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The Clairaut type equation

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Abstract. A characterization of first order differential equations with (classical) complete solutions is given. The (classical) Clairaut equation is also characterized as the special case.

0. INTRODUCTION

In this note we shall give a characterization of first order partial differential equations (briefly, equations) with (classical) complete solutions. Roughly speaking, we shall prove that this class of equations is equal to a class of equations with singular solutions which will be called Clairaut type equations. In classical treatises on equations (see Forsyth [1]) the discussions of equations with singular solutions are informal. In relation to this fact, we have shown that equations with singular solutions are non generic in the space of all equations [2]. Nevertheless, this class of equations is quite interesting. One of the typical examples of equations with singular solutions is the (classical) Clairaut equation (see Example 1.2) which has a (classical) complete solution consisting of hyperplanes. Moreover, the graph of the singular solution is an envelope of the family of graphs of the complete solution. But, we have no reasons why the complete solution must consist of hyperplanes. Then we shall give the notion of Clairaut type equations which is a genralization of the notion of (classical) Clairaut equations. Our main result is Theorem 1.1 which will be proved in §2. The method of the proof is surprisingly direct. We shall also give a characterization of the (classical) Clairaut equation as a special form of the Clairaut type equation in §3.

All maps considered here are class C^∞ unless stated otherwise.

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1. THE MAIN RESULT

In this section we shall state the main result. *An equation* is a submersion germ $F : (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0)) \to (\mathbb{R}, 0)$ on the 1-jet space of functions of n-variables. Let θ be the canonical contact form on $J^1(\mathbb{R}^n, \mathbb{R})$ which is given by $\theta = dy - \sum_{i=1}^n p_i dx_i$, where (x, y, p) is the canonical coordinate of $J^1(\mathbb{R}^n, \mathbb{R})$. We define *a geometric solution of* $F = 0$ to be an immersion $i : (L, q_0) \to (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0))$ of an n-dimensional manifold such that $i^*\theta = 0$ and $i(L) \subset F^{-1}(0)$ (i.e. a Legendrian submanifold which is contained in $F^{-1}(0)$). We say that (x_0, y_0, p_0) is *a contact singular point* if $\theta(T_{(x_0,y_0,p_0)} F^{-1}(0)) = 0$. It is easy to

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show that (x_0, y_0, p_0) is a contact singular point if and only if $F = F_{p_i} = F_{x_i} + p_i F_y = 0$ for $i = 1, \ldots, n$ at (x_0, y_0, p_0), where $F_{p_i} = \frac{\partial F}{\partial p_i}$ etc. We also say that (x_0, y_0, p_0) is *a π-singular point* if $F = F_{p_i} = 0$ for $i = 1, \ldots, n$ at (x_0, y_0, p_0). We denote the set of contact singular points by $\Sigma_c(F)$, the set of π-singular points by $\Sigma_\pi(F)$ and $\pi(\Sigma_\pi(F)) = D_F$, where $\pi(x, y, p) = (x, y)$. We call the set D_F *a discriminant set* of the equation $F = 0$.

An equation $F = 0$ is said to be *Clairaut type* if there exist smooth function germs $B_{ji}, A_i : (J^1(\mathbb{R}^n, \mathbb{R}, (x_0, y_0, p_0)) \to \mathbb{R}$ for $i, j = 1, \ldots, n$ such that

$$F_{x_i} + p_i F_y = \sum_{j=1}^{n} B_{ji} F_{p_j} + A_i F \qquad (i = 1, \ldots, n)$$

and satisfy that

$$(1) \qquad B_{ji} = B_{ij}$$

$$(2) \qquad \frac{\partial B_{jk}}{\partial x_i} + p_i \frac{\partial B_{jk}}{\partial y} + \sum_{\ell=1}^{n} B_{\ell i} \frac{\partial B_{jk}}{\partial p_\ell} = \frac{\partial B_{ji}}{\partial x_k} + p_k \frac{\partial B_{ji}}{\partial y} + \sum_{\ell=1}^{n} B_{\ell k} \frac{\partial B_{ji}}{\partial p_\ell}$$

at any $(x, y, p) \in (F^{-1}(0), (x_0, y_0, p_0))$ for $i, j, k = 1, \ldots, n$.

We also say that an n-parameter family of function germs $f : (\mathbb{R}^m \times \mathbb{R}^n, (t_0, x_0)) \to (\mathbb{R}, y_0)$ is *a (classical)complete solution of* $F = 0$ if $F(x, f(t, x), f_x(t, x)) = 0$ and $\mathrm{rank}(f_{t_i}, f_{t_i x_j}) = n$. Our main result is the following.

THEOREM 1.1. *For an equation germ $F = 0$, the following are equivalent.*
(1) $F = 0$ is the Clairaut type equation.
(2) $F = 0$ has a (classical) complete solution.

In this case, if $\Sigma_\pi(F) \neq \emptyset$, then $\Sigma_\pi(F)$ is a geometric solution (i.e. the singular solution) of $F = 0$ and the discriminant set D_F is the envelope of the family of graphs of the complete solution.

By the classical existence theorem (see [3]), if $F = 0$ is a π-regular equation, then there exists a (classical) complete solution. Then we can assert that π-regular equation is Clairaut type by the above theorem.

Here, we give three examples which describe the above assertion.

EXAMPLES 1.2. 1) Of course, one of examples of Clairaut type equations is the (classical) Clairaut equation. The Clairaut equation is given by $y = \sum_{i=1}^{n} x_i p_i + f(p_1, \ldots, p_n)$, where f is a smooth function. The complete solution is given by $y = \sum_{i=1}^{n} x_i t_i + f(t_1, \ldots, t_n)$ and we can easily verify that $F_{x_i} + p_i F_y = 0$ for $i = 1, \ldots, n$.
2) The second example is an equation for "Free particle" in the n-dimensional space. Consider the following equation ; $y^2 + \sum_{i=1}^{n} p_i^2 - 1 = 0$. Then we have

$$\Sigma_\pi(F) = \{(x_1, \ldots, x_n, \pm 1, 0, \ldots, 0) | (x_1, \ldots, x_n) \in \mathbb{R}^n\}.$$

We can calculate that $F_{x_i} + p_i F_y = y F_{p_i}$ for $i = 1, \ldots, n$. Then we have $B_{ij} = \pm(1 - \sum_{k=1}^{n} p_k^2)^{\frac{1}{2}}$ and $A_i = \frac{2p_i}{y \pm (1 - \sum_{k=1}^{n} p_k)^{\frac{1}{2}}}$, where $\pm$ corresponds to the point $(0, \pm 1, 0)$. The

complete solution is given by $y = \pm\cos(\frac{1}{(\sum_{i=1}^{n}(1+t_i)^2)^{\frac{1}{2}}}\sum_{i=1}^{n}(x_i + t_i x_i))$, which is defined on $(\mathbb{R}^n \times \mathbb{R}^n, (0,0))$.
3) Consider the following equation ; $y - 2p^3 = 0$. We can show that $\Sigma_\pi(F) = \Sigma_c(F) = \{(x,0,0)|x \in \mathbb{R}\}$ which is a singular solution. We also have an one-parameter family of geometric solution $s : (\mathbb{R}\times\mathbb{R}, 0) \to J^1(\mathbb{R},\mathbb{R})$ given by $s(u,t) = (3u^2 + t, 2u^3, u)$, where t is the parameter. Of course, each member have multi-valued near $\Sigma_\pi(F)$, then this solution is not the (classical) complete solution. This is the example which is an equation with singular solution but not Clairaut type.

In the classical books (see [1]), the notion of singular solution has been appeared accompany with the notion of complete solutions. In there the singular solution has been defined to be the envelope of the family of graphs of the complete solution. Theorem 1.1 gives a characterization of this class of equations as the class of Clairaut type equations.

2. Proof of Theorem 1.1

In this section we shall give a proof of Theorem 1.1. For our purpose, we need some preparations on the theory of Legendrian singularities. For a Legendrian immersion germ $i : (L, q_0) \to J^1(\mathbb{R}^n, \mathbb{R})$, $q_0 \in L$ is said to be *a Legendrian singular point* if $\pi \circ i$ is not an immersion at q_0. Then we have the following lemma.

LEMMA 2.1. *For an equation $F : (J^1(\mathbb{R}^n,\mathbb{R}), (x_0, y_0, p_0)) \to (\mathbb{R}, 0)$, the following are equivalent.*
(1) *$F = 0$ has a (classical) complete solution.*
(2) *There exists a foliation on $F^{-1}(0)$ by geometric solutions of $F = 0$ with leaves are Legendrian non singular.*

PROOF: Suppose that $f : (\mathbb{R}^n \times \mathbb{R}^n, (t_0, x_0)) \to (\mathbb{R}, y_0)$ is a (classical) complete solution of $F = 0$. Then we define a map germ $j^1_* f : (\mathbb{R}^n \times \mathbb{R}^n, (t_0, x_0)) \to (J^1(\mathbb{R}^n,\mathbb{R}), (x_0, y_0, p_0))$ by $j^1_*(t,x) = (x, f(t,x), f_x(t,x))$. We can easily show that $j^1_* f$ is an immersion if and only if $\mathrm{rank}(f_{t_i}, f_{t_i x_j}) = n$. It follows that $j^1_* f$ gives a local parameterization of $F^{-1}(0)$ and the family $\{\mathrm{Image} j^1_* f_t\}_{t\in(\mathbb{R}^n,t_0)}$ gives a desired foliation, where $f_t(x) = f(t,x)$.

For the converse, we remark that q_0 is a Legendrian non singular point of Legendrian immersion $i : (L, q_0) \to J^1(\mathbb{R}^n,\mathbb{R})$ if and only if $\tilde{\pi} \circ i$ is a local diffeomorphism at q_0, where $\tilde{\pi}(x,y,p) = x$.

Suppose that there exists a foliation which satisfies the condition (2). Then we have an n-parameter family of smooth sections $s : (\mathbb{R}^n \times \mathbb{R}^n, (t_0, x_0)) \to (J^1(\mathbb{R}^n,\mathbb{R}), (x_0, y_0, p_0))$ of $\tilde{\pi}$ (i.e. $\tilde{\pi} \circ s(t,x) = x$) such that s is an immersion, $s(\mathbb{R}^n \times \mathbb{R}^n) = F^{-1}(0)$ and $s_t^*\theta = 0$ for any $t \in (\mathbb{R}^n, t_0)$, where $s_t(x) = s(t,x)$. It follows that there exists a family of function germs $f : (\mathbb{R}^n \times \mathbb{R}^n, (t_0, x_0)) \to (\mathbb{R}^n, y_0)$ such that $j^1_* f(t,x) = s(t,x)$. Since s is an immersion, then f is a (classical) complete solution of $F = 0$.

Now we can give the proof that (1) implies (2).

PROOF OF THEOREM 1.1, (1) $\Rightarrow$ (2): By the assumption, there exist function germs $B_{ij}, A_i : (J^1(\mathbb{R}^n,\mathbb{R}), (x_0, y_0, p_0)) \to \mathbb{R}$ such that $F_{x_i} + p_i F_y = \sum_{j=1}^{n} B_{ji} F_{p_j} + A_i F$ for $i = 1, \ldots, n$, $B_{ji} = B_{ij}$ and $\frac{\partial B_{jk}}{\partial x_i} + p_i \frac{\partial B_{jk}}{\partial y} + \sum_\ell^n B_{\ell i} \frac{\partial B_{jk}}{\partial p_\ell} = \frac{\partial B_{ji}}{\partial x_k} + p_k \frac{\partial B_{ji}}{\partial y} + \sum_\ell^n B_{\ell k} \frac{\partial B_{ji}}{\partial p_\ell}$ at any $(x,y,p) \in (F^{-1}(0), (x_0, y_0, p_0))$ for $i, j, k = 1, \ldots, n$.

We consider linearly independent vector fields

$$V_i = \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial y} - \sum_{j=1}^{n} B_{ji} \frac{\partial}{\partial p_j} \qquad (i = 1, \dots, n)$$

on $(J^1(\mathrm{R}^n, \mathrm{R}), (x_0, y_0, p_0))$. Let $c(t)$ be an integral curve of V_i such that $c(0) \in F^{-1}(0)$. Then we can show that $\frac{dF(c(t))}{dt}|_{t=0} = F_{x_i} + p_i F_y - \sum_{j=1}^{n} B_{ji} F_{p_j} = 0$. It follows that $V_i(x, y, p) \in T_{(x,y,p)} F^{-1}(0)$ for any $(x, y, p) \in F^{-1}(0)$. Since V_i are linearly independent, then we can define an n-dimensional distribution E on $F^{-1}(0)$ which is generated by vectors $V_i(x, y, p)$ on each point $(x, y, p) \in F^{-1}(0)$. By the direct calculation, we have

$$[V_i, V_k] = \sum_{j=1}^{n} \left(\frac{\partial B_{ji}}{\partial x_k} - \frac{\partial B_{jk}}{\partial x_i} + p_k \frac{\partial B_{ji}}{\partial y} - p_i \frac{\partial B_{jk}}{\partial y} + \sum_{\ell=1}^{n} B_{\ell k} \frac{\partial B_{ji}}{\partial p_\ell} - \sum_{\ell=1}^{n} B_{\ell i} \frac{\partial B_{jk}}{\partial p_\ell} \right) \frac{\partial}{\partial p_j}$$

for any $i, k = 1, \dots, n$. By the assumption, $[V_i, V_k](x, y, p) \in E_{(x,y,p)}$ for any $(x, y, p) \in F^{-1}(0)$. Then the distribution E is integrable and there exists an n-dimensional foliation on $F^{-1}(0)$ by the Frobenius' theorem. Since $\theta(V_i) = 0$, then leaves of this foliation are Legendrian submanifold. By the definition of V_i, we have $d\tilde{\pi}(V_i) = \frac{\partial}{\partial x_i}$. It follows that leaves are Legendrian non singular. Then this foliation gives a (classical) complete solution by Lemma 2.1.

The converse direction is rather a direct.

PROOF OF THEOREM 1.1, (2) $\Rightarrow$ (1): Let $y = f(t, x)$ be a complete solution of $F = 0$. If we calculate the partial derivative of $F(x, f(t, x), f_x(t, x)) = 0$ with respect to x_i, then we have $F_{x_i} + f_{x_i} F_y + \sum_{j=1}^{n} f_{x_j x_i} F_{p_j} = 0$ at $(x, f(t, x), f_x(t, x)) \in F^{-1}(0)$.

Since the map germ $f_*^1 f$ is an immersion germ, then there exist function germs $B_{ji} : (J^1(\mathrm{R}^n, \mathrm{R}), (x_0, y_0, p_0)) \to \mathrm{R}$ such that $B_{ji} \circ j_*^1 f = f_{x_j x_i}$ for $i, j = 1, \dots n$.

For any $(x, y, p) \in F^{-1}(0)$, there exists $(t, x) \in (\mathrm{R}^n \times \mathrm{R}^n, (t_0, x_0))$ such that $(x, f(t, x), f_x(t, x)) = (x, y, p)$. Then we have $F_{x_i} + p_i F_y = \sum_{j=1}^{n} B_{ji} F_{p_j}$ on $F^{-1}(0)$. This means that there exists a function germ $A_i : (J^1(\mathrm{R}^n, \mathrm{R}), (x_0, y_0, p_0)) \to \mathrm{R}$ such that $F_{x_i} + p_i F_y = \sum_{j=1}^{n} B_{ji} F_{p_j} + A_i F$ for $i, j = 1, \dots, n$.

On the other hand, if we calculate the partial derivative of the equality $f_{x_j x_i}(t, x) = B_{ji}(x, f(t, x), f_x(t, x))$ with respect ro x_k, then we have $f_{x_j x_i x_k} = \frac{\partial B_{ji}}{\partial x_k} + \frac{\partial B_{ji}}{\partial y} f_{x_k} + \sum_{\ell=1}^{n} \frac{\partial B_{ji}}{\partial p_\ell} f_{x_\ell x_k}$. Since $f_{x_k}(t, x) = p_k$, $f_{x_\ell x_k} = B_{\ell k}$ and f is smooth, then $F = 0$ is Clairaut type. This completes the proof that (2) implies (1).

PROOF OF THE SECOND PART OF THEOREM 1.1: By the first part of the theorem, we may assume that there exists a (classical) complete solution $y = f(t, x)$ of $F = 0$ and $\Sigma_\pi(F) \neq \emptyset$. By the definition, $j_*^1(t, x) \in \Sigma_\pi(F)$ if and only if

$$\operatorname{rank} \begin{pmatrix} E & f_x \\ 0 & f_t \end{pmatrix} = n \quad \text{at } (t, x).$$

It is equivalent to the fact that $f_{t_i}(t, x) = 0$. Then the Jacobian matrix of this equation is given by $J(f_{t_1}, \dots, f_{t_n}) = (f_{t_i x_j}, f_{t_i t_k})$. Since $\operatorname{rank}(f_{t_i}, f_{t_i x_j}) = \operatorname{rank}(0, f_{t_i x_j}) = n$ at the point (t, x) with $j^1_* f(t, x) \in \Sigma_\pi(F)$, then we have rank $J(f_{t_1}, \ldots, f_{t_n}) = n$. It follows that $\Sigma_\pi(F) = j^1_* f(\{f_{t_i} = 0 | i = 1, \ldots, n\})$ is an n-dimensional submanifold.

On the other hand, we have $\Sigma_\pi(F) = \Sigma_c(F)$ by the definition of the Clairaut type equation. This means that $\Sigma_\pi(F)$ is a Legendrian submanifold.

Moreover, we now consider the family of graphs of complete solution which is given by the equation $f(t, x) - y = 0$. Then the set

$$\{(x, f(t, x)) | \text{There exists } t \in (\mathbb{R}^n, t_0) \text{ such that } f_{t_i}(t, x) = 0 \ (i = 1, \ldots n)\}$$

is the envelope of this family by the usual method of the elementary calculus. This set is equal to the discriminant set D_F by the previous arguments. This completes the proof of Theorem 1.1.

3. The Clairaut equation

In this section we shall give a characterization of the (classical) Clairaut equation by using the notion of the Legendre transformation. Let $(X_1, \ldots, X_n, Y, P_1, \ldots, P_n)$ be another coordinate system of $J^1(\mathbb{R}^n, \mathbb{R})$ whose contact structure is given by $\Theta = dY - \sum_{i=1}^n P_i dX_i$. We now define a contact diffeomorphism $*L$ by $X_i = p_i$, $Y = \sum_{i=1}^n x_i p_i - y$, $P_i = x_i$ for $i = 1, \ldots, n$. We call *$*L$ the Legendre transformation.* For an equation $F : (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0)) \to (\mathbb{R}, 0)$, we apply the Legendre trasformation $*L$ to the hypersurface $F = 0$. Then we obtain a new hypersurface $G = 0$ given by

$$G(X_1, \ldots, X_n, Y, P_1, \ldots, P_n) = F(P_1, \ldots, P_n, \sum_{i=1}^n P_i X_i - Y, X_1, \ldots, X_n).$$

If we calculate partial derivatives, we can obtain $G_{P_i} = (F_{x_i} + p_i F_y)$, $G_Y = -F_y$ and $G_{X_i} = P_{p_i} + x_i F_y$ for $i = 1, \ldots, n$. Then our characterization is the following.

Theorem 3.1. *For an equation $F : (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0)) \to (\mathbb{R}, 0)$, the following are equivalent.*
(1) There exist smooth function germs $A_i : (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0)) \to (\mathbb{R}, 0)$ such that

$$F_{x_i} + p_i F_y = A_i F \qquad \text{for } i = 1, \ldots, n$$

and $\Sigma_\pi(F) \neq \emptyset$.
(2) There exists a smooth function germ $f(p_1, \ldots, p_n)$ at p_0 such that

$$F^{-1}(0) = \{(x, y, p) | y = \sum_{i=1}^n x_i p_i - f(p_1, \ldots, p_n)\}.$$

Proof: Suppose that the equation $F = 0$ satisfies the condition (1). If $F_y = 0$ at (x_0, y_0, p_0), then $F = F_{p_i} = F_{x_i} = 0$ for $i = 1, \ldots, n$ at (x_0, y_0, p_0). This contradicts to the fact that F is a submersion. By the implicit function theorem, there exist a function germ $h : (\mathbb{R}^{2n}, (x_0, p_0)) \to (\mathbb{R}, y_0)$ and a function germ $\lambda : (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0)) \to \mathbb{R}$ with $\lambda(x_0, y_0, p_0) \neq 0$ such that $F(x, y, p) = \lambda(x, y, p)(h(x, p) - y)$.

We now consider the Legendre transform G of F. Then we have $G(X,Y,P) = \Lambda(X,Y,P)(H(X,P)-Y)$, where $\Lambda(X,Y,P) = -\lambda(P, \sum_{i=1}^{n} X_iP_i - Y, X)$ and $H(X,P) = \sum_{i=1}^{n} X_iP_i - h(P,X)$. It follows that $G_{P_i} = \Lambda \cdot H_{P_i}$ on $G^{-1}(0)$. Since $G_{P_i} = F_{x_i} + p_iF_y$ and $*L(\{(x,y,p)|y = h(x,p)\}) = \{(X,Y,P)|Y = H(X,P)\}$, we have the equality $H_{P_i} \equiv 0$. Then $G(X,Y,P) = \Lambda(X,Y,P)(f(X) - Y)$ for some function germ f at X_0. Pulling back by the Legendre transformation, we have

$$F^{-1}(0) = \{(x,y,p)|y = \sum_{i=1}^{n} x_ip_i - f(p_1,\ldots,p_n)\}.$$

The converse of the proof is given by a direct calculation.

REMARK. By the above theorem, we can assert that the (classical) Clairaut equation is the Clairaut type equation with $B_{ji} = 0$.

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