B,K)}. \end{aligned}$$

Let r be an integer satisfying

$$r \geq \frac{t}{\log(1+\epsilon)},$$

so that

$$e^{2t} \leq (1+\epsilon)^{2r}.$$

Then, for all $\theta \in \mathcal{P}(\bigwedge^n(K))$, we have by (3.4)

$$\begin{aligned}
||\hat{T}_t^{-1}\theta||^2_{\bigwedge^n(B,K)} &= (\theta, \hat{T}_t^{-2}\theta)_{\bigwedge^n(B,K)} \\
&= (\theta, \Gamma(e^{2t}) \otimes I\theta)_{\bigwedge^n(B,K)} \\
&\leq (\theta, \Gamma((A^*A)^{2r}) \otimes I\ \theta)_{\bigwedge^n(B,K)} \\
&= (\theta, \Gamma(A^*A)^{2r} \otimes I\ \theta)_{\bigwedge^n(B,K)} \\
&\leq (\theta, \Gamma_n^{2r}\theta)_{\bigwedge^n(B,K)} \\
&\leq (\theta, (1+\Gamma_n)^{2r}\theta)_{\bigwedge^n(B,K)} \\
&= ||(1+\Gamma_n)^r\theta||^2_{\bigwedge^n(B,K)}.
\end{aligned}$$

It is easy to see that for all s, $(1+\Gamma_n)^s$ leaves $\mathcal{P}(\bigwedge^n(K))$ invariant. Hence, taking $\theta = (1+\Gamma_n)^s\omega$, we obtain

$$||\hat{T}_t^{-1}(1+\Gamma_n)^s\omega||^2_{\bigwedge^n(B,K)} \leq ||(1+\Gamma_n)^\beta\omega||^2_{\bigwedge^n(B,K)}$$

with $\beta = r+s$. Thus (3.5) follows. ∎

Proof of Theorem 3.2:

(1) Let $\{g_i\}_{i=0}^\infty$ be a complete othonormal system (C.O.N.S.) of $\bigwedge^n(K)$ and $\theta \in \mathcal{P}(\bigwedge^n(K))$. In what follows, $C_j, j = 1, 2, \cdots$, denote constants.

Let

$$b_i(x) = ((I + \Delta_n)^s \theta(x), g_i)_{\bigwedge^n(K)}.$$

Then we have by (3.2)

$$\begin{aligned}
\|(I + \Delta_n)^s \theta(x)\|^p_{\bigwedge^n(K)} &= \left(\sum_{i=1}^{\infty} | b_i(x) |^2 \right)^{p/2} \\
&\leq C_1 E \left[|\sum_{i=1}^{\infty} \gamma_i(\omega) b_i(x)|^p \right].
\end{aligned}$$

By the hypercontractivity of T_t, there exists some t_0 such that

$$\int_B |\sum_{i=1}^{\infty} \gamma_i(\omega) b_j(x)|^p d\mu \leq \left(\int_B |\sum_{i=1}^{\infty} \gamma_i(\omega) T_{t_0}^{-1} b_j(x)|^2 d\mu \right)^{p/2}.$$

Hence we obtain

$$||\theta||^p_{p,s} \leq C_1 E \left[\left(\int_B | \sum_{i=1}^{\infty} \gamma_i(\omega) T_{t_0}^{-1} b_i(x) |^2 \, d\mu(x) \right)^{p/2} \right].$$

By using (3.3), we have

$$\begin{aligned}
&E \left[\left(\int_B | \sum_{i=1}^{\infty} \gamma_i(\omega) T_{t_0}^{-1} b_i(x) |^2 \, d\mu(x) \right)^{p/2} \right] \\
&= E \left[\left(\sum_{i,j=1}^{\infty} \gamma_i(\omega) \gamma_j(\omega) \int_B T_{t_0}^{-1} b_i(x) \overline{T_{t_0}^{-1} b_j(x)} d\mu(x) \right)^{p/2} \right] \\
&\leq C_2 \left(\sum_{i=1}^{\infty} \int_B | T_{t_0}^{-1} b_i(x) |^2 \, d\mu(x) \right)^{p/2}
\end{aligned}$$

Thus we obtain

$$\begin{aligned}||\theta||_{p,s} &\le C_3 \left(\sum_{i=1}^{\infty} \int_B |\, T_{t_0}^{-1} b_i(x)\,|^2 \, d\mu(x) \right)^{1/2} \\ &= C_3 ||\hat{T}_{t_0}^{-1} (I + \Delta_n)^s \theta||_{\bigwedge^n(B,K)},\end{aligned}$$

which, combined with Lemma 3.4, gives

$$||\theta||_{p,s} \le C_4 |||\theta|||_{2,\beta}.$$

This means the desired result.

(2) The method of the proof of this part is similar to that of part (1). Hence we give only the outline of the proof. Using Lemma 3.3, we first show that for all p and s, there exists a constant $t_0 > 0$ such that for all $\omega \in \mathcal{P}(\bigwedge^n(K))$,

$$|||\omega|||_{p,s} \le C_5 ||\hat{T}_{t_0}^{-1} (I + \Gamma_n)^s \omega||_{\bigwedge^n(B,K)}. \tag{3.9}$$

Then, in the same way as in the proof of Lemma 3.4, we can see that the RHS of (3.9) is dominated by

$$C_6 ||(I + \Gamma_n)^\beta \omega||_{\bigwedge^n(B,K)} = C_6 |||\omega|||_{2,\beta}$$

with some $\beta > 0$. This result implies that $H_{n,2}(B) \subset H_n(B)$. The opposite inclusion relation is obvious. Thus the desired result follows. ∎

4. Nucleearity

In this section we investigate the nuclearity of the complete countably normed spaces $W_2^\infty(\bigwedge^n(K))$ and $H_n(B)$. We prove the following theorems.

THEOREM 4.1. *Let dim* $H = \infty$. *Then, for any* $A \in C(H_c, K)$, $W_2^\infty(\bigwedge^n(K))$ *is not nuclear.*

THEOREM 4.2. *Let* $A \in C(H_c, K)$ *and (3.1) be satisfied. Suppose that the spectrum* $\sigma(A^*A)$ *of* A^*A *is purely discrete with* $\sigma(A^*A) = \{\lambda_k\}_{k=1}^\infty$ $(\lambda_1 \le \lambda_2 \le \cdots)$*(degenerate eigenvalues are counted repeatedly) and there exist constants* $C > 0$ *and* $\alpha > 0$ *such that for all sufficiently large* k,

$$\lambda_k \ge Ck^\alpha.$$

Then $H_n(B)$ *is nuclear.*

REMARK: The spectrum of a self-adjoint operator is said to be purely discrete if it consists of only isolated eigenvalues of finite multiplicity.

We first prove Theorem 4.1. For this purpose we prepare a lemma.

LEMMA 4.3. *Let dim* $H = \infty$. *Let* S *and* R *be nonnegative self-adjoint operators in* H_c *and* K*, respectively and*

$$L = d\Gamma(S) \otimes I + I \otimes d\Gamma_n(R).$$

Then, for any $\gamma > 0$, $(I+L)^{-\gamma}$ is not trace-class on $\bigwedge^n(B,K)$.

PROOF: Suppose that for some $\gamma > 0$, $(I+L)^{-\gamma}$ were trace-class on $\bigwedge^n(B,K)$. Then $\sigma(L)$ is purely discrete and hence so are $\sigma(S)$ and $\sigma(R)$ [14, §VIII.10]. It follows also that the number of the distinct eigenvalues of S is infinite (if it is finite, then the cardinality of the C.O.N.S. consisting of the normalized eigenvectors of S is finite, which implies dim $H < \infty$). Hence let $\sigma(S) = \{\lambda_k\}_{k=1}^{\infty}$ with $0 \le \lambda_1 < \lambda_2 < \cdots$ (each eigenvalue may be degenerate with a finite multiplicity). Then $\lambda_k \to \infty$ as $k \to \infty$. By the functional calculus for self-adjoint operators, one can easily show that for all $t > 0$

$$e^{-tL} \le \delta(I+L)^{-\gamma}$$

with

$$\delta = \frac{e^t}{t^\gamma}\sup(x^\gamma e^{-x}) < \infty.$$

This inequality implies that e^{-tL} is trace-class for all $t > 0$. Let $\sigma(R) = \{\nu_k\}_{k=1}^{N}(0 \le \nu_1 \le \nu_2 \le \cdots)(N \le \infty)$. By a general theorem on the spectrum of tensor products of self-adjoint operatros [14,§VIII.10], we have

$$\sigma(d\Gamma_n(R)) = \{\nu_{i_1} + \cdots + \nu_{i_n} | i_1, \cdots, i_n = 1, 2, \cdots, i_j \ne i_k \text{ for } j \ne k\}.$$

Hence

$$\mathrm{Tr}\; e^{-td\Gamma_n(R)} \ge e^{-t(\nu_1 + \cdots + \nu_n)} \equiv C_t. \tag{4.1}$$

Similarly we have

$$\begin{aligned}\sigma(d\Gamma(S)) &= \{0\} \cup (\cup_{n=1}^{\infty}\{\lambda_{i_1} + \cdots + \lambda_{i_n} | i_1, \cdots, i_n = 1, 2, \cdots, \}) \\ &= \cup_{N=1}^{\infty}\{k_1\lambda_1 + \cdots + k_N\lambda_N | k_j \ge 0, j = 1, 2, \cdots, N\}.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{Tr} e^{-d\Gamma(S)} &\geq \sum_{k_2,\cdots,k_{N+1}=0}^{\infty} e^{-t(k_2\lambda_2+\cdots+k_{N+1}\lambda_{N+1})}\\ &= \frac{1}{\prod_{n=2}^{N+1}(1-e^{-t\lambda_n})}.\end{aligned} \tag{4.2}$$

Using (4.1) and (4.2), we obtain

$$\begin{aligned}\operatorname{Tr} e^{-tL} &= \operatorname{Tr} e^{-td\Gamma(S)} \operatorname{Tr} e^{-td\Gamma_n(R)}\\ &\geq \frac{C_t}{\prod_{n=2}^{N+1}(1-e^{-t\lambda_n})}.\end{aligned} \tag{4.3}$$

By the functional calculus for self-adjoint operators, we have

$$(I+L)^{-\gamma} = \frac{1}{\Gamma(\gamma)}\int_0^\infty e^{-tL}e^{-t}t^{\gamma-1}dt,$$

where $\Gamma(z)$ is the gamma function and the integral is taken in the operator norm topology. Let $\{\Psi_m\}_{m=1}^\infty$ be a C.O.N.S. of $\bigwedge^n(B,K)$. Then we have for all $M \geq 1$

$$\sum_{m=1}^{M}(\Psi_m,(1+L)^{-\gamma}\Psi_m) = \frac{1}{\Gamma(\gamma)}\int_0^\infty \sum_{m=1}^{M}(\Psi_m, e^{-tL}\Psi_m)e^{-t}t^{\gamma-1}dt.$$

Hence, by Fatou's lemma and (4.3), we obtain

$$\begin{aligned}\operatorname{Tr}(I+L)^{-\gamma} &\geq \frac{1}{\Gamma(\gamma)}\int_0^\infty \operatorname{Tr} e^{-tL}e^{-t}t^{\gamma-1}dt\\ &\geq \frac{1}{\Gamma(\gamma)}\int_0^\infty \frac{e^{-t}t^{\gamma-1}C_t}{\prod_{n=2}^{N+1}(1-e^{-t\lambda_n})}.\end{aligned}$$

On the other hand, we have

$$\prod_{n=2}^{N+1}(1-e^{-t\lambda_n}) \leq D_N t^N,$$

where

$$D_N = \left(\prod_{n=2}^{N+1} \lambda_n\right) \left\{\sup_{x>0} \left(\frac{1-e^{-x}}{x}\right)\right\}^N < \infty.$$

Hence

$$\mathrm{Tr}\,(I+L)^{-\gamma} \geq \frac{1}{D_N \Gamma(\gamma)} \int_0^\infty C_t e^{-t} t^{-N+\gamma-1} dt.$$

Since N is arbitrary, we can take an N such that $N-\gamma > 0$. Then the integral of the RHS of the above inequality diverges to $+\infty$. Hence $\mathrm{Tr}(I+L)^{-\gamma} = +\infty$, which contradicts the assumption that $(I+L)^{-\gamma}$ is trace-class. ∎

PROOF OF THEOREM 4.1: Using formula (3.6) and applying Lemma 4.3 with $S = A^*A$ and $R = AA^*$, we conclude that for any $\gamma > 0$, $(I+\Delta_n)^{-\gamma}$ is not trace-class on $\bigwedge^n(B,K)$. This result implies that for any $\alpha > 0$, $(I+\Delta_n)^{-\alpha}$ is not Hilbert-Schmidt. Then an application of Theorem A in Appendix in the present paper gives the conclusion of Theorem 4.1. ∎

To prove Theorem 4.2, we need the following lemma.

LEMMA 4.4. *Let $\{\mu_k\}_{k=1}^\infty$ be a nondecreasing sequence with the following properties:*

(1) $\mu_1 > 1$.

(2) *there exists a number k_0 such that for all $k \geq k_0$, $\mu_k \geq Ck^\alpha$ with constants $C > 0$ and $\alpha > 0$.*

Then,

$$\lim_{t\to\infty} \sum_{k=1}^\infty \frac{1}{\mu_k^t} = 0. \tag{4.4}$$

In particular, there exists a constant t_0 such that for all $t > t_0$

$$\sum_{k=1}^{\infty} \frac{1}{\mu_k^t} < 1.$$

PROOF: Let $t > 1/\alpha$, $N > k_0$ and $\mu_1 \geq 1 + \epsilon$ with $\epsilon > 0$. Then we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{\mu_k^t} &\leq \sum_{k=1}^{N} \frac{1}{\mu_k^t} + \frac{1}{C^t} \sum_{k=N+1}^{\infty} \frac{1}{k^{t\alpha}} \\ &\leq \frac{N}{(1+\epsilon)^t} + \frac{1}{C^t} \int_N^{\infty} \frac{1}{x^{t\alpha}} dx \\ &\leq \frac{N}{(1+\epsilon)^t} + \frac{N}{(CN^{\alpha})^t (t\alpha - 1)}. \end{aligned} \tag{4.5}$$

We can take N such that $CN^{\alpha} > 1$. Then the RHS of (4.5) converges to 0 as $t \to \infty$. Thus (4.4) follows. ∎

PROOF OF THEOREM 4.2: By Theorem 3.2(2), we need only to prove that $H_{n,2}(B)$ is nuclear. We shall show that for some $\gamma > 0$, $\Gamma_n^{-\gamma}$ is trace-class on $\bigwedge^n(B, K)$ and hence Hilbert-Schmidt. Then the nuclearity of $H_{n,2}(B)$ follows from an application of Theorem A in Appendix in the present paper.

Under the present assumption, the spectrum of Γ_n is purely discrete. By (3.8) we have

$$\begin{aligned} \sigma(\Gamma_n) = \{1\} \cup_{m=1}^{\infty} \{\lambda_{i_1} \cdots \lambda_{i_m} \lambda_{j_1} \cdots \lambda_{j_n} | i_1, \cdots, i_m, \\ j_1, \cdots, j_n = 1, 2, \cdots, , j_k \neq j_\ell, k \neq \ell\}. \end{aligned}$$

Let $\sigma(\Gamma_n) = \{E_p\}_{p=1}^{\infty}(1 = E_1 < E_2 \leq \cdots)$. Then

$$
\begin{aligned}
&\sum_{p=1}^{\infty} \frac{1}{E_p^{-\gamma}} \\
&= 1 + \sum_{m=1}^{\infty} \sum_{i_1,\cdots,i_m=1}^{\infty} \frac{1}{(\lambda_{i_1}\cdots\lambda_{i_m})^{\gamma}} \sum_{j_k \neq j_\ell, k\neq \ell}^{\infty} \frac{1}{(\lambda_{j_1}\cdots\lambda_{j_n})^{\gamma}} \\
&\leq 1 + f(\gamma)^n \sum_{m=1}^{\infty} f(\gamma)^m, \qquad (4.6)
\end{aligned}
$$

where

$$f(\gamma) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{\gamma}}.$$

By Lemma 4.4, there exists a constant $\gamma_0 > 0$ such that for all $\gamma > \gamma_0$,

$$f(\gamma) < 1$$

Hence, for such a γ, the RHS of (4.6) is finite, which implies that $\Gamma_n^{-\gamma}$ is trace-class on $\bigwedge^n(B, K)$. ∎

Appendix. Nuclearity of a Countably Hilbert Space

Let T be an unbounded self-adjoint operator in a Hilbert space $\mathfrak{H}$ such that $T \geq 1$. For each $n = 0, 1, 2, \cdots$, the domain $\mathfrak{D}(T^n)$ becomes a Hilbert space with the inner product

$$(f, g)_n = (T^n f, T^n g)_{\mathfrak{H}} \ , \quad f, g \in \mathfrak{D}(T^n).$$

Obviously the sequence $\{\mathfrak{H}_n\}_{n=0}^{\infty}$ of Hilbert spaces is monotone decreasing and the vector space

$$\mathfrak{H}_{\infty} = \cap_{n=0}^{\infty} \mathfrak{H}_n$$

becomes a complete countably Hilbert space with the system $\{|| \cdot ||_n\}_{n=0}^{\infty}$ of norms [8]. The aim of this appendix is to give a necessary and sufficient condition for $\mathfrak{H}_{\infty}$ to be nuclear. We prove the following theorem.

THEOREM A. *The countably Hilbert space $\mathfrak{H}_{\infty}$ is nuclear if and only if*

(C) *there exists a constant $\alpha > 0$ such that $T^{-\alpha}$ is Hilbert-Schmidt on $\mathfrak{H}$.*

REMARK: This fact may be well-known, but we have been able to find no literatures which state it. For the sake of completeness, we give a proof of Theorem A.

PROOF: For each pair $\{m, n\}$ with $n > m$, we define $J_m^n : \mathfrak{H}_n \to \mathfrak{H}_m$ to be the natural embedding mapping:

$$J_m^n f = f, \quad f \in \mathfrak{H}_n.$$

Let m be fixed. It is easy to see that the adjoint $(J_m^n)^* : \mathfrak{H}_m \to \mathfrak{H}_n$ is given by

$$(J_m^n)^* = T^{2m-2n}.$$

Hence we have

$$(J_m^n)^* J_m^n f = T^{2m-2n} f, \quad f \in \mathfrak{H}_n. \tag{A.1}$$

Now suppose that $\mathfrak{H}_\infty$ is nuclear. Then, there exists an n such that the self-adjoint operator $(J_m^n)^* J_m^n$ on $\mathfrak{H}_n$ is Hilbert-Schmidt. Hence, by (A.1), $T^{-\alpha}$ with $\alpha = 2n - 2m$ is Hilbert-Schmidt on $\mathfrak{H}_n$. Therefore there exists an orthonormal system (O.N.S.) $\{\phi_k\}_{k=1}^\infty$ of $\mathfrak{H}_n$ and a sequence $\{\lambda_k\}_{k=1}^\infty$ of positive numbers such that

$$T^{-\alpha} f = \sum_{k=1}^\infty \lambda_k (\phi_k, f)_n \phi_k, \quad f \in \mathfrak{H}_n, \tag{A.2}$$

in the strong topology of $\mathfrak{H}_n$ and

$$\sum_{k=1}^\infty \lambda_k^2 < \infty. \tag{A.3}$$

Let

$$\psi_k = T^n \phi_k$$

Then $\{\psi_k\}_{k=1}^\infty$ is an O.N.S. of $\mathfrak{H}$. Since T^n is self-adjoint on $\mathfrak{H}$ with $T^n \geq 1$, the range of T^n equals $\mathfrak{H}$. It then follows from (A.2) that

$$T^{-\alpha} f = \sum_{k=1}^\infty \lambda_k (\psi_k, f)_{\mathfrak{H}} \ \psi_k, \quad f \in \mathfrak{H},$$

in the strong topology of $\mathfrak{H}$, which, together with (A.3), means that $T^{-\alpha}$ is Hilbert-Schmidt on $\mathfrak{H}$. Thus condition (C) is necessary.

We next show that condition (C) is sufficient. Suppose that for some $\alpha > 0$, $T^{-\alpha}$ is Hilbert-Schmidt on $\mathfrak{H}$. Take n such that $n > m + (\alpha/4)$. Then $T^{-(4n-4m-\alpha)}$ is bounded on $\mathfrak{H}$ and hence

$$T^{4(m-n)} = T^{-\alpha} T^{-(4n-4m-\alpha)}$$

is Hilbert-Schmidt on $\mathfrak{H}$. Therefore, for each C.O.N.S. $\{e_k\}_{k=1}^{\infty}$ of $\mathfrak{H}$, we have

$$\sum_{k=1}^{\infty} ||T^{4(m-n)} e_k||_{\mathfrak{H}}^2 < \infty.$$

Let $f_k = T^{-n} e_k$. Then $\{f_k\}_{k=1}^{\infty}$ is a C.O.N.S. of $\mathfrak{H}_n$ and we have

$$\sum_{k=1}^{\infty} ||(T^{2(m-n)})^2 f_k||_n^2 = \sum_{k=1}^{\infty} ||T^{4(m-n)} e_k||_{\mathfrak{H}}^2 < \infty,$$

which means that $T^{2(m-n)}$ is Hilbert-Schmidt on $\mathfrak{H}_n$. It follows from this fact and (A.1) that J_m^n is nuclear. Thus $\mathfrak{H}_\infty$ is nuclear. ∎

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