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L^p estimates for the Stokes system

MARIKO GIGA, YOSHIKAZU GIGA
AND HERMANN SOHR

1. Introduction.

This paper investigates the fractional powers $(A + B)^\alpha$, $0 \leq \alpha \leq 1$ of the sum $A + B$ of two closed (resolvent commuting) operators A and B in a ζ -convex Banach space X . We compare the domain $D((A + B)^\alpha)$ of $(A + B)^\alpha$ with the domain $D(A^\alpha + B^\alpha) = D(A^\alpha) \cap D(B^\alpha)$ of the sum $A^\alpha + B^\alpha$ and show in particular the relation

$$(1.1) \quad D((A + B)^\alpha) = D(A^\alpha) \cap D(B^\alpha)$$

with equivalent norms $\|(A + B)^\alpha u\|$ and $\|A^\alpha u\| + \|B^\alpha u\|$, under some assumptions on the pure imaginary powers of A and B .

Our results will be applied to L^p estimates for (generalized) solutions of the evolution equation

$$(1.2) \quad \frac{du}{dt} + Au = f \quad \text{in } (0, T), \quad 0 < T \leq \infty, \quad u(0) = 0.$$

Here we restrict ourselves to the Stokes operator $A = A_q$. Formally, we get such an equation if we apply the L^q Helmholtz projection P_q to the Stokes system

$$(1.3) \quad \begin{aligned} \frac{\partial u}{\partial t} - \Delta u + \nabla p &= f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega \times (0, T) \\ u|_{\partial\Omega} &= 0 \quad \text{on } \partial\Omega \times (0, T), \quad u = 0 \quad \text{on } \Omega \quad \text{at } t = 0, \end{aligned}$$

where Ω is a domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $1 < q < \infty$; see e.g. [GS1,2] for notations.

Our perturbation result is based on a theory recently developed by Dore and Venni [DV] which has been extended by Giga and Sohr [GS2] to the case that the inverse operators A^{-1} and B^{-1} need not be bounded. See also [PS] for another proof. The original theory of Dore and Venni is applicable to the evolution equation (1.2) only for a finite interval $[0, T]$; it yields a constant $C = C(\Omega, r, q, T) > 0$ such that

$$(1.4) \quad \int_0^T \left\| \frac{du}{dt} \right\|_q^r dt + \int_0^T \|A_q u\|_q^r dt \leq C \int_0^T \|f\|_q^r dt,$$

where $1 < r, q < \infty$ and $\|\cdot\|_q$ denotes the $L^q(\Omega)$ -norm. The extension by [GS2] strengthens the estimate (1.4) so that A_q^{-1} is allowed to be unbounded and that C may be chosen independently of T . Therefore, one may take $T = \infty$ in (1.4) which yields asymptotic properties of the solution u of (1.3) as $t \rightarrow \infty$ even when Ω need not be bounded [GS2]. In [GS2] the estimates applied to the nonlinear Navier-Stokes system. In [GGS] the estimate (1.4) has been extended to the case that $A = A(t)$ in (1.2) depends on t , and in [GS2] and [GGS] non zero initial values $u(0) = u_0$ are treated.

Recently Dore and Venni [DV2] applied their theory to get higher derivative estimates for solutions of (1.2).

The application of our abstract result (1.1) on fractional powers $(A + B)^\alpha$ to the evolution equation (1.2) yields now estimates of the form

$$(1.5) \quad \int_0^T \|(\frac{d}{dt})^{1-\alpha} u\|_q^r dt + \int_0^T \|A_q^{1-\alpha} u\|_q^r dt \leq C \int_0^T \|A_q^{-\alpha} f\|_q^r dt$$

with C independent of f and T , and $0 < \alpha < 1$. Here u is a generalized solution of (1.2) and f may be a distribution which is regularized by $A_q^{-\alpha}$. The case $\alpha = 1/2$ is especially important because (1.5) yields an a priori estimate

$$(1.6) \quad \int_0^T \|(\frac{d}{dt})^{1/2} u\|_q^r dt + \int_0^T \|\nabla u\|_q^r dt \leq C \int_0^T \|F\|_q^r dt$$

for solutions of (1.3) when $f = \operatorname{div} F$; here we restrict $n \geq 3$ and $n/(n-1) < q < n$ when Ω is an exterior domain. This estimate is considered as a nonstationary version of Cattabriga's estimate (see e.g. [BM]).

The class $BIP(a, K)$ of operators we consider here consists of nonnegative closed operators A in X which satisfy the estimate $\|A^{is} u\|_X \leq K e^{a|s|} \|u\|_X$ for all $s \in \mathbb{R}$ where $K \geq 1$ and $0 \leq a < \pi$ (independent of u and s). The well known application of this estimate of the pure imaginary powers A^{is} is the identification

$$[X, D(A)]_\alpha = D(A^\alpha),$$

where $[X, D(A)]_\alpha$ is the complex interpolation space; see e.g. [Tr]. The Dore-Venni theory gives now another important application of the above estimate. This theory requires the ζ -convexity of the Banach space. For various properties of ζ -convex space we refer to the nice review article [B]. For the theory of complex powers A^z , $z \in \mathbb{C}$ we refer to the comprehensive article [Ko].

Our main abstract result is given in Section 3; Section 2 contains preliminary lemmas and Section 4 is devoted to the application to the Stokes system.

2. Sum of operators with bounded imaginary powers.

Let A be a closed linear operator with dense domain $D(A)$ in a Banach space X equipped with norm $\|\cdot\|$. We say A is *nonnegative* if its resolvent set contains all negative real numbers and

$$\sup_{t>0} t \|(t + A)^{-1}\| < \infty,$$

where $\|\cdot\|$ denotes the operator norm in $\mathcal{L}(X)$, the space of all bounded linear operators. If a nonnegative operator has a dense range $R(A)$ in X , one can define its complex power A^z for every $z \in \mathbb{C}$ as a densely defined closed operator in X . (cf. [Ko]). For $a \geq 0$ and $K \geq 1$ we say a nonnegative operator A belongs to $BIP(a; K)$ if $A^{is} \in \mathcal{L}(X)$ and is estimated as

$$\|A^{is}\| \leq K e^{a|s|}, \quad s \in \mathbb{R}$$

where $D(A)$ and $R(A)$ are assumed to be dense in X . Let $BIP(a)$ denote the union of $BIP(a, K)$ for $K \geq 1$.

2.1. FUNDAMENTAL LEMMA. (i) If $A \in BIP(a; K)$, then $A^\alpha \in BIP(a\alpha; K)$ for $0 < \alpha < 1$.

(ii) If $A \in BIP(a)$, $0 \leq a < \pi$, then for each $\delta > 0$ with $\delta < \pi - a$ there is a constant M_δ independent of λ such that

$$\|(\lambda + A)^{-1}\| \leq M_\delta/|\lambda|, \quad |\arg \lambda| \leq \pi - a - \delta, \quad 0 \neq \lambda \in \mathbb{C}.$$

In particular, if $a < \pi/2$, then $-A$ generates an analytic semigroup e^{-tA} in X .

PROOF: (i) As well known, if A is nonnegative so is A^α ($0 < \alpha < 1$); see e.g. [Kr, p.119, (5.25)] or [Ka]. If $A \in BIP(a)$, then $A^\alpha \in BIP(a\alpha)$ since

$$\|(A^\alpha)^{i\theta}\| = \|A^{i\alpha\theta}\| \leq Ke^{a\alpha|\theta|}.$$

Here we use the property $(A^\alpha)^{i\theta} = A^{i\alpha\theta}$ which can be shown as follows. First we prove this property with A replaced by $(\varepsilon + A)^{-1}$, $\varepsilon > 0$; here we use the well known Dunford integral calculus. Then the assertion follows by letting $\varepsilon \rightarrow 0$ and using [PS, Theorem 3].

(ii) See [PS, Theorem 2].

2.2. SUMMATION LEMMA. Let X be a ζ -convex Banach space. Let A and B belong to $BIP(a, K)$ and $BIP(b, K)$, respectively. Suppose that A and B are resolvent commuting, i.e.,

$$(t + A)^{-1}(t + B)^{-1} = (t + B)^{-1}(t + A)^{-1} \quad \text{for all } t > 0.$$

Then $A + B \in BIP(a \vee b, K')$ provided that $a \neq b$, where $a \vee b = \max(a, b)$ and $K' = K'(a, b, K, X)$.

This is Theorem 5 in [PS], where the dependence of constants is not explicitly stated. For various properties of ζ -convex spaces there is the nice review article by Burkholder [B] so we do not touch them here.

We next recall the Dore-Venni theory [DV] on the inverse of $A + B$. Let T be an injective closed linear operator in a Banach space X . Let $\hat{D}(T)$ be the completion of $D(T)$ in the norm $\|Tu\|$. Since T may not have a bounded inverse, $\hat{D}(T)$ may not be a subspace of X . The element $Tv \in X$ for $v \in \hat{D}(T)$ is defined by $Tv = \lim_{j \rightarrow \infty} Tv_j$, where $\{v_j\}$ is a Cauchy sequence converging to v in $\hat{D}(T)$. The norm of v in $\hat{D}(T)$ is defined by

$$\|v\|_{\hat{D}(T)} = \|Tv\| = \lim_{j \rightarrow \infty} \|Tv_j\|.$$

Let T' be another injective closed linear operator in X . Let $T + T'$ be the operator defined on $D(T + T') = D(T) \cap D(T')$. By $D(T + T')^\wedge$ we represent the completion of $D(T + T')$ in the norm $\|Tu\| + \|T'u\|$. Clearly, this space is continuously embedded in $\hat{D}(T)$ and $\hat{D}(T')$. However, the intersection $\hat{D}(T) \cap \hat{D}(T')$ is not meaningful unless the norms $\|Tv\|$ and $\|T'v\|$ are consistent in the sense of the interpolation theory [RS, p.35]. Note that $D(T + T')^\wedge$ need not be equal to $\hat{D}(T + T')$.

2.3. THEOREM ON INVERSES. Let X be ζ -convex. Suppose that $A \in BIP(a; K)$ and $B \in BIP(b; K)$ are resolvent commuting and that $a + b < \pi$. Then the operator $A + B : D(A + B)^\wedge \rightarrow X$ is bijective and boundedly invertible. Moreover there is $C = C(a, b, K, X)$ such that

$$\|A(A + B)^{-1}\| \leq C, \|B(A + B)^{-1}\| \leq C.$$

REMARK: Observe as a consequence that $\|Au\| + \|Bu\|$ and $\|(A + B)u\|$ are equivalent norms on $D(A) \cap D(B)$ so that $D(A + B)^\wedge = \hat{D}(A + B)$.

This result was first proved by Dore and Venni [DV] under the assumption that both A and B have bounded inverses. The key observation is the following integral representation

$$(A + B)^{-1} = \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} \frac{A^{-z} B^{z-1}}{\sin \pi z} dz, \quad 0 < c < 1.$$

It turns out that the assumptions A^{-1} and $B^{-1} \in \mathcal{L}(X)$ can be removed. The first proof is given by Y. Giga and Sohr [GS2] by introducing appropriate dense subspaces of X so that the argument in [DV] can be justified without $A^{-1}, B^{-1} \in \mathcal{L}(X)$. Another proof is given by Prüss and Sohr [PS]. They established a functional calculus generated by the group A^{is} and proved that $A \in BIP(a)$ implies $A_\epsilon = \epsilon I + A \in BIP(a; L)$ with L independent of $\epsilon > 0$. This is considered as a special case of the summation lemma. Since A_ϵ has a bounded inverse, they applied the Dore-Venni estimate to A_ϵ and sent $\epsilon \rightarrow 0$ to get the desired estimates in Theorem 2.3. The first proof is more direct because it does not use the approximated operator A_ϵ .

The injectivity of the operators A, B is not explicitly assumed. It follows from the fact that these operators are nonnegative and have dense ranges; see [Ko, Theorem 3.2 and 3.7]. Indeed $Au = 0$ implies $u = t(t + A)^{-1}u$, so letting $t \rightarrow 0$ yields $u = 0$.

It is convenient to consider appropriate dense subspaces in X as in [GS2]. For $\xi = (\zeta, \eta)$ and $\Lambda = (h, j, k, \ell)$ with nonnegative integers h, j, k, ℓ we set

$$g_\Lambda(\xi) = I_A^j(t) J_A^h(\tau) I_B^\ell(s) J_B^k(\sigma) g, \quad g \in X$$

$$\zeta = (t, \tau^{-1}), \quad \eta = (s, \sigma^{-1}), \quad t, \tau, s, \sigma > 0$$

with $I_A(t) = A(t + A)^{-1}$ and $J_A(\tau) = \tau(\tau + A)^{-1}$. We introduce the subspace

$$G_\Lambda = \text{linear hull of } \{g_\Lambda(\xi); g \in X, \xi = (t, \tau^{-1}, s, \sigma^{-1}), t, \tau, s, \sigma > 0\}.$$

2.4. DENSITY LEMMA. Suppose that A and B are nonnegative and resolvent commuting with dense ranges and domains in X . Then G_Λ is dense in X . Moreover G_Λ is dense in $D(A) \cap D(B)$ under the norm $\|Av\| + \|Bv\|$.

PROOF: By a standard argument [Ko] we see $g_\Lambda(\xi) \rightarrow g$, $Ag_\Lambda(\xi) \rightarrow Ag$, $Bg_\Lambda(\xi) \rightarrow Bg$ in X as $\xi \rightarrow 0$, which proves the lemma. We give a proof for completeness. Since A is nonnegative, one observes

$$t(t + A)^{-1}f = t(A(t + A)^{-1}u) \rightarrow 0 \quad \text{as } t \rightarrow 0$$

for $f = Au \in R(A)$. Since $R(A)$ is dense in X and $\sup_t \|I_A(t)\| < \infty$, we conclude

$$I_A(t)f \rightarrow f \quad \text{in } X \quad \text{as } t \rightarrow 0.$$

A similar observation shows

$$J_A(\tau)f \rightarrow f \quad \text{in } X \quad \text{as } \tau \rightarrow \infty$$

and the same for B . Since all $I_A(t)$, $I_B(s)$, $J_A(t)$, $J_B(s)$ are bounded in $\mathcal{L}(X)$, these convergences for A and B imply that $g_\Lambda(\xi) \rightarrow g$ in X as $\xi \rightarrow 0$. The proofs of $Ag_\Lambda(\xi) \rightarrow Ag$ and $Bg_\Lambda(\xi) \rightarrow Bg$ under $g \in D(A) \cap D(B)$ are parallel, so they are omitted.

2.5. COMMUTATIVITY LEMMA. *Suppose that A and B are nonnegative and resolvent commuting with dense domains and ranges in X . Then*

$$A^z B^w A^u B^v f = B^w A^{z+u} B^v f \quad \text{for } f \in G_\Lambda$$

with $z, w, u, v \in \mathbb{C}$ and $\Lambda = (h, j, k, \ell)$

provided that h, j, k, ℓ are sufficiently large and the largeness only depends on the modulus of the real parts of z, w, u, v .

For the proof we use an integral representation of the complex powers of A and B by their resolvents [Ko, (1.3) and (4.11)]. Since A and B are resolvent commuting, it is not difficult to prove

$$(t + A)^{-1}(s + B)^{-1} = (s + B)^{-1}(t + A)^{-1}, \quad t, s > 0.$$

Applying this commutativity to the integral representation yields the commutativity of complex powers on G_Λ . The proof is straightforward, so we omit the details.

2.6. COROLLARY TO THE THEOREM ON INVERSES. *Assume the hypotheses of the theorem on inverses. Let m be a positive integer. Then*

$$A^m(A+B)^{-m} = (A(A+B)^{-1})^m$$

$$B^m(A+B)^{-m} = (B(A+B)^{-1})^m$$

on an appropriate dense subspace of X . In particular, $A^m(A+B)^{-m}$ and $B^m(A+B)^{-m}$ can be extended to bounded linear operators on X with a bound depending only on a, b, K, m, X .

PROOF: We give a proof for $m = 2$; the proof for general $m \geq 3$ is parallel, so it is omitted. We use the Dore-Venni representation of $(A+B)^{-1}$. Formally for $z \in \mathbb{C}$, $\operatorname{Re} z = c$ with $0 < c < 1$

$$AB^{z-1}(A+B)^{-1}f = AB^{z-1} \int_{c-i\infty}^{c+i\infty} \frac{A^{-w} B^{w-1} f}{2i \sin \pi w} dw$$

$$= B^{z-1} A(A+B)^{-1} f.$$

This calculation is justified by the commutativity lemma for $f \in G_\Lambda$, $\Lambda = (h, j, k, \ell)$ with h, j, k, ℓ sufficiently large. We thus observe

$$\begin{aligned} A^2(A+B)^{-2}f &= A \int_{c-i\infty}^{c+i\infty} \frac{A^{-z}AB^{z-1}(A+B)^{-1}f}{2i \sin \pi z} dz \\ &= A(A+B)^{-1}A(A+B)^{-1}f. \end{aligned}$$

Since G_Λ is dense in X and $A(A+B)^{-1}$ is bounded by the theorem on inverses, $A^2(A+B)^{-2}$ can be extended to a bounded linear operator $(A(A+B)^{-1})^2$. The same argument applies to $B^2(A+B)^{-2}$.

3. Spaces of fractional powers

For $A \in BIP(a)$ let $\hat{D}(A^\alpha)$ be the completion of the domain $D(A^\alpha)$ in the norm $\|A^\alpha u\|$, where $0 < \alpha < 1$. The space $\hat{D}(A^\alpha)$ can be characterized by a complex interpolation space, namely

$$\hat{D}(A^\alpha) = [X, \hat{D}(A)]_\alpha.$$

This follows from the general interpolation theory (see e.g. [Tr], [BB]). For the proof see e.g. [GS1, Proposition 6.1] or [BM]. In this section we compare various norms on $D(A) \cap D(B)$.

3.1. MAIN THEOREM. Suppose that X is ζ -convex. Suppose that $A \in BIP(a, K)$ and $B \in BIP(b, K)$ are resolvent commuting and that $a + b < \pi$. Then for $0 \leq \alpha \leq 1$

$$\begin{aligned} D(A^\alpha) \cap D(B^\alpha) &= D((A+B)^\alpha), \\ \hat{D}(A^\alpha + B^\alpha) &= D(A^\alpha + B^\alpha)^\wedge = \hat{D}((A+B)^\alpha) = [X, \hat{D}(A+B)]_\alpha \end{aligned}$$

and there are constants $C_j = C_j(a, b, \alpha, K, X) > 0$, $j = 1, 2, 3, 4$ such that

$$\begin{aligned} \|A^\alpha u\| + \|B^\alpha u\| &\leq C_1 \|(A^\alpha + B^\alpha)u\| \leq C_2 \|(A+B)^\alpha u\| \leq \\ &\leq C_3 \|u\|_{[X, \hat{D}(A+B)]_\alpha} \leq C_4 (\|A^\alpha u\| + \|B^\alpha u\|) \end{aligned}$$

for all $u \in D(A^\alpha) \cap D(B^\alpha)$.

PROOF: Since the summation lemma implies $A+B \in BIP(a \vee b + \delta, K')$, $\delta > 0$ with some $K' \geq 1$, it follows the identity

$$\hat{D}((A+B)^\alpha) = [X, \hat{D}(A+B)]_\alpha$$

with equivalent norms

$$\|(A+B)^\alpha u\| \quad \text{and} \quad \|u\|_{[X, \hat{D}(A+B)]_\alpha}.$$

Furthermore, since $A^\alpha \in BIP(a\alpha, K)$, $B^\alpha \in BIP(b\alpha, K)$ and $a\alpha + b\alpha < \pi$, by the theorem on inverses we observe that the norms

$$\|A^\alpha u\| + \|B^\alpha u\| \quad \text{and} \quad \|(A^\alpha + B^\alpha)u\|$$

are equivalent on $D(A^\alpha) \cap D(B^\alpha)$.

It remains to prove that $D(A^\alpha + B^\alpha) = D((A + B)^\alpha)$ and

$$(3.1) \quad \|(A + B)^\alpha u\| \leq C \|(A^\alpha + B^\alpha)u\|,$$

$$(3.2) \quad \|(A^\alpha + B^\alpha)u\| \leq C' \|(A + B)^\alpha u\|$$

for all $u \in D(A^\alpha + B^\alpha) = D(A^\alpha) \cap D(B^\alpha)$. Let us show the first inequality (3.1). To prove (3.1) it suffices to show that

$$(3.3) \quad \|(A + B)^\alpha (A^\alpha + B^\alpha)^{-1} v\| \leq C \|v\|$$

for all v belonging to an appropriate dense subspace of X . Let G_Λ be as in the density lemma with $\Lambda = (h, j, k, \ell)$. For sufficiently large h, j, k, ℓ the function

$$F(z) = e^{z^2} (A + B)^z (A^\alpha + B^\alpha)^{-z/\alpha} v, \quad v \in G_\Lambda$$

is holomorphic in a neighborhood of $0 \leq \operatorname{Re} z \leq 1$. Since $A + B \in BIP(a \vee b + \delta, K')$ and $A^\alpha + B^\alpha \in BIP((a \vee b + \delta)\alpha, K'')$ for all $\delta > 0$ with K', K'' depending on $K, a, b, \delta, \alpha, X$, estimating F on the imaginary axis yields

$$\begin{aligned} \|F(is)\| &\leq e^{-s^2} K' K'' e^{\rho|s|} e^{\rho|s|/\alpha} \|v\|, \quad \rho = a \vee b + \delta \\ &\leq M_0 \|v\| \quad \text{with} \quad M_0 = \sup_{s \in \mathbb{R}} K K'' \exp(\rho|s|(1 + 1/\alpha) - s^2) < \infty, \end{aligned}$$

where δ is now a fixed sufficiently small number. Similarly,

$$\begin{aligned} \|F(1 + is)\| &= e^{1-s^2} \|(A + B)^{is} (A + B)(A^\alpha + B^\alpha)^{-1/\alpha} (A^\alpha + B^\alpha)^{-is/\alpha} v\| \\ &\leq e^{1-s^2} K' K'' e^{\rho|s|} e^{\rho|s|/\alpha} \|(A + B)(A^\alpha + B^\alpha)^{-1/\alpha}\| \|v\| \\ &\leq e M_0 \|(A + B)(A^\alpha + B^\alpha)^{-1/\alpha}\| \|v\|. \end{aligned}$$

If $A(A^\alpha + B^\alpha)^{-1/\alpha}$ and $B(A^\alpha + B^\alpha)^{-1/\alpha}$ can be extended to bounded operators in X with

$$(3.4) \quad \|A(A^\alpha + B^\alpha)^{-1/\alpha}\| \leq c, \quad \|B(A^\alpha + B^\alpha)^{-1/\alpha}\| \leq c,$$

then

$$\|F(1 + is)\| \leq M_1 \|v\|, \quad M_1 = 2e M_0 c.$$

Applying the three line theorem [RS, p.33] yields

$$\|F(\alpha)\| \leq M_0^{1-\alpha} M_1^\alpha \|v\|, \quad v \in G_\Lambda.$$

This deduces (3.3), $D(A^\alpha + B^\alpha) \subset D((A + B)^\alpha)$ and (3.1) with $C = e^{-\alpha^2} M_0^{1-\alpha} M_1^\alpha$ since G_Λ is dense in X . The inequalities (3.4) are proved in the next lemma.

To prove the converse direction (3.2) we need that $A^\alpha(A + B)^{-\alpha}$ and $B^\alpha(A + B)^{-\alpha}$ extend to bounded operators in X , this is also proved in the next lemma. Similarly as above we then obtain $D((A + B)^\alpha) \subset D(A^\alpha + B^\alpha)$,

$$\|A^\alpha u\| + \|B^\alpha u\| \leq C \|(A + B)^\alpha u\|, \quad u \in D((A + B)^\alpha);$$

this implies (3.2) and the proof is complete.

3.2. LEMMA. Assume the hypotheses of the theorem on inverses.

(i) For $\sigma > 0$ the operators $A^\sigma(A+B)^{-\sigma}$ and $B^\sigma(A+B)^{-\sigma}$ can be extended to bounded linear operators in X with a bound depending only on a, b, K, σ, X .

(ii) For $0 < \alpha < 1$ the operators $A(A^\alpha + B^\alpha)^{-1/\alpha}$ and $B(A^\alpha + B^\alpha)^{-1/\alpha}$ can be extended to bounded linear operators in X with a bound depending only on a, b, K, α, X .

PROOF: Part (ii) follows from (i) by setting $A = A^\alpha, B = B^\alpha, \sigma = 1/\alpha$ so it remains to prove (i). In the corollary to the theorem on inverses, we have proved (i) when σ is a positive integer. For general σ we again appeal to the three line theorem. Let m be a nonnegative integer. If we take an appropriate dense subspace G_Λ of X , the function

$$H(z) = e^{z^2} A^{m+z} (A+B)^{-(m+z)} v, \quad v \in G_\Lambda$$

is holomorphic in a neighborhood of $0 \leq \operatorname{Re} z \leq 1$. Since $A+B \in BIP(a \vee b + \delta, K')$ for all $\delta > 0$ with some $K' = K'(K, a, b, \delta, \alpha, X)$, estimating on the imaginary axis yields

$$\|H(is)\| \leq e^{-s^2} K e^{a|s|} \|A^m(A+B)^{-m}\| K' e^{\rho|s|} \|v\|$$

with $\rho = a \vee b + \delta$, where δ is a fixed sufficiently small number. By the corollary to the theorem on inverses, $\|A^m(A+B)^{-m}\|$ is bounded by c_m ; we now observe

$$\|H(is)\| \leq c_m L \|v\|, \quad L = \sup_{s \in \mathbb{R}} K K' \exp(-s^2 + (a + \rho)|s|) < \infty.$$

Similarly, on $\operatorname{Re} z = 1$ we have

$$\|H(1+is)\| \leq c_{m+1} L e \|v\|.$$

Applying the three line theorem yields

$$\|H(\tau)\| \leq M \|v\|, \quad M = c_m^{1-\tau} c_{m+1}^\tau e^\tau L < \infty, \quad v \in G_\Lambda.$$

Since G_Λ is dense in X , we now obtain

$$\|A^{m+\tau}(A+B)^{-(m+\tau)}\| \leq e^{-\tau^2} M, \quad 0 < \tau < 1.$$

The proof for $B^\sigma(A+B)^{-\sigma}$ is parallel, so is omitted.

4. Application to the Stokes system.

Although our abstract result applies to a very general class of evolution equations (1.2), we consider here as an example only the Stokes system (1.3) on some domain Ω in $\mathbb{R}^n$.

Assumptions on the domain Ω .

In the following let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be either the whole space $\mathbb{R}^n$, a bounded domain, a halfspace or an exterior domain. The boundary $\partial\Omega$ is always assumed at least of class $C^{2+\mu}$ with $0 < \mu < 1$. If Ω is an exterior domain we suppose $n \geq 3$.

Stokes operator.

For $1 < q < \infty$ let L^q_σ denote the L^q closure of the space $C_{0,\sigma}^\infty$ of all smooth divergence-free vector fields with compact support in Ω . Let $P = P_q$ denote the projection operator from $L^q = (L^q(\Omega))^n$ to L^q_σ associated with the Helmholtz decomposition. The *Stokes operator* A_q is defined in L^q_σ by $A_q = -P\Delta$ with the dense domain

$$D(A_q) = \{u \in L^q_\sigma; \nabla^2 u \in L^q, u|_{\partial\Omega} = 0\},$$

where Δ denotes the Laplacian and $\nabla^2 u$ denotes the tensor of all second order derivatives. In [G] and [GS1] it is shown that for all $0 < a < \pi/2$, $A_q \in BIP(a, K)$ with K depending on a . For more information on the Stokes operator and the Helmholtz decomposition we refer to [GS1, 2] and [BM] and the references cited there.

Evolution equation.

Applying the projection P_q to the Stokes system (1.3), one formally obtains its abstract form

$$(4.1) \quad \frac{du}{dt} + A_q u = f \quad \text{in } (0, T), \quad u(0) = 0.$$

For $1 < r < \infty$, $0 < T \leq \infty$ let B denote the derivative operator on $X = L^r(0, T; L^q_\sigma)$ defined by $B = d/dt$ (weak derivative) with

$$D(B) = \{u \in X; du/dt \in X, u(0) = 0\}.$$

The operator A in X is defined by $(Au)(t) = A_q u(t)$ for a.e. $t \in (0, T)$ where

$$u \in D(A) = \{u \in X; u(t) \in D(A_q) \text{ for a.e. } t \in (0, T) \\ \text{and } \int_0^T \|A_q u(t)\|_q^r dt < \infty\}.$$

Using A and B we may rewrite (4.1) as

$$(4.2) \quad Bu + Au = f.$$

The space X is ζ -convex because L^q_σ is ζ -convex; see [GS2] and the references cited there. As shown in [DV] for each $\delta > 0$ the operator $B \in BIP(\pi/2 + \delta, K)$ with K depending on δ but independent of T , $0 < T \leq \infty$. The property $A_q \in BIP(a, K)$ yields $A \in BIP(a, K)$, where a is arbitrary $0 < a < \pi/2$ and K depends on a but is independent of T . Clearly, A and B are resolvent commuting. Applying the extended Dore-Venni theorem in [GS2] one observes that there is a unique solution $u \in D(A+B)^\wedge$ of (4.2) for each $f \in X$. If $T < \infty$, B^{-1} exists as a bounded operator so that

$$D(A+B)^\wedge = D(A) \cap D(B).$$

For $0 < T < \infty$ we call $u : (0, T) \rightarrow L^q_\sigma$ a *strong solution* of (4.2) if it satisfies (4.2) with $u \in D(A) \cap D(B)$. In case $T = \infty$ we call $u : (0, \infty) \rightarrow L^q_\sigma$ a *strong solution* if so is u

on each finite time interval $(0, T)$.

Generalized solutions.

In order to apply our abstract Theorem 3.1 to (4.2) we have to consider generalized solutions u of (4.1) for a class of distributions f . This is caused by the fractional powers $(B + A)^\alpha$. For simplicity we will avoid here the definition via test functions and prefer the definition via regularization. Roughly speaking, u is a generalized solution of (4.1) if the "regularization" $A_q^{-\alpha}u$ is a strong solution of (4.1) with f replaced by $A_q^{-\alpha}f$.

Let us give a precise definition. For $0 < \alpha < 1$ the space $D(A_q^{-\alpha}) = R(A_q^\alpha)$ is equipped with the norm $\|A_q^{-\alpha}u\|_q$ and $\hat{D}(A_q^{-\alpha})$ denotes the completion of $D(A_q^{-\alpha})$ under this norm. For $v = (v_j)_{j=1}^\infty \in \hat{D}(A_q^{-\alpha})$ we define $A_q^{-\alpha}v = (A_q^{-\alpha}v_j)$ and get $A_q^{-\alpha}v \in L_q^q$ for each $v \in \hat{D}(A_q^{-\alpha})$; $A_q^{-\alpha}v$ is called the regularization of $v \in \hat{D}(A_q^{-\alpha})$. In the case $T < \infty$ we say $u \in L^r(0, T; D(A_q^{1-\alpha}))$ is a *generalized solution* of (4.1) with $f \in L^r(0, T; \hat{D}(A_q^{-\alpha}))$ if $A_q^{-\alpha}u$ solves (4.2) as a strong solution with f replaced by $A_q^{-\alpha}f \in L^r(0, T; L_q^q)$. If $u : (0, \infty) \rightarrow D(A_q^{1-\alpha})$ is a generalized solution of (4.1) on each finite time interval $(0, T)$, u is called a *generalized solution* in case $T = \infty$.

4.1. UNIQUE EXISTENCE OF GENERALIZED SOLUTIONS. Let Ω be as above, $0 < T < \infty$, $1 < r < \infty$, $1 < q < \infty$, $0 < \alpha < 1$. Suppose $f \in L^r(0, T; \hat{D}(A_q^{-\alpha}))$. Then there exists a unique generalized solution $u \in L^r(0, T; D(A_q^{1-\alpha}))$ of (4.1). Moreover, $u \in D(B^{1-\alpha})$ and

$$(4.3) \quad \int_0^T \|(\frac{d}{dt})^{1-\alpha}u\|_q^r dt + \int_0^T \|A_q^{1-\alpha}u\|_q^r dt \leq C \int_0^T \|A_q^{-\alpha}f\|_q^r dt$$

with $C = C(\Omega, q, r, \alpha) > 0$ independent of T and f where $(d/dt)^{1-\alpha} = B^{1-\alpha}$.

REMARKS: a) The condition $u(0) = 0$ is implicitly contained in $u \in D(B^{1-\alpha})$ for small α (i.e. $0 < \alpha < 1 - 1/r$) while no condition is imposed on $u(0)$ for large α (i.e. $1 - 1/r < \alpha < 1$).

b) The case $T = \infty$ can be admitted in Theorem 4.1 if we replace $D(A_q^{1-\alpha})$ by $\hat{D}(A_q^{1-\alpha})$ and $D(B^{1-\alpha})$ by $\hat{D}(B^{1-\alpha})$. In this case (4.3) is

$$\int_0^\infty \|(\frac{d}{dt})^{1-\alpha}u\|_q^r dt + \int_0^\infty \|A_q^{1-\alpha}u\|_q^r dt \leq C \int_0^\infty \|A_q^{-\alpha}f\|_q^r dt$$

which yields asymptotic properties of u as $t \rightarrow \infty$.

c) Of course, this theorem extends to the class of all evolution equations for which Theorem 3.1 is applicable.

PROOF: We apply the extended Dore-Venni theorem in [GS2] to $A_q^{-\alpha}f \in X$ and obtain a unique solution $v \in D(B) \cap D(A)$ of $Bv + Av = A_q^{-\alpha}f$. The function $u = A_q^\alpha v$ is a generalized solution of (4.1) since $A_q^{-\alpha}u$ is a strong solution; the uniqueness of u is obvious.

To prove (4.3) we use the Yosida approximation $J_m = J_A(m) = m(m + A)^{-1}$ in Section 2 and obtain

$$\begin{aligned} BA^{-\alpha}J_mu + AA^{-\alpha}J_mu &= A^{-\alpha}(BJ_mu + AJ_mu) = A^{-\alpha}J_mf \\ BJ_mu + AJ_mu &= J_mf. \end{aligned}$$

Here J_mf is defined in the same way as $A_q^{-\alpha}f$. We know that $\lim_{m \rightarrow \infty} J_mu = u$ in $X = L^r(0, T; L^q_\sigma)$. Setting $u_m = J_mu$, $w = u_m - u$ and applying Theorem 3.1 yields

$$\begin{aligned} \|B^{1-\alpha}w\|_X + \|A^{1-\alpha}w\|_X &\leq C\|(B + A)^{1-\alpha}w\|_X \\ &= C\|(B + A)^{-\alpha}(B + A)w\|_X = C\|A^\alpha(B + A)^{-\alpha}(J_m - J_\infty)A^{-\alpha}f\|_X \\ &\leq C'\|(J_m - J_\infty)A^{-\alpha}f\|_X; \end{aligned}$$

here we used the fact that $A^\alpha(B + A)^{-\alpha}$ is bounded by Lemma 3.2. From this estimate we conclude $u \in D(B^{1-\alpha}) \cap D(A^{1-\alpha})$ since $B^{1-\alpha}$ and $A^{1-\alpha}$ are closed and $u \in X$. The same estimate with w replaced by u_m yields (4.3) by letting $m \rightarrow \infty$. This proves 4.1.

We next consider some concrete cases of distributions f in Theorem 4.1. In case a) of the following Corollary we consider a distribution of the form $f = \sum_{\nu=1}^n \partial_\nu f_\nu$ with $f_\nu \in X$ and $\partial_\nu = \partial/\partial x_\nu$, and in b) we let $f \in L^r(0, T; L^q_\sigma)$ with some exponent γ different from q .

4.2. COROLLARY. Suppose Ω as above and $0 < T < \infty$, $1 < q < \infty$, $1 < r < \infty$.

a) Let $f = \sum_{\nu=1}^n \partial_\nu f_\nu$ with $f_\nu \in X = L^r(0, T; L^q_\sigma)$, $\nu = 1, \dots, n$. If Ω is unbounded, suppose additionally $q > n/(n-1)$, $n \geq 3$. Then $A_q^{-1/2}f \in X$, $f \in L^q(0, T; \hat{D}(A_q^{-1/2}))$. There exists a unique generalized solution $u \in L^r(0, T; D(A_q^{1/2}))$ of (4.1) with $u \in D(B^{1/2})$ and

$$(4.4) \quad \int_0^T \|(\frac{d}{dt})^{1/2}u\|_q^r dt + \int_0^T \|A_q^{1/2}u\|_q^r dt \leq C \sum_{\nu=1}^n \int_0^T \|f_\nu\|_q^r dt$$

with $C = C(\Omega, q, r)$ independent of f and T .

b) For $1 < \alpha < 1$ let γ be defined by $2\alpha + n/q = n/\gamma$ and $f \in L^r(0, T; L^q_\sigma)$. If Ω is an exterior domain, suppose additionally $1 < \gamma < n/2$, $n \geq 3$. Then $A_q^{-\alpha}f \in L^r(0, T; L^q_\sigma)$, $f \in L^r(0, T; \hat{D}(A_q^{-\alpha}))$. There exists a unique generalized solution $u \in L^r(0, T; D(A_q^{1-\alpha}))$ of (4.1) with $u \in D(B^{1-\alpha})$ and

$$(4.5) \quad \int_0^T \|(\frac{d}{dt})^{1-\alpha}u\|_q^r dt + \int_0^T \|A_q^{1-\alpha}u\|_q^r dt \leq C \int_0^T \|f\|_\gamma^r dt$$

with $C = C(\Omega, q, r, \alpha)$ independent of f and T .

REMARKS: (i) To prove a) and b) it suffices to prove that $f \in L^r(0, T; \hat{D}(A_q^{-1/2}))$ and

$$\|A_q^{-1/2}f\|_X \leq C \sum_{\nu=1}^n \|f_\nu\|_X$$

in a) and that $f \in L^r(0, T; \hat{D}(A_q^{-\alpha}))$ and

$$\|A_q^{-\alpha} f\|_X \leq C \left(\int_0^T \|f\|_\gamma^r dt \right)^{1/r}$$

in b) respectively with C independent of f and T .

(ii) The estimate (4.4) yields (1.6) by applying of

$$\|\nabla u\|_q \leq C \|A_q^{1/2} u\|_q$$

which needs additionally the restriction $1 < q < n$, $n \geq 3$ when Ω is an exterior domain ([BM], [GS1]).

PROOF: a) In [GS1, p.123] it has been shown that $C_{0,\sigma}^\infty \subset R(A_q)$ if $q > n/(n-2)$ and Ω is the $\mathbb{R}^n$ or an exterior domain; the same proof works also for the half-space and the restriction becomes $q > n/(n-1)$ if A_q is replaced by $A_q^{1/2}$. If Ω is bounded, no restriction is needed.

So for each f_ν ($\nu = 1, 2, \dots, n$) we find a sequence $(f_{\nu j})_{j=1}^\infty$ in $L^r(0, T; C_{0,\sigma}^\infty) \subset L^r(0, T; D(A_q^{-1/2}))$ with $f_\nu = \lim_{j \rightarrow \infty} f_{\nu j}$ in $L^r(0, T; L_\sigma^q)$. It follows that $(\tilde{f}_j) = (\sum_{\nu=1}^n \partial_\nu f_{\nu j})$ is a sequence in $L^r(0, T; D(A_q^{-1/2}))$.

We next use the estimate

$$\|A_q^{-1/2} \nabla u\|_q \leq C \|u\|_q$$

(see [BM], [GS1]) which is valid in all cases for Ω but in exterior domains under the restriction $q > n/(n-1)$; observe that this estimate is equivalent to $\|\nabla u\|_{q'} \leq C \|A_q^{1/2} u\|_{q'}$, where by duality the restriction is now given by $1 < q' < n$. This leads to

$$\|A_q^{-1/2}(\tilde{f}_i - \tilde{f}_j)\|_X = \left\| \sum_{\nu=1}^n A_q^{-1/2} \partial_\nu (f_{\nu i} - f_{\nu j}) \right\|_X \leq C \sum_{\nu=1}^n \|f_{\nu i} - f_{\nu j}\|_X$$

which yields $f \in L^r(0, T; \hat{D}(A_q^{-1/2}))$. This estimate also yields

$$\|A_q^{-1/2} f\|_X \leq C \sum_{\nu=1}^n \|f_\nu\|_X$$

so Theorem 4.1 is applicable.

b) Since $R(A_q^\alpha) \subset L_\sigma^\gamma$ is dense in L_σ^γ , one can choose $f_j \in L^r(0, T; D(A_q^{-\alpha}))$, $j = 1, 2, \dots$ with $f = \lim_{j \rightarrow \infty} f_j$ in $L^r(0, T; L_\sigma^\gamma)$. Then we use the estimate

$$\|A_q^{-\alpha} u\|_q \leq C \|u\|_\gamma$$

in [GS1, p.104] which holds for $2\alpha + n/q = n/\gamma$; in exterior domains the restriction $1 < \gamma < n/2$, $n \geq 3$ is needed. This leads to

$$\|A_q^{-\alpha}(f_i - f_j)\|_X \leq C \left(\int_0^T \|f_i - f_j\|_\gamma^r dt \right)^{1/r}$$

which yields $f \in L^r(0, T; \hat{D}(A_q^{-\alpha}))$ and

$$\|A_q^{-\alpha} f\| \leq C \left(\int_0^T \|f\|_r^r dt \right)^{1/r},$$

so Theorem 4.1 is applicable.

Further applications. The estimates above can be applied to weak solutions of the nonlinear Navier-Stokes equations if we take the nonlinear term to the right hand side in (4.1). The procedure is completely analogous to that in [GS2].

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