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On Small Data Scattering for 2-Dimensional Semilinear Wave Equations

K. Kubota and K. Mochizuki

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On Small Data Scattering for 2-Dimensional Semilinear Wave Equations

by

Kôji KUBOTA and Kiyoshi MOCHIZUKI

Introduction

In this paper we consider a small data scattering for the nonlinear wave equation

$$(0.1) \qquad \partial_t^2 u - \Delta u = f(u)$$

in two space dimensios, where $f(u) = \Lambda|u|^{\rho-1}u$ or $f(u) = \Lambda|u|^{\rho}$, $\Lambda \in \mathrm{R}$, $\rho > \frac{3+\sqrt{17}}{2}$. The scattering theory compares asymptotic behaviors for $t \to \pm\infty$ of solutions of (0.1) with those of the free wave equation

$$(0.2) \qquad \partial_t^2 u_0 - \Delta u_0 = 0.$$

The comparison will be done in the energy norm:

$$(0.3) \qquad \|u(t)\|_e^2 = \frac{1}{2}\{\|\nabla_x u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2\}.$$

More precisely, we start with the solution $u_0^-(t)$ of (0.2) with initial data

$$u_0^-(0) \in C^3(\mathrm{R}^2), \quad \partial_t u_0^-(0) \in C^2(\mathrm{R}^2)$$

which are small and decay sufficiently rapidly at infinity. Then we construct a global solution $u(t)$ of (0.1) behaving like $u_0^-(t)$ near $t = -\infty$:

(0.4) $\qquad \|u(t)-u_0^-(t)\|_e \to 0 \quad \text{as } t \to -\infty.$

Moreover, there exists another solution $u_0^+(t)$ of (0.2) such that

(0.5) $\qquad \|u(t)-u_0^+(t)\|_e \to 0 \quad \text{as } t \to +\infty.$

Thus, the scattering operator S: $u_0^- \to u_0^+$ is shown to exist on a dense set of a neighborhood of 0 in the energy space.

The existence of the scattering operator has been proved by Strauss [13], Klainermann [6] and Mochizuki-Motai [8],[9] in the general n (≥ 2) dimensional problem. In these works the requirements on the power ρ of the nonlinear term are rather strong, and it is assumed in [9] that

(0.6) $$\rho > \frac{n^2+3n-2+\sqrt{(n^2+3n-2)^2-8n(n-1)}}{2n(n-1)} \equiv \gamma(n).$$

Note that $\gamma(2) = 2+\sqrt{3} \simeq 3.732$ and $\gamma(3) = \frac{4+\sqrt{13}}{3} \simeq 2.535$.

In the 3-dimensional case condition (0.6) was improved by Pecher [10] to $\rho > \rho(3) = 1+\sqrt{2} \simeq 2.414$, where

(0.7) $$\rho(n) = \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}.$$

His result is sharp in the sense that the Cauchy problem for (0.1) with non-trivial regular data does not have global regular solutions if $\rho \leq \rho(n)$ (see John [5], Schaeffer [11] and Asakura [2] for $n = 3$, Glassey [3] and Agemi-Takamura [1] for $n = 2$, and Sideris [12] for $n \geq 4$). Our aim of this paper is to obtain the corresponding sharp result in the 2-dimensional case. Thus, we require $\rho > \rho(2) = \frac{3+\sqrt{17}}{2} \simeq 3.561$.

Our problem (0.1),(0.4) will be reduced to the integral equation

(0.8) $$u(x,t) = u_0^-(x,t) + \int_{-\infty}^{t} \frac{t-\tau}{2\pi}\, d\tau \int_{|\xi|<1} \frac{f(u(x+(t-\tau)\xi,\tau))}{\sqrt{1-|\xi|^2}}\, d\xi.$$

To show the global solvability of (0.8), we use, as in the case of Pecher [10], a weighted norm in space and time. This kind of norm is originally introduced by John [5] to study a Cauchy problem of (0.1) at a finite time. In contrast to the Cauchy problem, (0.8) includes an integral over unbounded space-time region. In the 3-dimensional problem, the integral being simpler, we can estimate it by almost the same method as in the case of the Cauchy problem. In our 2-dimensional problem, however, to add an extra estimate which is not used in the Cauchy problem is really necessary.

The paper is organized as follows: In §1 we first summarize results of Kubota [7] (cf., also Glassey [4] and Tsutaya [14]) for the Cauchy problems at t = 0 for (0.1) and (0.2). In the last half of this section, asymptotics for solutions of (0.1) will be given as a direct result. In §2 we prove a basic a-priori estimate for the nonlinear term of (0.8), and construct a unique global solution of (0.8), from which the scattering operator is easily shown to exist.

Finally, we remark that a similar result for the 2-dimensional problem has been obtained independently by Tsutaya [15], where a basic estimate for nonlinear term is proved by a slightly different way.

§1. The Cauchy problems at $t = 0$

Let us first consider the Cauchy problem

$$(1.1)\qquad \begin{cases} \partial_t^2 u_0 - \Delta u_0 = 0, & x \in \mathrm{R}^2,\ t > 0, \\ u_0(x,0) = \varphi(x),\ \partial_t u_0(x,0) = \psi(x), & x \in \mathrm{R}^2. \end{cases}$$

For given $\varphi(x) \in C(\mathrm{R}^3)$ and $\psi(x) \in C(\mathrm{R}^2)$, a unique classical solution exists and is given for $t \geq 0$ by

$$(1.2)\qquad u_0(x,t) = \partial_t\left(\frac{t}{2\pi}\int_{|\xi|<1}\frac{\varphi(x+t\xi)}{\sqrt{1-|\xi|^2}}\,d\xi\right)+\frac{t}{2\pi}\int_{|\xi|<1}\frac{\psi(x+t\xi)}{\sqrt{1-|\xi|^2}}\,d\xi.$$

In the following we assume that for all $x \in \mathrm{R}^2$,

$$(1.3)\qquad |\varphi(x)| \le \frac{\varepsilon}{(1+|x|)^{k-1}}\ ,\quad |\nabla^j\psi(x)|+|\nabla^{j+1}\varphi(x)| \le \frac{\varepsilon}{(1+|x|)^k}$$

(j=0,1,2), where $\varepsilon > 0$ and $k > 1$.

Proposition 1.1. *Let* $k > \frac{3}{2}$ *in* (1.3). *Then there exists a constant* $C > 0$ *depending on* k *such that*

$$(1.4)\qquad |\nabla^j u_0(x,t)| \le \frac{C\varepsilon(1+|t-|x||)^{[-k+2]}}{\sqrt{1+t+|x|}\sqrt{1+|t-|x||}}\quad (0\le j\le 2)$$

for $x \in \mathrm{R}^2$ *and* $t \ge 0$. *Here and in the following we use the notation*: $[a] = \max\{0,a\}$ *for* $a \in \mathrm{R}$ *and* $A^{[0]} = (1+\log A)$ *for* $A \ge 1$.

We shall sketch a proof. As for the details, see e.g., Proposition 2.1 of [7]. Note that the first inequality of (1.3) is replaced by $|\varphi(x)| \le \dfrac{\varepsilon}{(1+|x|)^k}$ in [7]. However, by expression (1.2), we easily see that (1.3) is enough to obtain (1.4).

For the sake of simplicity, we only consider the term

$$u_{02}(x,t) \equiv \frac{t}{2\pi}\int_{|\xi|<1}\frac{\psi(x+t\xi)}{\sqrt{1-|\xi|^2}}\,d\xi = \frac{1}{2\pi}\int_{|\xi|<t}\frac{\psi(x+\xi)}{\sqrt{t^2-|\xi|^2}}\,d\xi.$$

In the last integral we use the polar coordinates $\xi = \eta\omega$, $|\omega|= 1$, and put $r = |x|$ and $\lambda = |x+\eta\omega|$. Then by means of (1.3),

$$(1.5)\qquad |u_{02}(x,t)| \le \frac{\varepsilon}{\pi}\int_0^t \eta\{t^2-\eta^2\}^{-1/2}d\eta\int_{|\omega|=1}(1+|x+\eta\omega|)^{-k}dS_\omega$$

$$= \frac{4\varepsilon}{\pi}\int_0^t \eta\{t^2-\eta^2\}^{-1/2}d\eta\int_{|\eta-r|}^{\eta+r}\lambda(1+\lambda)^{-k}h(\lambda,\eta,r)\,d\lambda,$$

where

$$(1.6)\qquad h(\lambda,\eta,r) = \{\eta^2-(\lambda-r)^2\}^{-1/2}\{(\lambda+r)^2-\eta^2\}^{-1/2}$$

(the equality in (1.5) is proved in Lemma 2.3 of [7]). Thus, changing the order of integrations, we have

(1.7) $|u_{02}(x,t)|$

$$\le \frac{4\varepsilon}{\pi}\int_{|t-r|}^{t+r} \lambda(1+\lambda)^{-k}d\lambda\int_{|\lambda-r|}^{t} \eta\{t^2-\eta^2\}^{-1/2}h(\lambda,\eta,r)d\eta$$

$$+ \frac{4\varepsilon}{\pi}\int_{0}^{[t-r]} \lambda(1+\lambda)^{-k}d\lambda\int_{|\lambda-r|}^{\lambda+r} \eta\{t^2-\eta^2\}^{-1/2}h(\lambda,\eta,r)d\eta.$$

Lemma 1.2. *Let* $a < b < c$. *Then*

$$\int_a^b (b-\sigma)^{-1/2}(\sigma-a)^{-1/2}(c-\sigma)^{-1/2}d\sigma \le \pi(c-b)^{-1/2}. \tag{1.8}$$

Proof. Obvious if we note that

$$\int_a^b (b-\sigma)^{-1/2}(\sigma-a)^{-1/2}d\sigma = \pi. \ \square$$

Lemma 1.3. *Let* $k > \frac{3}{2}$, $\ell > 0$, $a \in R$ and $b \ge |a|$. *Then*

$$\int_b^\infty (1+\sigma)^{-k+1}(\sigma-a)^{-1/2}d\sigma \le C(1+b)^{-k+3/2}. \tag{1.9}$$

$$\int_a^b (1+|\sigma|)^{-\ell}(b-\sigma)^{-1/2}d\sigma \le C(1+b)^{-1/2}(1+b)^{[-\ell+1]}. \tag{1.10}$$

Proof. We have only to show these inequalities in case $b > 1$. Then (1.9) follows since we have

$$\text{left of (1.9)} \le (1+b)^{-k+1}\int_b^{3b} (\sigma-b)^{-1/2}d\sigma$$

$$+ \sqrt{2}\int_{3b}^\infty (1+\sigma)^{-k+1/2}d\sigma \le C(1+b)^{-k+3/2}.$$

Next, we have

$$\text{left of (1.10)} \le (b/2)^{-1/2}\int_{-b}^{b/2} (1+|\sigma|)^{-\ell}d\sigma$$

$$+ (1+b/2)^{-\ell}\int_{b/2}^{b} (b-\sigma)^{-1/2}d\sigma,$$

which implies (1.10). $\square$

Proof of Proposition 1.1. We apply lemma 1.2 to (1.7). Note that

$$\int_{|\lambda-r|}^{t} \eta(t^2-\eta^2)^{-1/2}h(\lambda,\eta,r)d\eta$$
$$\le \tfrac{1}{2}\int_a^b (b-\sigma)^{-1/2}(\sigma-a)^{-1/2}(c-\sigma)^{-1/2}d\sigma,$$

where $a = (\lambda-r)^2$, $b = t^2$ and $c = (\lambda+r)^2$. Then by (1.8),

$$\int_{|\lambda-r|}^{t} \eta(t^2-\eta^2)^{-1/2}h(\lambda,\eta,r)d\eta \le \tfrac{\pi}{2}\{(\lambda+r)^2-t^2\}^{-1/2}.$$

Similarly, we have

$$\int_{|\lambda-r|}^{\lambda+r} \eta(t^2-\eta^2)^{-1/2}h(\lambda,\eta,r)d\eta \le \tfrac{\pi}{2}\{t^2-(\lambda+r)^2\}^{-1/2},$$

and thus, it follows that

$$|u_{02}(x,t)| \le 2\varepsilon\int_{|t-r|}^{t+r} \lambda(1+\lambda)^{-k}\{(\lambda+r)^2-t^2\}^{-1/2}d\lambda$$
$$+ 2\varepsilon\int_0^{[t-r]} \lambda(1+\lambda)^{-k}\{t^2-(\lambda+r)^2\}^{-1/2}d\lambda$$
$$\le 2\varepsilon(t+r)^{-1/2}\left\{\int_{|t-r|}^{t+r}+\int_0^{[t-r]}\right\}(1+\lambda)^{-k+1}|\lambda-t+r|^{-1/2}d\lambda$$
$$\le C\varepsilon(t+r)^{-1/2}(1+|t-r|)^{-1/2}(1+|t-r|)^{[-k+2]}.$$

Here we have used (1.9) with $a = t-r$, $b = |a|$ and (1.10) with $\ell = k-1$, $a = 0$, $b = |t-r|$ to obtain the last inequality. □

Next we consider the Cauchy problem

$$(1.11)\qquad \begin{cases} \partial_t^2 u - \Delta u = f(u), & x \in \mathrm{R}^2,\ t > 0, \\ u(x,0) = \varphi(x),\ \partial_t u(x,0) = \psi(x), & x \in \mathrm{R}^2, \end{cases}$$

where $f(u) = \Lambda|u|^{\rho-1}u$ or $f(u) = \Lambda|u|^{\rho}$, $\Lambda \in \mathrm{R}$, $\rho > \frac{3+\sqrt{17}}{2}$, and φ, ψ satisfies (1.3). As is well known, (1.11) is reduced to the integral equation

$$(1.12)\qquad u(x,t) = u_0(x,t) + L_0(f(u))(x,t),$$

where $u_0(x,t)$ is the solution of (1.1) and

$$(1.13)\qquad L_0(w)(x,t) = \frac{1}{2\pi}\int_0^t (t-\tau)d\tau \int_{|\xi|<1} \frac{w(x+(t-\tau)\xi,\tau)}{\sqrt{1-|\xi|^2}}\, d\xi.$$

Proposition 1.4. *Assume that for all* $x \in \mathrm{R}^2$ *and* $t > 0$,

$$(1.14)\qquad |u(x,t)| \le \frac{M}{\sqrt{1+t+|x|}\,(1+|t-|x||)^{\nu}},$$

where $M > 0$ *and*

$$(1.15)\qquad \nu > \left[\frac{5-\rho}{2\rho}\right] \text{ and } \rho\nu \ne 1.$$

Then there exists a $C > 0$ *depending on* ν *such that*

$$(1.16)\qquad |L_0(|u|^{\rho})(x,t)| \le \frac{CM^{\rho}(1+|t-|x||)^{[-\rho/2+[-\rho\nu+1]+2]}}{\sqrt{1+t+|x|}\sqrt{1+|t-|x||}}$$

for $x \in \mathrm{R}^2$ *and* $t > 0$.

As in the case of $u_{02}(x,t)$, a simple calculation with Lemma 1.2 shows that

$$(1.17)\qquad |L_0(|u|^{\rho})(x,t)|$$

$$\le 2M^{\rho}\int_0^t d\tau \int_{|t-\tau-r|}^{t-\tau+r} \lambda(1+\lambda+\tau)^{-\rho/2}(1+|\lambda-\tau|)^{-\rho\nu} \times\{(\lambda+r)^2-(t-\tau)^2\}^{-1/2}d\lambda$$

$$+ 2M^{\rho}\int_0^{[t-r]} d\tau \int_0^{t-\tau-r} \lambda(1+\lambda+\tau)^{-\rho/2}(1+|\lambda-\tau|)^{-\rho\nu} \times\{(t-\tau)^2-(\lambda+r)^2\}^{-1/2}d\lambda.$$

We choose the new independent variables

$$(1.18)\qquad \alpha = \lambda-\tau \text{ and } \beta = \lambda+\tau.$$

in the right integrals. Then it follows that

$$|L_0(|u|^{\rho})| \le M^{\rho}\int_{|t-r|}^{t+r} (1+\beta)^{-\rho/2+1}(\beta-t+r)^{-1/2}d\beta \times\int_{r-t}^{\beta} (1+|\alpha|)^{-\rho\nu}(\alpha+t+r)^{-1/2}d\alpha$$

$$+ M^{\rho}\int_0^{[t-r]} (1+\beta)^{-\rho/2+1}|t-r-\beta|^{-1/2}d\beta$$

$$\times\int_{-\beta}^{\beta} (1+|\alpha|)^{-\rho\nu}(\alpha+t+r)^{-1/2}d\alpha \equiv M^{\rho}\{I_1+ I_2\}.$$

Lemma 1.5. *Let* $a \in \mathrm{R}$, $b \geq |a|$ *and* $c \geq \mathrm{a}$. *Then*

$$(1.19) \quad \int_{-b}^{a} (1+|\sigma|)^{-\rho\nu}(c-\sigma)^{-1/2}d\sigma \leq C(1+|c|)^{-1/2}(1+b)^{[-\rho\nu+1]},$$

Proof. We denote the left of (1.19) = K. First assume $c > 1$. If $c \geq 2a$, obviously $K \leq C(c-a)^{-1/2}(1+b)^{[-\rho\nu+1]}$, and if $c \leq 2a$,

$$K = \int_{-b}^{a/2} + \int_{a/2}^{a}$$

$$\leq C(c- \tfrac{a}{2})^{-1/2}(1+b)^{[-\rho\nu+1]}+ C(1+ \tfrac{a}{2})^{-\rho\nu}(c- \tfrac{a}{2})^{1/2}.$$

Next assume $|c| \leq 1$. If $a \geq -2$,

$$K = \int_{-b}^{-2} +\int_{-2}^{a} \leq C(c+2)^{-1/2}(1+b)^{[-\rho\nu+1]}+C(1+|a|)^{-\rho\nu}(c+2)^{1/2},$$

and if $a \leq -2$,

$$K \leq \int_{-b}^{-2} \leq C(c+2)^{-1/2}(1+b)^{[-\rho\nu+1]}.$$

Finally assume $c \leq -1$. If $c \geq \frac{a}{2}$, obviously we have $K \leq C(c-a)^{-1/2}(1+b)^{[-\rho\nu+1]}$, if $c \leq \frac{a}{2}$ and $a \geq -\frac{b}{2}$,

$$K = \int_{-b}^{2a} + \int_{2a}^{a}$$

$$\leq C(c-2a)^{-1/2}(1+b)^{[-\rho\nu+1]}+ C(1+|a|)^{-\rho\nu}(c-2a)^{1/2},$$

and if $c \leq \frac{a}{2}$ and $a \leq -\frac{b}{2}$,

$$K \leq \int_{2\mathrm{a}}^{a} \leq C(1+|a|)^{-\rho\nu}(c-2a)^{1/2}.$$

These inequalities show (1.19). □

Proof of Proposition 1.4. Cf., e.g., Propositions 3.1 and 3.2 of [7]. First consider I_1. By use of (1.19) with $a = t-r$, $b = \beta$,

$c = t+r$,

$$I_1 \le C(1+t+r)^{-1/2} \int_{|t-r|}^{t+r} (1+|\beta|)^{-\rho/2+1+[-\rho\nu+1]} (\beta-t+r)^{-1/2} d\beta.$$

Since

$$(1.20) \qquad \frac{\rho}{2} -[-\rho\nu+1] = \begin{cases} \rho/2 & (\text{if } \rho\nu>1) \\ \rho/2+\rho\nu-1 & (\text{if } \rho\nu<1) \end{cases} > \frac{3}{2}$$

follows from (1.15), we can apply (1.9) with $k = \frac{\rho}{2} -[-\rho\nu+1]$, $a = t-r$, $b = |a|$ to obtain

$$I_1 \le C(1+t+r)^{-1/2}(1+|t-r|)^{-\rho/2+[-\rho\nu+1]+3/2}.$$

Next, we consider I_2 assuming $t-r > 0$ and $r+t \ge 3$ (the other cases are easier). By use of (1.19) with $a = b = \beta$, $c = t+r$ and then (1.10) with $\ell = \frac{\rho}{2} -[-\rho\nu+1]-1$, $a = 0$, $b = |t-r|$ we obtain

$$\begin{aligned} I_2 &\le C(1+t+r)^{-1/2} \int_0^{t-r} (1+\beta)^{-\rho/2+[-\rho\nu+1]+1} (t-r-\beta)^{-1/2} d\beta \\ &\le C(1+t+r)^{-1/2}(1+|t-r|)^{-1/2}(1+|t-r|)^{[-\rho/2+[-\rho\nu+1]+2]}. \end{aligned}$$

These inequalities show (1.16). □

Corollary 1.6. *If we choose*

$$(1.21) \qquad \left[\frac{5-\rho}{2(\rho-1)}\right] < \nu < \min\{\frac{1}{2}, \frac{\rho-3}{2}\} \ \textit{and} \ \rho\nu \ne 1$$

(*this is possible since we have required* $\rho > \frac{3+\sqrt{17}}{2}$), *then this* ν *satisfies* (1.15) *and we have*

$$(1.22) \qquad |L_0(|u|^\rho)(x,t)| \le \frac{CM^\rho}{\sqrt{1+t+|x|}(1+|t-|x||)^\nu}.$$

Proof. Assume first $\nu < \frac{1}{\rho}$. Then

$$m \equiv \frac{1}{2} -[-\frac{\rho}{2} +[-\rho\nu+1]+2] = \frac{1}{2} -[-\frac{\rho}{2} -\rho\nu+3].$$

By (1.21), $m = \rho\nu- \frac{5-\rho}{2} > \nu$ if $\nu < \frac{6-\rho}{2\rho}$, and $m = \frac{1}{2} > \nu$ if $\nu \ge \frac{6-\rho}{2\rho}$. Next assume $\nu > \frac{1}{\rho}$. Then $m = \frac{1}{2} -[-\frac{\rho}{2} +2] = \min\{\frac{1}{2}, \frac{\rho-3}{2}\} > \nu$ also by

(1.21). Hence we have

$$\frac{(1+|t-|x||)^{[-\rho/2+[-\rho\nu+1]+2]}}{\sqrt{1+|t-|x||}} \le \frac{C}{(1+|t-|x||)^{\nu}} . \square$$

Remark 1.7. If $\rho \neq 4$, we can choose

(1.23) $$\nu = \min\{\tfrac{1}{2}, \tfrac{\rho-3}{2}\}$$

in the above corollary.

Next, we consider the integral

(1.24) $$L_1(w)(x,t) = \int_t^{\infty} (\tau-t)d\tau \int_{|\xi|<1} \frac{w(x+(\tau-t)\xi,\tau)}{\sqrt{1-|\xi|^2}} d\xi .$$

Proposition 1.8. *Let $u(x,t)$ be as in Proposition* 1.4. *Then there exists a $C > 0$ depending on ν such that*

(1.25) $$|L_1(|u|^{\rho})(x,t)| \le \frac{CM^{\rho}(1+t+|x|)^{[-\rho\nu+1]}}{(1+t+|x|)^{\rho/2-1}}$$

for $x \in \mathrm{R}^2$ and $t > 0$.

Proof. As in the case of $L_0(|u|^{\rho})$, the use of Lemma 1.2 and (1.18) shows that

$$\begin{aligned} |L_1(|u|^{\rho})| \le CM^{\rho}\int_{r+t}^{\infty} &(1+\beta)^{-\rho/2+1}(\beta+r-t)^{-1/2}d\beta \\ &\times\int_{-r-t}^{r-t} (1+|\alpha|)^{-\rho\nu}(\alpha+r+t)^{-1/2}d\alpha \\ + CM^{\rho}\int_{r+t}^{\infty} &(1+\beta)^{-\rho/2+1}(\beta+r-t)^{-1/2}d\beta \\ &\times\int_{-\beta}^{-r-t}(1+|\alpha|)^{-\rho\nu}(-r-t-\alpha)^{-1/2}d\alpha \equiv CM^{\rho}\{\gamma_1+ \gamma_2\}. \end{aligned}$$

By use of (1.9) and (1.10) with with $k = \frac{\rho}{2}$, $\ell = \rho\nu$, $a = t-r$, $b = t+r$, it follows that

$$\gamma_1 \le C(1+t+r)^{-\rho/2+3/2}(1+t+r)^{-1/2}(1+t+r)^{[-\rho\nu+1]}.$$

Next, by use of (1.19) with $a = c = -t-r$, $b = \beta$,

$$\tilde{I}_2 \leq C(1+t+r)^{-1/2}\int_{t+r}^{\infty} (1+\beta)^{-\rho/2+1+[-\rho\nu+1]}(\beta-t+r)^{-1/2}d\beta.$$

By means of (1.20), we can apply (1.9) to obtain

$$\tilde{I}_2 \leq C(1+t+r)^{-1/2}(1+t+r)^{-\rho/2+[-\rho\nu+1]+3/2}.$$

These prove (1.25). □

In the rest of this section, to show the existence and properties of solutions of (1.11), we restrict k as

(1.26) $\qquad k > \max\{\frac{3}{2}, \frac{\rho+1}{\rho-1}\}$

and choose ν as follows:

(1.27) $\qquad \left[\frac{5-\rho}{2(\rho-1)}\right] < \nu < \min\{\frac{1}{2}, \frac{\rho-3}{2}, k-\frac{3}{2}\}$ and $\rho\nu \neq 1$.

Since $\frac{5-\rho}{2(\rho-1)} = \frac{\rho+1}{\rho-1} - \frac{3}{2} < k-\frac{3}{2}$, it is always possible to choose ν satisfying (1.27).

Define a Banach space

(1.28) $\qquad X = \{v \mid \nabla_x^j v(x,t) \in C(\mathrm{R}^2\times\overline{\mathrm{R}}_+)$ and $\|\nabla_x^j v\|_V < \infty$ for $0\leq j\leq 2\}$,

where

(1.29) $\qquad \|v\|_V = \sup_{(x,t)\in\mathrm{R}^2\times\overline{\mathrm{R}}_+} [(1+t+|x|)^{1/2}(1+|t-|x||)^{\nu}|v(x,t)|].$

Then, since $\nu < k-\frac{3}{2}$, Proposition 1.1 shows that the solution $u_0(x,t)$ of (1.1) belongs to X and

(1.30) $\qquad \|\nabla_x^j u_0\|_V \leq C\varepsilon, \quad 0\leq j\leq 2,$

where $C > 0$ depends on k and ν. Moreover, since ν satisfy (1.21), Corollary 1.6 shows that

(1.31) $\qquad \|L_0(f(u))\|_V \leq C\|u\|_V^{\rho}$

for any $u \in C(\mathrm{R}^2 \times \overline{\mathrm{R}}_+)$ such that $\|u\|_V < \infty$, where this $C > 0$ depends on ρ, k, ν and Λ.

With these inequalities with $\varepsilon > 0$ sufficiently small, we can follow a method of successive approximation (already used by John [5]) to establish the unique existence of solutions $u \in X$ of the integral equation (1.12). Moreover, asymptotics as $t \to +\infty$ of u can be derived since we have $f(u) \in L^1(\mathrm{R}^+; L^2(\mathrm{R}^2))$ for $u \in X$.

Our results are summarized in the following

Theorem 1.9. (1) *Let ρ, k and ν be as given above, and assume* (1.3). *Then there exists an $\varepsilon_0 > 0$ depending on these parameters and Λ such that the initial value problem* (1.11) *has a unique solution $u(x,t)$ in X provided $0 < \varepsilon \le \varepsilon_0$.*

(2) *There exists a solution $u_0^+(x,t) \in X$ of the linear wave equation* (0.2) *such that*

$$\|u(t)-u_0^+(t)\|_e \le C\|u\|_V^\rho\{(1+t)^{-\rho+3}(1+t)^{[-2\rho\nu+1]}\}^{1/2} \to 0 \text{ as } t \to +\infty. \tag{1.32}$$

Remark 1.10. As is proved e.g., in [7], [14] for the existence of a global small smooth solution of (1.11), it is "necessary" and sufficient to assume

$$\rho > \frac{3+\sqrt{17}}{2} \quad \text{and} \quad k \ge \frac{\rho+1}{\rho-1}. \tag{1.33}$$

In this paper we add the "needless" condition $k > \frac{3}{2}$ only for the sake of simplicity, and to concentrate on the power nonlinearities $\rho > \frac{3+\sqrt{17}}{2}$. Note also that the case $\rho > 5$ is already covered by the previous works [6],[8],[9] and [13].

Proof of Theorem 1.9. (1) With the above two inequalities (1.30) and (1.31), we can follow the same argument as given e.g., in Glassey [4] or Asakura [2] (see Kubota [7]).

(2) Note that

$$\nabla_x^j L_1(f(u))(x,t) = L_1(\nabla_x^j f(u))(x,t), \quad 0 \le j \le 2.$$

Then since $u \in X$, it follows from Proposition 1.8 that $L_1(f(u)) \in X$. Define

$$(1.34) \qquad u_0^+(x,t) = u(x,t) - L_1(f(u))(x,t) \in X.$$

Then we can directly prove that $u_0^+(x,t)$ satisfies the linear wave equation (0.2) (see, e.g., Pecher [10]) and moreover,

$$\|u(t)-u_0^+(t)\|_e \le \int_t^\infty \|f(u(\cdot,\tau))\|_{L^2} d\tau.$$

Thus, by means of (1.29), we have

$$\|u(t)-u_0^+(t)\|_e$$
$$\le C\|u\|_V^\rho \int_t^\infty \{\int_{R^2} (1+\tau+|x|)^{-\rho}(1+|\tau-|x||)^{-2\rho\nu} dx\}^{1/2} d\tau.$$

Put

$$\int_{R^2} \cdot\cdot\, dx = \int_{|x|<\tau/2} + \int_{\tau/2<|x|<3\tau/2} + \int_{|x|>3\tau/2} \equiv K_1+K_2+K_3.$$

Then

$$K_1 \le (1+\tau/2)^{-\rho-2\rho\nu} \int_{|x|<\tau/2} dx \le C(1+\tau)^{-\rho-2\rho\nu+2},$$

$$K_2 \le (1+\tau)^{-\rho+1} \int_{-\tau/2}^{\tau/2} (1+|s|)^{-2\rho\nu} ds \le C(1+\tau)^{-\rho+1}(1+\tau)^{[-2\rho\nu+1]},$$

$$K_3 \le \int_{|x|>\tau/2} C(1+|x|)^{-\rho-2\rho\nu} dx \le C(1+\tau)^{-\rho-2\rho\nu+2}.$$

Thus, we obtain

$$\|u(t)-u_0^-(t)\|_e \le C\|u\|_V^\rho \int_t^\infty \{(1+\tau)^{-(\rho-1)}(1+\tau)^{[-2\rho\nu+1]}\}^{1/2} d\tau$$
$$\le C\|u\|_V^\rho \{(1+t)^{-(\rho-3)}(1+t)^{[-2\rho\nu+1]}\}^{1/2}$$

and the proof is completed. □

§2. Basic estimates and the existence of the scattering operator

We begin with preparing a-priori estimates corresponding to Proposition 1.4, which guarantee the existence of the global solution of (0.8):

$$(2.1)\qquad u(x,t) = u_0^-(x,t) + L(f(u))(x,t),$$

where $u_0^-(x,t)$ is a solution of (1.1) with initial data $(\varphi^-(x),\psi^-(x))$ satisfying (1.3), $f(u) = \lambda|u|^{\rho-1}u$ or $f(u) = \lambda|u|^\rho$ with $\lambda \in \mathrm{R}$ and $\rho > \frac{3+\sqrt{17}}{2}$, and

$$(2.2)\qquad L(w)(x,t) = \frac{1}{2\pi}\int_{-\infty}^{t} (t-\tau)d\tau\int_{|\xi|<1} \frac{w(x+(t-\tau)\xi,\tau)}{\sqrt{1-|\xi|^2}}\, d\xi.$$

Proposition 2.1. *Let*

$$(2.3)\qquad |u(x,t)| \le \frac{M}{\sqrt{1+|t|+|x|}(1+||t|-|x||)^\nu},$$

where $M > 0$, and $\nu > 0$ satisfies (1.15). Then we have

$$(2.4)\qquad |L(|u|^\rho)(x,t)| \le \frac{CM^\rho(1+||t|-|x||)^{[-\rho/2+[-\rho\nu+1]+2+\delta]}}{\sqrt{1+|t|+|x|}\sqrt{1+||t|-|x||}}$$

for $x \in \mathrm{R}^2$, $t \in \mathrm{R}$ and $0 < \delta < \min\{\frac{1}{2},\frac{\rho-3}{2},\rho\nu-\frac{5-\rho}{2}\}$, where $C > 0$ depends on ν and δ.

In the following we shall prove this proposition in case $t > 0$. A stronger result is already proved for $t < 0$ by Proposition 1.8. In fact, if we change $\tau \to -\tau$ and $t \to -t$ in (2.2), we have

$$L(w)(x,t) = L_1(\hat{w})(x,-t), \text{ where } \hat{w}(x,t) = w(x,-t).$$

By definition

$$(2.5)\qquad |L(|u|^\rho)(x,t)| \le \frac{4M^\rho}{\pi}\{J_1+J_2+J_3\}+|L_0(|u|^\rho)(x,t)|;$$

$$(2.6)\quad J_1 \equiv \int_{-\infty}^{0} d\tau \int_{|t-\tau-r|}^{t-\tau+r} \lambda(1+\lambda+|\tau|)^{-\rho/2}(1+|\lambda-|\tau||)^{-\rho\nu} d\lambda$$

$$\times \int_{|\lambda-r|}^{t-\tau} \eta\{(t-\tau)^2-\eta^2\}^{-1/2} h(\lambda,\eta,r)d\eta,$$

$$(2.7)\quad J_2+J_3 \equiv \int_{-\infty}^{\min\{0,t-r\}} d\tau \int_0^{t-\tau-r} \lambda(1+\lambda+|\tau|)^{-\rho/2}(1+|\lambda-|\tau||)^{-\rho\nu} d\lambda$$

$$\times \int_{|\lambda-r|}^{\lambda+r} \eta\{(t-\tau)^2-\eta^2\}^{-1/2} h(\lambda,\eta,r)d\eta,$$

where $r = |x|$ and $h(\lambda,\eta,r)$ is as given in (1.6). Dividing the integral of (2.7), we define J_2 and J_3 as follows:

$$(2.8)\qquad J_j = \iint_{D_j} \cdots\, d\lambda d\tau \int_{|\lambda-r|}^{\lambda+r} \cdots\, d\eta \quad (j=2,3),$$

where

$$D_2 = \{-t+r<\tau<0,\, 0<\lambda<t-r+\tau\} \text{ and}$$

$$D_3 = \{-\infty<\tau<0,\, \max\{0,t-r+\tau\}<\lambda<t-r-\tau\}.$$

As for the term $|L_0(|u|^\rho)(x,t)|$, a corresponding estimate to (2.4) is already proved in Proposition 1.4. In order to obtain similar estimates for J_1 and J_2+J_3, we shall follow the line of proof of Proposition 1.4. In the present case, however, it is necessary to modify inequalities corresponding to (1.17). The main reason is in the fact that t and τ have different signs in both J_1 and J_2+J_3. Thus, we require other than Lemma 1.2 the following

Lemma 2.2. *Let $a < b < c$. Then*

$$(2.9)\qquad \int_a^b (b-\sigma)^{-1/2}(\sigma-a)^{-1/2}(c-\sigma)^{-1/2} d\sigma$$

$$\le \sqrt{2}\{\pi+\log(\frac{4(c-a)}{c-b})\}(b-a)^{-1/2}.$$

Proof. Note that

left of (2.9)

$$= [\int_a^{(a+b)/2} + \int_{(a+b)/2}^{b}](b-\sigma)^{-1/2}(\sigma-a)^{-1/2}(c-\sigma)^{-1/2}d\sigma$$

$$\le \sqrt{2}(b-a)^{-1/2}\{\int_a^c (\sigma-a)^{-1/2}(c-\sigma)^{-1/2}d\sigma + \int_a^b (b-\sigma)^{-1/2}(c-\sigma)^{-1/2}d\sigma\}.$$

Then since

$$\int_a^c (\sigma-a)^{-1/2}(c-\sigma)^{-1/2}d\sigma = \pi,$$

$$\int_a^b (b-\sigma)^{-1/2}(c-\sigma)^{-1/2}d\sigma = 2\{\log(\sqrt{b-a}+\sqrt{c-a})-\log\sqrt{c-b}\},$$

we have (2.9). □

Lemma 2.3. *We have*

$$J_2 \le \frac{\pi}{4} \int_0^{[t-r]} (1+\alpha)^{-\rho/2+1}(t+r+\alpha)^{-1/2}d\alpha \times\int_{-\alpha}^{\alpha} (1+|\beta|)^{-\rho\nu}(t-r-\beta)^{-1/2}d\beta, \tag{2.10}$$

$$J_3 \le \frac{\pi}{4} \int_{|t-r|}^{\infty} (1+\alpha)^{-\rho/2+1}(\alpha+t+r)^{-1/2}d\alpha \times\int_{-\alpha}^{t-r} (1+|\beta|)^{-\rho\nu}(t-r-\beta)^{-1/2}d\beta. \tag{2.11}$$

Proof. We use Lemma 1.2. Then it follows from (2.8) that

$$J_j \le \frac{\pi}{2} \iint_{D_j} \lambda(1+\lambda+|\tau|)^{-\rho/2}(1+|\lambda-|\tau||)^{-\rho\nu} \times(t-\tau-\lambda-r)^{-1/2}(t-\tau+\lambda+r)^{-1/2}d\lambda d\tau \quad (j=2,3).$$

Choosing the new variable (1.18) and noting $\tau < 0$, we obtain (2.10) and (2.11). □

Lewmma 2.4. *For any* $0 < \delta < \frac{\rho-3}{2}$, *we have*

$$J_1 \le C\int_{t-r}^{t+r} (1+|\beta|)^{-\rho\nu}(t+r-\beta)^{-1/2}\{1+(\beta-t+r)^{-\delta}\}d\beta \times\int_{\max\{r-t,\beta\}}^{\infty} (1+\alpha)^{-\rho/2+1+\delta}(\alpha-r+t)^{-1/2}d\alpha, \tag{2.12}$$

$$(2.13)\qquad J_3 \le Cr^{-1/2}\int_{|t-r|}^{\infty}(1+\alpha)^{-\rho/2+1/2+\delta}\,d\alpha$$
$$\times\int_{-\alpha}^{t-r}(1+|\beta|)^{-\rho\nu}\{1+(t-r-\beta)^{-\delta}\}\,d\beta.$$

Proof. We use Lemma 2.2. Then since

$$\log\frac{(\lambda+r)^2-(\lambda-r)^2}{(\lambda+r)^2-(t-\tau)^2} \le \log\frac{4\lambda}{\lambda+r-t+\tau} \le C+\lambda^{\delta}+(\lambda+r-t+\tau)^{-\delta}$$

for $\lambda+r > t-\tau$, (2.12) follows from (2.6). Next since

$$\log\frac{(t-\tau)^2-(\lambda-r)^2}{(t-\tau)^2-(\lambda+r)^2} \le \log\frac{2(\lambda-\tau)}{t-\tau-\lambda-r} \le C+(\lambda-\tau)^{\delta}+(t-\tau-\lambda-r)^{-\delta}$$

for $\lambda+r < t-\tau$ and $\lambda-\tau \ge |t-r|$, (2.13) follows from (2.8). □

Proof of Proposition 2.1. We choose δ as in Proposition 2.1. First we estimate J_1 using (2.12). Since $\frac{\rho}{2}-\delta > \frac{3}{2}$, we can use (1.9) with $k = \frac{\rho}{2}-\delta$, $a = r-t$, $b = |t-r|$ (or $b = \beta$) to obtain

$$J_1 \le C(1+|t-r|)^{-\rho/2+\delta+3/2}$$
$$\times\int_{t-r}^{|t-r|}(1+|\beta|)^{-\rho\nu}(t+r-\beta)^{-1/2}\{1+(\beta-t+r)^{-\delta}\}\,d\beta$$
$$+\ C\int_{|t-r|}^{t+r}(1+\beta)^{-\rho\nu-\rho/2+3/2+\delta}(t+r-\beta)^{-1/2}\{1+(\beta-t+r)^{-\delta}\}\,d\beta.$$

Note also that $\delta < \frac{1}{2}$. Then obviously $J_1 \le C$ if $t+r \le 3$. Thus, we have only to consider the case $t+r \ge 3$. If $t \ge 2r$,

$$J_1 \le C(1+t-r)^{-\rho\nu-\rho/2+3/2+\delta}\int_{t-r}^{t+r}(t+r-\beta)^{-1/2}\{1+(\beta-t+r)^{-\delta}\}\,d\beta$$
$$\le C(1+t-r)^{-\rho\nu-\rho/2+3/2+\delta}\{1+r^{1/2}\}.$$

If $r \le t \le 2r$,

$$J_1 \le Cr^{-1/2}\left\{\int_{t-r}^{t-r+1}+\int_{t-r+1}^{t+r}\right\}(1+\beta)^{-\rho\nu-\rho/2+3/2+\delta}\{1+(\beta-t+r)^{-\delta}\}\,d\beta$$
$$\le Cr^{-1/2}\Big\{(1+t-r)^{-\rho\nu-\rho/2+3/2+\delta}\int_{t-r}^{t-r+1}(\beta-t+r)^{-\delta}\,d\beta$$

$$\boxed{+C(1+t)^{-\rho\nu-\rho/2+3/2+\delta}\int_{t}^{t+r}(t+r-\beta)^{-1/2}\,d\beta}$$

$$\boxed{+\,Cr^{1/2}(1+t)^{-\rho\nu-\rho/2+3/2+\delta}}$$

$$+ \int_{t-r}^{t+r} (1+\beta)^{-\rho\nu-\rho/2+3/2+\delta} d\beta\Big\}$$

$$\le Cr^{-1/2}\{(1+t-r)^{-\rho\nu-\rho/2+3/2+\delta}+(1+t-r)^{-\rho\nu-\rho/2+5/2+\delta}\},$$

where we have used the fact that $-\rho\nu+ \frac{5-\rho}{2} +\delta < 0$. If $t \le r \le 2t$,

$$J_1 \le C(1+r-t)^{-\rho/2+3/2+\delta}$$

$$\times\Big\{r^{-1/2}\int_{t-r}^{t-r+1} (\beta-t+r)^{-\delta}d\beta + t^{-1/2}\int_{t-r}^{r-t} (1+|\beta|)^{-\rho\nu}d\beta\Big\}$$

$$+ Ct^{-1/2}\int_{r-t}^{t+r} (1+|\beta|)^{-\rho\nu-\rho/2+3/2+\delta}\{1+(\beta-t+r)^{-\delta}\}d\beta$$

$$\le C(1+r-t)^{-\rho/2+3/2+\delta}\{r^{-1/2}+t^{-1/2}(1+r-t)^{[-\rho\nu+1]}\}$$

$$+ Ct^{-1/2}\{(1+r-t)^{-\rho\nu-\rho/2+3/2+\delta}+(1+r-t)^{-\rho\nu-\rho/2+5/2+\delta}\}.$$

If $r \ge 2t$,

$$J_1 \le C(1+r-t)^{-\rho/2+3/2+\delta}\Big\{r^{-1/2}\int_{t-r}^{t-r+1} (\beta-t+r)^{-\delta}d\beta$$

$$+ \int_{t-r}^{r-t}(1+|\beta|)^{-\rho\nu}(t+r-\beta)^{-1/2}d\beta\Big\}$$

$$+ C(1+r-t)^{-\rho\nu-\rho/2+3/2+\delta}\int_{r-t}^{t+r} (t+r-\beta)^{-1/2}d\beta$$

$$\le C(1+r-t)^{-\rho/2+3/2+\delta}\{r^{-1/2}+(1+t+r)^{-1/2}(1+r-t)^{[-\rho\nu+1]}\}$$

$$+ C(1+r-t)^{-\rho\nu-\rho/2+3/2+\delta}(2t)^{1/2},$$

where we have used (1.19) with $\ell = \rho\nu$, $a = b = \cancel{t-r}\ r-t$, $c = t+r$ to obtain the last inequality. These inequalities show

(2.14) $$J_1 \le C(1+r+t)^{-1/2}(1+|t-r|)^{-\rho/2+3/2+[-\rho\nu+1]+\delta}.$$

Next, we estimate J_2 assuming $t > r$ in (2.10). Since $\alpha \le t-r$, with the help of (1.19) with $a = b = \alpha$, $c = t-r$,

(2.15) $$J_2 \le C(1+|t-r|)^{-1/2}\int_0^{t-r} (1+\alpha)^{-\rho/2+1+[-\rho\nu+1]}(\alpha+t+r)^{-1/2}d\alpha$$

$$\boxed{+\,Ct^{1/2}(1+r)^{-\rho\nu-\rho/2+3/2+\delta}}$$

$$\leq C(1+|t-r|)^{-1/2}(2t)^{-1/2}(1+|t-r|)^{[-\rho/2+2+[-\rho\nu+1]]}.$$

Finally we estimate J_3. If $t \geq 2r$ or $r+t \leq 3$, we use (2.11). By (1.19) with $a = c = r-t$, $b = \alpha$, and by (1.9) with $k = \frac{\rho}{2} -[-\rho\nu+1]$, $a = b = |t-r|$,

$$J_3 \leq C(1+|t-r|)^{-1/2}\int_{|t-r|}^{\infty} (1+\alpha)^{-\rho/2+1+[-\rho\nu+1]}(\alpha+t+r)^{-1/2}d\alpha$$

$$\leq C(1+|t-r|)^{-1/2}\boxed{\cancel{(1+|t-r|)^{-1/2}}}(1+|t-r|)^{-\rho/2+[-\rho\nu+1]+3/2}.$$

If $r \geq \frac{t}{2}$, we use (2.13). Then since

$$\int_{-\alpha}^{t-r} (1+|\beta|)^{-\rho\nu}\{1+(t-r-\beta)^{-\delta}\}d\beta \leq C\{1+(1+\alpha)^{[-\rho\nu+1]}\},$$

it follows that

$$J_3 \leq Cr^{-1/2}\int_{|t-r|}^{\infty} (1+\alpha)^{-\rho/2+1/2+\delta+[-\rho\nu+1]}d\alpha$$

$$\leq Cr^{-1/2}(1+|t-r|)^{-\rho/2+3/2+\delta+[-\rho\nu+1]}.$$

Summarizing these inequalities, we have

$$(2.16) \qquad J_3 \leq C(1+r+t)^{-1/2}(1+|t-r|)^{-\rho/2+[-\rho\nu+1]+3/2+\delta}.$$

Inequalities (2.14)~(2.16) and Proposition 1.4 prove (2.4). □

In the rest of this section, we choose k and ν to satisfy (1.26) and (1.27), respectively. And let δ be such that

$$(2.17) \qquad 0 < \delta < \min\{\tfrac{1}{2}, \tfrac{\rho-3}{2}, (\rho-1)\nu- \tfrac{5-\rho}{2}\}.$$

Define a Banach space

$$(2.18) \qquad Y = \{v \mid \nabla^j v(x,t) \in C(\mathrm{R}^2\times\mathrm{R}) \text{ and } \|\nabla_x^j v\|_W < \infty \text{ for } 0\leq j\leq 2\},$$

where

$$(2.19) \quad \|v\|_W = \sup_{(x,t)\in\mathrm{R}^2\times\mathrm{R}} [(1+|t|+|x|)^{1/2}(1+||t|-|x||)^{\nu}|v(x,t)|].$$

As for the free solution $u_0^-(x,t)$, the estimate of Proposition 1.1

holds for any $t \in \mathrm{R}$ if we replace t by $|t|$. Thus, $u_0^- \in Y$ and

(2.20) $$\|\nabla_x^j u_0^-\|_W \le C\varepsilon, \quad 0 \le j \le 2,$$

where $C > 0$ depends on k. On the other hand, Proposition 2.1 with δ satisfying (2.17) shows that

(2.21) $$\|L(f(u))\|_W \le C\|u\|_W^\rho$$

for any $u \in C(\mathrm{R}^2 \times \mathrm{R})$ such that $\|u\|_W < \infty$, where this $C > 0$ depends on ρ, k, ν, δ and Λ.

With these inequalities we can prove the following

Theorem 2.5. (1) *Let ρ, k, ν and δ be as above, and assume that (ψ^-, φ^-) satisfy* (1.3). *Then there exists an $\varepsilon_0 > 0$ depending on these parameters and Λ such that the integral equation* (2.1) *has a unique solution $u(x,t)$ in Y provided* $0 < \varepsilon \le \varepsilon_0$ *in* (1.3).

(2) *$u(t)$ is a classical global solution of the nonlinear wave equation* (0.1), *and we have*

(2.22) $$\|u(t)-u_0^-(t)\|_e$$
$$\le C\|u\|_W^\rho\{(1+|t|)^{-\rho+3}(1+|t|)^{[-2\rho\nu+1]}\}^{1/2} \to 0 \text{ as } t \to -\infty.$$

Furthermore, if we define

(2.23) $$u_0^+(x,t) = u(x,t) - L_1(f(u))(x,t),$$

then $u_0^+ \in Y$ satisfies the linear wave equation (0.2), *and we have*

(2.24) $$\|u(t)-u_0^+(t)\|_e$$
$$\le C\|u\|_W^\rho\{(1+t)^{-\rho+3}(1+t)^{[-2\rho\nu+1]}\}^{1/2} \to 0 \text{ as } t \to +\infty.$$

Proof. (1) With the two inequalities (2.20) and (2.21), a method of successive approximation is also applicable to show the existence of the unique solution $u(t)$ of (2.1) in Y for given small

$u_0^-(t)$ satisfying (2.20). See, e.g., Pecher [10].

(2) We note that

$$\|u(t)-u_0^-(t)\|_e \leq \int_{-\infty}^{t} \|f(u((\cdot,\tau))\|_{L^2} d\tau.$$

Then by means of (2.19), we can repeat the proof of Theorem 1.9 (2) to obtain (2.22). Moreover, (2.24) is exactly what we have proved in Theorem 1.9 (2). □

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Kôji Kubota: Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060, Japan

Kiyoshi Mochizuki: Department of Mathematics
Faculty of Science
Shinshu University
Matsumoto 390, Japan