WEIGHTED DEFORMATION THEOREM FOR NORMAL CURRENTS

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WEIGHTED DEFORMATION THEOREM FOR NORMAL CURRENTS

HIROYUKI TAKAMURA

Abstract

We are concerned with the deformation theorem in the geometric measure theory. We shall prove the theorem for "weighted" mass to know its essence and to apply it to some variational problem. Our basic idea is to devide the space into suitable cubes on which we can treat the weighted mass as usual one.

§1. Introduction

One of the most basic results in the geometric measure theory of currents is the deformation theorem, which states that any normal currents (i.e. currents with finite mass and boundary mass) can be deformed into real polyhedral chains with controled mass and boundary mass. The deformation theorem was first proved by H. Federer and W. Fleming [6]. Later, L. Simon [7] improved a bound of mass in this theorem. In this paper we prove the deformation theorem with "weighted" mass. The weighted mass is a mass with density changing from place to place.

There are two advantages to prove the weighted deformation theorem. First, by using weighted mass, the essential structure of the deformation theorem will be clarified. Second, it leads to extend the usual potential applications to variational problems. Let us give an example. We consider a functional $\mathcal{F}$ of C^1-mapping $u : \Omega \longrightarrow \mathbb{R}^m$ defined by

$$\mathcal{F}(u) = \int_\Omega f(x, u(x), Du(x))dx,$$

where Ω is an open set in $\mathbb{R}^n$ and the density function $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \longrightarrow [0, \infty)$ is continuous. Here Jacobian matrix $Du(x)$ of u at x is identified with an element of $\mathbb{R}^{nm}$. In constructing minimizers of $\mathcal{F}$ under some constraints as a limit of minimizing sequences of $\mathcal{F}$, we have to extend $\mathcal{F}$ to a class of functions wider than C^1. For example, we assume that $f = f(x, y, p)$ is homogeneous in p of degree 1 and $u \in BV(\Omega, \mathbb{R}^m)$, the space of mapping of bounded variation. However, in such situation, the meaning of $\mathcal{F}(u)$ is not at all clear because $Du(x)$ is no longer a function but a Radon measure. One should relax $\mathcal{F}$ on $BV(\Omega, \mathbb{R}^m)$. For $m > 1$, P. Aviles and Y. Giga [2], [3], [4] first solved this problem for isotropic f under the following assumptions on f.

$f(x, y, p)$ is convex in p.

f is coercive. i.e. there are positive constants c, C such that

$c|p| \leq f(x, y, p) \leq C|p|$.

But considering physical phenomena such as the phase-transition, we have to remove $c|p| \leq f(x, y, p)$ part from the above conditions. A typical example of f is of the form

$$f(x, y, p) = |y - \alpha||y - \beta||p|, \quad \alpha, \beta \in \mathbb{R}^m.$$

The weighted deformation theorem might be useful to solve the relaxation problems for $\mathcal{F}$ with such a density f. We also claim that our weighted theorem is to be an extension of ususal one. Usual version is for currents which are defined in full space. But weighted version is for one defined in an open set.

The key of the proof of weighted deformation theorem is to devide the open set into suitable cubes on which we can treat the weighted mass as usual one. In §2, we make the partition of an open set by suitable cubes. Constructing the retraction in such cubes in §3, we locally deform currents in §4. In final section, we state main theorem and prove it by summing up the local deformation. For the most part, we follow terminologies and notations of L. Simon [7] which slightly differ from those of H. Federer [5].

One other generalization of the deformation theorem is done by F. Almgren [1]. He has studied it for size bounded currents which is different from our direction.

The author is grateful to Professor Yosikazu Giga for his suggestion of this problem.

§2. Partition by suitable cubes

This section is devoted to constructing the screen on which we will deform currents. For convenience we use the following notation and terminology.

2.1. Definitions.

Throughout this paper we assume $m, n \in \mathbb{N}$. For each $x, y \in \mathbb{R}^{m+n}$, we define the edge metric $EM : \mathbb{R}^{m+n} \times \mathbb{R}^{m+n} \longrightarrow [0, \infty]$ by

$$EM(x, y) = \sup\{|x_i - y_i| \ ; \ i = 1, \cdots, m+n\}.$$

Moreover we set

$$EM(x,Y) = \inf\{EM(x,y)\ ;\ y \in Y\}$$

whenever $x \in \mathbb{R}^{m+n}, Y \subset \mathbb{R}^{m+n}$. Then it is clear that

$$\frac{|x-y|}{\sqrt{m+n}} \leq EM(x,y) \leq |x-y|, \quad x,y \in \mathbb{R}^{m+n}.$$

Let Ω be an open set in $\mathbb{R}^{m+n}$. Consider the cube $C = [a,b]^{m+n}$, $a,b \in \mathbb{R}$ in Ω. For given continuous function $f : \Omega \longrightarrow (0,\infty)$ and small positive constant ε such that $0 < \varepsilon < 1$, we say that C is (f,ε)-*admissible* cube if and only if the inequality

$$(1-\varepsilon)f(c) \leq f(x) \leq (1+\varepsilon)f(c)$$

is valid for each $x \in C$, where $c = [(a+b)/2, \cdots, (a+b)/2]$ is a center of C.

2.2. Lemma.

Now we state our key lemma. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

LEMMA 2. *Let Ω be an open subset in $\mathbb{R}^{m+n}$. For a given continuous function $f : \Omega \longrightarrow (0,\infty)$ and a small positive constant ε such that $0 < \varepsilon < 1$, there exists a family of sets $\{\Omega_\nu\}_{\nu \in \mathbb{N}_0}$ which has the following properties:*

(2.1) $$\Omega = \bigcup_{\nu \in \mathbb{N}_0} \Omega_\nu \quad \textit{(direct sum)}.$$

(2.2) $$EM(x, \partial\Omega) \leq 2^{-\mu} \quad \textit{for} \quad x \in \Omega \setminus \bigcup_{\nu=0}^{\mu} \Omega_\nu.$$

(2.3) $$\Omega_\nu = \bigcup_{k \in \mathbb{N}_0} \bigcup_{z \in A_k} C_k^z \quad \textit{(direct sum)},$$

where $A_k \subset \mathbf{Z}^{m+n}$. Here each C_k^z is of the form

$$C_k^z = [0, 2^{-k})^{m+n} + 2^{-k}z, \quad z \in A_k$$

and each $\overline{C_k^z}$ is (f, ε)-admissible with $\overline{C_k^z} \subset \Omega$.

Proof: First, set $C_k = [0, 2^{-k})^{m+n}$, $k \in \mathbf{N}_0$. We start with the following operation (0) for the first step to construct the partition of Ω.

(0) Consider the decomposition

$$\mathbf{R}^{m+n} = \bigcup_{z \in \mathbf{Z}^{m+n}} (z + C_0).$$

Throw away all cubes such that

$$\overline{z + C_0} \not\subset \Omega.$$

Denote the rest of all cubes in the above decomposotion by

$$\Omega_0 = \bigcup_{z \in A_0'} C_0^z, \quad A_0' \subset \mathbf{Z}^{m+n}.$$

We note in (0) that $\overline{C_0^z} \subset \Omega$ if and only if $z \in A_0'$. Moreover it holds that

$$EM(x, \partial\Omega) \leq 1 \quad \text{for} \quad x \in \Omega \setminus \Omega_0.$$

Assuming $\mu \in \mathbf{N}$, we define the next step as follows.

(μ) Consider the decomposition

$$\mathbf{R}^{m+n} \setminus \bigcup_{\nu=0}^{\mu-1} \Omega_\nu = \bigcup_{z \in \mathbf{Z}^{m+n}} (2^{-\mu}z + C_\mu).$$

Throw away all cubes such that

$$\overline{2^{-\mu}z + C_\mu} \not\subset \Omega.$$

Denote the rest of all cubes in the above decomposition by

$$\Omega_\mu = \bigcup_{z \in A'_\mu} C^z_\mu, \quad A'_\mu \subset \mathbf{Z}^{m+n}.$$

Proceeding the operation (μ) inductively, we obtain Borel partition of Ω because Ω is open. Hence (2.1), (2.2) hold. But each $\overline{C^z_k}$ need not be (f, ε)-admissible.

Now by continuity of f, we can define

$$k_\varepsilon(x) = \inf\left\{ k \in \mathbf{N}_0 \; ; \; \sup\left\{ \frac{|f(x) - f(q^z_k)|}{f(q^z_k)} \; ; \; EM(x, q^z_k) \le 2^{-(k+1)} \right\} \le \varepsilon \right\},$$

where $q^z_k = 2^{-k}(z+q)$, $q = (1/2, \cdots, 1/2) \in \mathbf{R}^{m+n}$. Consider the following (k) in each Ω_ν.

For $x \in C^z_k$,

$$\text{(k)} \qquad \begin{array}{l} C^z_k \quad \text{remians if} \quad k_\varepsilon(x) \le k, \quad \text{or} \\ C^z_k \quad \text{is decomposed into} \quad C^z_k = \cup C^{z'}_{k+1} \quad \text{if} \quad k_\varepsilon(x) > k. \end{array}$$

Proceeding (k) as $k = \nu, \nu+1, \cdots$, and taking the new index A_k, we see that (2.3) holds.

§3. Retraction in cubes

In this section we mention a tool to deform currents. For the local deformation in next section, we consider some fixed part of Ω in §2 :

$$\Omega_\nu = \bigcup_{k \in \mathbf{N}_0} \bigcup_{z \in A_k} C^z_k.$$

The special notation for this paper is as follows.

$$L_j^k = j\text{-skelton of the decomposition} \quad E_k = \bigcup_{z \in A_k} \overline{C_k^z},$$

$$L_{n-1}^k(a) = (2^{-k}a + L_{n-1}^k) \cap E_k, \quad \text{where}$$

$$a \in \mathsf{B}_{1/4}^{m+n}(q) \quad \text{and} \quad q = (1/2, \cdots, 1/2) \in \mathsf{R}^{m+n},$$

$$L_{n-1}^k(a;\rho) = \{x \in E_k \ ; \ \operatorname{dist}(x, L_{n-1}^k(a)) \le 2^{-k}\rho\}, \quad \text{where} \quad \rho \in (0, 1/4).$$

Note that

$$\operatorname{dist}(L_{n-1}^k(a), L_m^k) \ge \frac{1}{4} \cdot 2^{-k} \quad \text{for} \quad a \in \mathsf{B}_{1/4}^{m+n}(q).$$

Now we have

LEMMA 3. *For every $a \in \mathsf{B}_{1/4}^{m+n}(q)$ there is a locally Lipschitz map*

$$\psi_k : E_k \setminus L_{n-1}^k(a) \longrightarrow E_k \setminus L_{n-1}^k(a)$$

such that

$$\begin{aligned} &\psi_k(\overline{C_k^z} \setminus L_{n-1}^k(a)) = \overline{C_k^z} \cap L_m^k, \\ &\psi_k|_{\overline{C_k^z} \cap L_m^k} = id_{\overline{C_k^z} \cap L_m^k}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\psi_k(2^{-k}z + x) = 2^{-k}z + \psi_k(x) \\ &\text{for} \quad 2^{-k}z + x, x \in E_k \setminus L_{n-1}^k(a), z \in \mathsf{Z}^{m+n}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &|D\psi_k(x)| \le C/\rho, \quad \rho \in (0, 1/4), \quad C = C(m,n) \\ &\text{for} \quad \mathcal{L}^{m+n}\text{-a.e.} \quad x \in \overline{C_k^z} \setminus L_{n-1}^k(a;\rho). \end{aligned} \tag{3.3}$$

PROOF: We see that the lemma follows by setting

$$\psi_k(x) = 2^{-k}\psi(2^k x) \quad \text{for} \quad x \in E_k \setminus L_{n-1}^k(a),$$

where ψ is the one in [7, 29.4]. But, for the completeness of this paper, we shall prove the lemma directly.

First, $S_1, \cdots, S_N$ ($N = \binom{m+n}{m+1} = \binom{m+n}{n-1}$) denote the $(m+1)$-dimensional subspaces of $\mathbb{R}^{m+n}$ which contain $(m+1)$-face of $C_k = [0, 2^{-k})^{m+n}$. p_j denotes the orthogonal projection of $\mathbb{R}^{m+n}$ onto S_j ($j = 1, \cdots, N$). Without the loss of generality, we may assume that E_k contains $\overline{C_k}$. For each j-face F_j^k ($j \geq m+1$) of $\overline{C_k}$, let $a(F_j^k) \in F_j^k$ be the orthogonal projection of $a_k = 2^{-k}a$ onto F_j^k, and let $\psi_{F_j^k}$ denote the retraction of $\overline{F_j^k} \setminus \{a(F_j^k)\}$ onto ∂F_j^k which takes a point $x \in \overline{F_j^k} \setminus \{a(F_j^k)\}$ to the point $y \in \partial F_j^k$ such that $x \in \{a(F_j^k) + \lambda(y - a(F_j^k))\ ;\ \lambda \in (0,1]\}$. Note that

$$(3.4) \qquad \overline{a_k\ a(F_j^k)} \subset L_{n-1}^k(a) \cap \overline{C_k} \quad \text{for} \quad j \geq m+1.$$

In fact, setting $J_k = \{\ell\ ;\ S_\ell \text{ is parallel to } F_{m+1}^k\}$ we see that the line segment $\overline{a_k a(F_j^k)}$ is orthognal to F_j^k, hence orthogonal to S_ℓ, $\ell \in J_k$. Thus we have

$$\overline{a_k a(F_j^k)} \subset \bigcap_{\ell \in J_k} p_\ell^{-1}(p_\ell(a_k)).$$

On the other hand, we know that

$$(3.5) \qquad L_{n-1}^k(a) = \bigcup_{\ell=1}^{N} \bigcap_{z \in \mathbb{Z}^{m+n}} (2^{-k}z + p_\ell^{-1}(p_\ell(a_k))) \cap E_k.$$

Therefore we obtain (3.1).

Next, for each $j \geq m+1$, define

$$\psi_k^{(j)} : \cup \overline{F_j^k} \setminus \{a(F_j^k)\} \longrightarrow \cup F_{j-1}^k$$

by $\psi_k^{(j)}|_{\overline{F_j^k} \setminus \{a(F_j^k)\}} = \psi_{F_j^k}$. Since $\psi_{F_j^k}|_{\partial F_j^k} = \mathrm{id}_{\partial F_j^k}$, $\psi_k^{(j)}$ is locally Lipschitz on its domain. By (3.4) we can define $\overline{\psi_k}$ by

$$\overline{\psi_k} = \psi_k^{(m+1)} \circ \cdots \circ \psi_k^{(m+n)}|_{\overline{C_k} \setminus L_{n-1}^k(a)}.$$

Then $\overline{\psi_k}$ has the following property:

$$\overline{\psi_k}(2^{-k}z + x) = 2^{-k}z + \overline{\psi_k}(x) \quad \text{for} \quad x, 2^{-k}z + x \in \overline{C_k}, z \in \mathbb{Z}^{m+n}.$$

Indeed, $x, 2^{-k}z + x \in \overline{C_k}$ means that either $x, 2^{-k}z + x \in L^k_m \cap \overline{C_k}$ or else $x, 2^{-k}z + x \in F^k_j, \widetilde{F^k_j}$, where $\widetilde{F^k_j} = 2^{-k}z + F^k_j$ with z orthogonal to F^k_j and $a(\widetilde{F^k_j}) = 2^{-k}z + a(F^k_j)$. It follows that we can define a retraction

$$\psi_k : E_k \setminus L^k_{n-1}(a) \longrightarrow L^k_m$$

by setting

$$\psi_k(2^{-k}z + x) = 2^{-k}z + \overline{\psi_k}(x) \quad \text{for}$$
$$x \in \overline{C_k} \setminus L^k_{n-1}(a), \quad 2^{-k}z + x \in E_k \setminus L^k_{n-1}(a), \quad z \in \mathbb{Z}^{m+n}.$$

Therefore (3.1), (3.2) hold.

For (3.3) we have to show that

$$\text{(3.6)} \qquad \sup |D\psi_k| \leq C/\rho \quad \text{on} \quad E_k \setminus L^k_{n-1}(a;\rho), \quad C = C(m,n).$$

(3.6) can be proved by induction on n. In case $n = 1$ it is evident from its construction. Hence assume $n \geq 2$ and that (3.6) holds in case $(n-1)$ replaces n. Consider the retraction

$$\psi^{m+n}_k : \overline{C_k} \setminus \{a_k\} \longrightarrow \partial C_k$$

and let

$$y = \psi^{m+n}_k(x), \ x \in \operatorname{Int}(\overline{C_k} \setminus L^k_{n-1}(a;\rho)),$$

and let F denote F^k_{m+n-1} which containes y.

Taking new coodinates we assume that

$$F \subset \mathbb{R}^{m+n-1} \times \{0\} \subset \mathbb{R}^{m+n},$$

and set $\widetilde{L}^k_{n-2}(a) = L^k_{n-1}(a) \cap \mathbb{R}^{m+n} \times \{0\}$. By virtue of (3.5) we have $a(F) \in L^k_{n-1}(a)$, hence

$$|y - a(F)| \geq \operatorname{dist}(y, L^k_{n-1}(a)). \tag{3.7}$$

Let p_F be the orthogonal projection of $\mathbb{R}^{m+n}$ onto $\mathbb{R}^{m+n-1} \times \{0\}$, so that $a(F) = p_F(a_k)$. Clearly,

$$|p_F(x) - a(F)| \geq \operatorname{dist}(x, p_F^{-1}(p_F(a_k)))$$

holds and hence by (3.5) we have

$$|p_F(x) - a(F)| \geq \operatorname{dist}(x, L^k_{n-1}(a)). \tag{3.8}$$

Furthermore, by definition of y, we know that

$$y - a = \frac{|y-a|}{|x-a|}(x-a)$$

and hence, applying p_F, we have

$$y - a(F) = \frac{|y-a|}{|x-a|} p_F(x-a).$$

Thus, by $|y-a| \geq 3/4$, we obtain

$$|y - a(F)| \geq \frac{3}{4} \frac{|p_F(x-a)|}{|x-a|}. \tag{3.9}$$

Now, observing the argument of the construction of (3.1) and (3.2), we know that the retraction $\widetilde{\psi}$ of $F \backslash \widetilde{L}^k_{n-2}(a)$ onto N-face of F is already defined. It follows from the inductive hypothesis, together with (3.7), (3.8) and (3.9), that

$$
\begin{aligned}
|\overline{D}\widetilde{\psi}(y)| &\equiv \limsup_{z \to y} \frac{|\widetilde{\psi}(z) - \widetilde{\psi}(y)|}{|z-y|} \\
&\leq \frac{C}{\operatorname{dist}(y, \widetilde{L}^k_{n-2}(a))} \\
&\leq \frac{C}{|y - a(F)|} \qquad (3.10) \\
&\leq \frac{4}{3}\, C\, \frac{|x-a|}{|p_F(x-a)|} \\
&\leq \frac{4}{3}\, C\, \frac{|x-a|}{\operatorname{dist}(x, L^k_{n-1}(a))}.
\end{aligned}
$$

On the other hand, by definition of ψ_k^{m+n}, we have

$$
|\overline{D}\psi_k^{m+n}(x)| \leq \frac{C}{|x-a|} \tag{3.11}
$$

with different constant C from (3.10). Since $\psi(x) = \widetilde{\psi} \circ \psi^{m+n}(x)$, we obtain by (3.10) and (3.11) that

$$
\begin{aligned}
|\overline{D}\psi_k(x)| &\leq |\overline{D}\widetilde{\psi}(y)|\; |\overline{D}\psi_k^{m+n}(x)| \\
&\leq \frac{C}{\operatorname{dist}(x, L^k_{n-1}(a))},
\end{aligned}
$$

where C is a constant given by (3.10) and (3.11). This completes the proof Lemma 3.

§4. Local deformation

We now establish the local deformation theorem for normal currents. Local means the deformation only on Ω_ν in §2. With notations of previous sections we consider Ω_ν of fixed ν.

4.1. Special notations.

In this section we use the following notations.

$$\begin{aligned}
\mathcal{L}_j^k(\nu) &= \text{collection of } \ j\text{-faces in } \ L_j^k \\
&= \{2^{-k}z + F_j^k \subset E_k \ ; \ F_j^k \ \text{ is a closed } j\text{-face of } \overline{C_k^z}, z \in \mathbf{Z}^{m+n}\}, \\
L_{n-1}(a) &= \bigcup_{k \in \mathbf{N}_0} L_{n-1}^k(a), \\
L_{n-1}(a;\rho) &= \bigcup_{k \in \mathbf{N}_0} L_{n-1}^k(a;\rho), \\
d_k(x) &= \operatorname{dist}(x, L_{n-1}^k(a)) \quad \text{for} \quad x \in E_k, \\
d(x) &= 2^k d_k(x) \quad \text{for} \quad x \in \overline{\Omega_\nu}.
\end{aligned}$$

We define $\psi : \overline{\Omega_\nu} \setminus L_{n-1}(a) \longrightarrow \overline{\Omega_\nu} \setminus L_{n-1}(a)$ by

$$\psi(x) = \psi_k(x) \quad \text{for} \quad x \in E_k \setminus L_{n-1}^k(a),$$

where ψ_k is the one in §3.

4.2. Weighted mass.

Following L. Simon [7] we start with the definition of mass. The mass of $T \in \mathcal{D}_m(\Omega)$ is defined by

$$\underline{\underline{M}}(T) = \sup\{T(\omega) \ ; \ \omega \in \mathcal{D}^m(\Omega), \ |\omega| \le 1\}.$$

If $\underline{\underline{M}}(T) < \infty$, then Riesz representation theorem in general measure theory [7, 4.1] implies that there exists a positive Radon measure μ_T on Ω and a μ_T-measurable unit m-vectorfield $\overrightarrow{T}$ on Ω such that

$$T(\omega) = \int_\Omega \langle \omega(x), \overrightarrow{T}(x) \rangle d\mu_T(x), \quad \omega \in \mathcal{D}^m(\Omega).$$

Using Lusin's theorem to exhaust μ_T-almost all of Ω by an increasing sequence of compact sets on which $\overrightarrow{T}$ is continuous, we have

$$\underline{\underline{M}}(T) = \mu_T(\Omega) = \int_\Omega d\mu_T(x).$$

For such a finite mass current we define the weighted mass by

$$\underline{\underline{M}}(f,T) = \int_\Omega f(x) d\mu_T(x),$$

where weight function $f : \Omega \longrightarrow (0,\infty)$ is continuous. Set

$$\delta(x) = 2^{-k} \quad \text{for} \quad x \in \overline{C_k^z},\ z \in A_k$$

where C_k^z is the one in the patition of Ω in §2. For the finite boundary mass current $T \in \mathcal{D}_m(\Omega)$, we also define the weighted boundary mass by

$$\underline{\underline{M}}(f\delta, \partial T) = \int_\Omega f(x)\delta(x) d\mu_{\partial T}(x).$$

4.3. Local weighted deformation theorem.

Setting $T_\nu = T\lfloor_{\overline{\Omega_\nu}}$ and $(\partial T)_\nu = (\partial T)\lfloor_{\overline{\Omega_\nu}}$, we have

THEOREM 4. *Let Ω be an open subset in $\mathbb{R}^{m+n}$. Weight function $f : \Omega \longrightarrow (0,\infty)$ is continuous. If $T \in \mathcal{D}_m(\Omega)$ with $\underline{\underline{M}}(f,T) + \underline{\underline{M}}(f\delta, \partial T) < \infty$, then, for a given small constant ε such that $0 < \varepsilon < 1$, there exist*

$$P = \sum_{k \in \mathbb{N}_0} \sum_{F \in \mathcal{L}_m^k(\nu)} \beta_F [\![F]\!] \quad \in \mathcal{D}_m(\Omega) \quad \text{with} \quad \beta_F \in \mathbb{R},$$
$$S_\nu \in \mathcal{D}_m(\Omega), \quad R_\nu \in \mathcal{D}_{m+1}(\Omega)$$

such that

$$T_\nu = P_\nu + \partial R_\nu + S_\nu$$

with

$$\underline{\underline{M}}(f, P_\nu) \le C\underline{\underline{M}}(f, T_\nu), \quad \underline{\underline{M}}(f\delta, \partial P_\nu) \le C\underline{\underline{M}}(f\delta, (\partial T)_\nu),$$
$$\underline{\underline{M}}(f/\delta, R_\nu) \le C\underline{\underline{M}}(f, T_\nu), \quad \underline{\underline{M}}(f, S_\nu) \le C\underline{\underline{M}}(f\delta, (\partial T)_\nu),$$
$$C = C(m,n,\varepsilon),$$
$$spt\ P_\nu \cup spt\ R_\nu \subset \{x \in \overline{\Omega_\nu}\ ;\ EM(x, spt\ T_\nu) \le 2\delta(x)\},$$
$$spt\ \partial P_\nu \cup spt\ S_\nu \subset \{x \in \overline{\Omega_\nu}\ ;\ EM(x, spt\ (\partial T)_\nu) \le 2\delta(x)\},$$

Moreover P_ν and R_ν may be chosen to be integer multiplicity whenever T_ν is integer multiplicity, and also S_ν may be if so is $(\partial T)_\nu$.

PROOF: We proceed the proof with L. Simon's argument. It is divided into five steps. Constants C in the poof will change from line to line.

Step 1. Estimate in $L_{n-1}(a;\rho)$.

Set $F_j = \overline{C_0} \cap S_j$ so that F_j is a closed $(m+1)$-face of $\overline{C_0} = [0,1]^{m+n}$. Let x_j be the central point of F_j. For each $j = 1, \cdots, N$, we define a "good" subset $G_j \subset F_j \subset \mathrm{B}^{m+1}_{1/4}(x_j)$ by $g \in G_j \iff g \in F_j \cap \mathrm{B}^{m+1}_{1/4}(x_j)$ and

$$(4.1)\qquad \underline{\underline{M}}(T\lfloor_{L^k_{n-1}(g;\rho)\cap S_j \cap \overline{C^z_k}}) \le \beta\rho^{m+1}\underline{\underline{M}}(T\lfloor_{\overline{C^z_k}})$$

for $\rho \in (0,1/4)$, where β will be chosen.

We claim that the "bad" set $B_j = F_j \cap \mathrm{B}^{m+1}_{1/4}(x_j) \setminus G_j$ has small $\mathcal{L}^{m+1}$-measure such that

$$(4.2)\quad \mathcal{L}^{m+1}(B_j) \le 20^{m+1}\beta^{-1}\omega_{m+1}\left(\frac{1}{4}\right)^{m+1}, \quad \omega_{m+1} = \mathcal{L}^{m+1}(\mathrm{B}^{m+1}_1(0)).$$

In fact we see from (4.1) that, for $b \in B_j$, there is a $\rho_b \in (0,1/4)$ such that

$$(4.3)\qquad \underline{\underline{M}}(T\lfloor_{L^k_{n-1}(b_j;\rho_b)\cap S_j \cap \overline{C^z_k}}) \ge \beta\rho_b^{m+1}\underline{\underline{M}}(T\lfloor_{\overline{C^z_k}}).$$

By the covering theorem [7, 3.3], there is a pairwise disjoint subcollection $\{\mathrm{B}^{m+1}_{\rho_\ell}(b_\ell)\}_{\ell\in\mathbb{N}}$ of the collection $\{\mathrm{B}^{m+1}_{\rho_b}(b)\}_{b\in B_j}$ such that

$$(4.4)\qquad B_j \subset \bigcup_{\ell\in\mathbb{N}} \mathrm{B}^{m+1}_{5\rho_\ell}(b_\ell),$$

where $\rho_\ell = \rho_{b_\ell}$. Setting $b = b_\ell$ in (4.3) and summing up in ℓ, we get

$$\beta\sum_{\ell\in\mathbb{N}} \rho_{b_\ell}^{m+1}\underline{\underline{M}}(T\lfloor_{\overline{C^z_k}}) \le \underline{\underline{M}}(T\lfloor_{\overline{C^z_k}})$$

because $L^k_{n-1}(b_\ell;\rho_\ell)\cap S_j\cap\overline{C^z_k}$ is a pairwise disjoint collection for fixed j. Hence (4.2) follows from (4.4).

We now have

$$\mathcal{L}^{m+1}(G_j)\geq(1-20^{m+1}\beta^{-1})\omega_{m+1}\left(\frac{1}{4}\right)^{m+1},$$

and it follows that

$$\mathcal{L}^{m+n}(p_j^{-1}(G_j)\cap \mathbf{B}^{m+1}_{1/4}(q))\geq\left(1-\frac{\omega_{m+1}}{\omega_{m+n}}20^{m+1}\beta^{-1}\right)\omega_{m+n}\left(\frac{1}{4}\right)^{m+n}. \tag{4.5}$$

Then selecting β large enough so that $20^{m+1}\omega_{m+1}N\beta^{-1}<\omega_{m+n}/(m+n)$, we see from (4.5) that we can choose a point $a\in\bigcap_{j=1}^{N}p_j^{-1}(G_j)\cap\mathbf{B}^{m+n}_{1/4}(q)$. Since $p_j(a)\in G_j$, (4.1) gives

$$\underline{\underline{M}}(T\lfloor_{L^k_{n-1}(a;\rho)\cap\overline{C^z_k}})\leq N\beta\rho^{m+1}\underline{\underline{M}}(T\lfloor_{\overline{C^z_k}}) \tag{4.6}$$

for $\rho\in(0,1/4)$. Note that each $\overline{C^z_k}$ is (f,ε)-admissible, and we obtain from (4.6)

$$\underline{\underline{M}}(f,T\lfloor_{L_{n-1}(a;\rho)})\leq C\rho^{m+1}\underline{\underline{M}}(f,T_\nu) \tag{4.7}$$

because

$$\begin{aligned}\underline{\underline{M}}(f,T\lfloor_{L_{n-1}(a;\rho)})&\leq(1+\varepsilon)\sum_{k,z}f(q^z_k)\underline{\underline{M}}(T\lfloor_{L^k_{n-1}(a;\rho)\cap\overline{C^z_k}})\\&\leq C\rho^{m+1}(1+\varepsilon)\sum_{k,z}f(q^z_k)\underline{\underline{M}}(T\lfloor_{\overline{C^z_k}})\\&\leq C\rho^{m+1}\frac{1+\varepsilon}{1-\varepsilon}\underline{\underline{M}}(f,T_\nu),\end{aligned}$$

where q^z_k is the central point of $\overline{C^z_k}$. Note that the intersection of more than any $2n$ of $\overline{C^z_k}$ is empty. Here and hereafter we use this calculation in summing up in z and k.

Step 2. Estimate of deformed T.

Consider the retraction ψ_k in Lemma 3. By mass-estimate of the pushing forward [7, 26.25] and (4.6), we have

$$(4.8)\qquad \underline{\underline{M}}(\psi_{k\#}(T\lfloor_{\{\rho/2<d<\rho\}\cap\overline{C_k^z}})) \le \frac{C}{\rho^m}\cdot\rho^{m+1}\underline{\underline{M}}(T\lfloor_{\overline{C_k^z}}) \le C\rho\underline{\underline{M}}(T\lfloor_{\overline{C_k^z}}).$$

Summing up in z and k, we also get

$$(4.9)\qquad \underline{\underline{M}}(\psi_{\#}(T\lfloor_{\{\rho/2<d<d\}})) \le C\rho\underline{\underline{M}}(f,T_\nu).$$

Now, define the homotopy $h_k : \{E_k\setminus L^k_{n-1}(a)\}\times[0,1]\longrightarrow E_k\setminus L^k_{n-1}(a)$ by

$$h_k(x,t) = x + t(\psi_k(x)-x).$$

Notice that h_k is only Lipschitz on its domain. By mass-estimate of the homotopy [7, 26.23] and (4.6), we have

$$(4.10)\qquad \underline{\underline{M}}(h_{k\#}([\![(0,1)]\!]\times T\lfloor_{\{\rho/2<d<\rho\}\cap\overline{C_k^z}})) \le 2^{-k}C\rho^m\underline{\underline{M}}(T\lfloor_{\overline{C_k^z}}).$$

Defining $h : \{\overline{\Omega_\nu}\setminus L_{n-1}(a)\}\times[0,1]\longrightarrow\overline{\Omega_\nu}\setminus L_{n-1}(a)$ by $h(x,t)=h_k(x,t)$ for $x\in E_k\setminus L_{n-1}(a)$, we also get

$$(4.11)\qquad \underline{\underline{M}}(f/\delta,\ h_{\#}([\![(0,1)]\!]\times T_\nu\lfloor_{\{\rho/2<d<\rho\}})) \le C\rho^m\underline{\underline{M}}(f,T_\nu).$$

Step 3. Estimate of deformed ∂T.

Replacing T by ∂T in the argument of step 1, we obtain by (4.6) that

$$(4.12)\qquad \underline{\underline{M}}((\partial T)\lfloor_{L^k_{n-1}(a;\rho)\cap\overline{C_k^z}}) \le C\rho^{m+1}\underline{\underline{M}}((\partial T)\lfloor_{\overline{C_k^z}})$$

and hence

$$(4.13)\qquad \underline{\underline{M}}(f\delta,(\partial T)_\nu\lfloor_{L_{n-1}(a;\rho)}) \le C\rho^{m+1}\underline{\underline{M}}(f\delta,(\partial T)_\nu).$$

Furthermore by [7, 28. 10] we know that for each $\rho \in (0, 1/4)$ we can find $\rho^* \in (\rho/2, \rho)$ such that

$$\underline{\underline{M}}(\langle T\lfloor_{\overline{C_k^z}}, d_k, 2^{-k}\rho^*\rangle) \leq 2^k C\rho^m \underline{\underline{M}}(T\lfloor_{\overline{C_k^z}}), \tag{4.14}$$

so that

$$\underline{\underline{M}}(f\delta, \langle T_\nu, d, \rho^*\delta\rangle) \leq C\rho^m \underline{\underline{M}}(f, T_\nu). \tag{4.15}$$

On the other hand, as (4.8) and (4.9), we have

$$\underline{\underline{M}}(\psi_{k\#}((\partial T)\lfloor_{\{\rho/2<d<\rho\}\cap\overline{C_k^z}})) \leq C\rho\underline{\underline{M}}((\partial T)\lfloor_{\overline{C_k^z}}) \tag{4.16}$$

and

$$\underline{\underline{M}}(f\delta, \psi_\#((\partial T)_\nu\lfloor_{\{\rho/2<d<\rho\}})) \leq C\rho\underline{\underline{M}}(f\delta, (\partial T)_\nu). \tag{4.17}$$

It follows from the argument to drive (4.10) that

$$\underline{\underline{M}}(h_{k\#}([\![(0,1)]\!] \times (\partial T)\lfloor_{\{p/2<d<\rho\}\cap\overline{C_k^z}})) \leq 2^{-k}C\rho\underline{\underline{M}}((\partial T)\lfloor_{\overline{C_k^z}}) \tag{4.18}$$

and

$$\underline{\underline{M}}(f, h_\#([\![(0,1)]\!] \times (\partial T)_\nu\lfloor_{\{p/2<d<\rho\}})) \leq C\rho\underline{\underline{M}}(f\delta, (\partial T)_\nu). \tag{4.19}$$

By [7, 26.25] and (4.14) we see that

$$\underline{\underline{M}}(\psi_{k\#}\langle T\lfloor_{\overline{C_k^z}},\ d_k,\ 2^{-k}\rho^*\rangle) \leq 2^k C\rho\underline{\underline{M}}(T\lfloor_{\overline{C_k^z}}) \tag{4.20}$$

and

$$\underline{\underline{M}}(f\delta, \psi_\#\langle T_\nu,\ d,\ \rho^*\delta\rangle) \leq C\rho\underline{\underline{M}}(f, T_\nu). \tag{4.21}$$

Moreover it follows from [7, 26.23] and (4.14) that

$$\underline{\underline{M}}(h_{k\#}([\![(0,1)]\!] \times \langle T\lfloor_{\overline{C_k^z}}, d_k, 2^{-k}\rho^*\rangle) \leq C\rho^m \underline{\underline{M}}(T\lfloor_{\overline{C_k^z}}) \tag{4.22}$$

and

$$(4.23)\qquad \underline{\underline{M}}(f, h_{\#}([\![(0,1)]\!] \times \langle T_\nu, d, \rho^*\delta\rangle)) \le C\rho^m \underline{\underline{M}}(f, T_\nu).$$

Step 4. Deformation of T.

By iteration, (4.9) and (4.17) imply

$$(4.24)\qquad \begin{aligned} &\underline{\underline{M}}(f, \psi_{\#}(T_\nu \lfloor_{\{\rho/2^\ell < d < \rho\}})) \le 2C\rho \underline{\underline{M}}(f, T_\nu), \\ &\underline{\underline{M}}(f\delta, \psi_{\#}((\partial T)_\nu \lfloor_{\{\rho/2^\ell < d < \rho\}})) \le 2C\rho \underline{\underline{M}}(f\delta, (\partial T)_\nu) \end{aligned}$$

for each $\ell \in \mathbb{N}$, where C is in (4.9) and (4.17). Selecting $\rho = 1/4$ and using the abitrariness of $\ell \in \mathbb{N}$, we get

$$(4.25)\qquad \begin{aligned} &\underline{\underline{M}}(f, \psi_{\#}(T_\nu \lfloor_{\{\sigma < d\}})) \le C \underline{\underline{M}}(f, T_\nu), \\ &\underline{\underline{M}}(f\delta, \psi_{\#}((\partial T)_\nu \lfloor_{\{\sigma < d\}})) \le C \underline{\underline{M}}(f\delta, (\partial T)_\nu) \end{aligned}$$

for each $\sigma \in (0,1)$.

Now setting $\rho = \rho_\ell \equiv 2^{-\ell}$ and $\rho_\ell^* \in [2^{-(\ell+1)},\ 2^{-\ell}]$ such that (4.15), (4.21) and (4.23) hold with ρ_ℓ^* in place of ρ^*, we have that

$$\lim_{\ell \to \infty} [\underline{\underline{M}}(f\delta, \langle T_\nu,\ d,\ \rho_\ell^*\delta\rangle) + \underline{\underline{M}}(f\delta, \psi_{\#}\langle T_\nu, d, \rho_\ell^*\delta\rangle)] = 0.$$

Then it follows from the definition of the slice [7, 28.6, 28.7], together with (4.21), (4.23), (4.24) and (4.25), that

$$\psi_{\#}(T_\nu \lfloor_{\{\rho_\ell^* < d\}}), \quad h_{\#}([\![(0,1)]\!] \times \partial(T_\nu \lfloor_{\{\rho_\ell^* < d\}}))$$

are Cauchy sequences relative to $\underline{\underline{M}}(f, \cdot)$,

$$\psi_{\#}((\partial T)_\nu \lfloor_{\{\rho_\ell^* < d\}})$$

is so to $\underline{\underline{M}}(f\delta, \cdot)$ and

$$h_{\#}([\![(0,1)]\!] \times T_\nu \lfloor_{\{\rho_\ell^* < d\}})$$

is so to $\underline{\underline{M}}(f/\delta,\cdot)$. Hence there are currents $Q, S \in \mathcal{D}_m(\Omega)$ and $R \in \mathcal{D}_{m+1}(\Omega)$ such that

$$
\begin{aligned}
&\lim_{\ell\to\infty} \underline{\underline{M}}(f, Q - \psi_{\#}(T_\nu \llcorner_{\{\rho_\ell^* < d\}})) = 0, \\
&\lim_{\ell\to\infty} \underline{\underline{M}}(f, S - h_{\#}([\![(0,1)]\!] \times \partial(T_\nu \llcorner_{\{\rho_\ell^* < d\}}))) = 0, \\
&\lim_{\ell\to\infty} \underline{\underline{M}}(f/\delta, R - h_{\#}([\![(0,1)]\!] \times T_\nu \llcorner_{\{\rho_\ell^* < d\}})) = 0.
\end{aligned}
\tag{4.26}
$$

Furthermore, the homotopy formula [7, 26.22] gives

$$
\begin{aligned}
T_\nu \llcorner_{\{\rho_\ell^* < d\}} - \psi_{\#}(T_\nu \llcorner_{\{\rho_\ell^* < d\}}) =& \partial(h_{\#}([\![(0,1)]\!] \times T_\nu \llcorner_{\{\rho_\ell^* < d\}})) \\
&+ h_{\#}([\![(0,1)]\!] \times \partial(T_\nu \llcorner_{\{\rho_\ell^* < d\}})).
\end{aligned}
\tag{4.27}
$$

for each $\ell \in \mathbb{N}$. Therefore we obtain

$$
T_\nu - Q = \partial R + S. \tag{4.28}
$$

We note that Q need not be a polyhedral chain and that Q, R are integer multiplicity by [7, 28.4, 28.5, 27.5] and (4.26) in case T_ν is integer multiplicity. Similarly S is integer multiplicity if so is $(\partial T)_\nu$.

It follows from (3.1) that

$$
\mathrm{spt}\psi_{\#}(T_\nu \llcorner_{\{\rho_\ell^* < d\}}) \subset L_m,
$$

and hence

$$
\mathrm{spt} Q \subset L_m. \tag{4.29}
$$

(3.2) yields that

$$
\begin{aligned}
&\mathrm{spt}\ R \cup \mathrm{spt}\ Q \subset \{x \in \overline{\Omega_\nu}\ ;\ EM(x, \mathrm{spt}\ T_\nu) \leq \delta(x)\}, \\
&\mathrm{spt}\ S \subset \{x \in \overline{\Omega_\nu}\ ;\ EM(x,\ \mathrm{spt}(\partial T)_\nu) \leq \delta(x)\}
\end{aligned}
\tag{4.30}
$$

By (4.25) and (4.26), we also obtain

$$
\begin{aligned}
&\underline{\underline{M}}(f, Q) \leq C\underline{\underline{M}}(f, T_\nu), \\
&\underline{\underline{M}}(f/\delta, R) \leq C\underline{\underline{M}}(f, T_\nu), \\
&\underline{\underline{M}}(f, S) \leq C\underline{\underline{M}}(f\delta, (\partial T)_\nu).
\end{aligned}
\tag{4.31}
$$

From the semi-continuity of $\underline{\underline{M}}$ under weak convergence, we know that so is $\underline{\underline{M}}(f\delta,\cdot)$ and

$$
\begin{aligned}
\underline{\underline{M}}(f\delta,\partial Q) &\leq \liminf_{\ell\to\infty} \underline{\underline{M}}(f\delta,\partial(\psi_{\#}(T_\nu \lfloor_{\{\rho_\ell^*<d\}}))) \\
&= \liminf_{\ell\to\infty} \underline{\underline{M}}(f\delta,\psi_{\#}(\partial(T_\nu \lfloor_{\{\rho_\ell^*<d\}}))) \\
&\leq C\underline{\underline{M}}(f\delta,(\partial T)_\nu).
\end{aligned}
\tag{4.32}
$$

Step 5. Replacing Q by polyhedral chain.

Let F be a given m-face of $\overline{C_k^z}$ and $\mathring{F}$ be a interior of F. Suppose new coodinates are selected so that $F \subset \mathbb{R}^m \times \{0\} \subset \mathbb{R}^{m+n}$, and let p be the orthognal projection onto $\mathbb{R}^m \times \{0\}$. By construction of ψ_k, we know that $p \circ \psi_k = \psi_k$ in a neighbourhood of any point $y \in \mathring{F}$. Hence by (4.25) we have

$$
p_{\#}(Q\lfloor_{\mathring{F}}) = Q\lfloor_{\mathring{F}}.
\tag{4.33}
$$

It then follows, by the obvious modifications of the arguments in the proof of the constancy theorem [7, 26.27] and [7, 26.28], that

$$
(Q\lfloor_{\mathring{F}})(\omega) = \int_{\mathring{F}} \langle e_1 \wedge \cdots \wedge e_m, \omega(x)\rangle \theta_F(x)\,d\mathcal{L}^m(x)
\tag{4.34}
$$

for any $\omega \in \mathcal{D}^m(\Omega)$, for some $BV_{\mathrm{loc}}(\mathbb{R}^m)$, and

$$
\begin{aligned}
\underline{\underline{M}}(Q\lfloor_{\mathring{F}}) &= \int_{\mathring{F}} |\theta_F|\,d\mathcal{L}^m, \\
\underline{\underline{M}}((\partial Q)\lfloor_{\mathring{F}}) &= \int_{\mathring{F}} |D\theta_F|.
\end{aligned}
\tag{4.35}
$$

Since

$$
(Q\lfloor_{\mathring{F}} - \beta[\![F]\!])(\omega) = \int_{\mathring{F}} (\theta_F - \beta)\langle e_1 \wedge \cdots \wedge e_m, \omega(x)\rangle\,d\mathcal{L}^m(x)
$$

by (4.34), it follws from [7, 26.28] again that

$$
\begin{aligned}
\underline{\underline{M}}(Q\lfloor_{\mathring{F}} - \beta[\![F]\!]) &= \int_{\mathring{F}} |\theta_F - \beta|\,d\mathcal{L}^m, \\
\underline{\underline{M}}(\partial(Q\lfloor_{\mathring{F}} - \beta[\![F]\!])) &= \int_{\mathbb{R}^m} |D(\chi_{\mathring{F}}(\theta_F - \beta))|,
\end{aligned}
\tag{4.36}
$$

where $\chi_{\overset{\circ}{F}}$ is the characteristic function of $\overset{\circ}{F}$. Then we can take $\beta = \beta_F$ such that

$$\min\{\mathcal{L}^m\{x \in \overset{\circ}{F}; \theta_F \geq \beta\}, \mathcal{L}^m\{x \in \overset{\circ}{F}; \theta_F \leq \beta\}\} \geq 2^{-1}\mathcal{L}^m(\overset{\circ}{F}). \tag{4.37}$$

We note that $\beta_F \in \mathbb{Z}$ if θ_F is integer-valued. Using Poincaré inequality [7, 6.4, 6.6], together with (4.35) and (4.36), we obtain that

$$\begin{aligned} &\underline{\underline{M}}(Q\lfloor_{\overset{\circ}{F}} - \beta_F[\![F]\!]) \leq 2^{-k}C \int_{\overset{\circ}{F}} |D\theta_F| = 2^{-k}C\ \underline{\underline{M}}((\partial Q)\lfloor_{\overset{\circ}{F}}), \\ &\underline{\underline{M}}(\partial(Q\lfloor_{\overset{\circ}{F}} - \beta_F[\![F]\!])) \leq C \int_{\overset{\circ}{F}} |D\theta_F| = C\underline{\underline{M}}((\partial Q)\lfloor_{\overset{\circ}{F}}). \end{aligned} \tag{4.38}$$

We also have by [S, 26. 30]

$$Q\lfloor_{\partial F} = 0. \tag{4.39}$$

Then, summing over $F \in \mathcal{L}_m^k(\nu)$ and using (4.38) and (4.39), we have that

$$\begin{aligned} &\underline{\underline{M}}(f, Q - P) \leq C\underline{\underline{M}}(f\delta, \partial Q), \\ &\underline{\underline{M}}(f\delta, \partial Q - \partial P) \leq C\underline{\underline{M}}(f\delta, \partial Q), \end{aligned} \tag{4.40}$$

where we set

$$P = \sum_{k \in \mathbb{N}_0} \sum_{F \in \mathcal{L}_m^k(\nu)} \beta_F[\![F]\!].$$

Then it follows from (4.35) and (4.37) that

$$\underline{\underline{M}}(\beta_F[\![F]\!]) = |\beta_F|\mathcal{L}^m(\overset{\circ}{F}) \leq 2\int_{\overset{\circ}{F}} |\theta_F| d\mathcal{L}^m = 2\underline{\underline{M}}(Q\lfloor_{\overset{\circ}{F}}),$$

and hence

$$\underline{\underline{M}}(f, P) \leq C\underline{\underline{M}}(f, Q). \tag{4.41}$$

By second inequality of (4.40) we have

$$\underline{\underline{M}}(f\delta, \partial P) \leq C\underline{\underline{M}}(f\delta, \partial Q). \tag{4.42}$$

Finally, we note that (4.28) can be written

$$T_\nu - P = \partial R + (S + (Q - P)). \tag{4.43}$$

Setting $P_\nu = P$, $R_\nu = R$, $S_\nu = S + (Q - P)$, the theorem now follows immediately from (4.29), (4.30), (4.31) and (4.40), (4.41), (4.42); the fact that P_ν, R_ν are integer multiplicity if T_ν is should be evident from the remark during the course of the above proof, as should be the fact that S_ν is integer multiplicity if $T_\nu, (\partial T)_\nu$ are.

§5. Main theorem

With notations of previous sections we shall state the weighted deformation theorem. Moreover, we shall claim that the deformation theorem of unscaled version in [7, 29.1] is a direct consequence of the weighted one. At this point of view we may say that the weighted theorem is a generalization of unscaled version.

5.1. Result.

Let $\mathcal{L}_m$ be the m-skelton of the decomposition of Ω in §2;

$$\mathcal{L}_m = \sum_{\nu \in \mathbb{N}_0} \sum_{k \in \mathbb{N}_0} \mathcal{L}_m^k(\nu).$$

Here we newly define $\delta : \Omega \longrightarrow \mathbb{R}$ by

$$\delta(x) = 2^{-k} \quad \text{for} \quad x \in \overline{C_k^z}, \quad z \in A_k, \quad k \in \mathbb{N}_0,$$

where C_k^z is in §2.

THEOREM 5 (THE WEIGHTED DEFORMATION THEOREM). *Let Ω be an open subset in $\mathbb{R}^{m+n}$. Weight function $f : \Omega \longrightarrow (0, \infty)$ is continuous. Suppose $T \in \mathcal{D}_m(\Omega)$ with $\underline{\underline{M}}(f, T) + \underline{\underline{M}}(f\delta, \partial T) < \infty$. Then, for a small constant ε such that $0 < \varepsilon < 1$, we can write*

$$T - P = \partial R + S,$$

where

$$P = \sum_{F \in \mathcal{L}_m} \beta_F [\![F]\!] \quad \in \mathcal{D}_m(\Omega), \qquad \beta_F \in \mathbb{R},$$

$$R \in \mathcal{D}_{m+1}(\Omega), \quad S \in \mathcal{D}_m(\Omega)$$

which satisfy

$$\underline{\underline{M}}(f, P) \le C\underline{\underline{M}}(f, T), \quad \underline{\underline{M}}(f\delta, \partial P) \le C\underline{\underline{M}}(f\delta, \partial T),$$

$$\underline{\underline{M}}(f/\delta, R) \le C\underline{\underline{M}}(f, T), \quad \underline{\underline{M}}(f, S) \le C\underline{\underline{M}}(f\delta, \partial T),$$

$$C = C(m, n, \varepsilon),$$

$$spt\ P \cup\ spt\ R \subset \{x \in \Omega\ ;\ EM(x, spt\ T) \le 2\delta(x)\},$$

$$spt\ \partial P \cup\ spt\ S \subset \{x \in \Omega\ ;\ EM(x, spt\ \partial T) \le 2\delta(x)\}.$$

In case T is an integer multiplicity current, then P, R can be chosen to be integer multiplicity currents (and β_F appearing in the definition of P are integers). If in addition ∂T is integer multiplicity, then S can be chosen to be integer multiplicity.

PROOF: Theorem 5 follows immediately from Theorem 4 by setting $P = \sum_{\nu \in \mathbb{N}_0} P_\nu$, $R = \sum_{\nu \in \mathbb{N}_0} R_\nu$, $S = \sum_{\nu \in \mathbb{N}_0} S_\nu$ because the intersection of more than any two of sets $\overline{\Omega_\nu}$ is empty.

As a direct consequence of Theorem 5, we have

Corollary (The unscaled deformation theorem). *If $T \in \mathcal{D}_m(\mathbb{R}^{m+n})$ with $\underline{\underline{M}}(T) + \underline{\underline{M}}(\partial T) < \infty$, then we can write*

$$T = P + \partial R + S,$$

where $P, S \in \mathcal{D}_m(\Omega)$ and $R \in \mathcal{D}_{m+1}(\Omega)$ satisfy

$$P = \sum_{F \in \mathcal{L}_m} \beta_F [\![F]\!] \qquad \text{with} \quad \beta_F \in \mathbb{R},$$

with

$$\begin{aligned}
&\underline{\underline{M}}(P) \leq C\underline{\underline{M}}(T), \quad \underline{\underline{M}}(\partial P) \leq C\underline{\underline{M}}(\partial T),\\
&\underline{\underline{M}}(R) \leq C\underline{\underline{M}}(T), \quad \underline{\underline{M}}(S) \leq C\underline{\underline{M}}(\partial T),\\
&C = C(m,n),\\
&\text{spt } P \cup \text{ spt } R \subset \{x \in \mathbb{R}^{m+n} \ ; \ EM(x, \text{spt } T) \leq 2\},\\
&\text{spt } \partial P \cup \text{ spt } S \subset \{x \in \mathbb{R}^{m+n} \ ; \ EM(x, \text{spt } \partial T) \leq 2\}.
\end{aligned}$$

In case T is integer multiplicity, so are P, R; if ∂T is integer multiplicity then so is S.

Proof: Corollary readily follows from Theorem 5 by setting $\Omega = \mathbb{R}^{m+n}$ and $f \equiv 1$. We note that $f \equiv 1$ implies $k \equiv 0$ in Theorem 5, and hence $\delta \equiv 1$.

5.2. Special case.

The density function δ in the weighted deformation theorem is made by shape of Ω, weight function f and ε. It takes sharp form, but might be not directly applicable to all variational problems because of its discontinuity. So we give examples of continuous functions equivalent to δ for some special Ω, f, ε.

Ex.1: For

$$m = n = 1, \quad \Omega = \{x \in \mathbf{R}^2 \ ; \ |x| < 1\},$$
$$f(x) = 1 - |x|, \quad \varepsilon = 1/2\sqrt{2},$$

we readily check that there are positive constants C_1, C_2 which satisfy

$$C_1 f(x) \le \delta(x) \le C_2 f(x), \quad x \in \Omega.$$

Ex.2: For

$$m = n = 1, \quad \Omega = \mathbf{R}^2 \setminus \{(-1,0),(1,0)\},$$
$$f(x) = \mathrm{dist}(x, \partial\Omega), \quad \varepsilon = 1/\sqrt{2},$$

we readily check that

$$\delta_1(x) \le \delta(x) \le \delta_2(x), \quad x \in \Omega,$$

where both δ_1 and δ_2 are continuous functions defined by

$$\delta_1(x) = \begin{cases} f(x)/2\sqrt{2} & \text{if } \ \mathrm{dist}(x,\partial\Omega) \le 2\sqrt{2}, \\ 1 & \text{if } \ \mathrm{dist}(x,\partial\Omega) \ge 2\sqrt{2}, \end{cases}$$
$$\delta_2(x) = \begin{cases} f(x) & \text{if } \ \mathrm{dist}(x,\partial\Omega) \le 1, \\ 1 & \text{if } \ \mathrm{dist}(x,\partial\Omega) \ge 1. \end{cases}$$

Considering the above examples, we give the special weighted deformation theorem in which no δ appears in the weighted mass.

COROLLARY. *Let Ω be an open subset in $\mathbf{R}^{m+n}$. Weight function $f : \Omega \longrightarrow (0,\infty)$ is defined by*

$$f(x) = \min\{\mathrm{dist}(x,\partial\Omega), 1\}.$$

Suppose $T \in \mathcal{D}_m(\Omega)$ with $\underset{=}{M}(f,T) + \underset{=}{M}(f^2,\partial T) < \infty$. Then, for a small constant ε such that $0 < \varepsilon < 1$, we can write

$$T - P = \partial R + S,$$

where

$$P = \sum_{F \in \mathcal{L}_m} \beta_F [\![F]\!] \quad \in \mathcal{D}_m(\Omega), \qquad \beta_F \in \mathbb{R},$$
$$R \in \mathcal{D}_{m+1}(\Omega), \quad S \in \mathcal{D}_m(\Omega)$$

which satisfy

$$\begin{aligned}
&\underline{\underline{M}}(f,P) \le C\underline{\underline{M}}(f,T), \quad \underline{\underline{M}}(f^2,\partial P) \le C\underline{\underline{M}}(f^2,\partial T),\\
&\underline{\underline{M}}(R) \le C\underline{\underline{M}}(f,T), \quad \underline{\underline{M}}(f,S) \le C\underline{\underline{M}}(f^2,\partial T),\\
&\text{spt } P \cup \text{ spt } R \subset \{x \in \Omega \ ; \ EM(x, \text{spt } T) \le Cf(x)\},\\
&\text{spt } \partial P \cup \text{ spt } S \subset \{x \in \Omega \ ; \ EM(x, \text{spt } \partial T) \le Cf(x)\},\\
&C = C(m,n,\varepsilon,\Omega).
\end{aligned}$$

In case T is an integer multiplicity current, then P, R can be chosen to be integer multiplicity currents (and β_F appearing in the definition of P are integers). If in addition ∂T is integer multiplicity, then S can be chosen to be integer multiplicity.

PROOF: In Theorem 5, we see that there are positive constants C_1, C_2 ($C_1 > C_2$) depending on m, n, ε, Ω such that

$$M\delta_1(x) \le \delta(x) \le M\delta_2(x), \quad x \in \Omega$$

holds with $M = \sup\{\delta(x) \ ; \ x \in \Omega\}$, where each δ_j ($j = 1,2$) is defined by

$$\delta_j(x) = \begin{cases} \text{dist}(x,\partial\Omega)/C_j & \text{if} \quad \text{dist}(x,\partial\Omega) \le C_j, \\ 1 \quad \text{if} \quad \text{dist}(x,\partial\Omega) \ge C_j. \end{cases}$$

Hence, by definition of f, we know that

$$(M/C_1)f(x) \le \delta(x) \le (M/C_2)f(x), \quad x \in \Omega.$$

Therefore Corollary immediately follows from Theorem 5.

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