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Determinacy of Envelope of the Osculating Hyperplanes to a Curve.

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0. Introduction

For a curve in a projective space, the envelope of the one-parameter family of osculating hyperplanes to the curve is a singular hypersurface in general. Using the notion of dual curve, Shcherbak [S1] investigates the singularity of the envelope via Legendre singularity theory [AGV]. In this paper we pursue Shcherbak's method and give C^∞ local normal forms of envelopes for general cases (Theorems 1 and 2) beyond the results obtained in [A1], [S1], [S2]. The results in this paper also generalize the C^∞ classifications of tangent developables of space curves in [Cl], [GP], [M1], [M2], at least set germ level.

First we recall definitions and known facts needed to formulate our results ([P],[S1]).

Consider a C^∞ curve $\gamma : M \longrightarrow \mathbf{R}P^{n+1}$, where M is a one-dimensional manifold and $n \geqq 1$. We call the germ γ_p at a point $p \in M$ of finite osculation-type (or simply, of finite type) $\mathcal{A} = (a_1, a_2, \dots, a_{n+1})$ if there exist a C^∞ coordinate t of (M, p) and an affine coordinate $(x_1, \dots, x_{n+1})$ of $\mathbf{R}P^{n+1}$ centered at $\gamma(p)$ such that γ is represented by

$$x_1 = t^{a_1} + o(t^{a_1}), \quad \dots, \quad x_{n+1} = t^{a_{n+1}} + o(t^{a_{n+1}}),$$

where each a_i is a natural number and $1 \leqq a_1 < \dots < a_{n+1}$.

A point $p \in M$ is called an ordinary point if γ_p is of type $(1, 2, \dots, n, n+1)$, and, otherwise, it is called a special point. Special points of finite type are isolated in M.

For each $p \in M$ where γ_p is of finite type and for each i, $(0 \leqq i \leqq n+1)$, there exists the most osculating linear subspace to γ at p in $T_{\gamma(p)}\mathbf{R}P^{n+1}$ of dimension i. We call it the osculating i-subspace and denote by $O_i(\gamma, p)$. This subspace is identified with $\{x_{i+1} = \dots = x_{n+1} = 0\}$ under the above affine representation of γ_p. The corresponding projective subspace of $\mathbf{R}P^{n+1}$ through $\gamma(p)$ of dimension i is also denoted by $O_i(\gamma, p)$. We call $O_n(\gamma; p)$ the osculating hyperplane.

We mean by the envelope of the curve germ γ_p the germ at $\gamma(p)$ of the envelope of the family of hyperplanes $O_n(\gamma, q)$, q running over sufficiently small neighborhood of p.

The type of a curve describes the order of tangency to each osculating subspace, and it is the simplest local projective invariant of the curve. An interesting phenomenon is that the type of a curve determine the "shape" of the envelope. Precisely we intend to classify envelopes of curve germs of a fixed type under the C^∞ equivalence: The envelopes of $\gamma_p, \gamma'_{p'}$ respectively are called C^∞ equivalent if there exists a C^∞ diffeomorphism germ $\mathbf{R}P^{n+1}, \gamma(p) \longrightarrow \mathbf{R}P^{n+1}, \gamma'(p')$, mapping the envelope of γ_p to that of $\gamma'_{p'}$.

We say that a type $\mathcal{A} = (a_1, a_2, \dots, a_{n+1})$ of a curve germ in $\mathbf{R}P^{n+1}$ determines the C^∞ class of envelope if the envelopes of any two curve germs of type $\mathcal{A}$ are C^∞ equivalent.

The purpose of this paper is to prove the following:

THEOREM 1. *Let $n \geqq 1$ and let A be one of the following types:*

$(I)_{n,r}$ $\mathsf{A} = (1, 2, \ldots, n, n+r)$, $\quad r = 1, 2, \ldots,$

$(II)_{n,i}$ $\mathsf{A} = (1, 2, \ldots, i, i+2, \ldots, n+1, n+2)$, $\quad 0 \leqq i \leqq n-1,$

$(III)_n$ $\mathsf{A} = (3, 4, \ldots, n+2, n+3)$.

Then A determines the C^∞ class of the envelope of osculating hyperplanes. In particular, in this case, the envelope is C^∞ diffeomorphic to the germ of envelope of the family

$$t^{a_{n+1}} + x_1 t^{a_{n+1}-a_1} + \cdots + x_n t^{a_{n+1}-a_n} + x_{n+1} = 0,$$

of affine hyperplanes of $\mathbb{R}^{n+1}$ with the parameter t at 0.

Theorem 1 generalize the results on envelopes of curves of type $(\mathrm{I})_{n,2}$, [S2], $(\mathrm{II})_{n,0}$, [A1], [S1]. Remark that Shcherbak's proof in [S2] depends on the classification of projections of wave front sets [G], and it differs from the proof in this paper.

In the case $a_{n+1} - a_n = 1$, as we see below, the envelope of γ_p coincides with the developable of γ_p, that is, the germ at $\gamma(p)$ of hypersurface "ruled" by osculating $(n-1)$-subspaces to γ ([S1], [S2]). In the case $a_{n+1} - a_n > 1$, however, the envelope of γ_p is equal to the sum of the developable and $O_n(\gamma, p)$ itself. Therefore it is natural also to classify envelopes by diffeomorphisms preserving $O_n(\gamma, p)$. By the C^∞ class of full envelope, we mean the class of envelope under this equivalence.

In this paper we also prove the following:

THEOREM 2. *If A is one of types $(I)_{n,r}$, $(II)_{n,i}$ in Theorem 1, then A determines the C^∞ class of full envelope.*

Next we clarify the objects to study in this paper.

Assume γ_p is of finite type at each point $p \in M$. Denote the dual projective space of $\mathbb{R}P^{n+1}$ by $\mathbb{R}P^{n+1*}$, which is the space of all hyperplanes in $\mathbb{R}P^{n+1}$. Then we naturally identify $\mathbb{R}P^{n+1**}$ with $\mathbb{R}P^{n+1}$. We define the dual $\gamma^* : M \longrightarrow \mathbb{R}P^{n+1*}$ of γ by $p \mapsto O_n(\gamma, p)$, and describe the developable of γ by the dual curve γ^* in $\mathbb{R}P^{n+1*}$.

Then it is known the followings ([S1]):

(0) γ^* is a C^∞ map.

(1) Let $p \in M$. If γ_p is of type $\mathsf{A} = (a_1, \ldots, a_{n+1})$, then γ_p^* is also of finite type $\mathsf{A}^* = (a_{n+1} - a_n, a_{n+1} - a_{n-1}, \ldots, a_{n+1} - a_1, a_{n+1})$.

(2) Let $p \in M$. Then $O_i(\gamma^*, p) = O_{n-i}(\gamma, p)^*$, the dual of $O_{n-i}(\gamma, p)$, $0 \leqq i \leqq n$.

(3) $\gamma^{**} = \gamma$.

Set $\mathsf{A}^* = \mathsf{B} = (b_1, \ldots, b_{n+1})$ and $a_0 = 0$. Then $b_i = a_{n+1} - a_{n+1-j}, 1 \leqq i \leqq n+1$.

Take an affine representative of $\gamma_p^* : y_i = y_i(t)$ with the order of $y_i(t) = b_i$. We define the affine coordinate $x = (x_1, \ldots, x_{n+1})$ of $\mathbb{R}P^{n+1}$ at $\gamma(p)$ correspondingly such that the projective duality is described by $\sum_{j=0}^{n+1} x_j y_{n+1-j} = 0$, with $x_0 = y_0 = 1$. Set

$$F(x, t) = y_{n+1}(t) + x_1 y_n(t) + \cdots + x_n y_1(t) + x_{n+1} = \sum_{j=0}^{n+1} x_j y_{n+1-j}(t).$$

The one-parameter family of osculating hyperplanes of γ near p is defined by $F = 0$. The developable of γ_p is obtained when we solve the system of equations $F = \partial F/\partial t = 0$ first for $t \neq 0$, and then extend to $t = 0$. Thus we have a natural parametrization $f : \mathbb{R}^n, 0 \longrightarrow \mathbb{R}P^{n+1}, \gamma(p)$ of the form $f(x', t) = (x', x_n(x', t), x_{n+1}(x', t))$ where $x' = (x_1, \ldots, x_{n-1})$.

Simply solving the system of equations $F = \partial F/\partial t = 0$ we get the envelope of γ_p. If $b_1 = a_{n+1} - a_n > 1$, then the envelope contains the component $\{x_{n+1} = 0\} = O_n(\gamma, p)$ in addition to the developable.

According to the framework of the Legendre singularity theory [AGV], we consider the diagram

$$\{F = 0\} \overset{j}{\hookrightarrow} \mathbb{R}^{n+1} \times \mathbb{R}, 0 \xrightarrow{\pi} \mathbb{R}^{n+1}, 0,$$

where j is the inclusion and π is the projection to the first component. We call this diagram or simply F itself the generating family of γ^*. The envelope is regarded as the set of critical values of $\pi \circ j$, which can be seen the wave front sets of (singular) Legendre varieties [A2]. The determinacy of generating families under the $\mathcal{K}$-equivalence [AGV] relative to an osculation type leads that of envelopes.

We achieve to trivialize the type preserving deformations of generating families for the cases $(\mathrm{I})_{n,r}$, $(\mathrm{II})_{n,i}$, $(\mathrm{III})_n$, in the following section, and prove Theorems 1 and 2.

We notice that the generating family F is a Morse family, that is, $\{F = \partial F/\partial t = 0\}$ is non-singular in $\mathbb{R}^{n+1} \times \mathbb{R}, (0, 0)$ if and only if $b_1 = 1$. From the general results on the classifications of wave front sets [AGV], if $b_1 = 1$, then the classifications of envelopes under the C^∞ equivalence, them of parametrizations of developables under the C^∞ right left equivalence and them of generating families under the $\mathcal{K}$-equivalence coincide. Hence, by Theorem 1, we have

COROLLARY 1. *If A is one of types $(II)_{n,i}$ and $(III)_n$ in Theorem 1, then A determines the C^∞ class of the parametrization of developable.*

COROLLARY 2. *If A is one of types $(II)_{n,i}$ in Theorem 1, then A determines the C^∞ class of full developable.*

Here we mean by C^∞ class of full developable the class of developable under the equivalence preserving $O_n(\gamma, p)$. Corollary 2 generalizes Mond's result [M2] on space curves of type $(1, 3, 4)$.

It seems not so clear that merely the determinacy of generating families relative to osculation type implies that of parametrizations of developables, in the case $b_1 > 1$, for instance, in the case $(\mathrm{I})_{n,r}$, $r > 1$. In [I], we analyze directly the parametrizations as in [M1], [M2], and establish the characterization of types determining C^∞ class of developables.

For the general theory of envelope, see [T], [Bru], [Ca]. Discussions on not necessarily type preserving deformations can be seen in [M2], [K]. We also remark that the family of osculating hyperplanes defines "web structure" on the complement of envelope. For this subject, see [N].

In the next section, we fix the weight of each variable as follows: $w(x_j) = a_j, 1 \leqq j \leqq n+1$ and $w(t) = 1$. Then the function-germ F is weighted-semi-homogeneous, the initial part of which is weighted-homogeneous of weight a_{n+1}. This fact simplifies the subsequent discussions.

We denote by E_x the R-algebra of function-germs of variable x at a point fixed in a context and denote by m_x the unique maximal ideal of E_x. For instance, in the next section, $E_{x,t,\lambda}$ denotes the totality of function-germs on the (x,t,λ)-space $\mathbb{R}^{n+1}\times\mathbb{R}\times\mathbb{R}$ at $(0,0,\lambda_0)$.

We set $e(\ell)=t^\ell/\ell!$.

Hereafter all mappings, diffeomorphisms and vector fields are assumed of class C^∞.

1. Determinacy of a generating family

Let γ be a curve-germ of type A and γ^* the dual to γ of type $\mathrm{A}^*=\mathrm{B}$. Then there exists an affine representation of γ^*: $y_j=e(b_j)+v_j(t), 1\leqq j\leqq n+1$, with $v_j\in m_t^{b_j+1}$.

Consider an initial part preserving deformation

$$F_\lambda=\sum_{j=0}^{n+1}x_j(e(b_{n+1-j})+\lambda v_{n+1-j}),$$

$\lambda\in\mathbb{R}$, of the generating family F of γ^*, where $b_0=0$ and $v_0=0$.

For each λ_0, we seek to solve the equation for A_λ and Φ_λ :

$$F_{\lambda_0}=A_\lambda\cdot F_\lambda\circ\Phi_\lambda,$$

where A_λ (resp. Φ_λ) is a family of function-germs (resp. a family of π fiber preserving diffeomorphism germs) of $\mathbb{R}^{n+1}\times\mathbb{R}$ at the origin with $A_{\lambda_0}=1$ and Φ_{λ_0} the identity. Further we examine if Φ_λ preserves $\{t=0\}$ in the context of Theorem 2.

Now we introduce, as tangent spaces in our problem, an $E_{x,\lambda}$-module T (resp. T') consisting of function-germs in $E_{x,t,\lambda}$ of the form $h_\lambda\cdot F_\lambda+X_\lambda F_\lambda$, where $h_\lambda=h_\lambda(x,t)$ and X_λ is a π fiber preserving vector field germ over $\mathbb{R}^{n+1}\times\mathbb{R},(0,0)$ with $X_\lambda(0,0)=0$ (resp. and $X_\lambda t\subset tE_{x,t,\lambda}$).

As the corresponding infinitesimal condition, we need to check that $\partial F_\lambda/\partial\lambda\in T$ or T', for any $\lambda_0\in\mathbb{R}$. Remark that the spaces T and T' depend on $\lambda_0\in\mathbb{R}$.

The space of variations in our problem is

$$V=\sum_{j=0}^{n+1}x_jm_t^{b_{n+1-j}+1}=\sum_{j=0}^{n+1}x_jm_t^{a_{n+1}-a_j+1}\subset m_t^{b_1+1}E_{x,t},$$

where we set $a_0=0$. In fact we see $\partial F_\lambda/\partial\lambda=\sum_{j=0}^{n+1}x_jv_{n+1-j}\in V$. Thus we have

Proposition 1.1.

(1) *If $V\subset T$, for any $\lambda_0\in\mathbb{R}$, then the type A determines the C^∞ class of envelope.*

(2) *If $V \subset T'$, for any $\lambda_0 \in \mathsf{R}$, then the type A determines the C^∞ class of full envelope.*

We now examine the condition $V \subset T$ or $V \subset T'$. We set $V_m = \{h \in V | \text{ord } h \geqq a_{n+1} + m\}$, $m \geqq 1$, and $V_0 = \sum_{j=0}^{n+1} x_j m_t^{a_{n+1}-a_j}$. Then we have the order filtration $V_0 \supset V = V_1 \supset V_2 \supset \cdots$.

Remark that multiplying t maps V_m to V_{m+1} isomorphically for $m \geqq 0$.

We need the following in the proof of Proposition 1.3 below:

LEMMA 1.2. (1) *For any A,*

$$F_\lambda - F_0 = \lambda \sum_{j=0}^{n+1} x_j v_{n+1-j} \in E_{x,\lambda} V,$$

$$F_\lambda - e(a_{n+1}) = \lambda v_{n+1} + \sum_{j=1}^{n+1} x_j (\partial F_\lambda / \partial x_j) \in E_{x,\lambda} V + T',$$

$$t \partial F_\lambda / \partial t - t \partial F_0 / \partial t = \lambda \sum_{j=0}^{n+1} x_j t (d v_{n+1-j}/dt) \in E_{x,\lambda} V,$$

and

$$x_\ell \partial F_\lambda / \partial x_j - x_\ell e(b_{n+1-j}) = \lambda x_\ell v_{n+1-j} \in E_{x,\lambda} V_{a_\ell - a_j + 1}, \quad \textit{if } \ell \geqq j.$$

(2) *If $\mathsf{A}^* = \mathsf{B} = (1, 2, \ldots, n, n+r)$, then $\partial F_\lambda / \partial t - e(n+r-1) \in T' + E_{x,\lambda} V$.*

PROOF: (1) is clear.

To see (2), we remark that $F_\lambda = e(n+r) + \sum_{j=1}^{n+1} x_j (e(n+1-j) + \lambda v_{n+1-j})$. Then $x_j \partial F_\lambda / \partial x_{j+1} = x_j e(n-j) + \lambda x_j v_{n-j}$, $0 \leqq j \leqq n$. Therefore,

$$\begin{aligned} \partial F_\lambda / \partial t - e(n+r-1) &= \sum_{j=1}^{n} x_j (e(n-j) + \lambda d v_{n+1-j}/dt) \\ &\equiv \lambda \sum_{j=1}^{n} x_j ((d v_{n+1-j}/dt) - v_{n-j}) \quad \text{mod. } T'. \end{aligned}$$

Write $(d v_{n+1-j}/dt) - v_{n-j} = c_j e(n+1-j) + w_{n+1-j}$, with $w_{n+1-j} \in m_t^{n+2-j}, c_j \in \mathsf{R}, 1 \leqq j \leqq n$. Then we see

$$\partial F_\lambda / \partial t - e(n+r-1) \equiv \lambda \sum_{j=1}^{n} (c_j x_j \partial F_\lambda / \partial x_j - \lambda c_j x_j v_{n+1-j} + x_j w_{n+1-j}) \equiv 0,$$

modulo $T' + E_{x,\lambda} V$.

Q.E.D.

We define an $E_{x,\lambda}$-module Γ (resp. Γ') by the quotient $\Gamma = (E_{x,\lambda} V + T)/T$ (resp. $\Gamma' = (E_{x,\lambda} V + T')/T'$), and an R-module Λ (resp. Λ') by $\Lambda = \Gamma / m_{x,\lambda} \Gamma$ (resp. $\Lambda' = \Gamma' / m_{x,\lambda} \Gamma'$). Let $\rho : V \to \Lambda$ (resp. $\rho' : V \to \Lambda'$) denotes the canonical projection. Set $\Lambda_j = \mathsf{R} \cdot \rho(V_j) \subset \Lambda$ (resp. $\Lambda'_j = \mathsf{R} \cdot \rho(V'_j) \subset \Lambda'$). Then the following is the key to show the determinacy relative to a type:

PROPOSITION 1.3. *The $E_{x,\lambda}$-module Γ (resp. Γ') has a finite system of generators $H \subset V$ satisfying that*

(1) *any element $h \in H$ is of finite order, and*

(2) *$\rho(H \cap V_m)$ (resp. $\rho'(H \cap V_m)$) generates Λ_m (resp. Λ'_m) over $\mathbb{R}$, for $m \geqq 1$.*

Further, if A is one of types $(I)_{n,r}$, $(II)_{n,i}$ and $(III)_n$ in Theorem 1, then H can be chosen to satisfy also

(3) *$\rho(H \cap V_m) \subset \Lambda_{m+1}$, for $m \geqq 1$.*

If A is one of types $(I)_{n,r}$ and $(II)_{n,i}$, then H can be chosen to satisfy also

(4) *$\rho'(H \cap V_m) \subset \Lambda'_{m+1}$, for $m \geqq 1$.*

PROOF: Define functions $Y_0 = 1, Y_1, \ldots, Y_{a_{n+1}-1}$ by

$$Y_{b_j} = \partial F_\lambda / \partial x_j = e(n+1-j) + \lambda v_{n+1-j}, \quad 1 \leqq j \leqq n+1$$

and otherwise $Y_k = e(k)$. Then, by the Malgrange-Mather division theorem [Brö], they generate the $E_{x,\lambda}$-module $E_{x,t,\lambda}/(t\partial F_\lambda/\partial t)E_{x,t,\lambda}$. Therefore, each $h \in V$ can be written as

$$h = Q(x,t,\lambda) \cdot t\frac{\partial F_\lambda}{\partial t} + \sum_{j=0}^{a_{n+1}-1} R_j(x,\lambda) Y_j, \qquad \ldots (*),$$

where $Q \in E_{x,t,\lambda}, R_j \in E_{x,\lambda}, 1 \leqq j \leqq a_{n+1} - 1$.

Since $h \in m_t^{b_1+1} E_{x,t,\lambda}$ and $t\partial F_\lambda/\partial t \in m_t^{b_1} E_{x,t,\lambda}$, we see $R_j = 0$ for $j < b_1$, comparing the order of each term with respect to t. In general we have

$$R_k \in \langle x_n, x_{n-1}, \ldots, x_{n+1-j} \rangle E_{x,\lambda}, \quad \text{for } k < b_{j+1}.$$

To see this we define an $E_{x,\lambda}$-module homomorphism $\Phi_k : E_{x,t,\lambda} \longrightarrow E_{x,\lambda}$ by $\Phi_k(\rho) = \partial^k \rho / \partial t^k |_{t=0}$. Then $\Phi_k(h), \Phi_k(t\partial F_\lambda/\partial t)$ and $\Phi_k(Y_j)$, $j < k$, are in $\sum_{\ell=n+1-j}^{n} x_\ell E_{x,\lambda}$, $\Phi_k(Y_j) = 0, j > k$, and $\Phi_k(Y_k) = 1$. Thus $\Phi_k(R_k) = R_k$ is also in $\sum_{\ell=n+1-j}^{n} x_\ell E_{x,\lambda}$.

Since $Qt\partial F_\lambda/\partial t$ and $R_{b_j} \cdot \partial F_\lambda/\partial x_j, 1 \leqq j \leqq n+1$ belong to $T' \subset T$ we can take

$$H = \{x_\ell e(k) |\ b_j < k < b_{j+1},\ n+1-j \leqq \ell \leqq n, 1 \leqq j \leqq n\},$$

as a finite system of generators of Γ and Γ'. Remark that H consists of monomials and satisfies (1).

To see (2), we remark that $\mathrm{ord} R_j \geqq \mathrm{ord} h - \mathrm{ord} Y_j$, $b_1 \leqq j \leqq a_{n+1} - 1$, which follows from the formal uniqueness of the expression (*).

It remains to see (3) and (4). Remark that the number of elements of H is equal to $\sum_{j=1}^{n} (a_j - j)$.

If A is of type $(\mathrm{I})_{n,r}$, then $H = \emptyset$. Thus (3) and (4) are clear.

Let A be of type $(\mathrm{II})_{n,i}$. Then B is of type $(\mathrm{II})_{n,n-i}$ and therefore

$$H = \{x_\ell e(n-i+1) | i+1 \leqq \ell \leqq n\}.$$

Let A be of type $(\mathrm{III})_n$. Then

$$H = \{x_1 e(n+1), \ldots, x_n e(n+1), x_1 e(n+2), \ldots, x_n e(n+2)\}.$$

We denote by $V^{(m)}$ the R-vector space of weighted homogeneous polynomials in V of weight $a_{n+1} + m$. We see $V^{(m)} \cong V_m/V_{m+1}$ and $\dim V^{(m)} = n+2$, $m \geqq 1$.

Then the following simple facts are essential:

LEMMA 1.4. *(i) If A is $(II)_{n,i}$, then the system of elements in $V^{(m)}$*

$$S : t^m F_0, t^{m+1}\partial F_0/\partial t, x_1 e(n+m+1), \ldots, x_i e(n+m+2-i),$$

$$x_{i+1} e(n+m-i), \ldots, x_{i+m-1} e(n-i+2), x_{i+m+1} e(n-i), \ldots, x_{n+1} e(m),$$

forms a basis of the vector space $V^{(m)}$, $1 \leqq m \leqq n-i$.

(ii) If A is $(III)_n$, then the system of elements in $V^{(m)}$

$$S : t^m F_0, t^{m+1}\partial F_0/\partial t, x_1 e(n+m), \ldots, x_{m-1} e(n+2), x_{m+1} e(n), \ldots, x_{n+1} e(m)$$

forms a basis of the vector space $V^{(m)}$, $1 \leqq m \leqq n+1$.

Now let $h \in V^{(m)}$. Then, by Lemma 1.4, h is a linear combination of elements of S over R.

If each element of S belongs $E_{x,\lambda}V_{m+1} + m_{x,\lambda}V + T$ (resp. $E_{x,\lambda}V_{m+1} + m_{x,\lambda}V + T'$), then so is h. This means that $\rho(h) \in \Lambda_{m+1}$ (resp. $\rho'(h) \in \Lambda'_{m+1}$).

Let A be of type $(\mathrm{II})_{n,i}$. Then

$$t^m F_0 = t^m F_\lambda - t^m(F_\lambda - F_0) \subset T' + E_{x,\lambda}V_{m+1},$$

and

$$t^{m+1}\partial F_0/\partial t = t^m \cdot t\partial F_\lambda/\partial t - t^m(t\partial F_\lambda/\partial t - t\partial F_0/\partial t) \subset T' + E_{x,\lambda}V_{m+1}.$$

Further, consider an element of type $x_k e(\ell)$ in S. Then $a_k + \ell = a_{n+1} + m$. If $\ell > n+2 = a_{n+1}$, then $e(\ell) \in V$ and thus $x_k e(\ell) \in m_{x,\lambda}V$. If $\ell = n+2$, then $x_k e(n+2) = x_k F_\lambda - x_k(F_\lambda - e(n+2))$, which belongs to $T' + m_{x,\lambda}V$. If $\ell < n+2$, then $\ell \neq n-i+1$ and $e(\ell) = \partial F_0/\partial x_\alpha$ for some $\alpha \leqq k$. Then $\ell = b_{n+1-\alpha}$, therefore, $a_k - a_\alpha = m$. Therefore, $x_k e(\ell) = x_k \partial F_\lambda/\partial x_\alpha - x_k(\partial F_\lambda/\partial x_\alpha - \partial F_0/\partial x_\alpha) \in E_{x,\lambda}V_{m+1} + m_{x,\lambda}V + T'$.

From this we see $\rho'(H \cap V_m) \subset \Lambda'_{m+1}$.

Let A be of type $(\mathrm{III})_n$. Then we have $t^m F_0$ and $t^{m+1}\partial F_0/\partial t$ belong to $T' + E_{x,\lambda}V_{m+1}$ similarly as above. If $\ell > n+3 = a_{n+1}$, then $e(\ell) \in V$ and $x_k e(\ell) \in m_{x,\lambda}V$. If $\ell = n+3$, then $x_k e(n+3) = x_k F_\lambda - x_k(F_\lambda - e(n+3)) \in T' + m_{x,\lambda}V$. If $\ell = n+2$, then $x_k e(n+2) = x_k \partial F_\lambda/\partial t - x_k(\partial F_\lambda/\partial t - e(n+2))$. The first term belongs to T, and the second term belongs to $T' + m_{x,\lambda}V$ by Lemma 1.2.(2). If $\ell \leqq n$, then $e(\ell) = \partial F_0/\partial x_\alpha$ for some $\alpha \leqq k$. Thus $x_k e(\ell) \in E_{x,\lambda}V_{m+1} + m_{x,\lambda}V + T$.

From this we know that $\rho'(H \cap V_m) \subset \Lambda_{m+1}$.

Thus Proposition 1.3 is proved.

PROOF OF LEMMA 1.4: Set $P = e(n+2) + x_{i+m}e(n+1-i-m)$ (resp. $P = e(n+3) + x_m e(n+1)$), which is equal to F_0 restricted to $\{x_1 = \cdots = x_{i+m-1} = x_{i+m+1} = \cdots = x_{n+1} = 0\}$ (resp. $\{x_1 = \cdots = x_{m-1} = x_{m+1} = \cdots = x_{n+1} = 0\}$). Then Lemma follows from the simple fact that $t^m P$ and $t^{m+1}\partial P/\partial t$ are linearly independent over $\mathbf{R}$.

PROOF OF THEOREM 1: By (1) and (2) of Proposition 1.3, we see $\Lambda_j = 0$, for sufficiently large j. By (3), we have $\Lambda_1 = \Lambda_2 = \cdots$, therefore, $\Lambda = 0$. Hence $\Gamma = m_{x,\lambda}\Gamma$. Since Γ is finitely generated, Nakayama's lemma implies that $\Gamma = 0$. This means $V \subset T$. Thus Proposition 1.1.(1) implies Theorem 1. The normal forms are obtained when we choose γ_p and affine coordinates such that γ_p^* is represented by $y_1 = t^{a_{n+1}-a_n}, \cdots, y_{n+1} = t^{a_{n+1}}$.

PROOF OF THEOREM 2: Similarly as above, by (1), (2) and (4) of Proposition 1.3, we have $\Gamma' = 0$ and therefore $V \subset T'$. Then, by Proposition 2.1.(2), we have Theorem 2.

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