What is the Clairaut equation ?

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What is the Clairaut equation ?

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0. INTRODUCTION

What is the Clairaut equation ? We shall give some answers to this question in this article. We stick to ordinary differential equations of the form

$$F(x, y, \frac{dy}{dx}) = 0,$$

though some of our assertions can be generalized for partial differential equations. The first answer is given by the following :

EXAMPLE (0.1). $y = x \cdot \frac{dy}{dx} + f(\frac{dy}{dx})$; Alex Claude Clairaut (1734, [6]).

This is the Clairaut equation which is well known since old times. This equation is usually taught in the first or second year course of calculus in the university and it is treated as one of the typical examples of non-linear equations which are easily solved. Moreover it has a quite beautiful geometric structure as follows : There exists a "general solution" which consists of lines ; $y = t \cdot x + f(t)$, t is a parameter and the "singular solution" is the envelope of such a family (Fig. 1).

$$\begin{cases} y = & -t \cdot f'(t) + f(t) \\ x = & -f'(t). \end{cases}$$

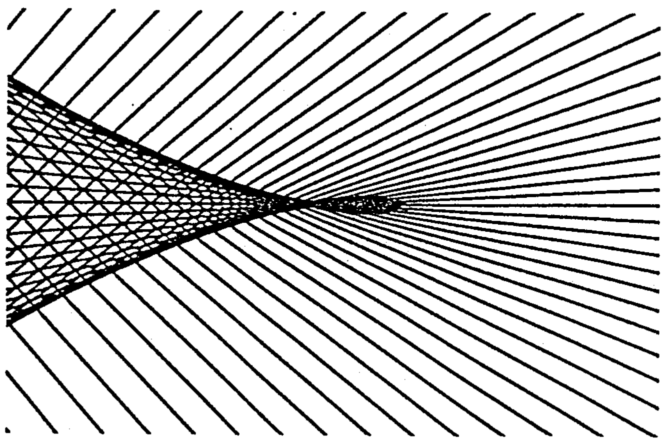

Fig. 1

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

By this reason, this equation is very interesting from the view point of the singularity theory (i.e. envelopes are subjects in the singularity theory [2],[3],[4]) and generic properties of subjects are usually studied in the singularity theory. However, we must refer the following result:

THEOREM (0.2) (R.THOM 1972, [14]). *Equations with singular solution are not generic in the space of all equations of the form* $F(x, y, \frac{dy}{dx}) = 0$.

Of course, the Clairaut equation is not generic. However, people have been interested in non-generic objects (i.e. circles, regular triangles etc) since the ancient time. In this article we shall argue about equations with a beautiful geometric structure like as the Clairaut equation. Here, we have no reason why "general solutions" must consist of lines (Fig. 2).

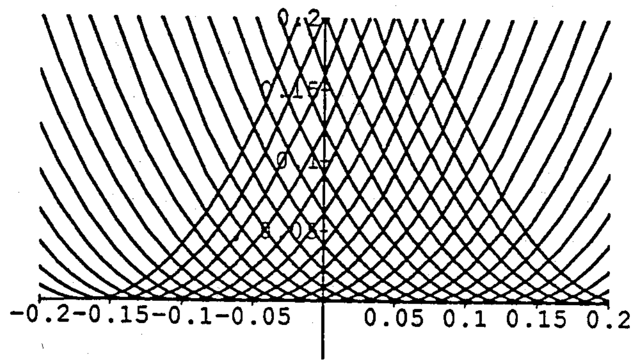

Fig.2

We will refer such an equation as a Clairaut type equation in §3. In §1 we shall prepare some basic notions and construct the framework. We must establish the notion of singular solutions, then we will give a rigorous definition in §2.

We shall use a purely elementary method, most of the arguments are contained in the course of advanced calculus in the university but these are original.

1. BASIC NOTIONS

We consider a first order ordinary differential equation of the form

$$F(x, y, \frac{dy}{dx}) = 0.$$

If we put $p = \frac{dy}{dx}$, we may consider that F is a function of (x, y, p)-variables and assume that F is a smooth function which is defined on an open subset U in $\mathbb{R}^3$ such that $\operatorname{grad} F \neq 0$ at any point $(x, y, p) \in U$. Then $S = F^{-1}(0)$ is a smooth surface in U.

We now define the notion of solutions. *A smooth solution* of $F = 0$ is a smooth function $y = f(x)$ defined on an interval $(a\ b) \subset \mathbb{R}$ such that $F(x, f(x), f'(x)) = 0$. This is the classical notion of solutions of the equation $F = 0$. However, we meet multivalued solutions in sometimes (cf. Fig. 1), then we need to generalize the notion of solutions. The following is the geometric generalization of the notion of solution due to Lie. *A geometric solution* of $F=0$ is a smooth regular curve $\gamma : (a\ b) \to F^{-1}(0)$ such that $y'(t) = p(t)x'(t)$, where $\gamma(t) = (x(t), y(t), p(t))$ in the canonical coordinate of $\mathbb{R}^3$. Here, we say that γ is *regular* if $\gamma'(t) = (x'(t), y'(t), p'(t)) \neq (0,0,0)$ at any $t \in (a\ b)$. In the terminology of contact geometry, the above curve is called a Legendrian curve (see [1]).

LEMMA 1.1. *Let $\gamma : (a\ b) \to F^{-1}(0)$ be a geometric solution. Suppose that $x'(t) \neq 0$ at any $t \in (a\ b)$. Then there exist real numbers c, d, a diffeomorphism $\phi : (c\ d) \to (a\ b)$ and a smooth function f defined on $(c\ d)$ such that $\gamma \circ \phi(x) = (x, f(x), f'(x))$.*

PROOF: By the assumption, there exists an interval $(c\ d)$ such that the mapping ψ : $(a\ b) \to (c\ d)$ defined by $\psi(t) = x(t)$ is a diffeomorphism. We put $\phi(x) = \psi^{-1}(x)$. Then we have $\gamma \circ \phi(x) = (x, y \circ \phi(x), p \circ \phi(x))$. If we define a function f by $f(x) = y \circ \phi(x)$, then we have

$$f'(x) = \frac{dy}{dx}(\phi(x)) \cdot \phi'(x) = y'(\phi(x)) \frac{1}{x'(\phi(x))} = p(\phi(x)).$$

This completes the proof.

According to the above property, we may define the notion of singular point of solutions. We say that t_0 is *a geometric singular point* of the solution γ if $x'(t_0) = 0$. Thus γ is multivalued around the geometric singular point. It is clear that t_0 is a geometric singular point of γ if and only if $(x'(t_0), y'(t_0)) = (0,0)$.

On the other hand, there exists a notion of the Legendrian transformation by which a dual relationship can be set up between one equation and another. We adopt another coordinate system (X, Y, P) of $\mathbb{R}^3$ by

$$X = p,\ Y = x \cdot p - y,\ P = x.$$

We refer a diffeomorphism

$$*L : \mathbb{R}^3 \to \mathbb{R}^3 \quad ; \quad *L(x,y,p) = (p, x \cdot p - y, x)$$

as *a Legendre transformation.* If we apply the Legendre transformation to our equation, we obtain a new equation

$$F^*(X,Y,P) = F(P, X \cdot P - Y, X) = 0$$

in the new coordinate system (X, Y, P).

If we calculate partial derivatives at the point (X_0, Y_0, P_0) corresponding to (x_0, y_0, p_0), we can show the following :

$$\begin{aligned} F^*_P(X_0,Y_0,P_0) &= (F_x + p \cdot F_y)(x_0, y_0, p_0) \\ F^*_Y(X_0,Y_0,P_0) &= -F_y(x_0, y_0, p_0) \\ F^*_X(X_0,Y_0,P_0) &= (F_p + x \cdot F_y)(x_0, y_0, p_0). \end{aligned}$$

The following lemma is quite simple but important in the later sections.

LEMMA 1.2. (1) *Let $\gamma : (a\ b) \to F^{-1}(0)$ be a geometric solution of $F = 0$. Then $*L \circ \gamma : (a\ b) \to F^{*-1}(0)$ is a geometric solution of $F^* = 0$.*
(2) *If t_0 is a geometric singular point of γ, then t_0 is a geometric non-singular point of $*L \circ \gamma$.*

PROOF: (1) Suppose that $\gamma(t) = (x(t), y(t), p(t))$, then

$$*L \circ \gamma(t) = (p(t), x(t) \cdot p(t) - y(t), x(t)),$$

so that

$$\frac{d}{dt}(x(t) \cdot p(t) - y(t)) = x'(t) \cdot p(t) + x(t) \cdot p'(t) - y'(t) = x(t) \cdot p'(t).$$

This is the condition that $*L \circ \gamma(t)$ is a geometric solution.
(2) If $x'(t_0) = 0$, then $x'(t_0) = y'(t_0) = 0$ by the definition. Since $\gamma(t)$ is a regular curve, then $p'(t_0) \neq 0$. This means that $*L \circ \gamma(t)$ is non-singular at t_0.

If the equation $F = 0$ satisfies $F_p \neq 0$ at (x_0, y_0, p_0), then we can locally rewrite this equation in the form $p = f(x, y)$, where f is a smooth function by the implicit function theorem. This form is far more convenient than the original one, because there exists the classical existence theorem of solutions. By the above argument, if $F = 0$ satisfies $F_x + p \cdot F_y \neq 0$ at (x_0, y_0, p_0), then the Legendre transform F^* of F satisfies $F_P^* \neq 0$ so that we get the solution of $F = 0$ pulling back by the Legendre transformation. Then the point (x_0, y_0, p_0) at which $F = F_p = 0$ or $F = F_p = F_x + p \cdot F_y = 0$ are satisfied has special meanings. We call (x_0, y_0, p_0) *a π-singular point* of $F = 0$ if $F = F_p = 0$ at (x_0, y_0, p_0) and *a contact singular point* of $F = 0$ if $F = F_p = F_x + p \cdot F_y = 0$. We denote $\Sigma_\pi(F)$ as the set of π-singular points, $\Sigma_c(F)$ as the set of contact singular points and $D_F = \pi(\Sigma_\pi(F))$ as the discriminant set of $F = 0$, where $\pi(x, y, p) = (x, y)$. The following lemma denotes the importance of the contact singular set $\Sigma_c(F)$.

LEMMA 1.3. *Let $\gamma : (a\ b) \to F^{-1}(0)$ be a smooth regular curve. If* Image $\gamma \subset \Sigma_c(F)$, *then γ is a geometric solution.*

PROOF: Denote $\gamma(t) = (x(t), y(t), p(t))$, then we have

$$(*) \qquad F(x(t), y(t), p(t)) = F_p(x(t), y(t), p(t)) = (F_x + p \cdot F_y)(x(t), y(t), p(t)) = 0.$$

If $F_y = 0$ at $\gamma(t)$, then $F_p = F_x = 0$ at $\gamma(t)$ by the above equality. This contradicts to the assumption that $\operatorname{grad} F \neq 0$, so that we have $F_y(x(t), y(t), p(t)) \neq 0$. Calculating the derivative of $F(x(t), y(t), p(t)) = 0$ with respect to t, we have

$$F_x(x(t), y(t), p(t)) \cdot x'(t) + F_y(x(t), y(t), p(t)) \cdot y'(t) = 0.$$

It follows that

$$y'(t) = -\frac{F_x(x(t), y(t), p(t))}{F_y(x(t), y(t), p(t))} \cdot x'(t).$$

By the relation (*), we have

$$p(t) = -\frac{F_x(x(t), y(t), p(t))}{F_y(x(t), y(t), p(t))},$$

so that we have a relation $y'(t) = p(t) \cdot x'(t)$.

2. SINGULAR SOLUTIONS

In classical textbooks ([8],[10],[13]), a "general solution" is defined to be an one parameter family of solutions and a singular solution is a solution which is not contained in the "general solution". But, this definition is very confused like as the following example :

EXAMPLE 2.1. Consider the equation $y = 2p \cdot x - p^2$. In [10] the "general solution" is given by

$$\begin{cases} x = & \frac{c}{p^2} + \frac{2}{3}p \\ y = & 2p \cdot x - p^2, \end{cases}$$

where c is a parameter. It is clear that $y = 0$ is also a solution but it is not contained in the "general solution". Then $y = 0$ must be the singular solution. On the other hand, we now define a smooth mapping $f : \mathrm{R}^2 \to \mathrm{R}^3$ by $f(x,p) = (x, 2p \cdot x - p^2, p)$ which gives a parameterization of the surface $y = 2p \cdot x - p^2$. If a regular curve $(x(t), p(t))$ is a corresponding solution on the (x,p)-plane via f, then it must satisfy

$$p(t) \cdot x'(t) = y'(t) = 2p(t) \cdot x'(t) + 2(x(t) - p(t)) \cdot p'(t).$$

It follows that we have a relation

$$p(t) \cdot x'(t) + 2(x(t) - p(t)) \cdot p'(t) = 0.$$

Then we may solve a system of ordinary differential equations :

$$\begin{cases} x'(t) = & 2(x(t) - p(t)) \\ p'(t) = & -p(t). \end{cases}$$

The solution is given by

$$\begin{cases} x(t) = & \frac{2}{3}c_1 e^{-t} + c_2 e^{2t} \\ p(t) = & c_1 e^{-t}, \end{cases}$$

where c_1, c_2 are arbitrary constants. Thus we have a two parameter family of geometric solutions :

$$\gamma_{(c_1,c_2)}(t) = (\frac{2}{3}c_1 e^{-t} + c_2 e^{2t}, \frac{1}{3}c_1^2 e^{-2t} + 2c_1 c_2 e^t, c_1 e^{-t}).$$

If we fix $p(0) = c_1 \neq 0$ and put $c = c_2 c_1^2$, then we have $x = \frac{2}{3}p + \frac{c}{p^2}$. But, if we fix $p(0) = c_1 = 0$, then we have $y = 0$. Moreover, if we consider this family of solutions around a point $(x_0, 0, 0)$, $x_0 \neq 0$, then we have an relation $x(0) = \frac{2}{3}c_1 + c_2 = x_0$, so that, we have an one-parameter family of geometric solutions around $(x_0, 0, 0)$:

$$\gamma_{c_1}(t) = (\frac{2}{3}c_1 e^{-t} + (x_0 - \frac{2}{3}c_1)e^{2t}, \frac{1}{3}c_1^2 e^{-2t} + 2c_1(x_0 - \frac{2}{3}c_1)e^t, c_1 e^{-t}).$$

Of course, $y = 0$ is contained in this family of solutions.

In order to avoid such confusions, we now introduce the following notion. Let $\Gamma : (a\ b) \times (\alpha\ \beta) \to F^{-1}(0)$ be one-parameter family of geometric solutions of $F = 0$. We say that Γ is *a complete solution* if

$$\operatorname{rank} \begin{pmatrix} x_t & y_t & p_t \\ x_c & y_c & p_c \end{pmatrix} = 2$$

at any point $(t, c) \in (a\ b) \times (\alpha\ \beta)$, where $\Gamma(t, c) = (x(t, c), y(t, c), p(t, c))$ and c is a parameter. In some classical textbooks (cf. [13]), the above term is used in the different sense. However, we adopt the above definition according to the terminology in the theory of first order partial differential equations ([5],[8]). We say that an equation $F = 0$ is *completely integrable around* (x_0, y_0, p_0) if there exists a complete solution of $F = 0$ around (x_0, y_0, p_0). The equation in the last example is not completely integrable around the origin. The uniqueness of the complete solution is the following :

PROPOSITION 2.2. *Let $\Gamma_i : (a_i\ b_i) \times (\alpha_i\ \beta_i) \to F^{-1}(0)$ $(i = 1, 2)$ be complete solutions of $F = 0$. If there exist $(t_i, c_i) \in (a_i\ b_i) \times (\alpha_i\ \beta_i)$ $(i = 1, 2)$ such that $\Gamma_1(t_1, c_1) = \Gamma_2(t_2, c_2) = (x_1, y_1, p_1)$ for any point $(x_1, y_1, p_1) \in U = \mathrm{Image}\,\Gamma_1 \cap \mathrm{Image}\,\Gamma_2$, then*

$$\mathrm{Image}\,\Gamma_{1,c_1} = \mathrm{Image}\,\Gamma_{2,c_2}$$

on U, where $\Gamma_{i,c_i}(t) = \Gamma_i(t, c_i)$.

PROOF: Suppose that the assertion does not hold. Since the solution is one-parameter submanifold in $F^{-1}(0)$, then there exists a point $(x_1, y_1, p_1) \in U$ such that Γ_{1,c_1} and Γ_{2,c_2} are transversal around (x_1, y_1, p_1) (Fig. 3).

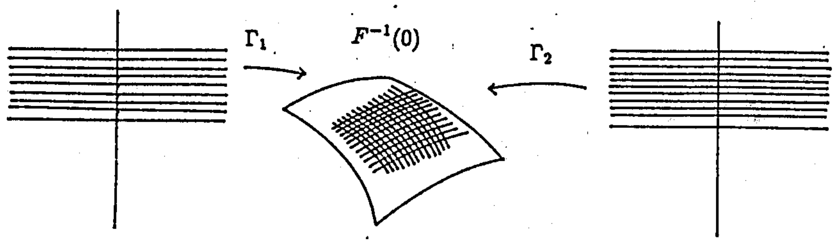

Fig. 3

Then we can construct an immersion $\Gamma : (a\ b) \times (\alpha\ \beta) \to F^{-1}(0)$ such that

$$\frac{\partial y}{\partial t} = p(t, s) \cdot \frac{\partial x}{\partial t}(s, t) \text{ and } \frac{\partial y}{\partial s} = p(t, s) \cdot \frac{\partial x}{\partial s}(t, s),$$

where $\Gamma(t, s) = (x(t, s), y(t, s), p(t, s))$. If we calculate second order partial derivatives of the both equalities, we get

$$\frac{\partial^2 y}{\partial t \partial s} = \frac{\partial p}{\partial s} \cdot \frac{\partial x}{\partial t} + p \cdot \frac{\partial^2 x}{\partial t \partial s} \text{ and } \frac{\partial^2 y}{\partial s \partial t} = \frac{\partial p}{\partial t} \cdot \frac{\partial x}{\partial s} + p \cdot \frac{\partial^2 x}{\partial s \partial t}.$$

Then we have $\frac{\partial p}{\partial s} \cdot \frac{\partial x}{\partial t} = \frac{\partial p}{\partial t} \cdot \frac{\partial x}{\partial s}$. This contradicts to the fact that Γ is an immersion.

If $F_p \neq 0$ at (x_0, y_0, p_0), it is already argued in §1 and that there exist a unique solution of $F = 0$ by the classical existence theorem. We can show that solutions give the unique complete solution of $F = 0$. We can also have a complete solution of $F = 0$ around (x_0, y_0, p_0) if it satisfies $F_x + p \cdot F_y \neq 0$ at (x_0, y_0, p_0) by the argument about the Legendre transformation in §1.

We now give a rigorous definition of singular solutions. Let $\gamma : (a\ b) \to F^{-1}(0)$ be a geometric solution. We say that γ is *a singular solution* of $F = 0$ if for any open subinterval $(c\ d) \subset (a\ b)$, $\gamma|(c\ d)$ is not a member of any complete solutions of $F = 0$. Then we have the following theorem.

THEOREM 2.2. *For an equation $F = 0$ and a geometric solution $\gamma : (a\ b) \to F^{-1}(0)$, the following are equivalent.*
(1) *γ is a singular solution of $F = 0$.*
(2) *$F = 0$ is completely integrable around any point of $\gamma((a\ b))$ and each members of the complete solution are transverse to γ.*
(3) Image $\gamma \subset \Sigma_c(F)$.

We remark that this theorem can be generalized in the case of partial differential equations [11]. In there, we need some techniques of contact (symplectic) geometry. Here, we shall give a proof by a purely elementary method.

PROOF: (3) $\Rightarrow$ (2). By the same reason as that of the proof of Lemma 1.3, we have $F_y \neq 0$ at $\gamma(0)$. By the implicit function theorem, the equation $F = 0$ can be rewritten as $y = h(x, p)$ around $\gamma(t_0)$, so that, we have

$$\Sigma_c(y - h(x,p)) = \{(x, h(x,p), p) | h_p(x,p) = h_x(x,p) - p = o\}.$$

Then we may distinguish two cases :
(a) $h_{xp}(x,p) - 1 = 0$,
(b) $h_{xp}(x,p) - 1 \neq 0$.
Case (a). Since $h_{xp} = h_{px} = 1 \neq 0$ at $(x(t_0), p(t_0))$, then the set $\Sigma_\pi(y - h(x,p)) = \{(x, h(x,p), p) | h_p = 0\}$ is a smooth curve. since γ is also a smooth curve, then we have $\gamma((a\ b)) = \Sigma_c(y - h(x,p)) = \Sigma_\pi(y - h(x,p))$. It follows that there exists a smooth function μ around $(x(t_0), p(t_0))$ such that μ does not vanish on such a neigbourhood and $h_x - p = \mu \cdot h_p$.
We now consider a vector field on the (x,p)-plane defined by

$$V = \frac{\partial}{\partial x} - \mu \cdot \frac{\partial}{\partial p}.$$

By the same reason as in Example 2.1, the flows of the vector field V gives a complete solution of $y - h(x,p) = 0$ around $\gamma(t_0)$. On the other hand, the curve $(x(t), p(t))$ is given by the equation $h_p = 0$ near $(x(t_0), p(t_0))$ by the previous arguments. If we calculate the canonical inner product of $\operatorname{grad} h_p$ and V, then we have

$$< \operatorname{grad} h_p, V >= 1 - \mu \cdot h_{pp}.$$

Differentiate the equation $h_x - p = \mu \cdot h_p$ with respect to p, then we have $\mu \cdot h_{pp} = 0$ at $(x(t_0), p(t_0))$. It follows that $< \operatorname{grad} h_p, V > \neq 0$ at $(x(t_0), p(t_0))$, so that $V \notin T\gamma$. This means that each members of the complete solution are transverse to γ around $\gamma(t_0)$.
Case (b). In this case the set $\{(x, h(x,p), p) | h_x(x,p) - p = 0\}$ is a smooth curve near $(x(t_0), y(t_0), p(t_0))$. By the same reason as that of the case (a), there exists a smooth function λ around $(x(t_0), p(t_0))$ which never vanish and satisfies $h_p = \lambda \cdot (h_x - p)$. We adopt a vector field $V = \lambda \cdot \frac{\partial}{\partial x} - \frac{\partial}{\partial p}$ on the (x,p)-plane, so that the flow of V gives a required complete solution.
(2) $\Rightarrow$ (1). If γ is not a singular solution, then there exist an interval $(a'\ b') \subset (a\ b)$ and a complete solution $\Gamma : (a'\ b') \times (\alpha\ \beta) \to F^{-1}(0)$ such that $\Gamma_{c_0} = \gamma$. By the uniqueness of the complete solution, this contradicts to the assumption (2).

(1) $\Rightarrow$ (3). If Image $\gamma \not\subset \Sigma_c(F)$, then there exists $t_0 \in (a\ b)$ such that $\gamma(t_0) \notin \Sigma_c(F)$. It follows that $F_p \neq 0$ or $F_x + p \cdot F_y \neq 0$ at $\gamma(t_0)$. In both cases, there exists a unique complete solution of $F = 0$ which contains γ around $\gamma(t_0)$. This contradicts to the definition of singular solutions.

3. The Clairaut type equation

Returning to our first question, we now consider a generalized subjects rather than the classical Clairaut equation (i.e. equations with smooth complete solutions). By the definition of smoothness of the solution and Lemma 1.1, *a smooth complete solution* of $F = 0$ is given by an one-parameter family of smooth function $y = f(x, c)$ defined on $(a\ b) \times (\alpha\ \beta)$ such that

$$F(x, f(x,c), \frac{\partial f}{\partial x}(x,c)) = 0$$

and a mapping

$$j^1_* f : (a\ b) \times (\alpha\ \beta) \to F^{-1}(0)$$

defined by

$$j^1_* f(x,c) = (x, f(x,c), \frac{\partial f}{\partial x}(x,c))$$

is an immersion. We remark that $j^1_* f$ is an immersion at (x_0, c_0) if and only if $(\frac{\partial f}{\partial c}, \frac{\partial^2 f}{\partial x \partial c}) \neq (0,0)$ at (x_0, c_0). In classical textbooks ([5],[8] etc) the complete solution is defined to be a family of functions which satisfies the last condition. So that the above definition gives a geometric characterization of the classical definition of complete solutions.

The following definition is due to Dara [7]. We say that an equation $F = 0$ is *Clairaut type* around (x_0, y_0, p_0) if there exist smooth functions $A(x,y,p)$, $B(x,y,p)$ such that $F_x + p \cdot F_y = A \cdot F + B \cdot F_p$ around (x_0, y_0, p_0). We now give some examples of Clairaut type equations.

Example 3.1. 1) Of course,one of the example is the Clairaut equation. In this case we can easily show that $F_x + p \cdot F_y = 0$. Then we may adopt $A = B = 0$.
2) Consider the following equation : $y - p^2 = 0$. Then we have $F_x + p \cdot F_y = p$ and $F_p = -2p$, so that we may adopt $A = 0$ and $B = -\frac{1}{2}$. Here, we can get the smooth complete solution as follows : $y = 2^{-\frac{3}{2}}(x+c)^2$.

Moreover,

$$\Sigma_\pi(F) = \{(x,0,0) | x \in \mathbb{R}\}$$

is the singular solution and D_F is the envelope of the family of graphs of the smooth complete solution.
3) "Free particle" on the line. Consider the following equation : $y^2 + p^2 - 1 = 0$. We can calculate that $F_x + p \cdot F_y = 2y \cdot p$ and $F_p = 2p$. Then we have $A = 0$ and $B = y$. The smooth complete solution around $(0, \pm 1, 0)$ is given by

$$y = \frac{\pm 1}{(c^2 + 2c + 2)^{\frac{1}{2}}} \cdot \cos(t + ct),$$

where (t, c) is a point near $(0, -1)$. In this case we can also show that

$$\Sigma_\pi(F) = \{(x, \pm 1, 0) | x \in \mathbb{R}\}$$

is the singular solution and D_F is the envelope of the family of graphs of the smooth complete solution.

The following theorem is our second answer to the question.

THEOREM 3.2. *For an equation $F = 0$, the followings are equivalent.*
(1) $F = 0$ is the Clairaut type equation around (x_0, y_0, p_0).
(2) $F = 0$ has a smooth complete solution around (x_0, y_0, p_0).
Moreover, in this case, if $\Sigma_\pi(F) \neq 0$, then $\Sigma_\pi(F)$ is the singular solution of $F = 0$ and D_F is the envelope of the family of the graphs of the complete solution.

We remark that this theorem can be also generalized in the case of partial differential equations [12].

PROOF: (1) $\Rightarrow$ (2). By the assumption, there exist smooth functions A, B such that $F_x + p \cdot F_y = A \cdot F + B \cdot F_p$ around (x_0, y_0, p_0). We now consider a vector field

$$V = \frac{\partial}{\partial x} + p \cdot \frac{\partial}{\partial y} - B \cdot \frac{\partial}{\partial p}$$

which is defined around (x_0, y_0, p_0). Let $c(t)$ be an integral curve of V such that $c(0) \in F^{-1}(0)$. Then we can calculate that $\frac{dF(c(t))}{dt}|_{t=0} = F_x + p \cdot F_y - B \cdot F_p = 0$. It follows that $V(x, y, p) \in T_{(x_0, y_0, p_0)} F^{-1}(0)$ for any (x, y, p) nearby (x_0, y_0, p_0). If we denote that $c(t) = (x(t), y(t), p(t))$, then we have $x'(t) = 1$, $y'(t) = p(c(t))$ and $p'(t) - B(c(t))$. These equalities guarantee that $c(t)$ is a smooth solution of $F = 0$ by Lemma 1.1. Then the flows of the vector field V gives the smooth complete solution of $F = 0$.
(2) $\Rightarrow$ (1). Let $y = f(x, c)$ be the complete solution of $F = 0$. If we calculate the partial derivative of $F(x, f(x, c), f_x(x, c)) = 0$ with respect to x, then we have

$$F_x + f_x \cdot F_y + f_{xx} \cdot F_p = 0$$

at $(x, f(x, c), f_x(x, c)) \in F^{-1}(0)$.

Since the map $j^1_* f$ is an immersion, then there exist a smooth function $B(x, y, p)$ which is defined nearby (x_0, y_0, p_0) such that $B \circ j^1_* f(x, c) = f_{xx}(x, c)$. For any $(x, y, p) \in F^{-1}(0)$ nearby (x_0, y_0, p_0), there exists (x, c) such that

$$(x, f(x, c), f_x(x, c)) = (x, y, p).$$

Then we have $F_x + p \cdot F_y = B \cdot F$ at $(x, y, p) \in F^{-1}(0)$ nearby (x_0, y_0, p_0). Since $\operatorname{grad} F \neq 0$ at (x_0, y_0, p_0), then the above equality means that there exists a function $A(x, y, p)$ defined nearby (x_0, y_0, p_0) such that $F_x + p \cdot F_y = B \cdot F_p + A \cdot F$. This completes the proof of the first part.

For the proof of the second part, we may assume that there exists a smooth complete solution $y = f(x, c)$ of $F = 0$ around (x_0, y_0, p_0). By the definition, $j^1_* f(x, c) \in \Sigma_\pi(F)$ if and only if

$$\operatorname{rank} \begin{pmatrix} 1 & f_x \\ 0 & f_c \end{pmatrix} < 2$$

at (x,c). It is equivalent to the fact that $f_c = 0$. Then the set $\Sigma_\pi(F)$ is given by the equation $f_c = 0$ around (x_0, y_0, p_0). Since $(f_c, f_{cx}) = (0, f_{cx}) \neq (0,0)$ at (x_0, c_0) with $j^1_* f(x,c) \in \Sigma_\pi(F)$, then we have $f_{cx} \neq 0$. It follows that $\Sigma_\pi(F) = j^1_* f(\{f_c = 0\})$ is a regular curve. On their other hand, $\Sigma_\pi(F) = \Sigma_c(F)$ by the definition of Clairaut type equations, so that it is a geometric solution by Lemma 1.3. Furthermore, we now consider the family of graphs of the smooth complete solution which is defined by the equation $f(x,c) - y = 0$ on the (x,y)-plane. Then the set

$$\{(x, f(x,c) | \text{There exist } c \text{ such that } f_c(x,c) = 0\}$$

is the envelope of this family by the usual method of the elementary calculus. This set is equal to the discriminant set D_F by the previous argument. This completes the proof.

In the classical textbooks ([6],[8] etc) the notion of singular solution has been appeared accompany with the notion of smooth complete solutions. In there the singular solution has been defined to be the envelope of the family of graphs of the smooth complete solution. The above theorem gives a characterization of this class of equations as the class of Clairaut type equations. In general the singular solution may not be in such a beautiful situation.

EXAMPLE 3.3. 1) The dual of the Clairaut equation. Consider the equation : $y = f(x)$. This equation is given by the Legendre transform of the Clairaut equation. We can calculate that $F_x + p \cdot F_y = f'(x) - p$ and $F_p = 0$. Then it is not Clairaut type. The geometric complete solution is given by

$$\{(c, f(c), x) | (c, x) \in \mathbb{R} \times \mathbb{R}\}.$$

The singular solution is given by

$$\Sigma_c(F) = \{(x, f(x), f'(x)) | x \in \mathbb{R}\}.$$

2) Consider the following equation : $y - 2p^3$. We can show that $\Sigma_\pi(F) = \Sigma_c(F) = \{(x, 0, 0) | x \in \mathbb{R}\}$ which is a singular solution. We can calculate that $F_x + p \cdot F_y = p$ and $F_p = -6p^2$. Then it is not Clairaut type around the origin. Moreover, we have the complete solution

$$\Gamma(t,c) = (3t^2 + c, 2t^3, t).$$

In this case the singular solution is a locus of cusps of the complete solution (Fig. 4).

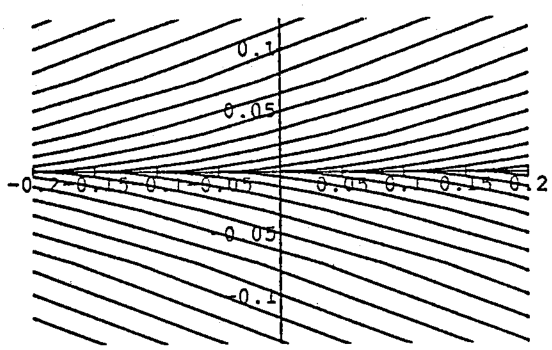

Fig. 4

In his textbook Forsyth [8] asserts that the locus of cusps of complete solutions never been a solution. We can understand that such a situation is correct in generic sense ([7],[9]).

We can also give a characterization of the Clairaut equation.

THEOREM 3.4. *For an equation $F = 0$, the followings are equivalent.*
(1) There exists a smooth function $A(x, y, p)$ around (x_0, y_0, p_0) such that

$$F_x + p \cdot F_y = A \cdot F$$

and $(x_0, y_0, p_0) \in \Sigma_\pi(F)$.
(2) There exists a smooth function $f(p)$ around p_0 such that

$$F^{-1}(0) = \{(x, y, p) | y = x \cdot p + f(p)\}.$$

This theorem has been proved by Dara [7], nevertheless, we now give an elementary proof.

PROOF: Suppose that $F = 0$ satisfies condition (1). If $F_y = 0$ at (x_0, y_0, p_0), then $F = F_x = F_p = 0$ at (x_0, y_0, p_0). This contradicts to the fact that $\operatorname{grad} F \neq 0$. Then $F_y \neq 0$ at (x_0, y_0, p_0). By the implicit function theorem, there exist smooth function $h(x, p)$ around (x_0, p_0) and a non vanishing smooth function $\lambda(x, y, p)$ around (x_0, y_0, p_0) such that $F(x, y, p) = \lambda(x, y, p) \cdot (h(x, p) - y)$. We now consider the Legendre transform F^* of F. Then $F^*(X, Y, P) = \Lambda(X, Y, P) \cdot (H(X, P) - Y)$, where $\Lambda(X, Y, P) = -\lambda(P, X \cdot P - Y, X)$ and $H(X, P) = X \cdot P - h(P, X)$. It follows that $F^*_P = \Lambda \cdot H_P$ on $F^{*-1}(0)$. Since $F^*_P = F_x + p \cdot F_y$ and

$$*L(\{(x, y, p) | y = h(x, p)\}) = \{(X, Y, P) | Y = H(X, P)\},$$

we have $H_P \equiv 0$. Then we can put $f(X) = -H(X, P)$. Pulling back by the Legendre transformation, we have

$$F^{-1}(0) = \{(x, y, p) | y = x \cdot p + f(p)\}.$$

The converse of the proof is given by a direct calculation.

4. THE PRINCIPLE OF DUALITY

The duality is one of the most beautiful properties in projective geometry. The modern theory of first order differential equations is described in contact geometry which can be considered as a generalization of projective geometry [1]. By this reason, we may expect that the duality holds among the first order differential equations.

As we already mentioned in §1, the dual relationships among the equations is given by the Legendre transformation in the classical theory of the ordinary differential equations. However, situations are confused in the classical theory as usual. Here, we shall establish the principle of duality among the completely integrable equations around (x_0, y_0, p_0).

Let $\mathcal{C}_I(x_0, y_0, p_0)$ be the set of completely integrable first order ordinary differential equations around (x_0, y_0, p_0). For any $F \in \mathcal{C}_I(x_0, y_0, p_0)$, we have a unique complete solution

$$\Gamma_F : (a\ b) \times (\alpha\ \beta) \to F^{-1}(0)$$

such that $\Gamma_F(t_0, c_0) = (x_0, y_0, p_0)$. We denote

$$\Gamma_F(t,c) = (x_F(t,c), y_F(t,c), p_F(t,c)).$$

We also define three subsets of $\mathcal{C}_I(x_0, y_0, p_0)$ as follows :

$$\begin{aligned}
\mathcal{C}_{I_0}(x_0,y_0,p_0) =& \{F \in \mathcal{C}_I(x_0,y_0,p_0) | \frac{dx_F}{dt} \neq 0 \text{ and } \frac{dp_F}{dt} \neq 0 \text{ at } (t_0,c_0)\},\\
\mathcal{C}_{I_1}(x_0,y_0,p_0) =& \{F \in \mathcal{C}_I(x_0,y_0,p_0) | \frac{dp_F}{dt} = 0 \text{ at } (t_0,c_0)\},\\
\mathcal{C}_{I_2}(x_0,y_0,p_0) =& \{F \in \mathcal{C}_I(x_0,y_0,p_0) | \frac{dx_F}{dt} = 0 \text{ at } (t_0,c_0)\}.
\end{aligned}$$

By the local uniqueness of the complete solution of $F = 0$, these subsets are well-defined. We denote $\mathcal{C}^*_I(X_0, Y_0, P_0)$ the set of complete integrable first order ordinary differential equations around (X_0, Y_0, P_0) in the coordinate system (X, Y, P). We also define sets $\mathcal{C}^*_{I_0}(X_0, Y_0, P_0,)$ $\mathcal{C}^*_{I_1}(X_0, Y_0, P_0)$ and $\mathcal{C}^*_{I_2}(X_0, Y_0, P_0)$ exactly the same definition as those of the above. Then we have the following duality theorem.

THEOREM 4.1. *We have an one-to-one correspondence*

$$\mathcal{D} : \mathcal{C}_I(x_0, y_0, p_0) \to \mathcal{C}^*_I(X_0, Y_0, P_0)$$

defined by $\mathcal{D}(F) = F^*$.

Furthermore, we have relations :

$$\mathcal{D}(\mathcal{C}_{I_0}(x_0,y_0,p_0) = \mathcal{C}^*_{I_0}(X_0,Y_0,P_0) \tag{1}$$

$$\mathcal{D}(\mathcal{C}_{I_1}(x_0,y_0,p_0) = \mathcal{C}^*_{I_2}(X_0,Y_0,P_0) \tag{2}$$

$$\mathcal{D}(\mathcal{C}_{I_2}(x_0,y_0,p_0) = \mathcal{C}^*_{I_1}(X_0,Y_0,P_0). \tag{3}$$

PROOF: By the definition, we have

$$\mathcal{D}(F) = F^* = F \circ (*L)^{-1},$$

where $*L$ is the Legendre transformation. For any $F \in \mathcal{C}_I(x_0, y_0, p_0)$, $*L \circ \Gamma_F$ is the unique complete solution of F^* by Lemma 1.2, (1). Then $\mathcal{D}$ is a well-defined and one-to-one correspondence. Since

$$*L \circ \Gamma_F(t,c) = (p_F(t,c), x_F(t,c) \cdot_F (t,c) - y_F(t,c), x_F(t,c)),$$

then we can easily show that the relations (1),(2) and (3).

We now give the final answer to our first question as a corollary of the above theorem.

COROLLARY 4.2. *Let $F = 0$ be an equation around (x_0, y_0, p_0). Then $F = 0$ is completely integrable around (x_0, y_0, p_0) if and only if $F = 0$ is Clairaut type around (x_0, y_0, p_0) or $F^* = 0$ is Clairaut type around (X_0, Y_0, P_0).*

We already presented two interesting examples which are integrable but not Clairaut type (see Example 3.3). We can easily verify that the duals of these examples are Clairaut type.

Acknowledgement. This work has been done during the author's stay at the University of Liverpool mainly supported by Anglo-Japanese Scientific Exchange programme (The Royal Society and JSPS). The author would like to thank all members of the Department of Pure Mathematics for they gave him a chance to talk about these subjects at the internal colloquium.

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