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# Local invariants of singular surfaces in an almost complex four-manifold

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## 0. Introduction.

Let  $(M^4, J)$  be an almost complex manifold of dimension 4,  $S$  an oriented closed surface, and  $f : S \rightarrow (M, J)$  a  $C^\infty$  mapping. Then  $f$  has two sorts of singularities: "complex points" and "non-immersive points". As well as these singularities, they appear also multi-singular points.

Associated to these singularities, we define in this paper two local invariants : the local self-intersection index  $i$  at a point of  $f(S)$  and the Maslov index  $m$  at a point of  $S$ , for a generic  $f$  belonging to the complement of an infinite codimensional subset in the space  $C^\infty(S, M)$  of  $C^\infty$  mappings from  $S$  to  $M$  endowed with the  $C^\infty$  topology. (See §1 for the precise definitions of the genericity and the invariants.)

Immersed surfaces in a four-space are studied by many authors from various aspects (e.g. [6],[5],[17],[11],[8],[9],[2], see also the reference of [4]). Also remark that similar local invariants as  $i$  and  $m$  are already defined and investigated in the contrary case, that is, for (pseudo-)holomorphic curves, ([18], [10]).

Then we show the following formulae

**THEOREM 1.** *Let  $f : S \rightarrow (M, J)$  be generic. Set  $V = f(S)$ . Then*

$$(1) \quad \sum_{y \in V} \{i(y) + 1\} = \chi(V) + V \cdot V, \quad (2) \quad \sum_{x \in S} m(x) = c.$$

Here  $\chi(V)$ ,  $V \cdot V$  and  $c$  are global numerical invariants;  $\chi(V)$  is the Euler characteristic of  $V$ ,  $V \cdot V$  is the self-intersection index of  $V$  in  $M$  and  $c = \langle c_1(f^*TM), [S] \rangle$  is the Chern number. Remark that  $M$  has the natural orientation from the almost complex structure  $J$ . Since we see below  $i(y) = -1$  and  $m(x) = 0$  except finite points, the left hand side of each formula has a meaning.

Theorem 1 turns out to unify and generalize two sorts of known formulae.

For an immersion  $f$  of  $S$  into  $M$ , it is known the following formula due to Lai [17], (see also [3],[4],[2]): If the complex points of  $f$  are all transverse, then

$$d_+ + d_- = \chi + \nu, \quad d_+ - d_- = c.$$

Here  $\chi$  is the Euler characteristic of  $S$  and  $\nu$  is the normal Euler number of  $f$ , whereas  $d_{\pm} = e_{\pm} - h_{\pm}$  with

$$e_{+} = \#(\text{positive elliptic point}), \quad e_{-} = \#(\text{negative elliptic point}),$$

$$h_{+} = \#(\text{positive hyperbolic point}), \quad h_{-} = \#(\text{negative hyperbolic point}).$$

See §1 for the notions.

The immersion  $f$  can be approximated so that  $V = f(S)$  has only transverse self-intersections on non-complex points. Then the invariants appeared in Lai's formula do not vary and Theorem 1 implies Lai's formula in a simple manner, if we calculate  $i$  and  $m$  for some special singular points.

In the symplectic situation, on the other hand, it is known a formula due to Givental' on the self-intersection index of a "Lagrange cycle" ([13],[14],[1]): Let  $(M^4, \omega)$  be a symplectic manifold of dimension 4 and  $f : S \rightarrow (M, \omega)$  be an isotropic  $C^{\infty}$  mapping, ( $f^*\omega = 0$ ). Remark that  $M$  has the orientation coming from  $\omega^2$ . If  $V = f(S)$  has the open Whitney umbrellas and the transverse self-intersections as singularities, then the formula is

$$-V \cdot V = \chi - 2\delta + T.$$

Here  $\delta$  is the sum of intersection indices of self-intersection points and  $T$  is the number of open Whitney umbrellas. An open Whitney umbrella has a local model  $f_{2,1} : \mathbb{R}^2, 0 \rightarrow \mathbb{R}^4, 0$  defined by

$$f_{2,1}(u, v) = (p_1, q_1, p_2, q_2) = (v^3/3, u, uv, v^2/2), \quad \omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2,$$

([16]). (The original form of the formula in [13] is  $V \cdot V = \chi + 2\# + T$ ,  $\# = \delta$ , because the orientation of  $M$  chosen in [13] differs by sign with the orientation chosen here.)

We remark that, for a symplectic manifold  $(M, \omega)$ , there exists an almost complex structure  $J$  unique up to homotopy such that  $\omega(\cdot, J\cdot)$  is positive definite (see [22]). Then an isotropic immersion has no complex points.

Thus we can apply Theorem 1 to this situation.

Theorem 1 follows also that, if  $f^*TM$  has a Lagrange subbundle, then  $c_1(f^*TM) = 0$  and therefore the sum of Maslov indices is equal to zero. This fact is first observed also by Givental' [13] in the simplest case.

The formula of Givental' is generalized in some sense to higher dimensional cases as formulae on "isotropic Thom polynomials" [20].

We also remark that, using Viro's integral formulation based on Euler characteristics [21], the formulae of Theorem 1 can be written in the following form:

$$(1) \int_{y \in V} i(y) d\chi(y) = V \cdot V, \quad (2) \int_{y \in V} m(y) d\chi(y) = c,$$

where  $m(y) = \sum_{x \in f^{-1}(y)} m(x)$ . Regarding the Chern number as the global counterpart of the Maslov index, we can observe each formula has a natural form that integrating a local invariant gives a global one.

The proof of Theorem 1 is simple if once the definitions of  $i$  and  $m$  are established.

Next we turn the local situation relatively to  $S$ . Let  $f : \mathbb{R}^2, 0 \rightarrow (M, J)$  be a generic map-germ. Then two invariants  $i(f)$  and  $m(f)$  can be defined as  $i(f) = i(f(0))$  and  $m(f) = m(0)$  respectively.

After taking a representative  $f : D^2 \rightarrow (M, J)$ ,  $D^2 = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < \epsilon^2\}$  for a sufficiently small  $\epsilon$ , we perturb  $f$  into an immersion  $\tilde{f}$  with transverse self-intersections such that all complex points of  $\tilde{f}$  are transverse. Then we have the following formula on perturbations:

**THEOREM 2.**  $i(f) = d_+ + d_- - 1 + 2\delta$ ,  $m(f) = d_+ - d_-$ .

Notice that the numbers  $d_+$ ,  $d_-$  and  $\delta$  depend on a perturbation of  $f$ . (See Example 2.1.)

As a corollary we see  $i(f) \equiv m(f) + 1, \text{ mod. } 2$ .

Beside the definitions of  $i$  and  $m$ , we need some calculations of them to prove Theorem 2, and also to show that Theorem 1 implies formulae of Lai and Givental' respectively. We gather the results into the following table:

Table 3.

type of the singularity	$i$	$m$
non-singular point	-1	0
open Whitney umbrella	-2	$\pm 1$
self-intersection of index +1	0	0
self-intersection of index -1	-4	0
positive elliptic point	0	+1
negative elliptic point	0	-1
positive hyperbolic point	-2	-1
negative hyperbolic point	-2	+1

We can deduce Theorem 1 contrary from Lai's formula, perturbing  $f$  and applying Theorem 2 and the results in Table 3. Thus Lai's formula is generalized to the simple formula (Theorem 1), the difficulty being pushed into the calculations of invariants.

In §1 we define  $i$  and  $m$ . In the next section we prove Theorem 2. The calculation of  $i$  and  $m$  (Table 3) are given in §§1 and 2. Theorem 1 is proved in §3.

In this paper manifolds and mappings are assumed of class  $C^\infty$ .

## 1. Genericity and local invariants.

Let  $f : S \rightarrow (M, J)$  be a mapping. An immersive point  $x \in S$  of  $f$  is called a *complex point* if  $f_*T_x S = J(f_*T_x S)$ .

DEFINITION 1.1:  $f$  is called *generic* if (1)  $f$  is finite to one and for any  $y \in V = f(S)$ , (2) the germ  $f : S, f^{-1}(y) \rightarrow M, y$  is an embedding with no complex points outside of  $f^{-1}(y)$  and (3) the pull-back by  $f$  of a positive definite Morse function around  $y$  is of finite multiplicity at  $f^{-1}(y)$ .

A map-germ  $f : S, x \rightarrow (M, J)$  is called *generic* if, for  $y = f(x)$ , (2) and (3) hold,  $f^{-1}(y)$  being replaced by  $x$ .

REMARK 1.2: Non generic mappings form an infinite codimensional subset in  $C^\infty(S, M)$ , even after more strict restrictions on genericity are imposed (see [12] for instance), since, in the 1-jet space  $J^1(S, M)$  the totality of immersive 1-jets corresponding to complex points is of codimension 2, (see [17]).

Let  $f$  be generic. Then  $V$  is a totally real submanifold of  $(M, J)$  except for finite points.

Now we intend to define  $i(y) \in \mathbb{Z}$  for  $y \in V$  as the self-intersection index  $V$  at  $y$ . To do this, we have to assign a perturbation of  $V$  near  $y$ .

For a sufficiently small sphere  $S^3$  centered  $y$  in  $M$  (with respect to some coordinate),  $f$  is transverse to  $S^3$  by the property (3) of Definition 1.1. Considering the link  $L = V \cap S^3$ , we take a tangent vector field  $v$  to  $V$  defined near  $L$  and directed outward. Then the field  $Jv$  does not tangent to  $V$  by the property (2). Thus we perturb  $f$  into  $f'$  along the direction  $Jv$  and count intersection indices of  $V$  and  $f'(S)$  near  $y$ . In other word, we adopt the following definition:

DEFINITION 1.2: (Local self-intersection index of  $f$  at  $y$ .) We set  $i(y) = \text{link}(L, L')$ , where  $L' = f'(S) \cap S^3$ .

Clearly  $i(y)$  does not depend on the choice of  $S^3$  and  $v$ .

Since, on an immersed surface without complex points, multiplying  $J$  maps the tangent bundle isomorphically to the normal bundle with the reverse orientation, it is easy to verify

LEMMA 1.3. *If  $y \in V$  is a non-singular point, then  $i(y) = -1$ . If  $y \in V$  is a transverse self-intersection (non-complex) point, then  $i(y) = 0, -4$ , according to the intersection index is  $+1, -1$ , respectively.*

REMARK 1.4: If  $J_t$  is a homotopy of complex structures such that  $J_0 = J$  and  $V$  has no complex points near  $L$  with respect to  $J_t$ . Then the number  $i(y)$  with respect to  $J_t$  does not depend on  $J_t$ . Similar result holds also for  $m$  defined below.

We next define the Maslov index  $m(x)$  for  $x \in S$ , generalizing the definition in [13].

Consider the  $\mathbb{C}^2$ -bundle  $E = f^*TM$  over  $S$ . Let  $\tilde{G}$  denote the space of oriented 2-planes in  $E$ , and  $\pi : \tilde{G} \rightarrow M$  the canonical projection. The fiber of  $\pi$  is  $G = G_{4,2}$ , the Grassmannian of oriented 2-planes in  $\mathbb{C}^2$ .

Let  $\tilde{C} \subset \tilde{G}$  be the totality of complex planes. We decompose  $\tilde{C} = \tilde{C}_+ \cup \tilde{C}_-$ , where an oriented plane  $\alpha \in \tilde{C}$  belongs to  $\tilde{C}_+$  if and only if the orientation of  $\alpha$  coincides with the

orientation as the complex plane. Then we see  $\tilde{C}_\pm$  are submanifolds of  $\tilde{G}$  of codimension 2 respectively.

We define orientations of  $\tilde{G}$ ,  $\tilde{C}_+$  and  $\tilde{C}_-$  as follows (cf. [17],[9],[3],[4]): For each  $x \in S$ , we take a local frame  $e_1, e_2, e_3, e_4$  of  $E$  as  $\mathbb{R}^4$ -bundle near  $x$  with  $Je_1 = e_2, Je_3 = e_4$ . Then we set

$$\begin{cases} x_1 = p_{12} + p_{34}, \\ x_2 = p_{23} + p_{14}, \\ x_3 = p_{31} + p_{24}, \end{cases} \quad \begin{cases} y_1 = p_{12} - p_{34}, \\ y_2 = p_{23} - p_{14}, \\ y_3 = p_{31} - p_{24}, \end{cases}$$

for the Plücker coordinate  $p_{ij}$ . Then we identify  $G$  with  $(\mathbb{R}^3 - 0)/\mathbb{R}_{>0} \times (\mathbb{R}^3 - 0)/\mathbb{R}_{>0} \cong S^2 \times S^2$  by these coordinates. The fiber of  $\tilde{C}_+$  (resp.  $\tilde{C}_-$ ) corresponds to  $C_+ = n \times S^2$  (resp.  $C_- = s \times S^2$ ), where  $n = (1, 0, 0), s = (-1, 0, 0)$ .

We orient  $\tilde{G}$  (resp.  $\tilde{C}_+, \tilde{C}_-$ ) from the orientations of  $S$  and  $G$  (resp.  $C_+, C_-$ ). We denote by  $-\tilde{C}_-$  the  $\tilde{C}_-$  with the reverse orientation.

Let  $\Sigma \subset S$  be the set of non-immersive points of  $f$ . Then we define the Gauss mapping  $g : S - \Sigma \rightarrow \tilde{G}$  by  $g(x) = f_*(T_x S), x \in S - \Sigma$ .

For  $x \in S$ , we take a small loop  $\ell$  around  $x$ . Then  $g \circ \ell$  extends to a section  $\tilde{g}$  over the disk, since  $\tilde{G}$  is a  $S^2 \times S^2$ -bundle. We count the intersection number of  $\tilde{g}$  with  $\tilde{C}_+ \cup (-\tilde{C}_-)$ . In other word, we adopt the following definition:

DEFINITION 1.5: (The Maslov index of  $f$  at  $x$ .) We set  $m(x) = \text{link}(g \circ \ell, \tilde{C}_+ \cup (-\tilde{C}_-))$ .

If  $x \in S$  is a complex point, then  $g(x) \in \tilde{C}$ .

DEFINITION 1.6: A complex point  $x \in S$  is called positive (resp. negative) if  $g(x) \in \tilde{C}_+$  (resp.  $g(x) \in \tilde{C}_-$ ). A complex point  $x \in S$  is called transverse if  $g$  is transverse to  $\tilde{C}$  at  $x$ . A transverse complex point  $x$  is elliptic (resp. hyperbolic) if the intersection index of  $g$  and  $\tilde{C} = \tilde{C}_+ \cup \tilde{C}_-$  at  $g(x)$  is equal to  $+1$  (resp.  $-1$ ).

Then the following is straightforward.

LEMMA 1.7. *Let  $x \in S - \Sigma$ . If  $x$  is not a complex point, then  $m(x) = 0$ . If  $x$  is a positive elliptic or negative hyperbolic point, then  $m(x) = +1$ . If  $x$  is a negative elliptic or positive hyperbolic point, then  $m(x) = -1$ .*

Next Lemma is used to show Theorem 1.(2).

LEMMA 1.8. *The homology class  $[\tilde{C}_+ \cup (-\tilde{C}_-)] \in H_4(\tilde{G}, \mathbb{Z})$  is the Pioncaré dual of  $\pi^*c_1(E) \in H^2(\tilde{G}, \mathbb{Z})$ .*

PROOF: Consider the complex line bundle  $\pi^*(E \wedge E)$  over  $\tilde{G}$ . Then  $c_1(\pi^*(E \wedge E)) = c_1(\pi^*E)$ , (see [H]). Taking a metric of  $E$  compatible with  $J$ , we define the section  $s$  of  $\pi^*(E \wedge E)$  over  $\tilde{G}$  by  $s(\alpha) = v \wedge w$ , where  $\alpha \in \tilde{G}$  and  $v, w$  are orthonormal basis of  $\alpha$  compatible with the orientation of  $\alpha$ . If, in above,  $e_1, e_2, e_3, e_4$  are orthonormal, then locally  $s$  is represented by  $s = (-x_3 + \sqrt{-1}x_2)e_1 \wedge e_3$ . Therefore we see that the zero locus of  $s$  with the induced orientation is equal to  $\tilde{C}_+ \cup (-\tilde{C}_-)$ . This shows the required result.

To end this section, we prove the following:



LEMMA 1.9. *If  $x$  is an elliptic (resp. hyperbolic) point, then  $i(x) = 0$  (resp.  $-2$ ).*

PROOF: First we follow the arguments in [4, §4.1]. Let  $x$  be a transverse complex point. Then, by [5], [19], there exist coordinates  $(u, v) : S, x \rightarrow \mathbb{R}^2, 0$  and  $(p_1, q_1; p_2, q_2) : M, f(x) \rightarrow \mathbb{C}^2, 0$  such that

$$f(u, v) = (u, v, (1 + 2\gamma)u^2 + (1 - 2\gamma)v^2 + \phi(u, v), \psi(u, v)),$$

with  $\gamma \in \mathbb{R}$ ,  $0 < \gamma \neq \frac{1}{2}$ ,  $\text{ord}_0 \phi \geq 3$ ,  $\text{ord}_0 \psi \geq 3$ , and  $J = J_0 +$  higher order terms, for the standard complex structure  $J_0$  on  $\mathbb{C}^2$ . If  $0 < \gamma < \frac{1}{2}$  (resp.  $\gamma > \frac{1}{2}$ ), then  $x$  is elliptic (resp. hyperbolic).

To compute  $i(x)$ , we take the Euler field  $E = u(\partial/\partial u) + v(\partial/\partial v)$ . Then

$$J(f_*E) = (-v + P_1)\frac{\partial}{\partial p_1} + (u + Q_1)\frac{\partial}{\partial q_1} + P_2\frac{\partial}{\partial p_2} + (2((1 + 2\gamma)u^2 + (1 - 2\gamma)v^2) + Q_2)\frac{\partial}{\partial q_2},$$

with  $\text{ord}_0 P_1 \geq 2$ ,  $\text{ord}_0 Q_1 \geq 2$ ,  $\text{ord}_0 P_2 \geq 3$ ,  $\text{ord}_0 Q_2 \geq 3$ . We set, for sufficiently small  $\epsilon$ ,  $0 < \epsilon^2 < |1 - 4\gamma^2|$ ,

$$f_\epsilon(u, v) = (u - \epsilon v + A, v + \epsilon u + B, (1 + 2\gamma)u^2 + (1 - 2\gamma)v^2 + C, 2\epsilon((1 + 2\gamma)u^2 + (1 - 2\gamma)v^2) + D),$$

where  $A = \epsilon P_1, B = \epsilon Q_1, C = \phi + \epsilon P_2, D = \psi + \epsilon Q_2$ . Consider the map-germ  $F : \mathbb{R}^4, 0 \rightarrow \mathbb{R}^4, 0$  defined by  $F(u, v, u', v') = f_\epsilon(u, v) - f(u', v')$ .

Let  $E_4$  denote the  $\mathbb{R}$ -algebra of function-germs on  $\mathbb{R}^4, 0$  and  $m$  the unique maximal ideal of  $E_4$ . For the ideal  $I(F) \subset E_4$  generated by the components of  $F$ , we easily see that  $m^3 \subset I(F) + m^4$ , therefore  $m^3 \subset I(F)$  by Nakayama's lemma. Hence  $F$  is a finite map-germ and we see  $i(x) = \text{deg}_0 F$ .

Following [7], we calculate  $\text{deg}_0 F$ . The algebra  $Q(F) = E_4/I(F)$  is generated by  $1, u, v$  and  $u^2$  over  $\mathbb{R}$ . The class  $s$  of Jacobian of  $F$  is equal to  $-64\epsilon^2\gamma(1 + 2\gamma + \frac{\epsilon^2}{1-2\gamma})u^2$  in  $Q(F)$ . Define the functional  $\varphi : Q(F) \rightarrow \mathbb{R}$  by  $\varphi(u^2) = -1, \varphi(1) = \varphi(u) = \varphi(v) = 0$ . Then we see  $\varphi(s) > 0$  and the matrix of the bilinear form  $\langle \cdot, \cdot \rangle_\varphi : Q(F) \times Q(F) \rightarrow \mathbb{R}, \langle a, b \rangle_\varphi = \varphi(ab)$ , is equal to

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1+2\gamma}{1-2\gamma} & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

If  $0 < \gamma < \frac{1}{2}$  (resp.  $\gamma > \frac{1}{2}$ ), then we have  $\text{deg}_0 F = \text{signature of } \langle \cdot, \cdot \rangle_\varphi = 0$  (resp.  $-2$ ).

## 2. Perturbations.

PROOF OF THEOREM 2: (1) As in §0, we denote by  $e_+, e_-, h_+, h_-$  the numbers of positive elliptic, negative elliptic, positive hyperbolic, negative hyperbolic complex points of  $\tilde{f}$  respectively. Further denote by  $\delta_+, \delta_-$  the numbers of self-intersection points of index  $+1, -1$  respectively. Then  $d_+ = e_+ - h_+, d_- = e_- - h_-$  and  $\delta = \delta_+ - \delta_-$ .

We may assume that the self-intersections do not occur on the complex points. We set  $W = \tilde{f}(D^2)$ . We take a tangent vector field  $v$  to  $W$  along  $\tilde{f}$  such that  $v$  are directed outward on  $\partial W$ , near all complex points and all self-intersection points. We perturb  $\tilde{f}$  to  $f'$  along the direction of  $Jv$ . Set  $W' = f'(D^2)$ . Then  $i(f)$  is equal to the sum of intersection indices of  $W$  and  $W'$ , which is equal to  $\sum i(y) - \chi(W_0)$ , where the sum runs over all complex points and self-intersection points, and  $W_0$  means  $W$  minus small balls centered at complex points and self-intersection points. Then  $\chi(W_0) = 1 - (e_+ + e_-) - (h_+ + h_-) - 2(\delta_+ + \delta_-)$ . By Lemmas 1.3 and 1.9, we have  $\sum i(y) = -2(h_+ + h_-) - 4\delta_-$ . Hence,

$$i(f) = e_+ - h_+ + e_- - h_- - 1 + 2(\delta_+ - \delta_-) = d_+ + d_- - 1 + 2\delta.$$

(2) The Maslov index  $m(f)$  is equal to  $\sum m(x)$ , the sum running over all complex points of  $\tilde{f}$ . Then by Lemma 1.7,  $m(f) = e_+ - e_- - h_+ + h_- = d_+ - d_-$ .

Q.E.D.

Now we apply Theorem 2 to calculate  $i$  and  $m$  for the open Whitney umbrella using concrete perturbations.

EXAMPLE 2.1: We perturb the local model  $f : \mathbb{R}^2, 0 \rightarrow \mathbb{R}^4, 0$  of the open Whitney umbrella into  $f_\epsilon : \mathbb{R}^2, 0 \rightarrow \mathbb{R}^4, 0$  defined by  $f_\epsilon(u, v) = (\frac{v^3}{3}, u, uv, \frac{v^2}{2} + \epsilon v)$ , for sufficiently small  $\epsilon > 0$ . Then we have  $\delta = 0, h_- = 1$  and  $e_+ = e_- = h_+ = 0$ . By Theorem 2, we see  $i(f) = -2$  and  $m(f) = 1$ . For the map-germ  $f'$  defined by  $f'(u, v) = f(u, v)$ , we see  $i(f) = -2$  and  $m(f) = -1$ .

For another perturbation  $f_\epsilon$  of  $f$ , for instance,  $f_\epsilon(u, v) = (\frac{v^3}{3} + \epsilon v, u, uv, \frac{v^2}{2})$ , we have  $\delta = 0, h_- = 1, e_+ = e_- = h_+ = 0$ , when  $\epsilon > 0$ , and  $\delta = -1, e_+ = 1, e_- = h_+ = h_- = 0$ , when  $\epsilon < 0$ .

Combined with Lemmas 1.3, 1.7, 1.9, Remark 1.4 and Example 2.1, we get Table 3.

### 3. Implications.

First we deduce the formulae of Lai and Givental' from Theorem 1.

THE FORMULA OF LAI: By Table 3, we have  $\sum_y \{i(y) + 1\} = d_+ + d_- + \delta_+ - 3\delta_-$ ,  $V \cdot V = \nu + 2(\delta_+ - \delta_-)$ ,  $\chi(V) = \chi(S) - (\delta_+ + \delta_-)$ ,  $\sum_x m(x) = d_+ - d_-$ , where  $\delta_+$  and  $\delta_-$  are similar numbers as in the proof of Theorem 2. Thus  $\chi(V) + V \cdot V = \nu + \chi + \delta_+ - 3\delta_-$ . By Theorem 1, we have  $d_+ + d_- = \nu + \chi$  and  $d_+ - d_- = c$ .

THE FORMULA OF GIVENTAL': By Table 3, we have

$$\sum_{y \in V} \{i(y) + 1\} = \delta_+ - 3\delta_- - T, \quad \chi(V) + V \cdot V = \chi - (\delta_+ + \delta_-) + V \cdot V.$$

By Theorem 1.(1), we have  $-V \cdot V = \chi - 2\delta + T$ .

PROOF OF THEOREM 1: (1) Denote by  $X$  the set of singular points of  $V$ . We remove from  $V$  small balls centered at points of  $X$ . Denote by  $V'$  the resulting surface with boundary. Let  $v$  be a vector field over  $V'$  directed inward (relatively to  $V'$ ) along  $\partial V'$ . Using  $Jv$ , we perturb  $V$ . Then we see

$$V \cdot V = \sum_{y \in X} i(y) - \chi(V'), \quad \chi(V') = \chi(V) - \#X.$$

Thus we have  $V \cdot V = \sum_{y \in X} \{i(y) + 1\} - \chi(V)$ .

(2) For  $x \in \Sigma$ , take a small disk  $D_x \subset S$  around  $x$ . We extend the Gauss map  $g : S - \bigcup_{x \in \Sigma} D_x \rightarrow \tilde{G}$  to a section  $\tilde{g} : S \rightarrow \tilde{G}$ . Then the sum  $\sum_{x \in S} m(x)$  is equal to the intersection number of  $\tilde{g}(S)$  and  $\tilde{C}_+ \cup (-\tilde{C}_-)$ . By Lemma 1.8, this number is equal to

$$\langle \pi^* c_1(E), \tilde{g}_*[S] \rangle = \langle \tilde{g}^* \pi^* c_1(E), [S] \rangle = \langle c_1(E), [S] \rangle = c.$$

Q.E.D.

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