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## $L^p$ estimates for the Stokes system

MARIKO GIGA, YOSHIKAZU GIGA  
AND HERMANN SOHR

### 1. Introduction.

This paper investigates the fractional powers  $(A + B)^\alpha$ ,  $0 \leq \alpha \leq 1$  of the sum  $A + B$  of two closed (resolvent commuting) operators  $A$  and  $B$  in a  $\zeta$ -convex Banach space  $X$ . We compare the domain  $D((A + B)^\alpha)$  of  $(A + B)^\alpha$  with the domain  $D(A^\alpha + B^\alpha) = D(A^\alpha) \cap D(B^\alpha)$  of the sum  $A^\alpha + B^\alpha$  and show in particular the relation

$$(1.1) \quad D((A + B)^\alpha) = D(A^\alpha) \cap D(B^\alpha)$$

with equivalent norms  $\|(A + B)^\alpha u\|$  and  $\|A^\alpha u\| + \|B^\alpha u\|$ , under some assumptions on the pure imaginary powers of  $A$  and  $B$ .

Our results will be applied to  $L^p$  estimates for (generalized) solutions of the evolution equation

$$(1.2) \quad \frac{du}{dt} + Au = f \quad \text{in } (0, T), \quad 0 < T \leq \infty, \quad u(0) = 0.$$

Here we restrict ourselves to the Stokes operator  $A = A_q$ . Formally, we get such an equation if we apply the  $L^q$  Helmholtz projection  $P_q$  to the Stokes system

$$(1.3) \quad \begin{aligned} \frac{\partial u}{\partial t} - \Delta u + \nabla p &= f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega \times (0, T) \\ u|_{\partial\Omega} &= 0 \quad \text{on } \partial\Omega \times (0, T), \quad u = 0 \quad \text{on } \Omega \quad \text{at } t = 0, \end{aligned}$$

where  $\Omega$  is a domain in  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$  and  $1 < q < \infty$ ; see e.g. [GS1,2] for notations.

Our perturbation result is based on a theory recently developed by Dore and Venni [DV] which has been extended by Giga and Sohr [GS2] to the case that the inverse operators  $A^{-1}$  and  $B^{-1}$  need not be bounded. See also [PS] for another proof. The original theory of Dore and Venni is applicable to the evolution equation (1.2) only for a finite interval  $[0, T]$ ; it yields a constant  $C = C(\Omega, r, q, T) > 0$  such that

$$(1.4) \quad \int_0^T \left\| \frac{du}{dt} \right\|_q^r dt + \int_0^T \|A_q u\|_q^r dt \leq C \int_0^T \|f\|_q^r dt,$$

where  $1 < r, q < \infty$  and  $\|\cdot\|_q$  denotes the  $L^q(\Omega)$ -norm. The extension by [GS2] strengthens the estimate (1.4) so that  $A_q^{-1}$  is allowed to be unbounded and that  $C$  may be chosen independently of  $T$ . Therefore, one may take  $T = \infty$  in (1.4) which yields asymptotic properties of the solution  $u$  of (1.3) as  $t \rightarrow \infty$  even when  $\Omega$  need not be bounded [GS2]. In [GS2] the estimates applied to the nonlinear Navier-Stokes system. In [GGS] the estimate (1.4) has been extended to the case that  $A = A(t)$  in (1.2) depends on  $t$ , and in [GS2] and [GGS] non zero initial values  $u(0) = u_0$  are treated.

Recently Dore and Venni [DV2] applied their theory to get higher derivative estimates for solutions of (1.2).

The application of our abstract result (1.1) on fractional powers  $(A + B)^\alpha$  to the evolution equation (1.2) yields now estimates of the form

$$(1.5) \quad \int_0^T \left\| \left( \frac{d}{dt} \right)^{1-\alpha} u \right\|_q^r dt + \int_0^T \|A_q^{1-\alpha} u\|_q^r dt \leq C \int_0^T \|A_q^{-\alpha} f\|_q^r dt$$

with  $C$  independent of  $f$  and  $T$ , and  $0 < \alpha < 1$ . Here  $u$  is a generalized solution of (1.2) and  $f$  may be a distribution which is regularized by  $A_q^{-\alpha}$ . The case  $\alpha = 1/2$  is especially important because (1.5) yields an a priori estimate

$$(1.6) \quad \int_0^T \left\| \left( \frac{d}{dt} \right)^{1/2} u \right\|_q^r dt + \int_0^T \|\nabla u\|_q^r dt \leq C \int_0^T \|F\|_q^r dt$$

for solutions of (1.3) when  $f = \operatorname{div} F$ ; here we restrict  $n \geq 3$  and  $n/(n-1) < q < n$  when  $\Omega$  is an exterior domain. This estimate is considered as a nonstationary version of Cattabriga's estimate (see e.g. [BM]).

The class  $BIP(a, K)$  of operators we consider here consists of nonnegative closed operators  $A$  in  $X$  which satisfy the estimate  $\|A^{is} u\|_X \leq K e^{a|s|} \|u\|_X$  for all  $s \in \mathbb{R}$  where  $K \geq 1$  and  $0 \leq a < \pi$  (independent of  $u$  and  $s$ ). The well known application of this estimate of the pure imaginary powers  $A^{is}$  is the identification

$$[X, D(A)]_\alpha = D(A^\alpha),$$

where  $[X, D(A)]_\alpha$  is the complex interpolation space; see e.g. [Tr]. The Dore-Venni theory gives now another important application of the above estimate. This theory requires the  $\zeta$ -convexity of the Banach space. For various properties of  $\zeta$ -convex space we refer to the nice review article [B]. For the theory of complex powers  $A^z$ ,  $z \in \mathbb{C}$  we refer to the comprehensive article [Ko].

Our main abstract result is given in Section 3; Section 2 contains preliminary lemmas and Section 4 is devoted to the application to the Stokes system.

## 2. Sum of operators with bounded imaginary powers.

Let  $A$  be a closed linear operator with dense domain  $D(A)$  in a Banach space  $X$  equipped with norm  $\|\cdot\|$ . We say  $A$  is *nonnegative* if its resolvent set contains all negative real numbers and

$$\sup_{t>0} t \|(t + A)^{-1}\| < \infty,$$

where  $\|\cdot\|$  denotes the operator norm in  $\mathcal{L}(X)$ , the space of all bounded linear operators. If a nonnegative operator has a dense range  $R(A)$  in  $X$ , one can define its complex power  $A^z$  for every  $z \in \mathbb{C}$  as a densely defined closed operator in  $X$ . (cf. [Ko]). For  $a \geq 0$  and  $K \geq 1$  we say a nonnegative operator  $A$  belongs to  $BIP(a; K)$  if  $A^{is} \in \mathcal{L}(X)$  and is estimated as

$$\|A^{is}\| \leq K e^{a|s|}, \quad s \in \mathbb{R}$$

where  $D(A)$  and  $R(A)$  are assumed to be dense in  $X$ . Let  $BIP(a)$  denote the union of  $BIP(a, K)$  for  $K \geq 1$ .

2.1. FUNDAMENTAL LEMMA. (i) If  $A \in BIP(a; K)$ , then  $A^\alpha \in BIP(a\alpha; K)$  for  $0 < \alpha < 1$ .

(ii) If  $A \in BIP(a)$ ,  $0 \leq a < \pi$ , then for each  $\delta > 0$  with  $\delta < \pi - a$  there is a constant  $M_\delta$  independent of  $\lambda$  such that

$$\|(\lambda + A)^{-1}\| \leq M_\delta/|\lambda|, \quad |\arg \lambda| \leq \pi - a - \delta, \quad 0 \neq \lambda \in \mathbb{C}.$$

In particular, if  $a < \pi/2$ , then  $-A$  generates an analytic semigroup  $e^{-tA}$  in  $X$ .

PROOF: (i) As well known, if  $A$  is nonnegative so is  $A^\alpha$  ( $0 < \alpha < 1$ ); see e.g. [Kr, p.119, (5.25)] or [Ka]. If  $A \in BIP(a)$ , then  $A^\alpha \in BIP(a\alpha)$  since

$$\|(A^\alpha)^{i\epsilon}\| = \|A^{i\alpha\epsilon}\| \leq Ke^{a\alpha|\epsilon|}.$$

Here we use the property  $(A^\alpha)^{i\epsilon} = A^{i\alpha\epsilon}$  which can be shown as follows. First we prove this property with  $A$  replaced by  $(\epsilon + A)^{-1}$ ,  $\epsilon > 0$ ; here we use the well known Dunford integral calculus. Then the assertion follows by letting  $\epsilon \rightarrow 0$  and using [PS, Theorem 3].

(ii) See [PS, Theorem 2].

2.2. SUMMATION LEMMA. Let  $X$  be a  $\zeta$ -convex Banach space. Let  $A$  and  $B$  belong to  $BIP(a, K)$  and  $BIP(b, K)$ , respectively. Suppose that  $A$  and  $B$  are resolvent commuting, i.e.,

$$(t + A)^{-1}(t + B)^{-1} = (t + B)^{-1}(t + A)^{-1} \quad \text{for all } t > 0.$$

Then  $A + B \in BIP(a \vee b, K')$  provided that  $a \neq b$ , where  $a \vee b = \max(a, b)$  and  $K' = K'(a, b, K, X)$ .

This is Theorem 5 in [PS], where the dependence of constants is not explicitly stated. For various properties of  $\zeta$ -convex spaces there is the nice review article by Burkholder [B] so we do not touch them here.

We next recall the Dore-Venni theory [DV] on the inverse of  $A + B$ . Let  $T$  be an injective closed linear operator in a Banach space  $X$ . Let  $\hat{D}(T)$  be the completion of  $D(T)$  in the norm  $\|Tu\|$ . Since  $T$  may not have a bounded inverse,  $\hat{D}(T)$  may not be a subspace of  $X$ . The element  $Tv \in X$  for  $v \in \hat{D}(T)$  is defined by  $Tv = \lim_{j \rightarrow \infty} Tv_j$ , where  $\{v_j\}$  is a Cauchy sequence converging to  $v$  in  $\hat{D}(T)$ . The norm of  $v$  in  $\hat{D}(T)$  is defined by

$$\|v\|_{\hat{D}(T)} = \|Tv\| = \lim_{j \rightarrow \infty} \|Tv_j\|.$$

Let  $T'$  be another injective closed linear operator in  $X$ . Let  $T + T'$  be the operator defined on  $D(T + T') = D(T) \cap D(T')$ . By  $D(T + T')^\wedge$  we represent the completion of  $D(T + T')$  in the norm  $\|Tu\| + \|T'u\|$ . Clearly, this space is continuously embedded in  $\hat{D}(T)$  and  $\hat{D}(T')$ . However, the intersection  $\hat{D}(T) \cap \hat{D}(T')$  is not meaningful unless the norms  $\|Tv\|$  and  $\|T'v\|$  are consistent in the sense of the interpolation theory [RS, p.35]. Note that  $D(T + T')^\wedge$  need not be equal to  $\hat{D}(T + T')$ .

**2.3. THEOREM ON INVERSES.** Let  $X$  be  $\zeta$ -convex. Suppose that  $A \in BIP(a; K)$  and  $B \in BIP(b; K)$  are resolvent commuting and that  $a + b < \pi$ . Then the operator  $A + B : D(A + B)^\wedge \rightarrow X$  is bijective and boundedly invertible. Moreover there is  $C = C(a, b, K, X)$  such that

$$\|A(A + B)^{-1}\| \leq C, \|B(A + B)^{-1}\| \leq C.$$

**REMARK:** Observe as a consequence that  $\|Au\| + \|Bu\|$  and  $\|(A + B)u\|$  are equivalent norms on  $D(A) \cap D(B)$  so that  $D(A + B)^\wedge = \hat{D}(A + B)$ .

This result was first proved by Dore and Venni [DV] under the assumption that both  $A$  and  $B$  have bounded inverses. The key observation is the following integral representation

$$(A + B)^{-1} = \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} \frac{A^{-z} B^{z-1}}{\sin \pi z} dz, \quad 0 < c < 1.$$

It turns out that the assumptions  $A^{-1}$  and  $B^{-1} \in \mathcal{L}(X)$  can be removed. The first proof is given by Y. Giga and Sohr [GS2] by introducing appropriate dense subspaces of  $X$  so that the argument in [DV] can be justified without  $A^{-1}, B^{-1} \in \mathcal{L}(X)$ . Another proof is given by Prüss and Sohr [PS]. They established a functional calculus generated by the group  $A^{is}$  and proved that  $A \in BIP(a)$  implies  $A_\varepsilon = \varepsilon I + A \in BIP(a; L)$  with  $L$  independent of  $\varepsilon > 0$ . This is considered as a special case of the summation lemma. Since  $A_\varepsilon$  has a bounded inverse, they applied the Dore-Venni estimate to  $A_\varepsilon$  and sent  $\varepsilon \rightarrow 0$  to get the desired estimates in Theorem 2.3. The first proof is more direct because it does not use the approximated operator  $A_\varepsilon$ .

The injectivity of the operators  $A, B$  is not explicitly assumed. It follows from the fact that these operators are nonnegative and have dense ranges; see [Ko, Theorem 3.2 and 3.7]. Indeed  $Au = 0$  implies  $u = t(t + A)^{-1}u$ , so letting  $t \rightarrow 0$  yields  $u = 0$ .

It is convenient to consider appropriate dense subspaces in  $X$  as in [GS2]. For  $\xi = (\zeta, \eta)$  and  $\Lambda = (h, j, k, \ell)$  with nonnegative integers  $h, j, k, \ell$  we set

$$g_\Lambda(\xi) = I_A^j(t) J_A^h(\tau) I_B^\ell(s) J_B^k(\sigma) g, \quad g \in X$$

$$\zeta = (t, \tau^{-1}), \quad \eta = (s, \sigma^{-1}), \quad t, \tau, s, \sigma > 0$$

with  $I_A(t) = A(t + A)^{-1}$  and  $J_A(\tau) = \tau(\tau + A)^{-1}$ . We introduce the subspace

$$G_\Lambda = \text{linear hull of } \{g_\Lambda(\xi); g \in X, \xi = (t, \tau^{-1}, s, \sigma^{-1}), t, \tau, s, \sigma > 0\}.$$

**2.4. DENSITY LEMMA.** Suppose that  $A$  and  $B$  are nonnegative and resolvent commuting with dense ranges and domains in  $X$ . Then  $G_\Lambda$  is dense in  $X$ . Moreover  $G_\Lambda$  is dense in  $D(A) \cap D(B)$  under the norm  $\|Av\| + \|Bv\|$ .

**PROOF:** By a standard argument [Ko] we see  $g_\Lambda(\xi) \rightarrow g, Ag_\Lambda(\xi) \rightarrow Ag, Bg_\Lambda(\xi) \rightarrow Bg$  in  $X$  as  $\xi \rightarrow 0$ , which proves the lemma. We give a proof for completeness. Since  $A$  is nonnegative, one observes

$$t(t + A)^{-1}f = t(A(t + A)^{-1}u) \rightarrow 0 \quad \text{as } t \rightarrow 0$$

for  $f = Au \in R(A)$ . Since  $R(A)$  is dense in  $X$  and  $\sup_t \|I_A(t)\| < \infty$ , we conclude

$$I_A(t)f \rightarrow f \quad \text{in } X \quad \text{as } t \rightarrow 0.$$

A similar observation shows

$$J_A(\tau)f \rightarrow f \quad \text{in } X \quad \text{as } \tau \rightarrow \infty$$

and the same for  $B$ . Since all  $I_A(t)$ ,  $I_B(s)$ ,  $J_A(t)$ ,  $J_B(s)$  are bounded in  $\mathcal{L}(X)$ , these convergences for  $A$  and  $B$  imply that  $g_\Lambda(\xi) \rightarrow g$  in  $X$  as  $\xi \rightarrow 0$ . The proofs of  $Ag_\Lambda(\xi) \rightarrow Ag$  and  $Bg_\Lambda(\xi) \rightarrow Bg$  under  $g \in D(A) \cap D(B)$  are parallel, so they are omitted.

**2.5. COMMUTATIVITY LEMMA.** *Suppose that  $A$  and  $B$  are nonnegative and resolvent commuting with dense domains and ranges in  $X$ . Then*

$$\begin{aligned} A^z B^w A^u B^v f &= B^w A^{z+u} B^v f \quad \text{for } f \in G_\Lambda \\ &\text{with } z, w, u, v \in \mathbb{C} \text{ and } \Lambda = (h, j, k, l) \end{aligned}$$

provided that  $h, j, k, l$  are sufficiently large and the largeness only depends on the modulus of the real parts of  $z, w, u, v$ .

For the proof we use an integral representation of the complex powers of  $A$  and  $B$  by their resolvents [Ko, (1.3) and (4.11)]. Since  $A$  and  $B$  are resolvent commuting, it is not difficult to prove

$$(t + A)^{-1}(s + B)^{-1} = (s + B)^{-1}(t + A)^{-1}, \quad t, s > 0.$$

Applying this commutativity to the integral representation yields the commutativity of complex powers on  $G_\Lambda$ . The proof is straightforward, so we omit the details.

**2.6. COROLLARY TO THE THEOREM ON INVERSES.** *Assume the hypotheses of the theorem on inverses. Let  $m$  be a positive integer. Then*

$$\begin{aligned} A^m(A + B)^{-m} &= (A(A + B)^{-1})^m \\ B^m(A + B)^{-m} &= (B(A + B)^{-1})^m \end{aligned}$$

on an appropriate dense subspace of  $X$ . In particular,  $A^m(A + B)^{-m}$  and  $B^m(A + B)^{-m}$  can be extended to bounded linear operators on  $X$  with a bound depending only on  $a, b, K, m, X$ .

**PROOF:** We give a proof for  $m = 2$ ; the proof for general  $m \geq 3$  is parallel, so it is omitted. We use the Dore-Venni representation of  $(A + B)^{-1}$ . Formally for  $z \in \mathbb{C}$ ,  $\text{Re } z = c$  with  $0 < c < 1$

$$\begin{aligned} AB^{z-1}(A + B)^{-1}f &= AB^{z-1} \int_{c-i\infty}^{c+i\infty} \frac{A^{-w} B^{w-1} f}{2i \sin \pi w} dw \\ &= B^{z-1} A(A + B)^{-1}f. \end{aligned}$$



This calculation is justified by the commutativity lemma for  $f \in G_A$ ,  $\Lambda = (h, j, k, \ell)$  with  $h, j, k, \ell$  sufficiently large. We thus observe

$$\begin{aligned} A^2(A+B)^{-2}f &= A \int_{c-i\infty}^{c+i\infty} \frac{A^{-z}AB^{z-1}(A+B)^{-1}f}{2i \sin \pi z} dz \\ &= A(A+B)^{-1}A(A+B)^{-1}f. \end{aligned}$$

Since  $G_A$  is dense in  $X$  and  $A(A+B)^{-1}$  is bounded by the theorem on inverses,  $A^2(A+B)^{-2}$  can be extended to a bounded linear operator  $(A(A+B)^{-1})^2$ . The same argument applies to  $B^2(A+B)^{-2}$ .

### 3. Spaces of fractional powers

For  $A \in BIP(a)$  let  $\hat{D}(A^\alpha)$  be the completion of the domain  $D(A^\alpha)$  in the norm  $\|A^\alpha u\|$ , where  $0 < \alpha < 1$ . The space  $\hat{D}(A^\alpha)$  can be characterized by a complex interpolation space, namely

$$\hat{D}(A^\alpha) = [X, \hat{D}(A)]_\alpha.$$

This follows from the general interpolation theory (see e.g. [Tr], [BB]). For the proof see e.g. [GS1, Proposition 6.1] or [BM]. In this section we compare various norms on  $D(A) \cap D(B)$ .

**3.1. MAIN THEOREM.** *Suppose that  $X$  is  $\zeta$ -convex. Suppose that  $A \in BIP(a, K)$  and  $B \in BIP(b, K)$  are resolvent commuting and that  $a + b < \pi$ . Then for  $0 \leq \alpha \leq 1$*

$$\begin{aligned} D(A^\alpha) \cap D(B^\alpha) &= D((A+B)^\alpha), \\ \hat{D}(A^\alpha + B^\alpha) &= D(A^\alpha + B^\alpha)^\wedge = \hat{D}((A+B)^\alpha) = [X, \hat{D}(A+B)]_\alpha \end{aligned}$$

and there are constants  $C_j = C_j(a, b, \alpha, K, X) > 0$ ,  $j = 1, 2, 3, 4$  such that

$$\begin{aligned} \|A^\alpha u\| + \|B^\alpha u\| &\leq C_1 \|(A^\alpha + B^\alpha)u\| \leq C_2 \|(A+B)^\alpha u\| \leq \\ &\leq C_3 \|u\|_{[X, \hat{D}(A+B)]_\alpha} \leq C_4 (\|A^\alpha u\| + \|B^\alpha u\|) \end{aligned}$$

for all  $u \in D(A^\alpha) \cap D(B^\alpha)$ .

**PROOF:** Since the summation lemma implies  $A+B \in BIP(a \vee b + \delta, K')$ ,  $\delta > 0$  with some  $K' \geq 1$ , it follows the identity

$$\hat{D}((A+B)^\alpha) = [X, \hat{D}(A+B)]_\alpha$$

with equivalent norms

$$\|(A+B)^\alpha u\| \quad \text{and} \quad \|u\|_{[X, \hat{D}(A+B)]_\alpha}.$$

Furthermore, since  $A^\alpha \in BIP(a\alpha, K)$ ,  $B^\alpha \in BIP(b\alpha, K)$  and  $a\alpha + b\alpha < \pi$ , by the theorem on inverses we observe that the norms

$$\|A^\alpha u\| + \|B^\alpha u\| \quad \text{and} \quad \|(A^\alpha + B^\alpha)u\|$$

are equivalent on  $D(A^\alpha) \cap D(B^\alpha)$ .

It remains to prove that  $D(A^\alpha + B^\alpha) = D((A + B)^\alpha)$  and

$$(3.1) \quad \|(A + B)^\alpha u\| \leq C \|(A^\alpha + B^\alpha)u\|,$$

$$(3.2) \quad \|(A^\alpha + B^\alpha)u\| \leq C' \|(A + B)^\alpha u\|$$

for all  $u \in D(A^\alpha + B^\alpha) = D(A^\alpha) \cap D(B^\alpha)$ . Let us show the first inequality (3.1). To prove (3.1) it suffices to show that

$$(3.3) \quad \|(A + B)^\alpha (A^\alpha + B^\alpha)^{-1} v\| \leq C \|v\|$$

for all  $v$  belonging to an appropriate dense subspace of  $X$ . Let  $G_\Lambda$  be as in the density lemma with  $\Lambda = (h, j, k, \ell)$ . For sufficiently large  $h, j, k, \ell$  the function

$$F(z) = e^{z^2} (A + B)^z (A^\alpha + B^\alpha)^{-z/\alpha} v, \quad v \in G_\Lambda$$

is holomorphic in a neighborhood of  $0 \leq \operatorname{Re} z \leq 1$ . Since  $A + B \in BIP(a \vee b + \delta, K')$  and  $A^\alpha + B^\alpha \in BIP((a \vee b + \delta)\alpha, K'')$  for all  $\delta > 0$  with  $K', K''$  depending on  $K, a, b, \delta, \alpha, X$ , estimating  $F$  on the imaginary axis yields

$$\begin{aligned} \|F(is)\| &\leq e^{-s^2} K' K'' e^{\rho|s|} e^{\rho|s|/\alpha} \|v\|, \quad \rho = a \vee b + \delta \\ &\leq M_0 \|v\| \quad \text{with} \quad M_0 = \sup_{s \in \mathbb{R}} K' K'' \exp(\rho|s|(1 + 1/\alpha) - s^2) < \infty, \end{aligned}$$

where  $\delta$  is now a fixed sufficiently small number. Similarly,

$$\begin{aligned} \|F(1 + is)\| &= e^{1-s^2} \|(A + B)^{is} (A + B)(A^\alpha + B^\alpha)^{-1/\alpha} (A^\alpha + B^\alpha)^{-is/\alpha} v\| \\ &\leq e^{1-s^2} K' K'' e^{\rho|s|} e^{\rho|s|/\alpha} \|(A + B)(A^\alpha + B^\alpha)^{-1/\alpha}\| \|v\| \\ &\leq e M_0 \|(A + B)(A^\alpha + B^\alpha)^{-1/\alpha}\| \|v\|. \end{aligned}$$

If  $A(A^\alpha + B^\alpha)^{-1/\alpha}$  and  $B(A^\alpha + B^\alpha)^{-1/\alpha}$  can be extended to bounded operators in  $X$  with

$$(3.4) \quad \|A(A^\alpha + B^\alpha)^{-1/\alpha}\| \leq c, \quad \|B(A^\alpha + B^\alpha)^{-1/\alpha}\| \leq c,$$

then

$$\|F(1 + is)\| \leq M_1 \|v\|, \quad M_1 = 2e M_0 c.$$

Applying the three line theorem [RS, p.33] yields

$$\|F(\alpha)\| \leq M_0^{1-\alpha} M_1^\alpha \|v\|, \quad v \in G_\Lambda.$$

This deduces (3.3),  $D(A^\alpha + B^\alpha) \subset D((A + B)^\alpha)$  and (3.1) with  $C = e^{-\alpha^2} M_0^{1-\alpha} M_1^\alpha$  since  $G_\Lambda$  is dense in  $X$ . The inequalities (3.4) are proved in the next lemma.

To prove the converse direction (3.2) we need that  $A^\alpha(A + B)^{-\alpha}$  and  $B^\alpha(A + B)^{-\alpha}$  extend to bounded operators in  $X$ , this is also proved in the next lemma. Similarly as above we then obtain  $D((A + B)^\alpha) \subset D(A^\alpha + B^\alpha)$ ,

$$\|A^\alpha u\| + \|B^\alpha u\| \leq C \|(A + B)^\alpha u\|, \quad u \in D((A + B)^\alpha);$$

this implies (3.2) and the proof is complete.

**3.2. LEMMA.** Assume the hypotheses of the theorem on inverses.

(i) For  $\sigma > 0$  the operators  $A^\sigma(A+B)^{-\sigma}$  and  $B^\sigma(A+B)^{-\sigma}$  can be extended to bounded linear operators in  $X$  with a bound depending only on  $a, b, K, \sigma, X$ .

(ii) For  $0 < \alpha < 1$  the operators  $A(A^\alpha + B^\alpha)^{-1/\alpha}$  and  $B(A^\alpha + B^\alpha)^{-1/\alpha}$  can be extended to bounded linear operators in  $X$  with a bound depending only on  $a, b, K, \alpha, X$ .

**PROOF:** Part (ii) follows from (i) by setting  $A = A^\alpha, B = B^\alpha, \sigma = 1/\alpha$  so it remains to prove (i). In the corollary to the theorem on inverses, we have proved (i) when  $\sigma$  is a positive integer. For general  $\sigma$  we again appeal to the three line theorem. Let  $m$  be a nonnegative integer. If we take an appropriate dense subspace  $G_\Lambda$  of  $X$ , the function

$$H(z) = e^{z^2} A^{m+z}(A+B)^{-(m+z)}v, \quad v \in G_\Lambda$$

is holomorphic in a neighborhood of  $0 \leq \operatorname{Re} z \leq 1$ . Since  $A+B \in BIP(a \vee b + \delta, K')$  for all  $\delta > 0$  with some  $K' = K'(K, a, b, \delta, \alpha, X)$ , estimating on the imaginary axis yields

$$\|H(is)\| \leq e^{-s^2} K e^{a|s|} \|A^m(A+B)^{-m}\| K' e^{\rho|s|} \|v\|$$

with  $\rho = a \vee b + \delta$ , where  $\delta$  is a fixed sufficiently small number. By the corollary to the theorem on inverses,  $\|A^m(A+B)^{-m}\|$  is bounded by  $c_m$ ; we now observe

$$\|H(is)\| \leq c_m L \|v\|, \quad L = \sup_{s \in \mathbb{R}} K K' \exp(-s^2 + (a + \rho)|s|) < \infty.$$

Similarly, on  $\operatorname{Re} z = 1$  we have

$$\|H(1+is)\| \leq c_{m+1} L e \|v\|.$$

Applying the three line theorem yields

$$\|H(\tau)\| \leq M \|v\|, \quad M = c_m^{1-\tau} c_{m+1}^\tau e^\tau L < \infty, \quad v \in G_\Lambda.$$

Since  $G_\Lambda$  is dense in  $X$ , we now obtain

$$\|A^{m+\tau}(A+B)^{-(m+\tau)}\| \leq e^{-\tau^2} M, \quad 0 < \tau < 1.$$

The proof for  $B^\sigma(A+B)^{-\sigma}$  is parallel, so is omitted.

#### 4. Application to the Stokes system.

Although our abstract result applies to a very general class of evolution equations (1.2), we consider here as an example only the Stokes system (1.3) on some domain  $\Omega$  in  $\mathbb{R}^n$ .

##### Assumptions on the domain $\Omega$ .

In the following let  $\Omega \subset \mathbb{R}^n$  ( $n \geq 2$ ) be either the whole space  $\mathbb{R}^n$ , a bounded domain, a halfspace or an exterior domain. The boundary  $\partial\Omega$  is always assumed at least of class  $C^{2+\mu}$  with  $0 < \mu < 1$ . If  $\Omega$  is an exterior domain we suppose  $n \geq 3$ .

### Stokes operator.

For  $1 < q < \infty$  let  $L^q_\sigma$  denote the  $L^q$  closure of the space  $C_{0,\sigma}^\infty$  of all smooth divergence-free vector fields with compact support in  $\Omega$ . Let  $P = P_q$  denote the projection operator from  $L^q = (L^q(\Omega))^n$  to  $L^q_\sigma$  associated with the Helmholtz decomposition. The *Stokes operator*  $A_q$  is defined in  $L^q_\sigma$  by  $A_q = -P\Delta$  with the dense domain

$$D(A_q) = \{u \in L^q_\sigma; \nabla^2 u \in L^q, u|_{\partial\Omega} = 0\},$$

where  $\Delta$  denotes the Laplacian and  $\nabla^2 u$  denotes the tensor of all second order derivatives. In [G] and [GS1] it is shown that for all  $0 < a < \pi/2$ ,  $A_q \in BIP(a, K)$  with  $K$  depending on  $a$ . For more information on the Stokes operator and the Helmholtz decomposition we refer to [GS1, 2] and [BM] and the references cited there.

### Evolution equation.

Applying the projection  $P_q$  to the Stokes system (1.3), one formally obtains its abstract form

$$(4.1) \quad \frac{du}{dt} + A_q u = f \quad \text{in } (0, T), \quad u(0) = 0.$$

For  $1 < r < \infty$ ,  $0 < T \leq \infty$  let  $B$  denote the derivative operator on  $X = L^r(0, T; L^q_\sigma)$  defined by  $B = d/dt$  (weak derivative) with

$$D(B) = \{u \in X; du/dt \in X, u(0) = 0\}.$$

The operator  $A$  in  $X$  is defined by  $(Au)(t) = A_q u(t)$  for a.e.  $t \in (0, T)$  where

$$u \in D(A) = \{u \in X; u(t) \in D(A_q) \text{ for a.e. } t \in (0, T) \\ \text{and } \int_0^T \|A_q u(t)\|_q^r dt < \infty\}.$$

Using  $A$  and  $B$  we may rewrite (4.1) as

$$(4.2) \quad Bu + Au = f.$$

The space  $X$  is  $\zeta$ -convex because  $L^q_\sigma$  is  $\zeta$ -convex; see [GS2] and the references cited there. As shown in [DV] for each  $\delta > 0$  the operator  $B \in BIP(\pi/2 + \delta, K)$  with  $K$  depending on  $\delta$  but independent of  $T$ ,  $0 < T \leq \infty$ . The property  $A_q \in BIP(a, K)$  yields  $A \in BIP(a, K)$ , where  $a$  is arbitrary  $0 < a < \pi/2$  and  $K$  depends on  $a$  but is independent of  $T$ . Clearly,  $A$  and  $B$  are resolvent commuting. Applying the extended Dore-Venni theorem in [GS2] one observes that there is a unique solution  $u \in D(A+B)^\wedge$  of (4.2) for each  $f \in X$ . If  $T < \infty$ ,  $B^{-1}$  exists as a bounded operator so that

$$D(A+B)^\wedge = D(A) \cap D(B).$$

For  $0 < T < \infty$  we call  $u : (0, T) \rightarrow L^q_\sigma$  a *strong solution* of (4.2) if it satisfies (4.2) with  $u \in D(A) \cap D(B)$ . In case  $T = \infty$  we call  $u : (0, \infty) \rightarrow L^q_\sigma$  a *strong solution* if so is  $u$

on each finite time interval  $(0, T)$ .

### Generalized solutions.

In order to apply our abstract Theorem 3.1 to (4.2) we have to consider generalized solutions  $u$  of (4.1) for a class of distributions  $f$ . This is caused by the fractional powers  $(B + A)^\alpha$ . For simplicity we will avoid here the definition via test functions and prefer the definition via regularization. Roughly speaking,  $u$  is a generalized solution of (4.1) if the "regularization"  $A_q^{-\alpha}u$  is a strong solution of (4.1) with  $f$  replaced by  $A_q^{-\alpha}f$ .

Let us give a precise definition. For  $0 < \alpha < 1$  the space  $D(A_q^{-\alpha}) = R(A_q^\alpha)$  is equipped with the norm  $\|A_q^{-\alpha}u\|_q$  and  $\hat{D}(A_q^{-\alpha})$  denotes the completion of  $D(A_q^{-\alpha})$  under this norm. For  $v = (v_j)_{j=1}^\infty \in \hat{D}(A_q^{-\alpha})$  we define  $A_q^{-\alpha}v = (A_q^{-\alpha}v_j)$  and get  $A_q^{-\alpha}v \in L_q^q$  for each  $v \in \hat{D}(A_q^{-\alpha})$ ;  $A_q^{-\alpha}v$  is called the regularization of  $v \in \hat{D}(A_q^{-\alpha})$ . In the case  $T < \infty$  we say  $u \in L^r(0, T; D(A_q^{1-\alpha}))$  is a *generalized solution* of (4.1) with  $f \in L^r(0, T; \hat{D}(A_q^{-\alpha}))$  if  $A_q^{-\alpha}u$  solves (4.2) as a strong solution with  $f$  replaced by  $A_q^{-\alpha}f \in L^r(0, T; L_q^q)$ . If  $u : (0, \infty) \rightarrow D(A_q^{1-\alpha})$  is a generalized solution of (4.1) on each finite time interval  $(0, T)$ ,  $u$  is called a *generalized solution* in case  $T = \infty$ .

**4.1. UNIQUE EXISTENCE OF GENERALIZED SOLUTIONS.** Let  $\Omega$  be as above,  $0 < T < \infty$ ,  $1 < r < \infty$ ,  $1 < q < \infty$ ,  $0 < \alpha < 1$ . Suppose  $f \in L^r(0, T; \hat{D}(A_q^{-\alpha}))$ . Then there exists a unique generalized solution  $u \in L^r(0, T; D(A_q^{1-\alpha}))$  of (4.1). Moreover,  $u \in D(B^{1-\alpha})$  and

$$(4.3) \quad \int_0^T \left\| \left( \frac{d}{dt} \right)^{1-\alpha} u \right\|_q^r dt + \int_0^T \|A_q^{1-\alpha} u\|_q^r dt \leq C \int_0^T \|A_q^{-\alpha} f\|_q^r dt$$

with  $C = C(\Omega, q, r, \alpha) > 0$  independent of  $T$  and  $f$  where  $(d/dt)^{1-\alpha} = B^{1-\alpha}$ .

REMARKS: a) The condition  $u(0) = 0$  is implicitly contained in  $u \in D(B^{1-\alpha})$  for small  $\alpha$  (i.e.  $0 < \alpha < 1 - 1/r$ ) while no condition is imposed on  $u(0)$  for large  $\alpha$  (i.e.  $1 - 1/r < \alpha < 1$ ).

b) The case  $T = \infty$  can be admitted in Theorem 4.1 if we replace  $D(A_q^{1-\alpha})$  by  $\hat{D}(A_q^{1-\alpha})$  and  $D(B^{1-\alpha})$  by  $\hat{D}(B^{1-\alpha})$ . In this case (4.3) is

$$\int_0^\infty \left\| \left( \frac{d}{dt} \right)^{1-\alpha} u \right\|_q^r dt + \int_0^\infty \|A_q^{1-\alpha} u\|_q^r dt \leq C \int_0^\infty \|A_q^{-\alpha} f\|_q^r dt$$

which yields asymptotic properties of  $u$  as  $t \rightarrow \infty$ .

c) Of course, this theorem extends to the class of all evolution equations for which Theorem 3.1 is applicable.

PROOF: We apply the extended Dore-Venni theorem in [GS2] to  $A_q^{-\alpha}f \in X$  and obtain a unique solution  $v \in D(B) \cap D(A)$  of  $Bv + Av = A_q^{-\alpha}f$ . The function  $u = A_q^\alpha v$  is a generalized solution of (4.1) since  $A_q^{-\alpha}u$  is a strong solution; the uniqueness of  $u$  is obvious.

To prove (4.3) we use the Yosida approximation  $J_m = J_A(m) = m(m + A)^{-1}$  in Section 2 and obtain

$$\begin{aligned} BA^{-\alpha}J_m u + AA^{-\alpha}J_m u &= A^{-\alpha}(BJ_m u + AJ_m u) = A^{-\alpha}J_m f \\ BJ_m u + AJ_m u &= J_m f. \end{aligned}$$

Here  $J_m f$  is defined in the same way as  $A_q^{-\alpha} f$ . We know that  $\lim_{m \rightarrow \infty} J_m u = u$  in  $X = L^r(0, T; L^q_\sigma)$ . Setting  $u_m = J_m u$ ,  $w = u_m - u$  and applying Theorem 3.1 yields

$$\begin{aligned} \|B^{1-\alpha}w\|_X + \|A^{1-\alpha}w\|_X &\leq C\|(B + A)^{1-\alpha}w\|_X \\ &= C\|(B + A)^{-\alpha}(B + A)w\|_X = C\|A^\alpha(B + A)^{-\alpha}(J_m - J_\infty)A^{-\alpha}f\|_X \\ &\leq C'\|(J_m - J_\infty)A^{-\alpha}f\|_X; \end{aligned}$$

here we used the fact that  $A^\alpha(B + A)^{-\alpha}$  is bounded by Lemma 3.2. From this estimate we conclude  $u \in D(B^{1-\alpha}) \cap D(A^{1-\alpha})$  since  $B^{1-\alpha}$  and  $A^{1-\alpha}$  are closed and  $u \in X$ . The same estimate with  $w$  replaced by  $u_m$  yields (4.3) by letting  $m \rightarrow \infty$ . This proves 4.1.

We next consider some concrete cases of distributions  $f$  in Theorem 4.1. In case a) of the following Corollary we consider a distribution of the form  $f = \sum_{\nu=1}^n \partial_\nu f_\nu$  with  $f_\nu \in X$  and  $\partial_\nu = \partial/\partial x_\nu$ , and in b) we let  $f \in L^r(0, T; L^q_\sigma)$  with some exponent  $\gamma$  different from  $q$ .

4.2. COROLLARY. Suppose  $\Omega$  as above and  $0 < T < \infty$ ,  $1 < q < \infty$ ,  $1 < r < \infty$ .

a) Let  $f = \sum_{\nu=1}^n \partial_\nu f_\nu$  with  $f_\nu \in X = L^r(0, T; L^q_\sigma)$ ,  $\nu = 1, \dots, n$ . If  $\Omega$  is unbounded, suppose additionally  $q > n/(n-1)$ ,  $n \geq 3$ . Then  $A_q^{-1/2} f \in X$ ,  $f \in L^q(0, T; \hat{D}(A_q^{-1/2}))$ . There exists a unique generalized solution  $u \in L^r(0, T; D(A_q^{1/2}))$  of (4.1) with  $u \in D(B^{1/2})$  and

$$(4.4) \quad \int_0^T \left\| \left( \frac{d}{dt} \right)^{1/2} u \right\|_q^r dt + \int_0^T \|A_q^{1/2} u\|_q^r dt \leq C \sum_{\nu=1}^n \int_0^T \|f_\nu\|_q^r dt$$

with  $C = C(\Omega, q, r)$  independent of  $f$  and  $T$ .

b) For  $1 < \alpha < 1$  let  $\gamma$  be defined by  $2\alpha + n/q = n/\gamma$  and  $f \in L^r(0, T; L^q_\sigma)$ . If  $\Omega$  is an exterior domain, suppose additionally  $1 < \gamma < n/2$ ,  $n \geq 3$ . Then  $A_q^{-\alpha} f \in L^r(0, T; L^q_\sigma)$ ,  $f \in L^r(0, T; \hat{D}(A_q^{-\alpha}))$ . There exists a unique generalized solution  $u \in L^r(0, T; D(A_q^{1-\alpha}))$  of (4.1) with  $u \in D(B^{1-\alpha})$  and

$$(4.5) \quad \int_0^T \left\| \left( \frac{d}{dt} \right)^{1-\alpha} u \right\|_q^r dt + \int_0^T \|A_q^{1-\alpha} u\|_q^r dt \leq C \int_0^T \|f\|_\gamma^r dt$$

with  $C = C(\Omega, q, r, \alpha)$  independent of  $f$  and  $T$ .

REMARKS: (i) To prove a) and b) it suffices to prove that  $f \in L^r(0, T; \hat{D}(A_q^{-1/2}))$  and

$$\|A_q^{-1/2} f\|_X \leq C \sum_{\nu=1}^n \|f_\nu\|_X$$

in a) and that  $f \in L^r(0, T; \hat{D}(A_q^{-\alpha}))$  and

$$\|A_q^{-\alpha} f\|_X \leq C \left( \int_0^T \|f\|_\gamma^r dt \right)^{1/r}$$

in b) respectively with  $C$  independent of  $f$  and  $T$ .  
(ii) The estimate (4.4) yields (1.6) by applying of

$$\|\nabla u\|_q \leq C \|A_q^{1/2} u\|_q$$

which needs additionally the restriction  $1 < q < n$ ,  $n \geq 3$  when  $\Omega$  is an exterior domain ([BM], [GS1]).

PROOF: a) In [GS1, p.123] it has been shown that  $C_{0,\sigma}^\infty \subset R(A_q)$  if  $q > n/(n-2)$  and  $\Omega$  is the  $\mathbb{R}^n$  or an exterior domain; the same proof works also for the half-space and the restriction becomes  $q > n/(n-1)$  if  $A_q$  is replaced by  $A_q^{1/2}$ . If  $\Omega$  is bounded, no restriction is needed.

So for each  $f_\nu$  ( $\nu = 1, 2, \dots, n$ ) we find a sequence  $(f_{\nu j})_{j=1}^\infty$  in  $L^r(0, T; C_{0,\sigma}^\infty) \subset L^r(0, T; D(A_q^{-1/2}))$  with  $f_\nu = \lim_{j \rightarrow \infty} f_{\nu j}$  in  $L^r(0, T; L_\sigma^q)$ . It follows that  $(\tilde{f}_j) = (\sum_{\nu=1}^n \partial_\nu f_{\nu j})$  is a sequence in  $L^r(0, T; D(A_q^{-1/2}))$ .

We next use the estimate

$$\|A_q^{-1/2} \nabla u\|_q \leq C \|u\|_q$$

(see [BM], [GS1]) which is valid in all cases for  $\Omega$  but in exterior domains under the restriction  $q > n/(n-1)$ ; observe that this estimate is equivalent to  $\|\nabla u\|_{q'} \leq C \|A_q^{1/2} u\|_{q'}$ , where by duality the restriction is now given by  $1 < q' < n$ . This leads to

$$\|A_q^{-1/2} (\tilde{f}_i - \tilde{f}_j)\|_X = \left\| \sum_{\nu=1}^n A_q^{-1/2} \partial_\nu (f_{\nu i} - f_{\nu j}) \right\|_X \leq C \sum_{\nu=1}^n \|f_{\nu i} - f_{\nu j}\|_X$$

which yields  $f \in L^r(0, T; \hat{D}(A_q^{-1/2}))$ . This estimate also yields

$$\|A_q^{-1/2} f\|_X \leq C \sum_{\nu=1}^n \|f_\nu\|_X$$

so Theorem 4.1 is applicable.

b) Since  $R(A_q^\alpha) \subset L_\sigma^\gamma$  is dense in  $L_\sigma^\gamma$ , one can choose  $f_j \in L^r(0, T; D(A_q^{-\alpha}))$ ,  $j = 1, 2, \dots$  with  $f = \lim_{j \rightarrow \infty} f_j$  in  $L^r(0, T; L_\sigma^\gamma)$ . Then we use the estimate

$$\|A_q^{-\alpha} u\|_q \leq C \|u\|_\gamma$$

in [GS1, p.104] which holds for  $2\alpha + n/q = n/\gamma$ ; in exterior domains the restriction  $1 < \gamma < n/2$ ,  $n \geq 3$  is needed. This leads to

$$\|A_q^{-\alpha} (f_i - f_j)\|_X \leq C \left( \int_0^T \|f_i - f_j\|_\gamma^r dt \right)^{1/r}$$

which yields  $f \in L^r(0, T; \hat{D}(A_q^{-\alpha}))$  and

$$\|A_q^{-\alpha} f\| \leq C \left( \int_0^T \|f\|_r^r dt \right)^{1/r},$$

so Theorem 4.1 is applicable.

**Further applications.** The estimates above can be applied to weak solutions of the nonlinear Navier-Stokes equations if we take the nonlinear term to the right hand side in (4.1). The procedure is completely analogous to that in [GS2].

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