SINGULAR SOLUTIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS

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Dedicated to the memory of Professor Giko Ikegami

0. Introduction

In classical treatises of equations (Carathéodory [2] , Courant-Hilbert [3], Forsyth [4] [5], Ince [8], Petrovski [14]) the discussions of equations with singular solutions are informal. In there, definitions of singular solutions are very confused. Even in modern articles ([9],[12]), it is studied under special assumptions. In this note we shall give a regorous definition of singular solutions of first order differential equations for real-valued functions (Theorem A).

On the other hand, the complete integrability is an important notion for the classical theory of first order differential equations. The notion of singular solutions has been usually appeared accompany with the notion of complete solutions in the above articles. Recently, we have studied some generic properties about completely integrable systems of first order differential equations as an application of the theory of Legendrian unfoldings ([7],[10],[11]). However, we have never seen a characterization of the complete integrability. Our another purpose is to give a characterization of complete integrability of first order differential equations (Theorem B).

In §1, we shall state our main results. The proof of Theorem A will be given in §2. We shall prove Theorem B in §3. Some typical examples will be given in §4.

All maps considered here are class C^∞ unless stated otherwise.

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1. Main results

In this section we shall state main results. *An equation* is a submersion germ $F : (J^1(\mathbb{R}^n,\mathbb{R}),(x_0,y_0,p_0)) \to (\mathbb{R},0)$ on the 1-jet space of functions of n-variables. Let θ be the canonical contact form on $J^1(\mathbb{R}^n,\mathbb{R})$ which is given by $\theta = dy - \sum_{i=1}^n p_i dx_i$, where (x,y,p) is the canonical coordinate of $J^1(\mathbb{R}^n,\mathbb{R})$. We define *a geometric solution of* $F=0$ to be an immersion $i : (L,q_0) \to (J^1(\mathbb{R}^n,\mathbb{R}),(x_0,y_0,p_0))$ of an n-dimensional manifold such that $i^*\theta = 0$ and $i(L) \subset F^{-1}(0)$ (i.e. a Legendrian submanifold which is contained in $F^{-1}(0)$). We say that (x_0,y_0,p_0) is *a contact singular point* if $\theta(T_{(x_0,y_0,p_0)}F^{-1}(0)) = 0$. It is easy to show that (x_0,y_0,p_0) is a contact singular point if and only if $F = F_{p_i} = F_{x_i} + p_iF_y = 0$ for $i = 1,\dots,n$ at (x_0,y_0,p_0), where $F_{p_i} = \frac{\partial F}{\partial p_i}$ etc. We also say that (x_0,y_0,p_0) is *a π-singular point* if $F = F_{p_i} = 0$ for $i = 1,\dots,n$ at (x_0,y_0,p_0). We denote the set of contact singular points by $\Sigma_c(F)$, the set of π-singular points by $\Sigma_\pi(F)$ and

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

$\pi(\Sigma_\pi(F)) = D_F$, where $\pi(x,y,p) = (x,y)$. We call the set D_F *a discriminant set* of the equation $F = 0$. An equation $F = 0$ is said to be *completely integrable* if there exists a foliation by geometric solutions on $F^{-1}(0)$. In this case such a foliation is called *a complete solution* of $F = 0$. Then we can define the notion of singular solutions. A geometric solution $i : (L, q_0) \to (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0))$ of $F = 0$ is called *a singular solution* of $F = 0$ if it satisfies the following condition :
$(*)$ For any representative $\tilde{i} : U \to F^{-1}(0)$ of i and any open subset $V \subset U$, $\tilde{i}(V)$ is not contained in a leaf of any complete solutions of $F = 0$.

Then we have the following theorem.

THEOREM A. *For an equation $F : (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0)) \to (\mathbb{R}, 0)$ and a geometric solution $i : (L, q_0) \to (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0))$ of $F = 0$, the following conditions are equivalent.*
(1) *i is a singular solution of $F = 0$.*
(2) *There exists a complete solution of $F = 0$ such that each leaves are transverse to i.*
(3) Image $i \subseteq \Sigma_c(F)$.

We can also state another theorem as follows.

THEOREM B. *For an equation $F : (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0)) \to (\mathbb{R}, 0)$, the following are equivalent.*
(1) *$F = 0$ is completely integrable.*
(2) *$\Sigma_c(F) = \emptyset$ or $\Sigma_c(F)$ is an n-dimensional submanifold.*

Theorem B gives a characterization of complete integrability of equations. By the definition, if $\Sigma_c(F)$ is an n-dimensional submanifold, it is automatically a geometric solution of $F = 0$. Then we have the following corollary of Theorems A and B.

COROLLARY. *An equation $F : (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0)) \to (\mathbb{R}, 0)$ has a singular solution if and only if $\Sigma_c(F)$ is an n-dimensional submanifold. Moreover, $\Sigma_c(F)$ is the singular solution of $F = 0$.*

2. PROOF OF THEOREM A

In this section we shall give a proof of Theorem A. At first we give the proof that (1) implies (3).

PROOF OF THEOREM A, (1) $\Rightarrow$ (3): Let $i : (L, q_0) \to (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0))$ be a singular solution of $F = 0$. Suppose that there exists a point $q_1 \in L$ such that $i(q_1) = (x_1, y_1, p_1) \notin \Sigma_c(F)$. This means that $F = 0$ is contact regular at (x_1, y_1, p_1). By the classification theorem of the geometric theory of first order differential equations (see [13], Corollary 2 of Theorem 2.2.7), we may assume that $F(x, y, p) = p_n$. Then we now define a submanifold E_0 by $x_n = p_n = 0$. It is easy to show that E_0 has a contact structure $\theta | E_0$. Since the contact Hamiltonian vector field of $p_n = 0$ is given by $X_{p_n} = -\frac{\partial}{\partial x_n}$ (see [13], Theorem 1.4.3) and i is a solution of $p_n = 0$, then $-\frac{\partial}{\partial x_n} \in Ti(L)$. It follows that $\ell_0 = i(L) \cap \{x_n = 0\}$ is an $(n-1)$-dimensional submanifold in E_0. Since i is a Legendrian immersion, then ℓ_0 is also a Legendrian submanifold of E_0. By the Darboux's theorem, we can easily show that there exists a foliation on E_0 with leaves are Legendrian submanifold of E_0 such that ℓ_0 is a leaf of this foliation. Since $-\frac{\partial}{\partial x_n} \notin TE_0$, then each leaves are isotropic submanifold of $J^1(\mathbb{R}^n, \mathbb{R})$ with the non-characteristic property. Thus we can solve the "parametrized"

Cauchy problem by ordinary characteristic method (see [13] Proposition 1.5.3), then we have a local complete solution of $p_n = 0$ such that $i(L)$ is a leaf of this solution. This contradicts to the definition of the singular solution.

We can also prove that (2) implies (1).

PROOF OF THEOREM A, (2) $\Rightarrow$ (1): Let $\tilde{i} : U \to F^{-1}(0)$ be a representative of i. If there exist an open subset $V \subset U$ and a complete solution such that $\tilde{i}(V)$ is contained in a leaf of such a foliation. Then there exists transversal foliations on $F^{-1}(0)$ around $\tilde{i}(V)$ whose leaves are Legendrian submanifolds. Then $F^{-1}(0)$ is an isotropic submanifold around $\tilde{i}(V)$. This contradicts to the fact that dimensions of isotropic submanifolds are at most n.

We need some preparations to prove that (3) implies (2).

Let $F : (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0)) \to (\mathbb{R}, 0)$ be an equation such that (x_0, y_0, p_0) is a contact singualr point. If $F_y = 0$ at (x_0, y_0, p_0), then $F_{x_i} = F_{p_i} = 0$ at (x_0, y_0, p_0) for any $i = 1, \dots, n$. This contradicts to the fact that F is a submersion germ. Then we have $F_y \neq 0$. By the implicit function theorem, there exists a function germ $h : (T^*\mathbb{R}^n, (x_0, p_0)) \to \mathbb{R}$ such that $F^{-1}(0) = \{(x, y, p) | y = h(x, p)\}$, where $T^*\mathbb{R}^n$ is the cotangent bundle of $\mathbb{R}^n$. We may consider that $J^1(\mathbb{R}^n, \mathbb{R}) \cong T^*\mathbb{R}^n \times \mathbb{R}$. In the terminology of Kossowski [12] the equation of the above form is called *a graphlike equation.* We now define a map germ

$$graph(h) : (T^*\mathbb{R}^n, (x_0, p_0)) \to (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0))$$

by

$$graph(h)(x, p) = (x, h(x, p), p).$$

We set a 1-form on $T^*\mathbb{R}^n$ by $\theta_h = graph(h)^*\theta = dh - \sum_{i=1}^n p_i dx_i$. Then we have the following one to one correspondence.

$$\{L \mid L \text{ is a solution of } y - h(x, p) = 0\}$$

$$graph(h) \uparrow\downarrow \Pi_*$$

$$\{L \mid i : L \subset T^*\mathbb{R}^n \text{ is a maximal integral submanifold of } \theta_h = 0\},$$

where $\Pi(x, y, p) = (x, p)$ and $\Pi_*(L) = \Pi(L)$. By this reason, a solution of graphlike equation $y - h(x, p) = 0$ may be regarded as a maximal isotropic submanifold of $(T^*\mathbb{R}^n, \theta_h)$. Since $-d\theta_h = \sum_{i=1}^n dp_i \wedge dx_i$ is the canonical symplectic two form, then a solution of $y - h(x, p) = 0$ is a Lagrangian submanifold of $(T^*\mathbb{R}^n, \omega)$, where $\omega = -d\theta_h$. For the definition and properties of Lagrangian submanifolds, see [1]. We now refer the following very important result.

THEOREM 2.1. (Kostant-Sternberg [6]) *Let (P, ω) be a symplectic manifold, L a Lagrangian submanifold and α a smooth 1-form on P with $\alpha \mid L = 0$ and $d\alpha = \omega$. Then there exists a tubular neighbourhood V of L in P, and a unique vector bundle isomorphism $K : V \to (T^*L, \theta_L)$ such that K is the identity on L and $K^*\theta_L = \alpha$. Here, θ_L is the canonical 1-form on T^*L.*

Now we can prove that (3) implies (2).

Proof of Theorem A, (3) $\Rightarrow$ (2): Since Image $i \subseteq \Sigma_c(F)$, we may assume that $F = y - h(x,p)$. In our case, $graph(h)^{-1} \circ i(L) = L_h$ is a Lagrangian submanifold of $T^*\mathbb{R}^n$. We may apply the Kostant-Sternberg theorem to conclude that there exists a tubular neighbourhood V of L_h in $T^*\mathbb{R}^n$ and a unique vector bundle isomorphism $K : V \to (T^*L_h, \theta_{L_h})$ such that K is the identity on L_h and $K^*\theta_{L_h} = -\theta_h$. Since the fibre of the cotangent bundle $T^*L_h \to L_h$ are maximal integral submanifolds of $\theta_{L_h} = 0$, these fibres make a foliation whose leaves are solution of $F = 0$ and tansverse to $i(L)$. This completes the proof.

3. Proof of Theorem B

In order to prove Theorem B, we need some preparations. We say that an n-parameter family of function germs $f : (\mathbb{R}^n \times \mathbb{R}^n, (t_0, x_0)) \to (\mathbb{R}, y_0)$ is *a classical complete solution* of $F = 0$ if $F(x, f(t,x), f_x(t,x)) = 0$ and $\operatorname{rank}(f_{t_i}, f_{t_i x_j}) = n$. Then we define a map germ $j^1_* f : (\mathbb{R}^n \times \mathbb{R}^n, (t_0, x_0)) \to (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0))$ by $j^1_* f(t,x) = (x, f(t,x), f_x(t,x))$. Since $f(t,x)$ is a classical complete solution, then $j^1_* f$ is a local parameterization of $F^{-1}(0)$. It follows that the family $\{\text{Image}\, j^1_* f_t\}_{t \in (\mathbb{R}^n, t_0)}$ gives a local foliation on $F^{-1}(0)$. For a Legendrian immersion germ $i : (L, q_0) \to J^1(\mathbb{R}^n, \mathbb{R})$, $q_0 \in L$ is said to be *a Legendrian singular point* if $\pi \circ i$ is not an immersion at q_0. We remark that q_0 is a Legendrian non-singular point if and only if $\tilde{\pi} \circ i$ is a local diffeomorphism at q_0, where $\tilde{\pi}(x,y,p) = x$. Then we have the following lemma.

Lemma 3.1. 1) *If $F = 0$ has a classical complete solution, then $\Sigma_c(F) = \emptyset$ or $\Sigma_c(F)$ is an n-dimensional submanifold.*
2) *If each leaves of a complete solution of $F = 0$ are Legendrian non-singular, then it is a classical complete solution.*

Proof: 1) By the above argument, $j^1_* f$ is a local parameterization of $F^{-1}(0)$. Since $j^1_* f^* \theta = \sum_{i=1}^n f_{t_i}(t,x) dt_i$, then $j^1_* f(t,x) \in \Sigma_c(F)$ if and only if $f_{t_i}(t,x) = 0$ for $i = 1, \ldots, n$. Since $\operatorname{rank}(0, f_{t_i x_j}) = \operatorname{rank}(f_{t_i}, f_{t_i x_j}) = n$ at $j^1_* f(t,x) \in \Sigma_c(F)$, then we have $\operatorname{rank}(f_{t_i x_j}, f_{x_i x_j}) = n$. It follows that $\Sigma_c(F)$ is an n-dimensional submanifold.

2) Suppose that there exists a complete solution with leaves are Legendrian non-singular. Then we have an n-parameter family of smooth sections

$$s : (\mathbb{R}^n \times \mathbb{R}^n, (t_0, x_0)) \to (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0))$$

of $\tilde{\pi}$ (i.e. $\tilde{\pi} \circ s(t,x) = x$) such that s is an immersion, $s(\mathbb{R}^n \times \mathbb{R}^n) = F^{-1}(0)$ and $s_t^* \theta = 0$ for any $t \in (\mathbb{R}^n, t_0)$, where $s_t(x) = s(t,x)$. It follows that there exists a family of function germs $f : (\mathbb{R}^n \times \mathbb{R}^n, (t_0, x_0)) \to (\mathbb{R}^n, y_0)$ such that $j^1_* f(t,x) = s(t,x)$. Since s is an immersion, then f is a (classical) complete solution of $F = 0$.

Now we can prove Theorem B.

Proof of Theorem B: If $\Sigma_c(F) = \emptyset$, then $F = 0$ is contact regular at (x_0, y_0, p_0). It follows that the classical existence theorem (see [13]) that there exists a complete solution of $F = 0$. If $\Sigma_c(F)$ is an n-dimensional submanifold, then it is also a solution of $F = 0$. In this case it has been already proved in the proof of Theorem A ((3) $\Rightarrow$ (2)).

Suppose that there exists a complete solution of $F = 0$. Let $(L, (x_0, y_0, p_0))$ be a germ of a leaf of the complete solution at the point (x_0, y_0, p_0). By Arnol'd-Zakalyukin's theory

([1], Corollary 20.2), there exist a partition (I, J) of the set $\{1, \ldots, n\}$ and a function germ $S(x_I, p_J)$ such that

$$L = \{(x_I, -\frac{\partial S}{\partial p_J}, S(x_I, p_J) - < \frac{\partial S}{\partial p_J}, p_J >, \frac{\partial S}{\partial x_I}, p_J)\},$$

where $< x_J, p_J >$ is the canonical inner product. We now define a contact diffeomorphism germ by

$$C_{(I,J)}(x, y, p) = (x_I, p_J, y - < x_J, p_J >, p_I, -x_J).$$

Then we have $C_{(I,J)}(L) = \{(x_I, p_J, S(x_I, p_J), \frac{\partial S}{\partial x_I}, \frac{\partial S}{\partial p_J})$. It follows that $C_{(I,J)}(L)$ is Legendrian non-singular at $C_{(I,J)}((x_0, y_0, p_0))$. By Lemma 3.1, we have a classical complete solution on $F \circ C_{(I,J)}^{-1} = 0$. Then $\Sigma_c(F) = C_{(I,J)}^{-1}(\Sigma_c(F \circ C_{(I,J)}^{-1}))$ is also an n-dimensional submanifold. This complete the proof.

4. Examples

In this section we shall give typical examples of equations with singular solution.

Examples 4.1, The Clairaut equation. The following is the classical example of an equation with singular solution : $y = \sum_{i=1}^{n} x_i p_i + f(p_1, \ldots, p_n)$, where f is a smooth function. The complete solution is given by $y = \sum_{i=1}^{n} x_i t_i + f(t_1, \ldots, t_n)$ and the singular solution is the envelope of graphs of complete solution.

Example 4.2, The dual of the Clairaut equation. Consider the equation : $y = f(x_1, \ldots, x_n)$. This equation is given by the Legendre transform (see [9]) of the Clairaut equation. The complete solution is given by $\{(t, f(t), u) | (t, u) \in \mathbb{R}^n \times \mathbb{R}^n\}$, where $t = (t_1, \ldots, t_n)$ is the parameter.

The singular solution is given by $\Sigma_c(F) = \{(x, f(x), f_x(x)) | x \in \mathbb{R}^n\}$. We can observe that $F^{-1}(0) = \Sigma_\pi(F) \supset \Sigma_c(F)$.

Example 4.3. Consider the following equation : $y - 2p^3 = 0$. We can show that $\Sigma_\pi(F) = \Sigma_c(F) = \{(x, 0, 0) | x \in \mathbb{R}\}$ which is a singular solution. We also have a complete solution $s : (\mathbb{R} \times \mathbb{R}, 0) \to J^1(\mathbb{R}, \mathbb{R})$ given by $s(u, t) = (3u^2 + t, 2u^3, u)$, where t is the parameter. In this case the singular solution is a locus of cusps of the complete solution (not an envelope!).

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