The Euler characteristic of the image of a stable mapping from a closed n-manifold to a $(2n-1)$-manifold

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The Euler characteristic of the image of a stable mapping from a closed n-manifold to a $(2n-1)$-manifold

S.IZUMIYA AND W.L.MARAR

INTRODUCTION

One of the themes in the global theory of singularities of mappings $f: N \to P$ between manifolds is to study the relationship among the topology of N, P and $f(N)$ in the case when $\dim N < \dim P$ ([3]).

Recently, there appeared a considerable progress in the local theory of singularities of mappings ([4],[5],[6],[7]) mainly due to the work of David Mond. In [4] a method has been introduced to compute the Euler characteristic of the image of a stable perturbation of an $\mathcal{A}$-finite map-germ. Here we shall apply this method to compute the Euler characteristic of the image of a stable mapping from a closed n-manifold to a $(2n-1)$-manifold. We also determine the set of Euler characteristics of images of stable mappings from a closed n-manifold to a $(2n-1)$-manifold as an application of our main theorem.

All mappings considered here are differentiable class C^∞ unless stated otherwise.

1. THE MAIN RESULT

It is well-known that a mapping $f: N \to P$ from an n-manifold to a $(2n-1)$-manifold is stable if and only if it is an immersion with normal crossings except at the isolated singularities of cross-caps ([8], fig.1). It follows that the number of cross-caps is finite and we denote it by $C(f)$. There also exist finitely many three-to-one points in $f(N)$ at where three sheets of regular images are in general position. Such a point (fig.2) is called triple point of f and the number of triple points is denoted by $T(f)$.

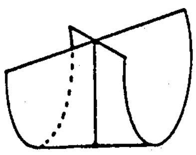

fig.1

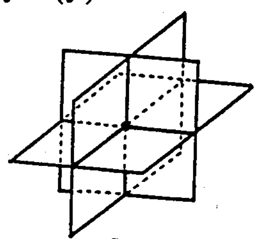

fig.2

We denote the Euler characteristic of a topological space X by $\chi(X)$. Our main result is the following:

THEOREM. (i) $\chi(f(N)) = \chi(N) + T(f) + C(f)/2$, *if* $n = 2$.
(ii) $\chi(f(N)) = \chi(N) + C(f)/2$, *if* $n \geq 3$.

PROOF: (i) Let us consider the following sets:
$D^2(f) = cl\{x \in S | \# f^{-1}f(x) \geq 2\}$,
$D^3(f) = \{x \in D^2(f) | \# f^{-1}f(x) = 3\}$ and
$D^2(f,(2)) = \{x \in D^2(f) | \# f^{-1}f(x) = 1\}$,
where clX is the topological closure of X. Then we have the following diagram:

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

$$
\begin{tikzcd}
 & D^3(f) \arrow[d, "k"] & \\
D^2(f,(2)) \arrow[r, "j"] & D^2(f) \arrow[d, "i"] & \\
 & N \arrow[r, "f"] & f(N) \subset P
\end{tikzcd}
$$

where i, j, k are inclusions.

By the characterization of stable mappings ([8]), $D^2(f)$ is a union of closed curves on the n-manifold N whose set of self-intersections is $D^3(f)$, which is the inverse image of triple points, and $D^2(f,(2))$ is the set of cross-cap points of f. It follows that these are immersed submanifolds of N with dim $D^2(f) = 1$ and dim $D^3(f) =$ dim $D^2(f,(2)) = 0$, if not empty.

In order to prove the theorem, we consider the following problem: find real numbers α, β, γ and δ such that

$$\chi(f(N)) = \alpha\,\chi(N) + \beta\,\chi(D^2(f)) + \gamma\,\chi(D^2(f,(2)) + \delta\;\chi(D^3(f)). \tag{1.1}$$

We shall solve this by a purely combinatorial method.

Initially we construct a triangulation K_f of the set $f(N)$ as follows: we start to triangulate $f(N)$ by including the image of $D^2(f,(2))$ and the image of $D^3(f)$ among the vertices of K_f. After this, we build up the one-skeleton $K_f^{(1)}$ of K_f so that the image of $D^2(f)$ is a subcomplex of $K_f^{(1)}$. We complete our procedure by constructing the 2-skeleton $K_f^{(2)}$ of K_f.

Since f and its restrictions to $D^2(f)$, $D^2(f,(2))$ and $D^3(f)$ are proper and finite-to-one mappings, then we can pull back K_f to obtain triangulations for N, $D^2(f)$, $D^2(f,(2))$ and $D^3(f)$ respectively. Let C_i^X be the number of i-cells in X, where $X = f(N)$, N, $D^2(f)$, $D^2(f,(2))$ or $D^3(f)$. Then the equation (1.1) can be written by $\sum_i(-1)^i C_i^{f(N)} = \alpha\sum_i(-1)^i C_i^N + \beta\sum_i(-1)^i C_i^{D^2(f)} + \gamma\sum_i(-1)^i C_i^{D^2(f,(2))} + \delta\sum_i(-1)^i C_i^{D^3(f)}$, where $C_i^X = 0$ if $i > dimX$. So, if we can find real numbers α, β, γ and δ such that

$$C_i^{f(N)} = \alpha\,C_i^N + \beta\,C_i^{D^2(f)} + \gamma\,C_i^{D^2(f,(2))} + \delta\,C_i^{D^3(f)} \tag{1.2}$$

for any i, then we have an answer for the problem. By the construction of the triangulation, we may concentrate on solving (1.2) in the case when $i = 0$. We remark that f is 3 to 1 over the points in the image of $D^3(f)$, 1 to 1 over the points in the image of $D^2(f,(2))$, 2 to 1 over the points in the image of $D^2(f) - (D^2(f,(2)) \cup D^3(f))$, and 1 to 1 over the points in the image of $N - D^2(f)$. It follows that the equation

$$C_0^{f(N)} = \alpha\,C_0^N + \beta\,C_0^{D^2(f)} + \gamma\,C_0^{D^2(f,(2))} + \delta\,C_0^{D^3(f)}$$

is equivalent to the system of linear equations :

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\2&2&0&0\\1&1&1&0\\3&3&0&3\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix}.$$

Then we have the solutions $\alpha = 1$, $\beta = -1/2$, $\gamma = 1/2$ and $\delta = -1/6$ so that

$$\chi(f(S)) = \chi(S) - \chi(D^2(f))/2 + \chi(D^2(f,(2)))/2 - \chi(D^3(f))/6. \tag{1.3}$$

By the definition, $\chi(D^2(f,(2))) = C(f)$ and $\chi(D^3(f)) = 3T(f)$. Since $D^2(f)$ is a union of closed curves on the surface N with $3T(f)$ crossings and circles, then we can triangulate it with $3T(f)+n$ 0-cells and $6T(f)+n$ 1-cells, where n is the number of circles. It follows that $\chi(D^2(f)) = -3T(f)$. Finally, substituting these on the equation (1.3), we get

$$\chi(f(N)) = \chi(N) + T(f) + C(f)/2.$$

This completes the proof of (i).
(ii) When $n \geq 3$ then $D^k(f) = \emptyset$, for any $k \geq 3$. So, following the same arguments as above we get

$$\chi(f(N)) = \chi(N) + C(f)/2.$$

2. An application

In this section we shall determine the set of Euler characteristcs of images of stable mappings from a connected closed n-manifold to a $(2n-1)$-manifold as an application of the theorem.

We now define $\chi(N,P) = \{\chi(f(N)) | f : N \to P \text{ is stable}\}$. Then we have the following :

Proposition 2.1. (1) *Suppose that $n = 2$.*

(i)*If N is not homeomorphic to the connected sum of a projective plane and an orientable surface, then*

$$\chi(N,P) = \{n \in \mathbb{Z} | n \geq \chi(N)\}$$

.

(ii) *If N is homeomorphic to the connected sum of a projective plane and an orientable surface, then*

$$\chi(N,P) = \{n \in \mathbb{Z} | n \geq \chi(N) + 1\}.$$

(2) *Suppose that $n \geq 3$, then*

$$\chi(N,P) = \{n \in \mathbb{Z} | n \geq \chi(N)\}.$$

Proof: (1) (i) In this case we can always construct an immersion $f : N \to P$ with normal crossings without triple points. Then we have $\chi(f(N)) = \chi(N)$. We now define a stable mapping $g : D \to P$ by $g(x,y) = (x, y^2, yx^2 + y^3 - r^2y)$ in suitable local coordinates, where

D is a disc centred at the origin of $\mathbb{R}^2$ and r is any positive number smaller than the radius of D. Then g has two cross-caps (fig.3).

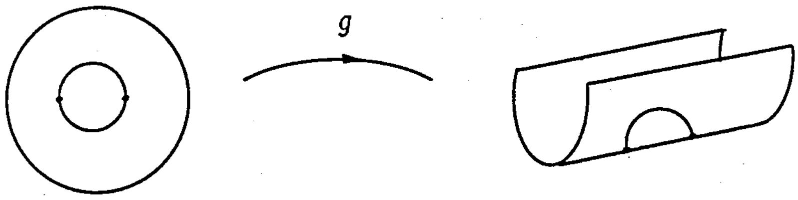

fig.3

If we consider the connected sum of f and g, then we have a stable mapping $f\#g : N \to P$ with $C(f\#g) = 2$ and $T(f\#g) = 0$. It folows that $\chi(f\#g(N)) = \chi(N) + 1$. By this procedure, we can construct a stable mapping $h : N \to P$ such that $\chi(h(N)) = n$, for any $n \geq \chi(N)$.

(ii) It is enough to consider the case when $N = \mathbb{P}^2$. In this case we cannot construct an immersion with normal crossings without triple points [1]. If we consider $f(\mathbb{P}^2)$ as the Boy surface, then the number of triple points is 1 ([2]) and $\chi(f(\mathbb{P}^2)) = \chi(\mathbb{P}^2) + 1$. Now, by exactly the same procedure as that of case (i), we can get the result.

(2) By the immersion theorem [8], we have an immersion with normal clossing $f : N \to P$. Since $n \geq 3$, then f has not triple points. Then, if we use the mapping

$$g : D^n \to P \; ; \; g(x_1, \ldots, x_n) = (x_1, x_2^2, x_3, \ldots, x_n, (x_1^2 + x_2^2 - r)x_2, x_1x_3, \ldots, x_1x_n)$$

in suitable local coordinates like as in the case (1) (ii), we can complete the proof.

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