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L^p estimates for the Stokes system

MARIKO GIGA, YOSHIKAZU GIGA
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1. Introduction.

This paper investigates the fractional powers $(A+B)^\alpha$, $0 \le \alpha \le 1$ of the sum $A+B$ of two closed (resolvent commuting) operators A and B in a ζ-convex Banach space X. We compare the domain $D((A+B)^\alpha)$ of $(A+B)^\alpha$ with the domain $D(A^\alpha + B^\alpha) = D(A^\alpha) \cap D(B^\alpha)$ of the sum $A^\alpha + B^\alpha$ and show in particular the relation

$$D((A+B)^\alpha) = D(A^\alpha) \cap D(B^\alpha) \tag{1.1}$$

with equivalent norms $||(A+B)^\alpha u||$ and $||A^\alpha u|| + ||B^\alpha u||$, under some assumptions on the pure imaginary powers of A and B.

Our results will be applied to L^p estimates for (generalized) solutions of the evolution equation

$$\frac{du}{dt} + Au = f \quad \text{in} \quad (0,T), \quad 0 < T \le \infty, \quad u(0) = 0. \tag{1.2}$$

Here we restrict ourselves to the Stokes operator $A = A_q$. Formally, we get such an equation if we apply the L^q Helmholtz projection P_q to the Stokes system

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u + \nabla p = f, \ \operatorname{div} u = 0 \ \text{ in } \ \Omega \times (0,T) \\ &u|_{\partial\Omega} = 0 \ \text{ on } \ \partial\Omega \times (0,T), \ u = 0 \ \text{ on } \ \Omega \ \text{ at } \ t = 0, \end{aligned} \tag{1.3}$$

where Ω is a domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ and $1 < q < \infty$; see e.g. [GS1,2] for notations.

Our perturbation result is based on a theory recently developed by Dore and Venni [DV] which has been extended by Giga and Sohr [GS2] to the case that the inverse operators A^{-1} and B^{-1} need not be bounded. See also [PS] for another proof. The original theory of Dore and Venni is applicable to the evolution equation (1.2) only for a finite interval [0,T); it yields a constant $C = C(\Omega, r, q, T) > 0$ such that

$$\int_0^T ||\frac{du}{dt}||_q^r dt + \int_0^T ||A_q u||_q^r dt \le C \int_0^T ||f||_q^r dt, \tag{1.4}$$

where $1 < r$, $q < \infty$ and $||\cdot||_q$ denotes the $L^q(\Omega)$-norm. The extension by [GS2] strengthens the estimate (1.4) so that A_q^{-1} is allowed to be unbounded and that C may be chosen independently of T. Therefore, one may take $T = \infty$ in (1.4) which yields asymptotic properties of the solution u of (1.3) as $t \to \infty$ even when Ω need not be bounded [GS2]. In [GS2] the estimates applied to the nonlinear Navier-Stokes system. In [GGS] the estimate (1.4) has been extended to the case that $A = A(t)$ in (1.2) depends on t, and in [GS2] and [GGS] non zero initial values $u(0) = u_0$ are treated.

Recently Dore and Venni [DV2] applied their theory to get higher derivative estimates for solutions of (1.2).

The application of our abstract result (1.1) on fractional powers $(A+B)^\alpha$ to the evolution equation (1.2) yields now estimates of the form

$$\int_0^T \|(\frac{d}{dt})^{1-\alpha}u\|_q^r dt + \int_0^T \|A_q^{1-\alpha}u\|_q^r dt \le C\int_0^T \|A_q^{-\alpha}f\|_q^r dt \tag{1.5}$$

with C independent of f and T, and $0<\alpha<1$. Here u is a generalized solution of (1.2) and f may be a distribution which is regularized by $A_q^{-\alpha}$. The case $\alpha = 1/2$ is especially important because (1.5) yields an a priori estimate

$$\int_0^T \|(\frac{d}{dt})^{1/2}u\|_q^r dt + \int_0^T \|\nabla u\|_q^r dt \le C\int_0^T \|F\|_q^r dt \tag{1.6}$$

for solutions of (1.3) when $f = \operatorname{div} F$; here we restrict $n \ge 3$ and $n/(n-1) < q < n$ when Ω is an exterior domain. This estimate is considered as a nonstationary version of Cattabriga's estimate (see e.g. [BM]).

The class $BIP(a,K)$ of operators we consider here consists of nonnegative closed operators A in X which satisfy the estimate $\|A^{is}u\|_X \le Ke^{a|s|}\|u\|_X$ for all $s \in \mathbf{R}$ where $K \ge 1$ and $0 \le a < \pi$ (independent of u and s). The well known application of this estimate of the pure imaginary powers A^{is} is the identification

$$[X, D(A)]_\alpha = D(A^\alpha),$$

where $[X,D(A)]_\alpha$ is the complex interpolation space; see e.g. [Tr]. The Dore-Venni theory gives now another important application of the above estimate. This theory requires the ζ-convexity of the Banach space. For various properties of ζ-convex space we refer to the nice review article [B]. For the theory of complex powers A^z, $z \in \mathbf{C}$ we refer to the comprehensive article [Ko].

Our main abstract result is given in Section 3; Section 2 contains preliminary lemmas and Section 4 is devoted to the application to the Stokes system.

2. Sum of operators with bounded imaginary powers.

Let A be a closed linear operator with dense domain $D(A)$ in a Banach space X equipped with norm $\|\cdot\|$. We say A is *nonnegative* if its resolvent set contains all negative real numbers and

$$\sup_{t>0} t\|(t+A)^{-1}\| < \infty,$$

where $\|\cdot\|$ denotes the operator norm in $\mathcal{L}(X)$, the space of all bounded linear operators. If a nonnegative operator has a dense range $R(A)$ in X, one can define its complex power A^z for every $z \in \mathbf{C}$ as a densely defined closed operator in X. (cf. [Ko]). For $a \ge 0$ and $K \ge 1$ we say a nonnegative operator A belongs to $BIP(a;K)$ if $A^{is} \in \mathcal{L}(X)$ and is estimated as

$$\|A^{is}\| \le Ke^{a|s|}, \quad s \in \mathbf{R}$$

where $D(A)$ and $R(A)$ are assumed to be dense in X. Let $BIP(a)$ denote the union of $BIP(a,K)$ for $K \ge 1$.

2.1. FUNDAMENTAL LEMMA. (i) *If $A \in BIP(a;K)$, then $A^\alpha \in BIP(a\alpha;K)$ for $0 < \alpha < 1$.*
(ii) *If $A \in BIP(a)$, $0 \le a < \pi$, then for each $\delta > 0$ with $\delta < \pi - a$ there is a constant M_δ independent of λ such that*

$$\|(\lambda + A)^{-1}\| \le M_\delta/|\lambda|, \quad |\arg \lambda| \le \pi - a - \delta, \ 0 \ne \lambda \in \mathbf{C}.$$

In particular, if $a < \pi/2$, then $-A$ generates an analytic semigroup e^{-tA} in X.

PROOF: (i) As well known, if A is nonnegative so is A^α $(0 < \alpha < 1)$; see e.g. [Kr, p.119, (5.25)] or [Ka]. If $A \in BIP(a)$, then $A^\alpha \in BIP(a\alpha)$ since

$$\|(A^\alpha)^{is}\| = \|A^{i\alpha s}\| \le K e^{a\alpha|s|}.$$

Here we use the property $(A^\alpha)^{is} = A^{i\alpha s}$ which can be shown as follows. First we prove this property with A replaced by $(\varepsilon + A)^{-1}$, $\varepsilon > 0$; here we use the well known Dunford integral calculus. Then the assertion follows by letting $\varepsilon \to 0$ and using [PS, Theorem 3].
(ii) See [PS, Theorem 2].

2.2. SUMMATION LEMMA. *Let X be a ζ-convex Banach space. Let A and B belong to $BIP(a,K)$ and $BIP(b,K)$, respectively. Suppose that A and B are resolvent commuting, i.e.,*

$$(t + A)^{-1}(t + B)^{-1} = (t + B)^{-1}(t + A)^{-1} \quad \text{for all} \quad t > 0.$$

Then $A + B \in BIP(a \vee b, K')$ provided that $a \ne b$, where $a \vee b = \max(a,b)$ and $K' = K'(a,b,K,X)$.

This is Theorem 5 in [PS], where the dependence of constants is not explicitly stated. For various properties of ζ-convex spaces there is the nice review article by Burkholder [B] so we do not touch them here.

We next recall the Dore-Venni theory [DV] on the inverse of $A + B$. Let T be an injective closed linear operator in a Banach space X. Let $\hat{D}(T)$ be the completion of $D(T)$ in the norm $\|Tu\|$. Since T may not have a bounded inverse, $\hat{D}(T)$ may not be a subspace of X. The element $Tv \in X$ for $v \in \hat{D}(T)$ is defined by $Tv = \lim_{j\to\infty} Tv_j$, where $\{v_j\}$ is a Cauchy sequence converging to v in $\hat{D}(T)$. The norm of v in $\hat{D}(T)$ is defined by

$$\|v\|_{\hat{D}(T)} = \|Tv\| = \lim_{j\to\infty} \|Tv_j\|.$$

Let T' be another injective closed linear operator in X. Let $T + T'$ be the operator defined on $D(T + T') = D(T) \cap D(T')$. By $D(T + T')^\wedge$ we represent the completion of $D(T + T')$ in the norm $\|Tu\| + \|T'u\|$. Clearly, this space is continuously embedded in $\hat{D}(T)$ and $\hat{D}(T')$. However, the intersection $\hat{D}(T) \cap \hat{D}(T')$ is not meaningful unless the norms $\|Tv\|$ and $\|T'v\|$ are consistent in the sense of the interpolation theory [RS, p.35]. Note that $D(T + T')^\wedge$ need not be equal to $\hat{D}(T + T')$.

2.3. THEOREM ON INVERSES. *Let X be ζ-convex. Suppose that $A \in BIP(a;K)$ and $B \in BIP(b;K)$ are resolvent commuting and that $a+b<\pi$. Then the operator $A+B : D(A+B)^\wedge \to X$ is bijective and boundedly invertible. Moreover there is $C = C(a,b,K,X)$ such that*

$$\|A(A+B)^{-1}\| \leq C, \ \|B(A+B)^{-1}\| \leq C.$$

REMARK: Observe as a consequence that $\|Au\|+\|Bu\|$ and $\|(A+B)u\|$ are equivalent norms on $D(A) \cap D(B)$ so that $D(A+B)^\wedge = \hat{D}(A+B)$.

This result was first proved by Dore and Venni [DV] under the assumption that both A and B have bounded inverses. The key observation is the following integral representation

$$(A+B)^{-1} = \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} \frac{A^{-z}B^{z-1}}{\sin \pi z} dz, \quad 0<c<1.$$

It turns out that the assumptions A^{-1} and $B^{-1} \in \mathcal{L}(X)$ can be removed. The first proof is given by Y. Giga and Sohr [GS2] by introducing appropriate dense subspaces of X so that the argument in [DV] can be justified without $A^{-1}, B^{-1} \in \mathcal{L}(X)$. Another proof is given by Prüss and Sohr [PS]. They established a functional calculus generated by the group A^{is} and proved that $A \in BIP(a)$ implies $A_\varepsilon = \varepsilon I + A \in BIP(a;L)$ with L independent of $\varepsilon > 0$. This is considered as a special case of the summation lemma. Since A_ε has a bounded inverse, they applied the Dore-Venni estimate to A_ε and sent $\varepsilon \to 0$ to get the desired estimates in Theorem 2.3. The first proof is more direct because it does not use the approximated operator A_ε.

The injectivity of the operators A, B is not explicitly assumed. It follows from the fact that these operators are nonnegative and have dense ranges; see [Ko, Theorem 3.2 and 3.7]. Indeed $Au = 0$ implies $u = t(t+A)^{-1}u$, so letting $t \to 0$ yields $u = 0$.

It is convenient to consider appropriate dense subspaces in X as in [GS2]. For $\xi = (\zeta, \eta)$ and $\Lambda = (h,j,k,\ell)$ with nonnegative integers h, j, k, ℓ we set

$$g_\Lambda(\xi) = I_A^j(t) J_A^h(\tau) I_B^\ell(s) J_B^k(\sigma) g, \quad g \in X$$
$$\zeta = (t, \tau^{-1}), \ \eta = (s, \sigma^{-1}), \ t, \tau, s, \sigma > 0$$

with $I_A(t) = A(t+A)^{-1}$ and $J_A(\tau) = \tau(\tau + A)^{-1}$. We introduce the subspace

$$G_\Lambda = \text{linear hull of } \{g_\Lambda(\xi); g \in X, \xi = (t, \tau^{-1}, s, \sigma^{-1}), t, \tau, s, \sigma > 0\}.$$

2.4. DENSITY LEMMA. *Suppose that A and B are nonnegative and resolvent commuting with dense ranges and domains in X. Then G_Λ is dense in X. Moreover G_Λ is dense in $D(A) \cap D(B)$ under the norm $\|Av\| + \|Bv\|$.*

PROOF: By a standard argument [Ko] we see $g_\Lambda(\xi) \to g$, $Ag_\Lambda(\xi) \to Ag$, $Bg_\Lambda(\xi) \to Bg$ in X as $\xi \to 0$, which proves the lemma. We give a proof for completeness. Since A is nonnegative, one observes

$$t(t+A)^{-1}f = t(A(t+A)^{-1}u) \to 0 \quad \text{as} \quad t \to 0$$

for $f = Au \in R(A)$. Since $R(A)$ is dense in X and $\sup_t \|I_A(t)\| < \infty$, we conclude

$$I_A(t)f \to f \quad \text{in} \quad X \quad \text{as} \quad t \to 0.$$

A similar observation shows

$$J_A(\tau)f \to f \quad \text{in} \quad X \quad \text{as} \quad \tau \to \infty$$

and the same for B. Since all $I_A(t)$, $I_B(s)$, $J_A(t)$, $J_B(s)$ are bounded in $\mathcal{L}(X)$, these convergences for A and B imply that $g_\Lambda(\xi) \to g$ in X as $\xi \to 0$. The proofs of $Ag_\Lambda(\xi) \to Ag$ and $Bg_\Lambda(\xi) \to Bg$ under $g \in D(A) \cap D(B)$ are parallel, so they are omitted.

2.5. Commutativity Lemma. *Suppose that A and B are nonnegative and resolvent commuting with dense domains and ranges in X. Then*

$$A^z B^w A^u B^v f = B^w A^{z+u} B^v f \quad \text{for} \quad f \in G_\Lambda$$
$$\text{with } z, w, u, v \in \mathbf{C} \text{ and } \Lambda = (h, j, k, \ell)$$

provided that h, j, k, ℓ are sufficiently large and the largeness only depends on the modulus of the real parts of z, w, u, v.

For the proof we use an integral representation of the complex powers of A and B by their resolvents [Ko, (1.3) and (4.11)]. Since A and B are resolvent commuting, it is not difficult to prove

$$(t+A)^{-1}(s+B)^{-1} = (s+B)^{-1}(t+A)^{-1}, \quad t, s > 0.$$

Applying this commutativity to the integral representation yields the commutativity of complex powers on G_Λ. The proof is straightforward, so we omit the details.

2.6. Corollary to The Theorem on inverses. *Assume the hypotheses of the theorem on inverses. Let m be a positive integer. Then*

$$A^m(A+B)^{-m} = (A(A+B)^{-1})^m$$
$$B^m(A+B)^{-m} = (B(A+B)^{-1})^m$$

on an appropriate dense subspace of X. In particular, $A^m(A+B)^{-m}$ and $B^m(A+B)^{-m}$ can be extended to bounded linear operators on X with a bound depending only on a, b, K, m, X.

Proof: We give a proof for $m = 2$; the proof for general $m \geq 3$ is parallel, so it is omitted. We use the Dore-Venni representation of $(A+B)^{-1}$. Formally for $z \in \mathbf{C}$, Re $z = c$ with $0 < c < 1$

$$AB^{z-1}(A+B)^{-1}f = AB^{z-1}\int_{c-i\infty}^{c+i\infty} \frac{A^{-w}B^{w-1}f}{2i \sin \pi w} dw$$
$$= B^{z-1}A(A+B)^{-1}f.$$

This calculation is justified by the commutativity lemma for $f \in G_\Lambda$, $\Lambda = (h, j, k, \ell)$ with h, j, k, ℓ sufficiently large. We thus observe

$$\begin{aligned} A^2(A+B)^{-2}f &= A\int_{c-i\infty}^{c+i\infty} \frac{A^{-z}AB^{z-1}(A+B)^{-1}f}{2i\sin\pi z}dz \\ &= A(A+B)^{-1}A(A+B)^{-1}f. \end{aligned}$$

Since G_Λ is dense in X and $A(A+B)^{-1}$ is bounded by the theorem on inverses, $A^2(A+B)^{-2}$ can be extended to a bounded linear operator $(A(A+B)^{-1})^2$. The same argument applies to $B^2(A+B)^{-2}$.

3. Spaces of fractional powers

For $A \in BIP(a)$ let $\hat{D}(A^\alpha)$ be the completion of the domain $D(A^\alpha)$ in the norm $||A^\alpha u||$, where $0 < \alpha < 1$. The space $\hat{D}(A^\alpha)$ can be characterized by a complex interpolation space, namely

$$\hat{D}(A^\alpha) = [X, \hat{D}(A)]_\alpha.$$

This follows from the general interpolation theory (see e.g. [Tr], [BB]). For the proof see e.g. [GS1, Proposition 6.1] or [BM]. In this section we compare various norms on $D(A) \cap D(B)$.

3.1. Main Theorem. *Suppose that X is ζ-convex. Suppose that $A \in BIP(a,K)$ and $B \in BIP(b,K)$ are resolvent commuting and that $a+b<\pi$. Then for $0 \le \alpha \le 1$*

$$\begin{aligned} &D(A^\alpha) \cap D(B^\alpha) = D((A+B)^\alpha), \\ &\hat{D}(A^\alpha + B^\alpha) = D(A^\alpha + B^\alpha)^\wedge = \hat{D}((A+B)^\alpha) = [X, \hat{D}(A+B)]_\alpha \end{aligned}$$

and there are constants $C_j = C_j(a,b,\alpha,K,X) > 0$, $j = 1,2,3,4$ such that

$$\begin{aligned} ||A^\alpha u|| + ||B^\alpha u|| &\le C_1||(A^\alpha + B^\alpha)u|| \le C_2||(A+B)^\alpha u|| \le \\ &\le C_3||u||_{[X,\hat{D}(A+B)]_\alpha} \le C_4(||A^\alpha u|| + ||B^\alpha u||) \end{aligned}$$

for all $u \in D(A^\alpha) \cap D(B^\alpha)$.

Proof: Since the summation lemma implies $A + B \in BIP(a \vee b + \delta, K')$, $\delta > 0$ with some $K' \ge 1$, it follows the identity

$$\hat{D}((A+B)^\alpha) = [X, \hat{D}(A+B)]_\alpha$$

with equivalent norms

$$||(A+B)^\alpha u|| \quad \text{and} \quad ||u||_{[X,\hat{D}(A+B)]_\alpha}.$$

Furthermore, since $A^\alpha \in BIP(a\alpha, K)$, $B^\alpha \in BIP(b\alpha, K)$ and $a\alpha + b\alpha < \pi$, by the theorem on inverses we observe that the norms

$$||A^\alpha u|| + ||B^\alpha u|| \quad \text{and} \quad ||(A^\alpha + B^\alpha)u||$$

are equivalent on $D(A^\alpha) \cap D(B^\alpha)$.

It remains to prove that $D(A^\alpha + B^\alpha) = D((A+B)^\alpha)$ and

$$\|(A+B)^\alpha u\| \leq C\|(A^\alpha + B^\alpha)u\|, \tag{3.1}$$

$$\|(A^\alpha + B^\alpha)u\| \leq C'\|(A+B)^\alpha u\| \tag{3.2}$$

for all $u \in D(A^\alpha + B^\alpha) = D(A^\alpha) \cap D(B^\alpha)$. Let us show the first inequality (3.1). To prove (3.1) it suffices to show that

$$\|(A+B)^\alpha (A^\alpha + B^\alpha)^{-1} v\| \leq C\|v\| \tag{3.3}$$

for all v belonging to an appropriate dense subspace of X. Let G_Λ be as in the density lemma with $\Lambda = (h, j, k, \ell)$. For sufficiently large h, j, k, ℓ the function

$$F(z) = e^{z^2}(A+B)^z (A^\alpha + B^\alpha)^{-z/\alpha} v, \quad v \in G_\Lambda$$

is holomorphic in a neighborhood of $0 \leq \mathrm{Re}\ z \leq 1$. Since $A + B \in BIP(a \vee b + \delta, K')$ and $A^\alpha + B^\alpha \in BIP((a \vee b + \delta)\alpha, K'')$ for all $\delta > 0$ with K', K'' depending on K, a, b, δ, α, X, estimating F on the imaginary axis yields

$$\begin{aligned} \|F(is)\| &\leq e^{-s^2} K' K'' e^{\rho|s|} e^{\rho|s|/\alpha} \|v\|, \quad \rho = a \vee b + \delta \\ &\leq M_0 \|v\| \quad \text{with} \quad M_0 = \sup_{s \in \mathbb{R}} K K'' \exp(\rho|s|(1 + 1/\alpha) - s^2) < \infty, \end{aligned}$$

where δ is now a fixed sufficiently small number. Similarly,

$$\begin{aligned} \|F(1+is)\| &= e^{1-s^2} \|(A+B)^{is}(A+B)(A^\alpha + B^\alpha)^{-1/\alpha}(A^\alpha + B^\alpha)^{-is/\alpha} v\| \\ &\leq e^{1-s^2} K' K'' e^{\rho|s|} e^{\rho|s|/\alpha} \|(A+B)(A^\alpha + B^\alpha)^{-1/\alpha}\| \, \|v\| \\ &\leq e M_0 \|(A+B)(A^\alpha + B^\alpha)^{-1/\alpha}\| \, \|v\|. \end{aligned}$$

If $A(A^\alpha + B^\alpha)^{-1/\alpha}$ and $B(A^\alpha + B^\alpha)^{-1/\alpha}$ can be extended to bounded operators in X with

$$\|A(A^\alpha + B^\alpha)^{-1/\alpha}\| \leq c, \quad \|B(A^\alpha + B^\alpha)^{-1/\alpha}\| \leq c, \tag{3.4}$$

then

$$\|F(1+is)\| \leq M_1 \|v\|, \quad M_1 = 2eM_0 c.$$

Applying the three line theorem [RS, p.33] yields

$$\|F(\alpha)\| \leq M_0^{1-\alpha} M_1^\alpha \|v\|, \quad v \in G_\Lambda.$$

This deduces (3.3), $D(A^\alpha + B^\alpha) \subset D((A+B)^\alpha)$ and (3.1) with $C = e^{-\alpha^2} M_0^{1-\alpha} M_1^\alpha$ since G_Λ is dense in X. The inequalities (3.4) are proved in the next lemma.

To prove the converse direction (3.2) we need that $A^\alpha(A+B)^{-\alpha}$ and $B^\alpha(A+B)^{-\alpha}$ extend to bounded operators in X, this is also proved in the next lemma. Similarly as above we then obtain $D((A+B)^\alpha) \subset D(A^\alpha + B^\alpha)$,

$$\|A^\alpha u\| + \|B^\alpha u\| \leq C\|(A+B)^\alpha u\|, \quad u \in D((A+B)^\alpha);$$

this implies (3.2) and the proof is complete.

3.2. LEMMA. *Assume the hypotheses of the theorem on inverses.*
(i) *For $\sigma > 0$ the operators $A^\sigma(A+B)^{-\sigma}$ and $B^\sigma(A+B)^{-\sigma}$ can be extended to bounded linear operators in X with a bound depending only on a, b, K, σ, X.*
(ii) *For $0 < \alpha < 1$ the operators $A(A^\alpha + B^\alpha)^{-1/\alpha}$ and $B(A^\alpha + B^\alpha)^{-1/\alpha}$ can be extended to bounded linear operators in X with a bound depending only on a, b, K, α, X.*

PROOF: Part (ii) follows from (i) by setting $A = A^\alpha$, $B = B^\alpha$, $\sigma = 1/\alpha$ so it remains to prove (i). In the corollary to the theorem on inverses, we have proved (i) when σ is a positive integer. For general σ we again appeal to the three line theorem. Let m be a nonnegative integer. If we take an appropriate dense subspace G_Λ of X, the function

$$H(z) = e^{z^2} A^{m+z}(A+B)^{-(m+z)}v, \quad v \in G_\Lambda$$

is holomorphic in a neighborhood of $0 \le \mathrm{Re}\ z \le 1$. Since $A+B \in BIP(a \vee b + \delta, K')$ for all $\delta > 0$ with some $K' = K'(K, a, b, \delta, \alpha, X)$, estimating on the imaginary axis yields

$$||H(is)|| \le e^{-s^2} K e^{a|s|} ||A^m(A+B)^{-m}|| K' e^{\rho|s|} ||v||$$

with $\rho = a \vee b + \delta$, where δ is a fixed sufficiently small number. By the corollary to the theorem on inverses, $||A^m(A+B)^{-m}||$ is bounded by c_m; we now observe

$$||H(is)|| \le c_m L ||v||, \quad L = \sup_{s \in \mathbf{R}} KK' \exp(-s^2 + (a+\rho)|s|) < \infty.$$

Similarly, on $\mathrm{Re}\, z = 1$ we have

$$||H(1+is)|| \le c_{m+1} Le ||v||.$$

Applying the three line theorem yields

$$||H(\tau)|| \le M||v||, \quad M = c_m^{1-\tau} c_{m+1}^{\tau} e^{\tau} L < \infty, \quad v \in G_\Lambda.$$

Since G_Λ is dense in X, we now obtain

$$||A^{m+\tau}(A+B)^{-(m+\tau)}|| \le e^{-\tau^2} M, \quad 0 < \tau < 1.$$

The proof for $B^\sigma(A+B)^{-\sigma}$ is parallel, so is omitted.

4. Application to the Stokes system.

Although our abstract result applies to a very general class of evolution equations (1.2), we consider here as an example only the Stokes system (1.3) on some domain Ω in $\mathbf{R}^n$.

Assumptions on the domain Ω.

In the following let $\Omega \subset \mathbf{R}^n$ ($n \ge 2$) be either the whole space $\mathbf{R}^n$, a bounded domain, a halfspace or an exterior domain. The boundary $\partial\Omega$ is always assumed at least of class $C^{2+\mu}$ with $0 < \mu < 1$. If Ω is an exterior domain we suppose $n \ge 3$.

Stokes operator.

For $1 < q < \infty$ let L^q_σ denote the L^q closure of the space $C^\infty_{0,\sigma}$ of all smooth divergence-free vector fields with compact support in Ω. Let $P = P_q$ denote the projection operator from $L^q = (L^q(\Omega))^n$ to L^q_σ associated with the Helmholtz decomposition. The *Stokes operator* A_q is defined in L^q_σ by $A_q = -P\Delta$ with the dense domain

$$D(A_q) = \{u \in L^q_\sigma; \nabla^2 u \in L^q,\ u|_{\partial\Omega} = 0\},$$

where Δ denotes the Laplacian and $\nabla^2 u$ denotes the tensor of all second order derivatives. In [G] and [GS1] it is shown that for all $0 < a < \pi/2$, $A_q \in BIP(a, K)$ with K depending on a. For more information on the Stokes operator and the Helmholtz decomposition we refer to [GS1, 2] and [BM] and the references cited there.

Evolution equation.

Applying the projection P_q to the Stokes system (1.3), one formally obtains its abstract form

$$\frac{du}{dt} + A_q u = f \quad \text{in} \quad (0,T), \quad u(0) = 0. \tag{4.1}$$

For $1 < r < \infty$, $0 < T \leq \infty$ let B denote the derivative operator on $X = L^r(0,T;L^q_\sigma)$ defined by $B = d/dt$ (weak derivative) with

$$D(B) = \{u \in X; du/dt \in X, u(0) = 0\}.$$

The operator A in X is defined by $(Au)(t) = A_q u(t)$ for a.e. $t \in (0,T)$ where

$$u \in D(A) = \{u \in X; u(t) \in D(A_q) \text{ for a.e. } t \in (0,T) \text{ and } \int_0^T \|A_q u(t)\|_q^r dt < \infty\}.$$

Using A and B we may rewrite (4.1) as

$$Bu + Au = f. \tag{4.2}$$

The space X is ζ-convex because L^q_σ is ζ-convex; see [GS2] and the references cited there. As shown in [DV] for each $\delta > 0$ the operator $B \in BIP(\pi/2 + \delta, K)$ with K depending on δ but independent of T, $0 < T \leq \infty$. The property $A_q \in BIP(a, K)$ yields $A \in BIP(a, K)$, where a is arbitrary $0 < a < \pi/2$ and K depends on a but is independent of T. Clearly, A and B are resolvent commuting. Applying the extended Dore-Venni theorem in [GS2] one observes that there is a unique solution $u \in D(A+B)^\wedge$ of (4.2) for each $f \in X$. If $T < \infty$, B^{-1} exists as a bounded operator so that

$$D(A + B)^\wedge = D(A) \cap D(B).$$

For $0 < T < \infty$ we call $u : (0,T) \to L^q_\sigma$ a *strong* solution of (4.2) if it satisfies (4.2) with $u \in D(A) \cap D(B)$. In case $T = \infty$ we call $u : (0,\infty) \to L^q_\sigma$ a *strong* solution if so is u

on each finite time interval $(0,T)$.

Generalized solutions.
In order to apply our abstract Theorem 3.1 to (4.2) we have to consider generalized solutions u of (4.1) for a class of distributions f. This is caused by the fractional powers $(B+A)^\alpha$. For simplicity we will avoid here the definition via test functions and prefer the definition via regularization. Roughly speaking, u is a generalized solution of (4.1) if the "regularization" $A_q^{-\alpha}u$ is a strong solution of (4.1) with f replaced by $A_q^{-\alpha}u$.

Let us give a precise definition. For $0<\alpha<1$ the space $D(A_q^{-\alpha}) = R(A_q^\alpha)$ is equipped with the norm $||A_q^{-\alpha}u||_q$ and $\hat{D}(A_q^{-\alpha})$ denotes the completion of $D(A_q^{-\alpha})$ under this norm. For $v=(v_j)_{j=1}^\infty \in \hat{D}(A_q^{-\alpha})$ we define $A_q^{-\alpha}v = (A_q^{-\alpha}v_j)$ and get $A_q^{-\alpha}v \in L_\sigma^q$ for each $v \in \hat{D}(A_q^{-\alpha})$; $A_q^{-\alpha}v$ is called the regularization of $v \in \hat{D}(A_q^{-\alpha})$. In the case $T<\infty$ we say $u \in L^r(0,T;D(A_q^{1-\alpha}))$ is a *generalized solution* of (4.1) with $f \in L^r(0,T;\hat{D}(A_q^{-\alpha}))$ if $A_q^{-\alpha}u$ solves (4.2) as a strong solution with f replaced by $A_q^{-\alpha}f \in L^r(0,T;L_\sigma^q)$. If $u:(0,\infty)\to D(A_q^{1-\alpha})$ is a generalized solution of (4.1) on each finite time interval $(0,T)$, u is called a *generalized solution* in case $T=\infty$.

4.1. UNIQUE EXISTENCE OF GENERALIZED SOLUTIONS. *Let Ω be as above, $0<T<\infty$, $1<r<\infty$, $1<q<\infty$, $0<\alpha<1$. Suppose $f \in L^r(0,T;\hat{D}(A_q^{-\alpha}))$. Then there exists a unique generalized solution $u \in L^r(0,T;D(A_q^{1-\alpha}))$ of (4.1). Moreover, $u \in D(B^{1-\alpha})$ and*

$$\int_0^T ||(\frac{d}{dt})^{1-\alpha}u||_q^r dt + \int_0^T ||A_q^{1-\alpha}u||_q^r dt \le C\int_0^T ||A_q^{-\alpha}f||_q^r dt \tag{4.3}$$

with $C=C(\Omega,q,r,\alpha)>0$ independent of T and f where $(d/dt)^{1-\alpha}=B^{1-\alpha}$.

REMARKS: a) The condition $u(0)=0$ is implicitly contained in $u \in D(B^{1-\alpha})$ for small α (i.e. $0<\alpha<1-1/r$) while no condition is imposed on $u(0)$ for large α (i.e. $1-1/r<\alpha<1$).
b) The case $T=\infty$ can be admitted in Theorem 4.1 if we replace $D(A_q^{1-\alpha})$ by $\hat{D}(A_q^{1-\alpha})$ and $D(B^{1-\alpha})$ by $\hat{D}(B^{1-\alpha})$. In this case (4.3) is

$$\int_0^\infty ||(\frac{d}{dt})^{1-\alpha}u||_q^r dt + \int_0^\infty ||A_q^{1-\alpha}u||_r^q dt \le C\int_0^\infty ||A_q^{-\alpha}f||_q^r dt$$

which yields asymptotic properties of u as $t\to\infty$.
c) Of course, this theorem extends to the class of all evolution equations for which Theorem 3.1 is applicable.

PROOF: We apply the extended Dore-Venni theorem in [GS2] to $A_q^{-\alpha}f \in X$ and obtain a unique solution $v \in D(B)\cap D(A)$ of $Bv+Av = A_q^{-\alpha}f$. The function $u=A_q^\alpha v$ is a generalized solution of (4.1) since $A_q^{-\alpha}u$ is a strong solution; the uniqueness of u is obvious.

To prove (4.3) we use the Yosida approximation $J_m = J_A(m) = m(m+A)^{-1}$ in Section 2 and obtain

$$\begin{aligned} &BA^{-\alpha}J_m u + AA^{-\alpha}J_m u = A^{-\alpha}(BJ_m u + AJ_m u) = A^{-\alpha}J_m f \\ &BJ_m u + AJ_m u = J_m f. \end{aligned}$$

Here $J_m f$ is defined in the same way as $A_q^{-\alpha} f$. We know that $\lim_{m\to\infty} J_m u = u$ in $X = L^r(0,T;L^q_\sigma)$. Setting $u_m = J_m u$, $w = u_m - u_\ell$ and applying Theorem 3.1 yields

$$\begin{aligned} ||B^{1-\alpha}w||_X + ||A^{1-\alpha}w||_X &\le C||(B+A)^{1-\alpha}w||_X \\ &= C||(B+A)^{-\alpha}(B+A)w||_X = C||A^\alpha(B+A)^{-\alpha}(J_m - J_\ell)A^{-\alpha}f||_X \\ &\le C'||(J_m - J_\ell)A^{-\alpha}f||_X; \end{aligned}$$

here we used the fact that $A^\alpha(B+A)^{-\alpha}$ is bounded by Lemma 3.2. From this estimate we conclude $u \in D(B^{1-\alpha}) \cap D(A^{1-\alpha})$ since $B^{1-\alpha}$ and $A^{1-\alpha}$ are closed and $u \in X$. The same estimate with w replaced by u_m yields (4.3) by letting $m \to \infty$. This proves 4.1.

We next consider some concrete cases of distributions f in Theorem 4.1. In case a) of the following Corollary we consider a distribution of the form $f = \sum_{\nu=1}^n \partial_\nu f_\nu$ with $f_\nu \in X$ and $\partial_\nu = \partial/\partial x_\nu$ and in b) we let $f \in L^r(0,T;L^\gamma_\sigma)$ with some exponent γ different from q.

4.2. Corollary. *Suppose Ω as above and $0 < T < \infty$, $1 < q < \infty$, $1 < r < \infty$.*
a) Let $f = \sum_{\nu=1}^n \partial_\nu f_\nu$ with $f_\nu \in X = L^r(0,T;L^q_\sigma)$, $\nu = 1,\cdots,n$. If Ω is unbounded, suppose additionally $q > n/(n-1)$, $n \ge 3$. Then $A_q^{-1/2}f \in X$, $f \in L^q(0,T;\hat{D}(A_q^{-1/2}))$. There exists a unique generalized solution $u \in L^r(0,T;D(A_q^{1/2}))$ of (4.1) with $u \in D(B^{1/2})$ and

$$(4.4)\qquad \int_0^T ||(\frac{d}{dt})^{1/2}u||_q^r dt + \int_0^T ||A_q^{1/2}u||_q^r dt \le C\sum_{\nu=1}^n \int_0^T ||f_\nu||_q^r dt$$

with $C = C(\Omega,q,r)$ independent of f and T.
b) For $1 < \alpha < 1$ let γ be defined by $2\alpha + n/q = n/\gamma$ and $f \in L^r(0,T;L^\gamma_\sigma)$. If Ω is an exterior domain, suppose additionally $1 < \gamma < n/2$, $n \ge 3$. Then $A_q^{-\alpha}f \in L^r(0,T;L^q_\sigma)$, $f \in L^r(0,T;\hat{D}(A_q^{-\alpha}))$. There exists a unique generalized solution $u \in L^r(0,T;D(A_q^{1-\alpha}))$ of (4.1) with $u \in D(B^{1-\alpha})$ and

$$(4.5)\qquad \int_0^T ||(\frac{d}{dt})^{1-\alpha}u||_q^r dt + \int_0^T ||A_q^{1-\alpha}u||_q^r dt \le C\int_0^T ||f||_\gamma^r dt$$

with $C = C(\Omega,q,r,\alpha)$ independent of f and T.

Remarks: (i) To prove a) and b) it suffices to prove that $f \in L^r(0,T;\hat{D}(A_q^{-1/2}))$ and

$$||A_q^{-1/2}f||_X \le C\sum_{\nu=1}^n ||f_\nu||_X$$

in a) and that $f \in L^r(0,T;\hat{D}(A_q^{-\alpha}))$ and

$$||A_q^{-\alpha} f||_X \leq C(\int_0^T ||f||_\gamma^r dt)^{1/r}$$

in b) respectively with C independent of f and T.
(ii) The estimate (4.4) yields (1.6) by applying of

$$||\nabla u||_q \leq C||A_q^{1/2} u||_q$$

which needs additionally the restriction $1 < q < n$, $n \geq 3$ when Ω is an exterior domain ([BM], [GS1]).

PROOF: a) In [GS1, p.123] it has been shown that $C_{0,\sigma}^\infty \subset R(A_q)$ if $q > n/(n-2)$ and Ω is the $\mathbb{R}^n$ or an exterior domain; the same proof works also for the half-space and the restriction becomes $q > n/(n-1)$ if A_q is replaced by $A_q^{1/2}$. If Ω is bounded, no restriction is needed.

So for each f_ν ($\nu = 1, 2, \cdots, n$) we find a sequence $(f_{\nu j})_{j=1}^\infty$ in $L^r(0,T;C_{0,\sigma}^\infty) \subset L^r(0,T;D(A_q^{-1/2}))$ with $f_\nu = \lim_{j\to\infty} f_{\nu j}$ in $L^r(0,T;L_\sigma^q)$. It follows that $(\tilde{f}_j) = (\sum_{\nu=1}^n \partial_\nu f_{\nu j})$ is a sequence in $L^r(0,T;D(A_q^{-1/2}))$.

We next use the estimate

$$||A_q^{-1/2} \nabla u||_q \leq C||u||_q$$

(see [BM], [GS1]) which is valid in all cases for Ω but in exterior domains under the restriction $q > n/(n-1)$; observe that this estimate is equivalent to $||\nabla u||_{q'} \leq C||A_{q'}^{1/2} u||_{q'}$, where by duality the restriction is now given by $1 < q' < n$. This leads to

$$||A_q^{-1/2}(\tilde{f}_i - \tilde{f}_j)||_X = ||\sum_{\nu=1}^n A_q^{-1/2} \partial_\nu (f_{\nu i} - f_{\nu j})||_X \leq C \sum_{\nu=1}^n ||f_{\nu i} - f_{\nu j}||_X$$

which yields $f \in L^r(0,T;\hat{D}(A_q^{-1/2}))$. This estimate also yields

$$||A_q^{-1/2} f||_X \leq C \sum_{\nu=1}^n ||f_\nu||_X$$

so Theorem 4.1 is applicable.
b) Since $R(A_q^\alpha) \subset L_\sigma^\gamma$, is dense in L_σ^γ, one can choose $f_j \in L^r(0,T;D(A_q^{-\alpha}))$, $j = 1, 2, \cdots$ with $f = \lim_{j\to\infty} f_j$ in $L^r(0,T;L_\sigma^\gamma)$. Then we use the estimate

$$||A_q^{-\alpha} u||_q \leq C||u||_\gamma$$

in [GS1, p.104] which holds for $2\alpha + n/q = n/\gamma$; in exterior domains the restriction $1 < \gamma < n/2$, $n \geq 3$ is needed. This leads to

$$||A_q^{-\alpha}(f_i - f_j)||_X \leq C(\int_0^T ||f_i - f_j||_\gamma^r dt)^{1/r}$$

which yields $f \in L^r(0,T;\hat{D}(A_q^{-\alpha}))$ and

$$||A_q^{-\alpha} f|| \leq C(\int_0^T ||f||_{\tilde{q}}^r dt)^{1/r},$$

so Theorem 4.1 is applicable.

Further applications. The estimates above can be applied to weak solutions of the nonlinear Navier-Stokes equations if we take the nonlinear term to the right hand side in (4.1). The procedure is completely analogous to that in [GS2].

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