DUALITY FORMULAS
AND VARIATIONAL INTEGRALS

P.Aviles, Y. Giga and N. Komuro

Series ♯134. January 1992

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

♯ 109: Y.-G. Chen, Y. Giga and S. Goto, Analysis toward snow crystal growth, 18 pages. 1991.

♯ 110: T. Hibi, M. Wakayama, A q-analogue of Capelli's identity for $GL(2)$, 7 pages. 1991.

♯ 111: T. Nishimori, A qualitative theory of similarity pseudogroups and an analogy of Sacksteder's theorem, 13 pages. 1991.

♯ 112: K. Matsuda, An analogy of the theorem of Hector and Duminy, 10 pages. 1991.

♯ 113: S. Takahashi, On a regularity criterion uo to the boundary for weak solutions of the Navier-Stokes equations, 23 pages. 1991.

♯ 114: T. Nakazi, Sum of two inner functions and exposed points in H^1, 18 pages. 1991.

♯ 115: A. Arai, De Rham operators, Laplacians, and Dirac operators on topological vector spaces, 27 pages. 1991.

♯ 116: T. Nishimori, A note on the classification of non-singular flows with transverse similarity structures, 17 pages. 1991.

♯ 117: T. Hibi, A lower bound theorem for Ehrhart polynomials of convex polytopes, 6 pages. 1991.

♯ 118: R. Agemi, H. Takamura, The lifespan of classical solutions to nonlinear wave equations in two space dimensions, 30 pages. 1991.

♯ 119: S. Altschuler, S. Angenent and Y. Giga, Generalized motion by mean curvature for surfaces of rotation, 15 pages. 1991.

♯ 120: T. Nakazi, Invariant subspaces in the bidisc and commutators, 20 pages. 1991.

♯ 121: A. Arai, Commutation properties of the partial isometries associated with anticommuting self-adjoint operators, 25 pages. 1991.

♯ 122: Y.-G. Chen, Blow-up solutions to a finite difference analogue of $u_t = \Delta u + u^{1+\alpha}$ in N-dimensional balls, 31 pages. 1991.

♯ 123: A. Arai, Fock-space representations of the relativistic supersymmetry algebra in the two-dimensional space-time, 13 pages. 1991.

♯ 124: S. Izumiya, The theory of Legendrian unfoldings and first order differential equations, 16 pages. 1991.

♯ 125: T. Hibi, Face number inequalities for matroid complexes and Cohen-Macaulay types of Stanley-Reisner rings of distributive lattices, 17 pages. 1991.

♯ 126: S. Izumiya, Completely integrable holonomic systems of first order differential equations, 35 pages. 1991.

♯ 127: G. Ishikawa, S. Izumiya and K. Watanabe, Vector fields near a generic submanifold, 9 pages. 1991.

♯ 128: A. Arai, I. Mitoma, Comparison and nuclearity of spaces of differential forms on topological vector spaces, 27 pages. 1991.

♯ 129: K. Kubota, Existence of a global solution to a semi-linear wave equation with initial data of non-compact support in low space dimensions, 53 pages. 1991.

♯ 130: S. Altschuler, S. Angenent and Y. Giga, Mean curvature flow through singularities for surfaces of rotation, 62 pages. 1991.

♯ 131: M. Giga, Y. Giga and H. Sohr, L^p estimates for the Stokes system, 13 pages. 1991.

♯ 132: Y. Okabe, T. Ootsuka, Applications of the theory of KM_2O-Langevin equations to the non-linear prediction problem for the one-dimensional strictly stationary time series, 27 pages. 1992.

♯ 133: Y. Okabe, Applications of the theory of KM_2O-Langevin equations to the linear prediction problem for the multi-dimensional weakly stationary time series, 22 pages. 1992.

DUALITY FORMULAS
AND VARIATIONAL INTEGRALS

Patricio Aviles
Yoshikazu Giga
Naoto Komuro

Abstract. A duality representation of a measure $f(x,\mu)$ for a finite dimensional vector valued Radon measure μ is established for a continuous function $f = f(x,p)$ which is convex and of linear growth in p. Besides technical new aspects of the result the proof given here is simpler than those based on the theory of convex functionals. As an application, a new proof for a representation $f(x,\nabla u)$ by the graph of u is given when u is a mapping of bounded variation.

1. Introduction. In convex analysis a duality representation of convex functionals is very important [6], [13]. Let us begin with a simple example. Suppose that f is a continuous convex function on $\mathbf{R}^d$. Then f has a representation

$$f(p) = \sup_q(< q,p > -f^*(q)), \tag{1.1}$$

where f^* is the conjugate convex function of f defined by

$$f^*(q) = \sup_p(< p,q > -f(p)) \tag{1.2}$$

and $<\,,\,>$ denotes the inner product in $\mathbf{R}^d$. If f is positively homogeneous of degree one f^* becomes the indicator function of the subdifferential $K = \partial f(0)$ at 0. In other words $f^*(q) = 0$ for $q \in K$ otherwise $f^*(q) = \infty$ and

$$\partial f(0) = \{q; < q,p > \leqslant f(p) \quad \text{for all} \quad p \in \mathbf{R}^d\}. \tag{1.3}$$

The duality formula (1.1) now yields

$$(1.4) \qquad f(p) = \sup_{q \in K} < q, p >$$

for homogeneous f.

In this paper we give such a representation for a convex functional $\int_X f(x,\mu)$ defined on the space of finite $\mathbf{R}^d$ valued Borel regular measure μ on a compact set X in $\mathbf{R}^N$ (or more generally a compact metric space). The integrand $f(x,\mu)$ is well-defined as a Borel regular measure if $f = f(x,p)$ is continuous on $X \times \mathbf{R}^d$ and grows linearly in p ([8],[10]). When f is positively homogeneous of degree one in p, we derive a duality formula (corresponding to (1.4)) which reads:

$$(1.5) \qquad \int_X \varphi(x) f(x,\mu) = \sup \int_X \varphi < v, \mu > .$$

Here the supremum is taken over all $\mathbf{R}^d$ valued continuous funciton v on X such that $v(x) \in K_x$ for all $x \in X$ where K_x is the subdifferential of $f_x(p) = f(x,p)$ at $p = 0$ (cf. (1.3)). The function φ is a nonnegative bounded Borel measurable function. The bracket means the canonical pairing of $\mathbf{R}^d$ valued measures and functions. We also extend a duality formula for inhomogeneous f growing linearly in p as an application of (1.5). Under an assumption of uniform continuity of $f/(1+|p|)$ in x (uniform in p) the duality formula (corresponding to (1.1)) reads:

$$(1.6) \qquad \int_X \varphi(x) f(x,\mu) = \sup \left(\int_X \varphi < v, \mu > - \int_X \varphi f^*(x,v) d\mathcal{L}^N \right).$$

Here the supremum is taken over all $\mathbf{R}^d$ valued continuous function v on X such that $f^*(x,v)$ is integrable with respect to the Lebesgue measure $\mathcal{L}^N$. Here $f^*(x,q) = f_x^*(q)$ is defined by (1.2).

Our results (1.5) and (1.6) are not entirely new. In [11] Rockafellar proved (1.5) and (1.6) with $\varphi \equiv 1$ under assumptions different from ours. Neither sets of assumptions do not imply the other. For example for (1.5) Rockafellar needs to assume that K_x has an interior point which does not necessarily hold in our case. This restriction on K_x does not allow us to apply (1.5) and (1.6) to our results in Section 4. His assumption on the regularity of $f(x,p)$ in x seems slightly weaker than ours although it is not easy to compare because his assumptions are given for f^* rather than f.

The proof of Rockafellar [11] depends on a couple of duality formulas of convex functionals on other spaces so it is long. Our proof of (1.5), (1.6) is rather simple and easy to access. For example to show (1.5) we apply the measurable selection (due to Castaing) and (1.4) to get (1.5) for all μ-measurable function v with $v(x) \in K_x$. We next approximate v by a continuous function. A key observation is the continuity of K_x in x. No nontrivial theorems rather than Lusin's theorem are used. The formula (1.6) is proved by extending (1.5) to f which takes ∞ outside $X \times C$ where C is a closed convex cone. For $f \geqslant 0$ we do not need any expensive argument but for general f we use Michael's continuous selection theorem [9]. The formula (1.6) is proved by Demengel and Temam [7] when f is independent of x and φ is a continuous function with compact support or φ is a characteristic function of an open set. However, their method does not seem to apply to the case when f depends on x.

As an application of the duality formulas in Section 4 we give a new and direct proof of the representation [2, Theorem 8.2] of

$$\int_\Omega f(x, \nabla u)$$

for a mapping of bounded variation $u : \Omega \to \mathbf{R}^m$, where Ω is a domain in $\mathbf{R}^n$. Their representation is interpreted as a variant of area formula. Roughly speaking it represents $\int_\Omega f(x, \nabla u)$ by an integration of a current associated with a graph of u in $\Omega \times \mathbf{R}^m$. We present our proof readable for those who are not familiar with the wording of [2].

Throughout this paper, by a nonnegative measure we mean an outer measure. If $\mu = \mu_1 - \mu_2$ with two nonnegative measures we simply say μ is a measure. The measure $|\mu| = \mu_1 + \mu_2$ is called the total variation measure of μ. If $\mu = (\mu_1, \cdots \mu_d)$ and each μ_j is a measure, we say μ is a $\mathbf{R}^d$ valued measure.

This work was initiated while the first author visited the Mathematics Department of Hokkaido University, supported by a grant from the Japan Association of Mathematical Sciences.

2. Duality formulas — homogeneous version. Let μ be a $\mathbf{R}^d$ valued finite Borel regular measure on a compact metric space X. Let $f = f(x,p)$ be a continuous function on $X \times \mathbf{R}^d$ which is positively homogeneous of degree one in p, i.e., for each $x \in X$ the function $f_x(p) = f(x,p)$ has a scaling property

$$f_x(\lambda p) = \lambda f_x(p) \quad \text{for all} \quad \lambda > 0, \quad p \in \mathbf{R}^d.$$

The finite Borel measure $f(x,\mu)$ on X is defined by

$$\int_A f(x,\mu) = \int_A f(x,\vec{\mu}(x))d|\mu| \quad \text{for a Borel set} \quad A \subset X,$$

where $|\mu|$ is the total variation measure of μ; for an open set A the valve of $|\mu|(A)$ is defined by

$$|\mu|(A) = \sup\{\sum_{i=1}^{d} \int_A v_i d|\mu_i|; \ v_i : X \to \mathbf{R}$$

$$\text{is a continuous function compactly supported in } A \text{ and } \sum_{i=1}^{d} v_i^2 \leq 1\}.$$

Here $\vec{\mu}(x) = (d\mu/d|\mu|)(x)$ is the Radon-Nikodym derivative of μ with respect to $|\mu|$ (cf. [10]). The measure $f(x,\mu)$ is independent of the choice of a norm $|\cdot|$ in $\mathbf{R}^d$. When X is a compact set in $\mathbf{R}^N$ and $|\mu|$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^N$ with a density h, the homogenuity in p yields

$$\int_A f(x,\mu) = \int_A f(x,h)d\mathcal{L}^N$$

so the definition of $f(x,\mu)$ is very natural. We are interested in deriving a duality representation of $f(x,\mu)$ when f_x is convex in p.

2.1. Duality formula. Let K_x be the subdifferential of f_x at 0 defined by (1.3), i.e.,

$$K_x = \{q \in \mathbf{R}^d; < q,p > \leqslant f_x(p) \quad \text{for all} \quad p \in \mathbf{R}^d\}.$$

Clearly, K_x is a closed convex subset of $\mathbf{R}^d$. Let f_x^* denote the conjugate function of f_x in (1.2), i.e.,

$$f_x^*(q) = \sup_{p \in \mathbf{R}^d} (< p,q > - f_x(p)).$$

When f_x is positively homogeneous of degree one, f_x^* is the indicator function of K_x. Let $C(X, \mathbf{R}^d)$ denote the space of all $\mathbf{R}^d$ valued continuous functions on X. We are now in position to state our duality formula. Note that all Borel sets are $|\mu|$-measurable and that for each $A \subset X$ there is a Borel set $B \supset A$ such that $|\mu|(A) = |\mu|(B)$ if and only if μ is a Borel regular measure on X.

THEOREM. *Let $f = f(x,p)$ be a real valued continuous function on $X \times \mathbf{R}^d$, where X is a compact metric space. Suppose that $f_x(p) = f(x,p)$ is positively homogeneous of degree one and convex in p. Let μ be a $\mathbf{R}^d$ valued finite Borel regular measure on X. Let $\varphi \geqslant 0$ be a bounded $|\mu|$-measurable function on X. Then*

$$\int_X f(x,\mu)\varphi = \sup\left\{ \int_X < \vec{\mu}(x), v(x) > \varphi(x) d|\mu|(x); v \in C(X, \mathbf{R}^d) \right. \tag{2.1}$$

$$\left. v(x) \in K_x \quad \text{for all} \quad x \in X \right\}.$$

REMARK: If we write each side of this identity by G and H respectively, $G \geqslant H$ is easy to prove. Indeed, since (1.4) is

$$f_x(p) = \sup_{q \in K_x} < p, q >,$$

we see

$$\begin{aligned} G = \int_X f(x,\mu)\varphi &= \int_X f(x, \vec{\mu}(x))\varphi(x) d|\mu| \\ &= \int_X \sup_{q \in K_x} < \vec{\mu}(x), q > \varphi(x) d|\mu| \\ &\geqslant \int_X < \vec{\mu}(x), v(x) > \varphi(x) d|\mu| \end{aligned}$$

holds for every $v \in C(X, \mathbf{R}^d)$ with $v(x) \in K_x$ for all $x \in X$. This yields $G \geqslant H$.

To prove the theorem it remains to show the inequality $G \leqslant H$ which we shall prove in several steps in the following subsections.

REMARK: The formula (2.1) is the same as (1.5) since

$$\int_X \varphi < v, \mu > = \int_X < \vec{\mu}(x), v(x) > \varphi(x) d\mu(x).$$

In (2.1) φ is allowed to be simply $|\mu|$-measurable (not necessarily Borel measurable).

2.2 Continuity of K_x. We say a closed set valued function $\Gamma_x \subset \mathbf{R}^d$ defined in $X(\ni x)$ is *upper semicontinuous* at $x_0 \in X$ if $x_n \to x_0, q_n \to q_0$ $(n \to \infty)$ with $q_n \in \Gamma_{x_n}$ implies $q_0 \in \Gamma_{x_0}$. We say Γ_x is *lower semicontinuous* at $x_0 \in X$ if for any $q_0 \in \Gamma_{x_0}$ and $x_n \to x_0$ there is a sequence $\{q_n\}$ converging to q_0 with $q_n \in \Gamma_{x_n}$. We say Γ_x is *continuous* at $x_0 \in X$ if it is both upper and lower semicontinuous at $x_0 \in X$. If it is continuous at all $x_0 \in X$, we simply say Γ_x is continuous on X. We refer to [1] for various properties of set valued functions.

LEMMA. *Suppose that $f = f(x,p)$ is a real valued continuous function on $X \times \mathbf{R}^d$. Suppose that $f_x(p) = f(x,p)$ is convex and positively homogeneous of degree one. Then K_x is continuous on X.*

PROOF: For $x_0 \in X$ suppose that $x_n \to x_0$ as $n \to \infty$. If $q_n \in K_{x_n}$, then $< q_n, p > \leqslant f(x_n, p)$ for all $p \in \mathbf{R}^d$. Suppose that $q_n \to q_0$. Since f is continuous, letting $n \to \infty$ yields $< q_0, p > \leqslant f(x_0, p)$ for all p. In other words $q_0 \in K_{x_0}$ so K_x is upper semicontinuous at any $x_0 \in X$.

It remains to prove that K_x is lower semicontinuous at $x_0 \in X$. Suppose that K_x were not lower semicontinuous at x_0. Then there would exist $q_0 \in K_{x_0}$, a positive number $\varepsilon_0 > 0$ and a sequence $\{x_n\}$ converging to x_0 such that

$$K_{x_n} \cap B_{\varepsilon_0}(q_0) = \varnothing$$

for every n, where $B_{\varepsilon_0}(q_0)$ is a closed ball of radius ε_0 centered at q_0. By the separation theorem [13, Theorem 11.3], for each n there exists $p_n \in \mathbf{R}^d$ such that $|p_n| = 1$ and

$$\sup_{q \in K_{x_n}} < q, p_n > \leqslant \inf_{q \in B_{\varepsilon_0}(q_0)} < q, p_n > .$$

By (1.4) we now observe that

$$\begin{aligned} f(x_n, p_n) &= \sup_{q \in K_{x_n}} < q, p_n > \leqslant \inf_{q \in B_{\varepsilon_0}(q_0)} < q, p_n > \\ &= < \bar{q}, p_n > = < q_0, p_n > - < q_0 - \bar{q}, p_n > = < q_0, p_n > - \varepsilon_0 \\ &\leqslant \sup_{q \in K_{x_0}} < q, p_n > - \varepsilon_0 = f(x_0, p_n) - \varepsilon_0 \end{aligned}$$

for every n, where $\bar{q}$ is the supporting point of $B_{\varepsilon_0}(q_0)$. Since $x_n \to x_0$, this contradicts the equicontinuity of the family of functions

$$\{f(\cdot, p) \; ; \; |p| = 1, \; p \in \mathbf{R}^d\}$$

at x_0; the equicontinuity is trivial since f is continuous on a compact set $X \times \{|p| = 1\}$. We thus prove the lower semicontinuity of K_x at x_0. Q.E.D.

REMARK: When f_x is continuous on $\mathbf{R}^d$, the set K_x is compact. Looking over the proof, we see this fact is never used. Actually, one can conclude the continuity of K_x even if f is continuous only on $X \times C$ and $f = \infty$ on $X \times (\mathbf{R}^d \backslash C)$, where C is a closed convex cone in $\mathbf{R}^d$ provided that f_x is convex and positively homogeneous of degree one.

2.3. Weak duality formula. We apply the measurable selection theorem to get a weaker version of the duality formula in §2.1.

PROPOSITION. *Assume the hypotheses of the duality theorem in* §2.1. *Then*

$$\int_X f(x, \mu)\varphi = \sup \Big\{ \int_X < \vec{\mu}(x), w(x) > \varphi(x) d|\mu|;$$
$$w \text{ is } |\mu| - \text{measurable and } w(x) \in K_x \text{ for } |\mu|\text{-a.e. } x \in X \Big\}.$$

We recall measurability of a closed set valued function. Let Y be a set and let ν be an outer measure on Y. Let Γ be a closed-valued multifunction from Y to $\mathbf{R}^d$, i.e.,

$$\Gamma : Y \to 2^{\mathbf{R}^d}$$

has the property that $\Gamma(y)$ is closed in $\mathbf{R}^d$ for all $y \in Y$. We say Γ is *ν-measurable* if

$$\Gamma^{-1}(U) = \{y \in Y; \Gamma(y) \cap U \neq \emptyset\}$$

is ν-measurable provided that U is open in $\mathbf{R}^d$.

For example the mapping $x \mapsto K_x$ in §2.2 is $|\mu|$-measurable since K_x is lower semicontinuous. Indeed, for an open set U in $\mathbf{R}^d$ the set

$$K^{-1}(U) = \{x \in X; K_x \cap U \neq \emptyset\}$$

is open in X if (and actually only if) K_x is lower semicontinuous. Let us give its proof. For $x_0 \in K^{-1}(U)$ suppose that $x_n \to x_0$. We take $q_0 \in K_{x_0} \cap U$. Since K_x is lower semicontinuous, there is $q_n \in K_{x_n}$ converging to q_0. For sufficiently large n we see $q_n \in U \cap K_{x_n}$ since U is open. This implies $x_n \in K^{-1}(U)$ for sufficiently large n. Therefore $K^{-1}(U)$ is open in X. Since every open set in X is a Borel set, so $K^{-1}(U)$ is $|\mu|$-measurable.

There is a nice article by Rockafellar [11] on measurable multifunctions. We often refer to [11] for their various properties. One of the important properties is the following selection theorem due to C. Castaing [4] (see also [11, Theorem 1B]).

THE MEASURABLE SELECTION THEOREM. *Let Γ be ν-measurable. Then there is a ν-measurable function $\psi : \operatorname{dom}\Gamma \to \mathbf{R}^d$ with $\psi(y) \in \Gamma(y)$, where $\operatorname{dom}\Gamma = \{y \in Y; \Gamma(y) \neq \varnothing\}$. The function ψ is called a (ν-)measurable selection of Γ.*

LEMMA. *Assume the hypotheses of the duality theorem in §2.1 concerning f and μ. For any $\varepsilon > 0$ there is a $|\mu|$-measurable function $w : X \to \mathbf{R}^d$ such that $w(x) \in K_x$ for all $x \in X$ and $< \vec{\mu}(x), w(x) > \geqslant f_x(\vec{\mu}(x)) - \varepsilon$.*

PROOF: We define a multifunction Γ by

$$\Gamma(x) = \{p \in K_x; < \vec{\mu}(x), p > \geqslant f_x(\vec{\mu}(x)) - \varepsilon\}$$

for each $x \in X$. Since

$$f_x(\vec{\mu}(x)) = \sup_{p \in K_x} < \vec{\mu}(x), p >,$$

the set $\Gamma(x)$ is a nonempty (closed) set in $\mathbf{R}^d$ for every $x \in X$. Since $x \longmapsto K_x$ is $|\mu|$-mesurable and $\Gamma(x) = K_x \cap \Gamma_0(x)$ with

$$\Gamma_0(x) = \{p \in \mathbf{R}^d; < \vec{\mu}(x), p > \geqslant f_x(\vec{\mu}(x)) - \varepsilon\},$$

we see, by [11, Theorem 1M or Theorem 1E(b)], Γ is $|\mu|$-measurable provided that so is Γ_0. The measurable selection theorem yields the desired w as a $|\mu|$-measurable selection of Γ.

It remains to prove that Γ_0 is $|\mu|$-measurable. Let U be an open set in $\mathbf{R}^d$. Let D be a countable dense subset of U. Since $\Gamma_0(x)$ is an affine half space, $\Gamma_0(x) \cap U \neq \varnothing$ if and only if $\Gamma_0(x) \cap D \neq \emptyset$. In other words

$$\Gamma_0^{-1}(U) = \Gamma_0^{-1}(D) = \bigcup_{p \in D} A_p$$

$$\text{with } A_p = \{x \in X; < \vec{\mu}(x), p > \geqslant f_x(\vec{\mu}(x)) - \varepsilon\}.$$

Since $< \vec{\mu}(x), p >$ and $f_x(\vec{\mu}(x))$ are $|\mu|$-measurable, so is A_p. Thus $\Gamma_0^{-1}(U)$ is $|\mu|$-measurable since D is countable. Q.E.D.

Proof of Proposition: For $\varepsilon > 0$ let w be as in the lemma. Then

$$\int_X < \vec{\mu}(x), w(x) > \varphi(x) d|\mu| \geqslant \int_X f_x(\vec{\mu}(x))\varphi(x)d|\mu| - \varepsilon \int_X \varphi d|\mu|$$

$$= \int_X f(x,\mu)\varphi - \varepsilon \int_X \varphi d|\mu|$$

The last term is finite since $|\mu|$ is a finite measure and φ is bounded. Since $\varepsilon > 0$ can be chosen arbitrarily, we obtain "$\leqslant$" in the weak duality formula. As remarked in §2.1, the opposite inequality is trivial so one obtains the weak duality formula. Q.E.D.

2.4. Approximation. We shall approximate $|\mu|$-measurable function by continuous functions with values in K_x.

Lemma. *Assume the hypotheses of the duality theorem in §2.1 concerning f. Suppose that $u \in C(X, \mathbf{R}^d)$. Then there is a function $v \in C(X, \mathbf{R}^d)$ such that $v(x) \in K_x$ for all $x \in X$ and that $v(x) = u(x)$ for x with $u(x) \in K_x$.*

Proof: We may assume that u is identically equal to zero on X by replacing $f_x(p)$ by $f_x(p)-$
$< u(x), p >$. Let $v(x)$ denote the smallest vector in K_x, i.e.,

$$|v(x)| = \min_{p \in K_x} |p|.$$

Since K_x is a nonempty closed convex set, such $v(x)$ is uniquely determined. If $u(x) = 0 \in K_x$, clearly $v(x) = 0 = u(x)$.

It remains to prove that v is continuous. For $x_0 \in X$ let $x_n \to x$ in X as $n \to \infty$. We set

$$\sigma(x) = \inf_{p \in K_x} < v(x_0), p >$$

and observe that $\sigma(x_0) = |v(x_0)|^2$ by the convexity of K_x. Since

$$\begin{aligned}\sigma(x) &= -\sup_{p \in K_x} < -v(x_0), p > \\ &= -f(x, -v(x_0)),\end{aligned}$$

the function σ is continuous in x. Hence for $\varepsilon > 0$

$$< v(x_0), v(x_n) > \geqslant \sigma(x_n) \geqslant |v(x_0)|(|v(x_0)| - \varepsilon)$$

for sufficiently large n. The lower semicontinuity of K_x implies that there is $p_n \in K_{x_n}$ with $|p_n - v(x_0)| < \varepsilon$ for sufficiently large n. In particular,

$$|v(x_n)| \leqslant |p_n| < |v(x_0)| + \varepsilon.$$

Applying above two inequalities, we see

$$\begin{aligned}|v(x_n) - v(x_0)|^2 &= |v(x_n)|^2 + |v(x_0)|^2 - 2 < v(x_n), v(x_0) > \\ &< (|v(x_0)| + \varepsilon)^2 + |v(x_0)|^2 - 2|v(x_0)|(|v(x_0)| - \varepsilon) \\ &= 4|v(x_0)|\varepsilon + \varepsilon^2\end{aligned}$$

for sufficiently large n. This implies that $v(x_n) \longrightarrow v(x_0)$ as $n \longrightarrow \infty$ so v is continuous at each $x_0 \in X$. Q.E.D.

PROPOSITION. *Assume the hypotheses of the duality theorem in §2.1 on f and μ. Suppose that $w : X \to \mathbf{R}^d$ is $|\mu|$-measurable with $w(x) \in K_x$ for all $x \in X$. For each $\delta > 0$ there exist $v \in C(X, \mathbf{R}^d)$ and a closed set Y in X such that* (i) $|\mu|(X \backslash Y) < \delta$ (ii) $v = w$ *on Y* (iii) $v(x) \in K_x$ *for all $x \in X$.*

PROOF: By Lusin's theorem (and Tietze's theorem) there exist $u \in C(X, \mathbf{R}^d)$ and a closed set Y such that $u = w$ on Y and $|\mu|(X \backslash Y) < \delta$. Applying the preceeding lemma to u yields $v \in C(X, \mathbf{R}^d)$ satisfying (ii) and (iii). Q.E.D.

2.5. Proof of the duality formula. As remarked in §2.2 it suffices to prove $G \leqslant H$. For $\varepsilon > 0$ we take a $|\mu|$-measurable function w as in the lemma in §2.3 so that

$$G \leqslant \int_X < \vec{\mu}(x), w(x) > \varphi(x) d|\mu| + \varepsilon \int_X \varphi d|\mu|.$$

We approximate w by a continuous function v in the proposition in §2.4 so that

$$\begin{aligned}\int_X < \vec{\mu}(x), w(x) > \varphi(x)d|\mu| &= \int_X < \vec{\mu}(x), v(x) > \varphi(x)d|\mu| \\ &+ \int_{X\setminus Y} < \vec{\mu}(x), w(x) > \varphi(x)d|\mu| - \int_{X\setminus Y} < \vec{\mu}(x), v(x) > \varphi(x)d|\mu| \\ &\leqslant H + \int_{X\setminus Y} f(x, \vec{\mu}(x))\varphi(x)d|\mu| + \|v\| \int_{X\setminus Y} \varphi d|\mu|, \quad \|v\| = \sup_X |v|,\end{aligned}$$

since $w(x) \in K_x$ for all $x \in X$. By the compactness of S the continuity of f implies

$$M = \sup\{|f(x,p)| \ ; \ |p| = 1, x \in X\} < \infty.$$

Combining above inequalities yields

$$G \leqslant H + (M + \|v\|)L|\mu|(X\setminus Y) + L\varepsilon|\mu|(X)$$

with $L = \operatorname{ess.sup}_X |\varphi|$. If $\|v\|$ is bounded by some $R > 0$ independent of δ and ε, this inequality yields

$$G \leqslant H + (M + R)L\delta + L\varepsilon|\mu|(X).$$

Since $\delta, \varepsilon > 0$ are arbitrary and $\mu(X)$ is finite, this now yields $G \leqslant H$.

It remains to prove that $\|v\| \leqslant R$ independent of δ and ε. Since f_x is continuous in $\mathbf{R}^d$ the set K_x is compact. Since K_x is continuous in x and X is compact, there is $R > 0$ such that

$$\bigcup_{x \in X} K_x \subset B_R(0).$$

By the choice of v its value $v(x)$ is in K_x so $\|v\| \leqslant R$ and R depends only on f through K_x.

Q.E.D.

2.6. Generalization to integrands defined on conic domains. We now derive a duality formula when $f \geqslant 0$ is continuous only on $X \times C$ with a closed convex cone C and $f = \infty$ outside this set.

THEOREM. *Let $f : X \times \mathbf{R}^d \to [0, \infty]$ be continuous on $X \times C$ and $f = \infty$ on $X \times (\mathbf{R}^d \setminus C))$, where X is a compact metric space and C is a closed convex cone in $\mathbf{R}^d$. Suppose that $f_x(p) = f(x,p)$ is positively homogeneous of degree one and convex in p. Let μ be a*

C-valued finite Borel regular measure on X in the sense that $\vec{\mu}(x) \in C$ for $|\mu|$-a.e. x. Let $\varphi \geqslant 0$ be a bounded $|\mu|$-measurable function on X. Then the duality formula (2.1) holds.

PROOF: Looking over the proofs we observe that the compactness of K_x is not invoked in §2.2-§2.4. Thus all statements in §2.2-§2.4 holds even for f which is continuous on $X \times C$ and $f = \infty$ on $X \times (\mathbf{R}^d \backslash C)$.

The proof is similar to that in §2.5 but it becomes more technical because K_x is no longer bounded in $\mathbf{R}^d$. For $\varepsilon > 0$ we take a $|\mu|$-measurable function w as in the lemma in §2.3 so that

$$G \leqslant \int_X < \vec{\mu}(x), w(x) > \varphi(x) d|\mu| + \varepsilon L |\mu|(X)$$

with

$$L = \sup_X |\varphi|$$

$$-M - \varepsilon \leqslant < \vec{\mu}(x), w(x) > \leqslant M, \quad x \in X$$

$$M = \sup\{|f(x,p)| \; ; \; |p| = 1, p \in C, x \in X\} < \infty.$$

Since $< \vec{\mu}(x), w(x) > \varphi(x)$ is $|\mu|$-integrable, the Lebesgue convergence theorem yields

$$\lim_{R \to \infty} \int_X < \vec{\mu}(x), w_R(x) > \varphi(x) d|\mu| = \int_X < \vec{\mu}(x), w(x) > \varphi(x) d|\mu|$$

with

$$w_R(x) = \begin{cases} w(x) & \text{if } |w(x)| \leqslant R \\ 0 & \text{otherwise.} \end{cases}$$

Taking R sufficiently large, say $R \geqslant R_0$, we now have

$$G \leqslant \int_X < \vec{\mu}(x), w_R(x) > \varphi(x) d|\mu| + \varepsilon L |\mu|(X) + \varepsilon. \tag{2.2}$$

We approximate w_R as in §2.4 but we should be more careful on a bound of v. By Lusin's theorem and Tietze's theorem (for real valued functions) for every $\delta > 0$ there is $u \in C(X, B_{cR}(0))$ and a closed set Y such that $u = w_R$ on Y and $|\mu|(X \backslash Y) < \delta$ with some $c = c(d) > 0$. Actually one may take $c = 1$ since $B_R(0)$ is a convex closed set in $\mathbf{R}^d$

[1]. However, we need not this property of c. Since $0 \in K_x$ by $f \geqslant 0$, $w_R(x) \in K_x$ for all $x \in X$. Let $v \in C(X, \mathbf{R}^d)$ be defined in the proof of the lemma in §2.4. Since K_x is convex and $0 \in K_x$ it follows that $v(x) \in B_{cR}(0)$. In other words

$$||v|| = \sup_X |v| \leqslant cR.$$

By the construction of v we see $v = w_R$ on Y and $v(x) \in K_x$ for $x \in X$. As in §2.5 the inequality (2.2) yields

$$\begin{aligned} G &\leqslant H + (M + ||v||)L|\mu|(X \backslash Y) + L\varepsilon|\mu|(X) + \varepsilon \\ &\leqslant H + (M + cR)L\delta + L\varepsilon|\mu|(X) + \varepsilon. \end{aligned} \tag{2.3}$$

For $\varepsilon > 0$ we fix $R = R_0$ and take δ small so that $(M + cR_0)L\delta < \varepsilon$. With this choice (2.3) yields

$$G \leqslant H + 2\varepsilon + L\varepsilon|\mu|(X).$$

Since $|\mu|(X) < \infty$ and ε is arbitrary we conclude $G \leqslant H$. Q.E.D.

REMARK: The assumption $f \geqslant 0$ in the theorem can be removed. Since K_x is lower semicontinuous in x, by the continuous selection theorem [9] there is $a \in C(X, \mathbf{R}^d)$ such that $a(x) \in K_x$ for $x \in X$, which implies

$$\tilde{f}(x,p) = f(x,p) - < p, a(x) > \geqslant 0 \quad \text{for all} \quad p \in \mathbf{R}^d.$$

We apply the theorem for $\tilde{f}$ and obtain

$$\begin{aligned} \int_X \tilde{f}(x,\mu)\varphi &= \int_X f(x,\mu)\varphi - \int < \vec{\mu}(x), a(x) > \varphi d|\mu| \\ &= \sup_{\substack{v(x) \in K_x(\tilde{f}) \\ v \in C(X, \mathbf{R}^d)}} \int < \vec{\mu}(x), v(x) > \varphi d|\mu| \end{aligned}$$

which yields

$$\begin{aligned} \int_X f(x,\mu)\varphi &= \sup_{\substack{v(x) \in K_x(\tilde{f}) \\ v \in C(X, \mathbf{R}^d)}} \int < \vec{\mu}(x), v(x) + a(x) > \varphi d|\mu| \\ &= \sup_{\substack{v \in K_x(f) \\ v \in C(X, \mathbf{R}^d)}} \int < \vec{\mu}(x), v(x) > \varphi d|\mu|. \end{aligned}$$

Here $K_x(\tilde{f})$ denote the subdifferential of $\tilde{f}_x$ at zero so that $K_x(\tilde{f}) = K_x(f) - a(x)$.

3. Duality formula — inhomogeneous version. Our goal is to derive a duality formula when the integrand f is not necessarily positively homogeneous. We begin by listing assumptions on f.

Let X be a compact metric space. We consider a real valued function $f = f(x,p)$ defined on $X \times \mathbf{R}^d$ which satisfies

$$f \text{ is continuous on } X \times \mathbf{R}^d \tag{3.1}$$

$$f_x(p) = f(x,p) \text{ is convex in } p \in \mathbf{R}^d. \tag{3.2}$$

On the growth of f we assume :

$$f(x,p) \leqslant C(1+|p|) \tag{3.3}$$

with some constant $C > 0$ independent of $(x,p) \in X \times \mathbf{R}^d$. We further assume some equicontinuity in x :

(3.4) For every $x_0 \in X$ and $\varepsilon > 0$ there is $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|f(x,p) - f(x_0,p)| < \varepsilon(1+|p|) \quad \text{for all} \quad p \in \mathbf{R}^d.$$

3.1. Definition of $f(x,\mu)$. We consider the homogenization $F(x,p_0,p)$ of $f(x,p)$ defined by

$$F(x,p_0,p) = \begin{cases} f_\infty(x,p) & \text{if} \quad p_0 = 0 \\ f(x,p/p_0)p_0 & \text{if} \quad p_0 > 0 \\ \infty & \text{if} \quad p_0 < 0 \end{cases}$$

where f_∞ is the recession function of f, i.e.,

$$f_\infty(x,p) = \lim_{t \downarrow 0} f(x,p/t)t.$$

As shown in [5], F is a well-defined real valued continuous function on $X \times C$ with $C = [0,\infty) \times \mathbf{R}^d$ and $F = \infty$ on $X \times (\mathbf{R}^{d+1} \backslash C)$ provided that f satisfies (3.1)-(3.4). Moreover, it is easy to see that F is convex in $(p_0, p) \in \mathbf{R}^{d+1}$ and positively homogeneous in the sense that

$$F(x, \lambda p_0, \lambda p) = \lambda F(x, p_0, p)$$

for all $\lambda \geqslant 0$ and $(x, p_0, p) \in X \times C$.

We next give a meaning of $f(x, \mu)$ by using F. Let α be a nonnegative finite Borel regular measure on X. We fix this measure and for a $\mathbf{R}^d$ valued finite Borel regular measure μ on X we define

$$f(x, \mu) = F(x, \nu), \quad \nu = (\alpha, \mu).$$

Here F is the homogenization of f and ν is a $C = [0, \infty) \times \mathbf{R}^d$ valued Borel regular measure. Since F is positively homogeneous of degree one, $f(x, \mu)$ is a finite Borel regular measure. It is easy to see that

$$\begin{aligned} f(x, \mu) &= F(x, \alpha, \mu) \\ &= F(x, 1, \vec{h}(x))\alpha + F(x, 0, \mu^s) \\ &= f(x, \vec{h}(x))\alpha + f_\infty(x, \vec{\mu}^s(x))|\mu^s| \end{aligned}$$

where $\vec{h}(x)\alpha$ is the absolutely continuous part of μ and μ^s is the singular part with respect to α. We refer to [8] for the definition of $f(x, \mu)$ and its fundamental properties.

3.2. Duality formula. *Suppose that $f = f(x, p)$ satisfies (3.1)-(3.4). Let μ be a $\mathbf{R}^d$-valued finite Borel regular measure on X. Let $\varphi \geqslant 0$ be a bounded $|\mu|$-measurable function on X. Then*

$$\begin{aligned} \int_X f(x, \mu)\varphi = \sup \Big\{ &\int_X < \vec{\mu}(x), v(x) > \varphi(x) d|\mu|(x) - \int_X \varphi(x) f^*(x, v(x)) d\alpha; \\ &v \in C(X, \mathbf{R}^d), f^*(x, v(x)) \in L^1(X, d\alpha) \Big\} \end{aligned} \tag{3.5}$$

where $f^*(x, p) = f_x^*(p)$ and $L^1(X, d\alpha)$ is the space of α-integrable functions.

PROOF: Since F is positively homogeneous on $C = [0, \infty) \times \mathbf{R}^d$ one can apply the homogeneous version of the duality formula for F in §2.6 (theorem and remark). For every $\varepsilon > 0$ the duality formula yields the existence of

$$v_0 \in C(X, \mathbf{R}) \quad \text{and} \quad v \in C(X, \mathbf{R}^d)$$

such that

$$\text{(3.6)} \qquad \int_X \varphi(x)F(x,\alpha,\mu) \leqslant \int_X \varphi v_0 d\alpha + \int_X \varphi < v, \vec{\mu} > d|\mu| + \varepsilon$$

$$(v_0(x), v(x)) \in K_x$$

where $K_x = \partial F_x(0) = \{(p_0, p) \in \mathbf{R}^{d+1}; p_0 \cdot q_0 + < p, q > \leqslant F(x, q_0, q)$ for all $q_0 \geqslant 0$ and $q \in \mathbf{R}^d\}$.

We next prove that $f^*(x, v(x))$ is α-integrable on X. By the definition of K_x and F one observes that

$$v_0(x)p_0 + < v(x), p > \leqslant F(x, p_0, p) = \begin{cases} f_\infty(x,p) & \text{if } p_0 = 0 \\ f(x, p/p_0) \cdot p_0 & \text{if } p_0 > 0 \end{cases}$$

which yields

$$< v(x), p/p_0 > -f(x, p/p_0) \leqslant -v_0(x).$$

Since this inequality holds for every $p \in \mathbf{R}^d$ and $p_0 \geqslant 0$, it yields

$$\text{(3.7)} \qquad f^*(x, v(x)) \leqslant -v_0(x).$$

By the definition of f^* we see

$$f^*(x, v(x)) = \sup_{p \in \mathbf{R}^d} (< v(x), p > -f(x,p)) \geqslant -f(x, 0)$$

Thus the function $f^*(x, v(x)) \in L^1(X, d\alpha)$.

Multiplying φ with (3.7) and integrating it against $d\alpha$ yields

$$\int_X \varphi(x) v_0(x) d\alpha(x) \leqslant -\int_X \varphi(x) f^*(x, v(x)) d\alpha(x).$$

From (3.6) it now follows that

$$\int_X \varphi(x)F(x,\alpha,\mu) \leqslant \int_X \varphi < v, \vec{\mu} > d|\mu| - \int_X \varphi f^*(x, v) d\alpha + \varepsilon$$

which yields (3.5) since the inequality $\geqslant$ in (3.5) is trivial by (1.1) (cf. remark in §2.1).

Q.E.D.

REMARK: The definition of $f(x,\mu)$ may depend on the choice of α unless f is positively homogeneous of degree one in p. When X is a compact subset of $\mathbf{R}^N$ we usually take the Lebesgue measure as α.

REMARK: Our duality formulas (2.1) and (3.5) still hold even if X is merely a compact Hausdorff space. The assumption that X is a metric space is invoked only in the proofs in §2.2 and §2.4. However, both statements in §2.2 and §2.4 can be proved when X is a compact Hausdorff space, though we have to modify the definition of continuity of K_x.

REMARK: In [11] Rockafellar showed results similar to (2.1) and (3.5) with $\varphi \equiv 1$ under assumptions different from ours. Although neither sets of assumptions do not imply the other, there is a drawback in the results of [11] for our purpose because K_x is always assumed to have a nonempty interior. This restriction is serious to apply the duality formula in the next section.

4. A variant of area formulas. An area formula says that the area $\mathcal{H}^n(G)$ of the graph G of a scalar C^1 function u defined on $\Omega \subset \mathbf{R}^n$ is given by

$$\mathcal{H}^n(G) = \int_G 1 d\mathcal{H}^n(x,y) = \int_\Omega (1+|\nabla u|^2(x))^{1/2} d\mathcal{L}^n(x),$$

where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure in $\Omega \times \mathbf{R}$ and $\mathcal{L}^n$ denotes the Lebesgue measure on $\mathbf{R}^n$. More generally it gives an expression of

$$I = \int_\Omega f(x, u(x), \nabla u(x)) d\mathcal{L}^n(x)$$

through a measure supported on the graph G even if u is a mapping : $\Omega \to \mathbf{R}^m$ for $m > 1$. We are interested in deriving a similar formula for I when u is merely in $BV(\Omega, \mathbf{R}^m)$, the space of mappings of (essentially) bounded variation. Since ∇u is a (finite) $\mathbf{R}^{nm}$ valued Radon measure, I is well-defined, provided that f does not depend on u explicitly (under the assumptions in Sect.3). If f does depend on u, the value of I itself is even unclear. Indeed, Dal Maso [5] gave a reasonable definition of I through area formulas but only for $m = 1$. Aviles and Giga [2, 3] introduced several new ideas and gave the meaning for I through area formulas for general $m > 1$ for a certain class of f. In [2, Theorem 8.2] an variant of area formula is given for I when f does not depend on u. The proof is rather long and depends on various properties of minimal graphs. In this section we give a simple proof depending on the duality formula in Sect. 3.

Let D be an bounded open set in $\mathbf{R}^n$. Let $T = (T_0, T_i^j)_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$ be a $C = [0,\infty) \times \mathbf{R}^{nm}$ valued Radon measure on $U = D \times \mathbf{R}^m$. For a smooth n-forms ω with compact support in U, we define

$$T(\omega) = < T_0, \omega_0 > + \sum_{i=1}^{n} \sum_{j=1}^{m} < T_i^j, \omega_{ij} >$$

so that T is a n-current on U. Here ω is of form

$$\omega = \omega_0(x,y) dx_1 \wedge \cdots \wedge dx_n + \sum_{i=1}^{n} \sum_{j=1}^{m} \omega_{ij}(x,y) dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dy_j \wedge dx_{i+1} \wedge \cdots \wedge dx_n$$
$$+ \sigma \qquad (x,y) \in D \times \mathbf{R}^m$$

with σ contains at least two dy's in each terms of its expression by dx_α and dy_β; the bracket means a pairing of measures and functions. We need subspaces of n-forms. Let $\mathcal{D}^{n,0}$ denotes the set of n-forms ω compactly supported in U of form

$$\omega = \omega_0(x,y) dx_1 \wedge \cdots \wedge dx_n;$$

let $\mathcal{B}^{n,1}$ denote the set of n-forms ω compactly supported in U of form

$$\omega = \omega' + \sum_{i=1}^{n} \sum_{j=1}^{m} dx_1 \wedge \cdots \wedge dx_{i-1} \wedge d_y \psi_j \wedge dx_{i+1} \wedge \cdots \wedge dx_n$$

with $\omega' \in \mathcal{D}^{n,0}$ and smooth $\psi_j = \psi_j(x,y)$ compactly supported in U, where d_y denotes the exterior derivative in y variables. For $\varphi_j, \varphi_{ij} \in C(\overline{D}, \mathbf{R})$ we set

$$\Phi_0 = \varphi_0(x) dx_1 \wedge \cdots \wedge dx_n \qquad 1 \leqslant j \leqslant m$$
$$\Phi_{ij} = \varphi_{ij}(x) dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dy_j \wedge \cdots \wedge dx_n, \quad 1 \leqslant i \leqslant n.$$

Unfortunately both forms do not belong to $\mathcal{D}^{n,0}$ and $\mathcal{B}^{n,1}$ because of support conditions and regularity. However, one can approximate elements in $\mathcal{D}^{n,0}$ and $\mathcal{B}^{n,1}$ by a standard argument.

LEMMA 4.1 (APPROXIMATION). *For Φ_0 there is a sequence $\{\omega_\ell\}$ in $\mathcal{D}^{n,0}$ such that $\omega_\ell \to \Phi_0$ uniformly on every compact sets in U (as $\ell \to \infty$) and that $\sup_\ell \sup_U |\omega_\ell| < \infty$*

(ii) For each Φ_{ij} there is a sequence $\{\omega_\ell\}$ in $\mathcal{B}^{n,1}$ such that $\omega_\ell \to \Phi_{ij}$ uniformly on every compact sets in U (as $\ell \to \infty$) and that $\sup_\ell \sup_U |\omega_\ell| < \infty$.

We next associate $u \in BV(D, \mathbf{R}^m)$ with current T. Let T be a C-valued Radon measure on U. We say T is a *stitched graph* of u if

$$T(\omega) = G_u^*(\omega) \quad \text{for all} \quad \omega \in \mathcal{B}^{n,1},$$

where G_u^* is the *graph cycle* of u in [2, §4].

Note that the graph of u may have huge gaps since u merely belongs to $BV(\Omega, \mathbf{R}^m)$. The graph cycle lives in a quotient space of currents. It associates with the graph of u with some "stitches" at gaps but the way of stitchings is in moduls class. A stitched graph is one of its representatives.

Instead of giving the precise definition of the graph cycle we recall the value of $G_u^*(\omega)$ for special ω. For $\omega' \in \mathcal{D}^{n,0}$

$$G_u^*(\omega') = \int_D \omega_0(x, u(x))d\mathcal{L}^n(x), \quad \omega' = \omega_0(x,y)dx_1 \wedge \cdots \wedge dx_n.$$

Since G_u^* satisfies growth conditions [2, Theorem 4.2], one can extend the definition of G_u^* on Φ_0 and Φ_{ij} by the approximation lemma 4.1.

LEMMA 4.2. *Let Φ_0 and Φ_{ij} be as above for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Let $u = (u^1, \cdots u^m) \in BV(D, \mathbf{R}^m)$. Then*

$$G_u^*(\Phi_0) = \int_D \varphi_0(x)d\mathcal{L}^n(x)$$
$$G_u^*(\Phi_{ij}) = \int_D \varphi_{ij}(x)\nabla_i u^j,$$

where $\nabla_i u^j$ is the distributional partial derivative $\partial u^j/\partial x_i$.

The first identy follows from the value of $G_u^*(\omega')$ for $\omega' \in \mathcal{D}^{n,0}$ and lemma 4.1. The second identity is included in [2, Lemma 4.7] except that φ_{ij} is now taken in $C(\overline{D}, \mathbf{R}^{nm+1})$. This alternation is possible because ∇u is now a bounded measure on D so that [2, Theorem 4.2] holds with $D = \Omega$.

If T satisfy $\int_U d|T| < \infty$, then the values $< T_0, \varphi_0 >, < T_i^j, \varphi_{ij} >$ are finite and approximable by the element of $\mathcal{D}^{n,0}$ and $\mathcal{B}^{n,1}$.

LEMMA 4.3. *Let $T = (T_0, T_i^j)$ be a stitched graph of $u \in BV(D, \mathbf{R}^m)$. Suppose that*

$$\int_U d|T_i^j| < \infty \quad \text{for} \quad 1 \leqslant i \leqslant n, \ \ 1 \leqslant j \leqslant m.$$

Then

$$< T_0, \varphi_0 > = \int_D \varphi_0(x) d\mathcal{L}^n(x)$$
$$< T_i^j, \varphi_{ij} > = \int_D \varphi_{ij}(x) \nabla_i u^j.$$

PROOF: Since $G_u^*(\omega') = < T_0, \omega_0 >$ for $\omega' = \omega_0 dx_1 \wedge \cdots \wedge dx_n \in \mathcal{D}^{n,0}$, the expression of $G_u^*(\omega')$ yields

$$< T_0, \omega_0 > = \int_D \omega_0(x, u(x)) d\mathcal{L}^n(x).$$

Since $\mathcal{L}^n(D) < \infty$, this implies that T_0 is a finite measure on $D \times \mathbf{R}^m$ so that $\int_U d|T| < \infty$. By Lemma 4.1 one now extends $T(\omega)$ for $\omega = \Phi_0$ or Φ_{ij}. Since T is a stitched graph, Lemma 4.2 yields desired formulas. Q.E.D.

THEOREM 4.4. *Let D be a bounded open set in $\mathbf{R}^n$. Suppose that $f = f(x,p)$ is continuous on $\overline{D} \times \mathbf{R}^{nm}$ and that f is convex in p. Suppose that there is $0 < c \leqslant C$ such that*

$$c|p| \leqslant f(x,p) \leqslant C(|p| + 1).$$

For each $x_0 \in D$ and $\varepsilon > 0$ there is a positive constant C such that $|x - x_0| < \delta$ implies

$$|f(x,p) - f(x_0,p)| \leqslant \varepsilon(1 + |p|) \quad \text{for all} \quad p \in \mathbf{R}^{nm}.$$

Then for $u \in BV(D, \mathbf{R}^m)$

$$\int_D f(x, \nabla u) = \inf_T \left\{ \int_U F(x,T); T \text{ is a stitched graph of } u \right\}$$

where F is the homogenization of f and $U = D \times \mathbf{R}^m$.

PROOF: It suffices to prove

$$\int_U F(x,T) \geqslant \int_D f(x, \nabla u)$$

for all stitched graph of u since the equality holds for the affinely stitched graph of u by a direct calculation; see [2, the last paragraph of the proof of Theorem 8.2] and [8, Remark 1.5]. We may assume $\int_U F(x,T) < \infty$ otherwise the above inequality is trivial since $\int_D f(x,\nabla u)$ is finite by $u \in BV(D,\mathbf{R}^m)$ and the growth condition of f. Let K_x be the subdifferential of $F_x(p_0,p) = F(x,p_0,p)$ at $\xi = (p_0,p) = 0$, i.e.,

$$K_x = \{\xi = (p_0,p) \in C = [0,\infty) \times \mathbf{R}^{nm}; \sup_{\eta \in C}(<\xi,\eta> - F(x,\eta)) \leqslant 0\}.$$

Then by the duality of convex functions we have

$$F(x,\xi) = \sup_{\eta \in K_x} <\xi,\eta>.$$

Since $T = (T_0, T_i^j)$ is a measure

$$\int_U F(x,T) \geqslant \sup\{<T,v>; \quad v(x) \in K_x, \quad v \in C(\overline{D},C)\} \equiv J$$

Since $c|p| \leqslant f(x,p)$ implies $c|p| \leqslant F(x,\xi)$, $\xi = (p_0,p)$, $\int_U F(x,T) < \infty$ yields

$$\int_U d|T_i^j| < \infty.$$

Applying Lemma 4.3 we obtain, for $v = (\varphi_0, \varphi_{ij})$,

$$\begin{aligned} <T,v> &= <T_0,\varphi_0> + \sum_{i=1}^n \sum_{j=1}^m <T_i^j, \varphi_{ij}> \\ &= \int_D \varphi_0(x) d\mathcal{L}^n(x) + \sum_{i=1}^n \sum_{j=1}^m \int_D \varphi_{ij}(x) \nabla_i u^j. \end{aligned}$$

Taking supremum over $v = (\varphi_0, \varphi_{ij})$ such that $v(x) \in K_x$ and $v \in C(\overline{D}, C)$ yields

$$J = \int_D F(x, \mathcal{L}^n, \nabla u)$$

by the duality formula in §2.6 for F. Note that F is continuous only on $\overline{D} \times [0,\infty) \times \mathbf{R}^{nm}$ and outside $F = \infty$ so the formula in §2.1 is not applicable. The right hand side of the last identity equals $\int_D f(x,\nabla u)$ (see §3.1), so we conclude

$$\int_U F(x,T) \geqslant J = \int_D f(x,\nabla u)$$

for any stitched graph of T and this is the desired inequality. Q.E.D.

REMARK: Theorem 4.4 appears to be weaker than [2, Theorem 8.2] where u is assumed to be merely a mapping of locally bounded variation in an arbitrary open set Ω. However, one can derive [2, Theorem 8.2] easily be a localization. Indeed, we observe that Ω is expressed as a countable disjoint union of bounded open sets D_i up to a $|\nabla u|$-measure zero set. Since an f-minimal graph can be localized [2, Theorem 6.1], adding the representation formula of $\int f(x, \nabla u)$ on D_i in Theorem 4.4 yields

$$\int_{\Omega\times\mathbf{R}^m} F(x,T) \geqslant \int_{\Omega} f(x, \nabla u)$$

for all stitched graph of u on Ω. This deduces [2, Theorem 8.2].

REFERENCES

[1] J. P. Aubin and H. Frankowska, *Set-valued analysis*, Birkhäuser, Boston-Basel-Berlin.

[2] P. Aviles and Y. Giga, *Variational integrals on mappings of bounded variation and their lower semicontinuity*, Arch. Rational Mech. Anal. **115** (1991), 201-255.

[3] ——————, *Announcement: Minimal currents, geodesics and relaxation of variational integrals on mappings of bounded variation*, Duke Math. J. (to appear); *Minimal currents and relaxation of variational integrals on mappings of bounded variation*, Proc. Japan Acad. Ser.A. Math. **66** (1990), 68-71.

[4] C. Castaing, *Sur les multi-applications mesurables*, thèse Caën (1967).

[5] G. Dal Maso, *Integral representation of $BV(\Omega)$ of Γ-limits of variational integrals*, Manuscripta Math. **30** (1980), 387-417.

[6] I. Ekeland and R. Temam, *Convex analysis and variational problems*, North-Holland, Amsterdam (1976).

[7] F. Demengel and R. Temam, *Convex functions of a measure and applications*, Indiana Univ. Math. J. **33** (1984), 673-709.

[8] M. Giaguinta, G. Modica and J. Souček, *Functionals with linear growth in the calculus of variation I*, Comment. Math. Univ. Carolinae **20** (1979), 143-156.

[9] E. Michael, *Continuous selections. I*, Ann. of Math. **63** (1956), 361-382.

[10] Yu. G. Reshetnyak, *Weak convargence of completely additive vector functions on a set*, Siberian Math. J. **9** (1968), 1039-1045; (translation of Sibirsk. Math. Ž. **9** (1968), 1386-1394).

[11] R. T. Rockafellar, *Integrals which are convex functionals*, II, Pacific J. Math. **39** (1971), 439-469.

[12] R. T. Rockafellar, *Integral functionals, normal integrands and measurable selections, in Nonlinear operators and the calculus of variations*, Springer Lecture Notes in Math. 543(1976), 157-207.

[13] R. T. Rockafellar, *Convex analysis*, Princeton University Press (1970).

Patricio Aviles
Department of Mathematics
University of Illinois
Urbana, Illinois 61801, U.S.A.

Yoshikazu Giga
Department of Mathematics
Hokkaido University
Sapporo 060, JAPAN

Naoto Komuro
Hokkaido University of Education
at Asahikawa
9 Hokumoncho
Asahikawa 070, JAPAN