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Goo Ishikawa and Toru Ohmoto

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Local invariants of singular surfaces in an almost complex four-manifold

GOO ISHIKAWA† AND TORU OHMOTO‡

† Department of Mathematics, Hokkaido University, Sapporo 060, Japan,
‡ Department of Mathematics, Tokyo Institute of Technology, Ohokayama 152, Japan, and Department of Mathematics, Hokkaido University, Sapporo 060, Japan.

0. Introduction.

Let (M^4, J) be an almost complex manifold of dimension 4, S an oriented closed surface, and $f : S \to (M, J)$ a C^∞ mapping. Then f has two sorts of singularities: "complex points" and "non-immersive points". As well as these singularities, they appear also multi-singular points.

Associated to these singularities, we define in this paper two local invariants : the local self-intersection index i at a point of $f(S)$ and the Maslov index m at a point of S, for a generic f belonging to the complement of an infinite codimensional subset in the space $C^\infty(S, M)$ of C^∞ mappings from S to M endowed with the C^∞ topology. (See §1 for the precise definitions of the genericity and the invariants.)

Immersed surfaces in a four-space are studied by many authors from various aspects (e.g. [6],[5],[17],[11],[8],[9],[2], see also the reference of [4]). Also remark that similar local invariants as i and m are already defined and investigated in the contrary case, that is, for (pseudo-)holomorphic curves, ([18], [10]).

Then we show the following formulae

THEOREM 1. *Let $f : S \to (M, J)$ be generic. Set $V = f(S)$. Then*

$$(1) \quad \sum_{y \in V} \{i(y) + 1\} = \chi(V) + V \cdot V, \qquad (2) \quad \sum_{x \in S} m(x) = c.$$

Here $\chi(V)$, $V \cdot V$ and c are global numerical invariants; $\chi(V)$ is the Euler characteristic of V, $V \cdot V$ is the self-intersection index of V in M and $c = \langle c_1(f^*TM), [S] \rangle$ is the Chern number. Remark that M has the natural orientation from the almost complex structure J. Since we see below $i(y) = -1$ and $m(x) = 0$ except finite points, the left hand side of each formula has a meaning.

Theorem 1 turns out to unify and generalize two sorts of known formulae.

For an immersion f of S into M, it is known the following formula due to Lai [17],(see also [3],[4],[2]): If the complex points of f are all transverse, then

$$d_+ + d_- = \chi + \nu, \qquad d_+ - d_- = c.$$

Here χ is the Euler characteristic of S and ν is the normal Euler number of f, whereas $d_\pm = e_\pm - h_\pm$ with

$e_+ = \sharp$(positive elliptic point), $e_- = \sharp$(negative elliptic point),

$h_+ = \sharp$(positive hyperbolic point), $h_- = \sharp$(negative hyperbolic point).

See §1 for the notions.

The immersion f can be approximated so that $V = f(S)$ has only transverse self-intersections on non-complex points. Then the invariants appeared in Lai's formula do not vary and Theorem 1 implies Lai's formula in a simple manner, if we calculate i and m for some special singular points.

In the symplectic situation, on the other hand, it is known a formula due to Givental' on the self-intersection index of a "Lagrange cycle" ([13],[14],[1]): Let (M^4, ω) be a symplectic manifold of dimension 4 and $f : S \to (M, \omega)$ be an isotropic C^∞ mapping, $(f^*\omega = 0)$. Remark that M has the orientation coming from ω^2. If $V = f(S)$ has the open Whitney umbrellas and the transverse self-intersections as singularities, then the formula is

$$-V \cdot V = \chi - 2\delta + T.$$

Here δ is the sum of intersection indices of self-intersection points and T is the number of open Whitney umbrellas. An open Whitney umbrella has a local model $f_{2,1} : \mathbf{R}^2, 0 \to \mathbf{R}^4, 0$ defined by

$$f_{2,1}(u, v) = (p_1, q_1, p_2, q_2) = (v^3/3, u, uv, v^2/2), \quad \omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2,$$

([16]). (The original form of the formula in [13] is $V \cdot V = \chi + 2\sharp + T, \sharp = \delta$, because the orientation of M chosen in [13] differs by sign with the orientation chosen here.)

We remark that, for a symplectic manifold (M, ω), there exists an almost complex structure J unique up to homotopy such that $\omega(\,\cdot\,, J\,\cdot\,)$ is positive definite (see [22]). Then an isotropic immersion has no complex points.

Thus we can apply Theorem 1 to this situation.

Theorem 1 follows also that, if f^*TM has a Lagrange subbundle, then $c_1(f^*TM) = 0$ and therefore the sum of Maslov indices is equal to zero. This fact is first observed also by Givental' [13] in the simplest case.

The formula of Givental' is generalized in some sense to higher dimensional cases as formulae on "isotropic Thom polynomials" [20].

We also remark that, using Viro's integral formulation based on Euler characteristics [21], the formulae of Theorem 1 can be written in the following form:

$$(1)\ \int_{y \in V} i(y) d\chi(y) = V \cdot V, \qquad (2)\ \int_{y \in V} m(y) d\chi(y) = c,$$

where $m(y) = \sum_{x \in f^{-1}(y)} m(x)$. Regarding the Chern number as the global counterpart of the Maslov index, we can observe each formula has a natural form that integrating a local invariant gives a global one.

The proof of Theorem 1 is simple if once the definitions of i and m are established.

Next we turn the local situation relatively to S. Let $f : \mathbf{R}^2, 0 \to (M, J)$ be a generic map-germ. Then two invariants $i(f)$ and $m(f)$ can be defined as $i(f) = i(f(0))$ and $m(f) = m(0)$ respectively.

After taking a representative $f : D^2 \to (M, J)$, $D^2 = \{(u, v) \in \mathbf{R}^2 \mid u^2 + v^2 < \epsilon^2\}$ for a sufficiently small ϵ, we perturb f into an immersion $\tilde{f}$ with transverse self-intersections such that all complex points of $\tilde{f}$ are transverse. Then we have the following formula on perturbations:

THEOREM 2. $\quad i(f) = d_+ + d_- - 1 + 2\delta, \quad m(f) = d_+ - d_-.$

Notice that the numbers d_+, d_- and δ depend on a perturbation of f. (See Example 2.1.)

As a corollary we see $i(f) \equiv m(f) + 1$, mod. 2.

Beside the definitions of i and m, we need some calculations of them to prove Theorem 2, and also to show that Theorem 1 implies formulae of Lai and Givental' respectively. We gather the results into the following table:

Table 3.

type of the singularity	i	m
non-singular point	-1	0
open Whitney umbrella	-2	± 1
self-intersection of index $+1$	0	0
self-intersection of index -1	-4	0
positive elliptic point	0	$+1$
negative elliptic point	0	-1
positive hyperbolic point	-2	-1
negative hyperbolic point	-2	$+1$

We can deduce Theorem 1 contrary from Lai's formula, perturbing f and applying Theorem 2 and the results in Table 3. Thus Lai's formula is generalized to the simple formula (Theorem 1), the difficulty being pushed into the calculations of invariants.

In §1 we define i and m. In the next section we prove Theorem 2. The calculation of i and m (Table 3) are given in §§1 and 2. Theorem 1 is proved in §3.

In this paper manifolds and mappings are assumed of class C^∞.

1. Genericity and local invariants.

Let $f : S \to (M, J)$ be a mapping. An immersive point $x \in S$ of f is called *a complex point* if $f_* T_x S = J(f_* T_x S)$.

DEFINITION 1.1: f is called *generic* if (1) f is finite to one and for any $y \in V = f(S)$, (2) the germ $f : S, f^{-1}(y) \to M, y$ is an embedding with no complex points outside of $f^{-1}(y)$ and (3) the pull-back by f of a positive definite Morse function around y is of finite multiplicity at $f^{-1}(y)$.

A map-germ $f : S, x \to (M, J)$ is called *generic* if, for $y = f(x)$, (2) and (3) hold, $f^{-1}(y)$ being replaced by x.

REMARK 1.2: Non generic mappings form an infinite codimensional subset in $C^\infty(S, M)$, even after more strict restrictions on genericity are imposed (see [12] for instance), since, in the 1-jet space $J^1(S, M)$ the totality of immersive 1-jets corresponding to complex points is of codimension 2, (see [17]).

Let f be generic. Then V is a totally real submanifold of (M, J) except for finite points.

Now we intend to define $i(y) \in \mathbb{Z}$ for $y \in V$ as the self-intersection index V at y. To do this, we have to assign a perturbation of V near y.

For a sufficiently small sphere S^3 centered y in M (with respect to some coordinate), f is transverse to S^3 by the property (3) of Definition 1.1. Considering the link $L = V \cap S^3$, we take a tangent vector field v to V defined near L and directed outward. Then the field Jv does not tangent to V by the property (2). Thus we perturb f into f' along the direction Jv and count intersection indices of V and $f'(S)$ near y. In other word, we adopt the following definition:

DEFINITION 1.2: (Local self-intersection index of f at y.) We set $i(y) = \operatorname{link}(L, L')$, where $L' = f'(S) \cap S^3$.

Clearly $i(y)$ does not depend on the choice of S^3 and v.

Since, on an immersed surface without complex points, multiplying J maps the tangent bundle isomorphically to the normal bundle with the reverse orientation, it is easy to verify

LEMMA 1.3. *If $y \in V$ is a non-singular point, then $i(y) = -1$. If $y \in V$ is a transverse self-intersection (non-complex) point, then $i(y) = 0, -4$, according to the intersection index is $+1$, -1, respectively.*

REMARK 1.4: If J_t is a homotopy of complex structures such that $J_0 = J$ and V has no complex points near L with respect to J_t. Then the number $i(y)$ with respect to J_t does not depend on J_t. Similar result holds also for m defined below.

We next define the Maslov index $m(x)$ for $x \in S$, generalizing the definition in [13].

Consider the $\mathbb{C}^2$-bundle $E = f^*TM$ over S. Let $\tilde{G}$ denote the space of oriented 2-planes in E, and $\pi : \tilde{G} \to M$ the canonical projection. The fiber of π is $G = G_{4,2}$, the Grassmannian of oriented 2-planes in $\mathbb{C}^2$.

Let $\tilde{C} \subset \tilde{G}$ be the totality of complex planes. We decompose $\tilde{C} = \tilde{C}_+ \cup \tilde{C}_-$, where an oriented plane $\alpha \in \tilde{C}$ belongs to $\tilde{C}_+$ if and only if the orientation of α coincides with the orientation as the complex plane. Then we see $\tilde{C}_\pm$ are submanifolds of $\tilde{G}$ of codimension 2 respectively.

We define orientations of $\tilde{G}, \tilde{C}_+$ and $\tilde{C}_-$ as follows (cf. [17],[9],[3],[4]): For each $x \in S$, we take a local frame e_1, e_2, e_3, e_4 of E as $\mathbb{R}^4$-bundle near x with $Je_1 = e_2, Je_3 = e_4$. Then we set

$$\begin{cases} x_1 = p_{12} + p_{34}, \\ x_2 = p_{23} + p_{14}, \\ x_3 = p_{31} + p_{24}, \end{cases} \qquad \begin{cases} y_1 = p_{12} - p_{34}, \\ y_2 = p_{23} - p_{14}, \\ y_3 = p_{31} - p_{24}, \end{cases}$$

for the Plücker coordinate p_{ij}. Then we identify G with $(\mathbb{R}^3 - 0)/\mathbb{R}_{>0} \times (\mathbb{R}^3 - 0)/\mathbb{R}_{>0} \cong S^2 \times S^2$ by these coordinates. The fiber of $\tilde{C}_+$ (resp. $\tilde{C}_-$) corresponds to $C_+ = n \times S^2$ (resp. $C_- = s \times S^2$), where $n = (1,0,0), s = (-1,0,0)$.

We orient $\tilde{G}$ (resp. $\tilde{C}_+$, $\tilde{C}_-$) from the orientations of S and G (resp. C_+, C_-). We denote by $-\tilde{C}_-$ the $\tilde{C}_-$ with the reverse orientation.

Let $\Sigma \subset S$ be the set of non-immersive points of f. Then we define the Gauss mapping $g : S - \Sigma \to \tilde{G}$ by $g(x) = f_*(T_x S)$, $x \in S - \Sigma$.

For $x \in S$, we take a small loop ℓ around x. Then $g \circ \ell$ extends to a section $\tilde{g}$ over the disk, since $\tilde{G}$ is a $S^2 \times S^2$-bundle. We count the intersection number of $\tilde{g}$ with $\tilde{C}_+ \cup (-\tilde{C}_-)$. In other word, we adopt the following definition:

DEFINITION 1.5: (The Maslov index of f at x.) We set $m(x) = \mathrm{link}(g \circ \ell, \tilde{C}_+ \cup (-\tilde{C}_-))$.

If $x \in S$ is a complex point, then $g(x) \in \tilde{C}$.

DEFINITION 1.6: A complex point $x \in S$ is called positive (resp. negative) if $g(x) \in \tilde{C}_+$ (resp. $g(x) \in \tilde{C}_-$). A complex point $x \in S$ is called transverse if g is transverse to $\tilde{C}$ at x. A transverse complex point x is elliptic (resp. hyperbolic) if the intersection index of g and $\tilde{C} = \tilde{C}_+ \cup \tilde{C}_-$ at $g(x)$ is equal to $+1$ (resp. -1).

Then the following is straightforward.

LEMMA 1.7. *Let $x \in S - \Sigma$. If x is not a complex point, then $m(x) = 0$. If x is a positive elliptic or negative hyperbolic point, then $m(x) = +1$. If x is a negative elliptic or positive hyperbolic point, then $m(x) = -1$.*

Next Lemma is used to show Theorem 1.(2).

LEMMA 1.8. *The homology class $[\tilde{C}_+ \cup (-\tilde{C}_-)] \in H_4(\tilde{G}, \mathbb{Z})$ is the Pioncaré dual of $\pi^* c_1(E) \in H^2(\tilde{G}, \mathbb{Z})$.*

PROOF: Consider the complex line bundle $\pi^*(E \wedge E)$ over $\tilde{G}$. Then $c_1(\pi^*(E \wedge E)) = c_1(\pi^* E)$, (see [H]). Taking a metric of E compatible with J, we define the section s of $\pi^*(E \wedge E)$ over $\tilde{G}$ by $s(\alpha) = v \wedge w$, where $\alpha \in \tilde{G}$ and v, w are orthonormal basis of α compatible with the orientation of α. If, in above, e_1, e_2, e_3, e_4 are orthonormal, then locally s is represented by $s = (-x_3 + \sqrt{-1}x_2)e_1 \wedge e_3$. Therefore we see that the zero locus of s with the induced orientation is equal to $\tilde{C}_+ \cup (-\tilde{C}_-)$. This shows the required result.

To end this section, we prove the following:

LEMMA 1.9. *If x is an elliptic (resp. hyperbolic) point, then $i(x) = 0$ (resp. -2).*

PROOF: First we follow the arguments in [4, §4.1]. Let x be a transverse complex point. Then, by [5], [19], there exist coordinates $(u, v) : S, x \to \mathbf{R}^2, 0$ and $(p_1, q_1; p_2, q_2) : M, f(x) \to \mathbf{C}^2, 0$ such that

$$f(u,v) = (u,\ v,\ (1+2\gamma)u^2 + (1-2\gamma)v^2 + \phi(u,v),\ \psi(u,v)),$$

with $\gamma \in \mathbf{R},\ 0 < \gamma \neq \frac{1}{2}$, $\mathrm{ord}_0\phi \geq 3$, $\mathrm{ord}_0\psi \geq 3$, and $J = J_0 +$ higher order terms, for the standard complex structure J_0 on $\mathbf{C}^2$. If $0 < \gamma < \frac{1}{2}$ (resp. $\gamma > \frac{1}{2}$), then x is elliptic (resp. hyperbolic).

To compute $i(x)$, we take the Euler field $E = u(\partial/\partial u) + v(\partial/\partial v)$. Then

$$J(f_*E) = (-v + P_1)\frac{\partial}{\partial p_1} + (u + Q_1)\frac{\partial}{\partial q_1} + P_2\frac{\partial}{\partial p_2} + (2((1+2\gamma)u^2 + (1-2\gamma)v^2) + Q_2)\frac{\partial}{\partial q_2},$$

with $\mathrm{ord}_0 P_1 \geq 2$, $\mathrm{ord}_0 Q_1 \geq 2$, $\mathrm{ord}_0 P_2 \geq 3$, $\mathrm{ord}_0 Q_2 \geq 3$. We set, for sufficiently small ϵ, $0 < \epsilon^2 < |1 - 4\gamma^2|$,

$$f_\epsilon(u,v) = (u - \epsilon v + A, v + \epsilon u + B, (1+2\gamma)u^2 + (1-2\gamma)v^2 + C, 2\epsilon((1+2\gamma)u^2 + (1-2\gamma)v^2) + D),$$

where $A = \epsilon P_1, B = \epsilon Q_1, C = \phi + \epsilon P_2, D = \psi + \epsilon Q_2$. Consider the map-germ $F : \mathbf{R}^4, 0 \to \mathbf{R}^4, 0$ defined by $F(u, v, u', v') = f_\epsilon(u, v) - f(u', v')$.

Let E_4 denote the $\mathbf{R}$-algebra of function-germs on $\mathbf{R}^4, 0$ and m the unique maximal ideal of E_4. For the ideal $I(F) \subset E_4$ generated by the components of F, we easily see that $m^3 \subset I(F) + m^4$, therefore $m^3 \subset I(F)$ by Nakayama's lemma. Hence F is a finite map-germ and we see $i(x) = \deg_0 F$.

Following [7], we calculate $\deg_0 F$. The algebra $Q(F) = E_4/I(F)$ is generated by $1, u, v$ and u^2 over $\mathbf{R}$. The class s of Jacobian of F is equal to $-64\epsilon^2\gamma(1 + 2\gamma + \frac{\epsilon^2}{1-2\gamma})u^2$ in $Q(F)$. Define the functional $\varphi : Q(F) \to \mathbf{R}$ by $\varphi(u^2) = -1, \varphi(1) = \varphi(u) = \varphi(v) = 0$. Then we see $\varphi(s) > 0$ and the matrix of the bilinear form $\langle\ ,\ \rangle_\varphi : Q(F) \times Q(F) \to \mathbf{R}$, $\langle a, b\rangle_\varphi = \varphi(ab)$, is equal to

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1+2\gamma}{1-2\gamma} & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

If $0 < \gamma < \frac{1}{2}$ (resp. $\gamma > \frac{1}{2}$), then we have $\deg_0 F =$ signature of $\langle\ ,\ \rangle_\varphi = 0$ (resp. -2).

2. Perturbations.

Proof of Theorem 2: (1) As in §0, we denote by e_+, e_-, h_+, h_- the numbers of positive elliptic, negative elliptic, positive hyperbolic, negative hyperbolic complex points of $\tilde{f}$ respectively. Further denote by δ_+, δ_- the numbers of self-intersection points of index $+1, -1$ respectively. Then $d_+ = e_+ - h_+, d_- = e_- - h_-$ and $\delta = \delta_+ - \delta_-$.

We may assume that the self-intersections do not occur on the complex points. We set $W = \tilde{f}(D^2)$. We take a tangent vector field v to W along $\tilde{f}$ such that v are directed outward on ∂W, near all complex points and all self-intersection points. We perturb $\tilde{f}$ to f' along the direction of Jv. Set $W' = f'(D^2)$. Then $i(f)$ is equal to the sum of intersection indices of W and W', which is equal to $\sum i(y) - \chi(W_0)$, where the sum runs over all complex points and self-intersection points, and W_0 means W minus small balls centered at complex points and self-intersection points. Then $\chi(W_0) = 1-(e_+ + e_-)-(h_+ + h_-) - 2(\delta_+ + \delta_-)$. By Lemmas 1.3 and 1.9, we have $\sum i(y) = -2(h_+ + h_-) - 4\delta_-$. Hence,

$$i(f) = e_+ - h_+ + e_- - h_- - 1 + 2(\delta_+ - \delta_-) = d_+ + d_- - 1 + 2\delta.$$

(2) The Maslov index $m(f)$ is equal to $\sum m(x)$, the sum running over all complex points of $\tilde{f}$. Then by Lemma 1.7, $m(f) = e_+ - e_- - h_+ + h_- = d_+ - d_-$.

Q.E.D.

Now we apply Theorem 2 to calculate i and m for the open Whitney umbrella using concrete perturbations.

Example 2.1: We perturb the local model $f : \mathbb{R}^2, 0 \to \mathbb{R}^4, 0$ of the open Whitney umbrella into $f_\epsilon : \mathbb{R}^2, 0 \to \mathbb{R}^4, 0$ defined by $f_\epsilon(u,v) = (\frac{v^3}{3}, u, uv, \frac{v^2}{2} + \epsilon v)$, for sufficiently small $\epsilon > 0$. Then we have $\delta = 0, h_- = 1$ and $e_+ = e_- = h_+ = 0$. By Theorem 2, we see $i(f) = -2$ and $m(f) = 1$. For the map-germ f' defined by $f'(u,v) = f(u,v)$, we see $i(f) = -2$ and $m(f) = -1$.

For another perturbation f_ϵ of f, for instance, $f_\epsilon(u,v) = (\frac{v^3}{3} + \epsilon v, u, uv, \frac{v^2}{2})$, we have $\delta = 0, h_- = 1, e_+ = e_- = h_+ = 0$, when $\epsilon > 0$, and $\delta = -1, e_+ = 1, e_- = h_+ = h_- = 0$, when $\epsilon < 0$.

Combined with Lemmas 1.3, 1.7, 1.9, Remark 1.4 and Example 2.1, we get Table 3.

3. Implications.

First we deduce the formulae of Lai and Givental' from Theorem 1.

THE FORMULA OF LAI: By Table 3, we have $\sum_y \{i(y)+1\} = d_+ + d_- + \delta_+ - 3\delta_-$, $V \cdot V = \nu + 2(\delta_+ - \delta_-)$, $\chi(V) = \chi(S) - (\delta_+ + \delta_-)$, $\sum_x m(x) = d_+ - d_-$, where δ_+ and δ_- are similar numbers as in the proof of Theorem 2. Thus $\chi(V) + V \cdot V = \nu + \chi + \delta_+ - 3\delta_-$. By Theorem 1, we have $d_+ + d_- = \nu + \chi$ and $d_+ - d_- = c$.

THE FORMULA OF GIVENTAL': By Table 3, we have

$$\sum_{y \in V} \{i(y)+1\} = \delta_+ - 3\delta_- - T, \quad \chi(V) + V \cdot V = \chi - (\delta_+ + \delta_-) + V \cdot V.$$

By Theorem 1.(1), we have $-V \cdot V = \chi - 2\delta + T$.

PROOF OF THEOREM 1: (1) Denote by X the set of singular points of V. We remove from V small balls centered at points of X. Denote by V' the resulting surface with boundary. Let v be a vector field over V' directed inward (relatively to V') along $\partial V'$. Using Jv, we perturb V. Then we see

$$V \cdot V = \sum_{y \in X} i(y) - \chi(V'), \quad \chi(V') = \chi(V) - \sharp X.$$

Thus we have $V \cdot V = \sum_{y \in X} \{i(y)+1\} - \chi(V)$.

(2) For $x \in \Sigma$, take a small disk $D_x \subset S$ around x. We extend the Gauss map $g : S - \bigcup_{x \in \Sigma} D_x \to \tilde{G}$ to a section $\tilde{g} : S \to \tilde{G}$. Then the sum $\sum_{x \in S} m(x)$ is equal to the intersection number of $\tilde{g}(S)$ and $\tilde{C}_+ \cup (-\tilde{C}_-)$. By Lemma 1.8, this number is equal to

$$\langle \pi^* c_1(E), \tilde{g}_*[S] \rangle = \langle \tilde{g}^* \pi^* c_1(E), [S] \rangle = \langle c_1(E), [S] \rangle = c.$$

Q.E.D.

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