



Title	Determinacy of envelope of the osculating hyperplanes to a curve
Author(s)	Ishikawa, Goo
Citation	Hokkaido University Preprint Series in Mathematics, 150, 2-9
Issue Date	1992-05
DOI	10.14943/83294
Doc URL	http://hdl.handle.net/2115/68896
Type	bulletin (article)
File Information	pre150.pdf



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Series #150. May 1992

HOKKAIDO UNIVERSITY
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Determinacy of Envelope of the Osculating Hyperplanes to a Curve.

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0. Introduction

For a curve in a projective space, the envelope of the one-parameter family of osculating hyperplanes to the curve is a singular hypersurface in general. Using the notion of dual curve, Shcherbak [S1] investigates the singularity of the envelope via Legendre singularity theory [AGV]. In this paper we pursue Shcherbak's method and give C^∞ local normal forms of envelopes for general cases (Theorems 1 and 2) beyond the results obtained in [A1], [S1], [S2]. The results in this paper also generalize the C^∞ classifications of tangent developables of space curves in [Cl], [GP], [M1], [M2], at least set germ level.

First we recall definitions and known facts needed to formulate our results ([P],[S1]).

Consider a C^∞ curve $\gamma : M \rightarrow \mathbb{R}P^{n+1}$, where M is a one-dimensional manifold and $n \geq 1$. We call the germ γ_p at a point $p \in M$ of finite osculation-type (or simply, of finite type) $A = (a_1, a_2, \dots, a_{n+1})$ if there exist a C^∞ coordinate t of (M, p) and an affine coordinate (x_1, \dots, x_{n+1}) of $\mathbb{R}P^{n+1}$ centered at $\gamma(p)$ such that γ is represented by

$$x_1 = t^{a_1} + o(t^{a_1}), \quad \dots, \quad x_{n+1} = t^{a_{n+1}} + o(t^{a_{n+1}}),$$

where each a_i is a natural number and $1 \leq a_1 < \dots < a_{n+1}$.

A point $p \in M$ is called an ordinary point if γ_p is of type $(1, 2, \dots, n, n+1)$, and, otherwise, it is called a special point. Special points of finite type are isolated in M .

For each $p \in M$ where γ_p is of finite type and for each i , ($0 \leq i \leq n+1$), there exists the most osculating linear subspace to γ at p in $T_{\gamma(p)}\mathbb{R}P^{n+1}$ of dimension i . We call it the osculating i -subspace and denote by $O_i(\gamma, p)$. This subspace is identified with $\{x_{i+1} = \dots = x_{n+1} = 0\}$ under the above affine representation of γ_p . The corresponding projective subspace of $\mathbb{R}P^{n+1}$ through $\gamma(p)$ of dimension i is also denoted by $O_i(\gamma, p)$. We call $O_n(\gamma, p)$ the osculating hyperplane.

We mean by the envelope of the curve germ γ_p the germ at $\gamma(p)$ of the envelope of the family of hyperplanes $O_n(\gamma, q)$, q running over sufficiently small neighborhood of p .

The type of a curve describes the order of tangency to each osculating subspace, and it is the simplest local projective invariant of the curve. An interesting phenomenon is that the type of a curve determine the "shape" of the envelope. Precisely we intend to classify envelopes of curve germs of a fixed type under the C^∞ equivalence: The envelopes of γ_p, γ'_p respectively are called C^∞ equivalent if there exists a C^∞ diffeomorphism germ $\mathbb{R}P^{n+1}, \gamma(p) \rightarrow \mathbb{R}P^{n+1}, \gamma'(p')$, mapping the envelope of γ_p to that of γ'_p .

We say that a type $A = (a_1, a_2, \dots, a_{n+1})$ of a curve germ in $\mathbb{R}P^{n+1}$ determines the C^∞ class of envelope if the envelopes of any two curve germs of type A are C^∞ equivalent.

The purpose of this paper is to prove the following:

THEOREM 1. Let $n \geq 1$ and let A be one of the following types:

$$(I)_{n,r} \quad A = (1, 2, \dots, n, n+r), \quad r = 1, 2, \dots,$$

$$(II)_{n,i} \quad A = (1, 2, \dots, i, i+2, \dots, n+1, n+2), \quad 0 \leq i \leq n-1,$$

$$(III)_n \quad A = (3, 4, \dots, n+2, n+3).$$

Then A determines the C^∞ class of the envelope of osculating hyperplanes. In particular, in this case, the envelope is C^∞ diffeomorphic to the germ of envelope of the family

$$t^{a_{n+1}} + x_1 t^{a_{n+1}-a_1} + \dots + x_n t^{a_{n+1}-a_n} + x_{n+1} = 0,$$

of affine hyperplanes of \mathbb{R}^{n+1} with the parameter t at 0.

Theorem 1 generalize the results on envelopes of curves of type $(I)_{n,2}$, [S2], $(II)_{n,0}$, [A1], [S1]. Remark that Shcherbak's proof in [S2] depends on the classification of projections of wave front sets [G], and it differs from the proof in this paper.

In the case $a_{n+1} - a_n = 1$, as we see below, the envelope of γ_p coincides with the developable of γ_p , that is, the germ at $\gamma(p)$ of hypersurface "ruled" by osculating $(n-1)$ -subspaces to γ ([S1], [S2]). In the case $a_{n+1} - a_n > 1$, however, the envelope of γ_p is equal to the sum of the developable and $O_n(\gamma, p)$ itself. Therefore it is natural also to classify envelopes by diffeomorphisms preserving $O_n(\gamma, p)$. By the C^∞ class of full envelope, we mean the class of envelope under this equivalence.

In this paper we also prove the following:

THEOREM 2. If A is one of types $(I)_{n,r}$, $(II)_{n,i}$ in Theorem 1, then A determines the C^∞ class of full envelope.

Next we clarify the objects to study in this paper.

Assume γ_p is of finite type at each point $p \in M$. Denote the dual projective space of $\mathbb{R}P^{n+1}$ by $\mathbb{R}P^{n+1*}$, which is the space of all hyperplanes in $\mathbb{R}P^{n+1}$. Then we naturally identify $\mathbb{R}P^{n+1**}$ with $\mathbb{R}P^{n+1}$. We define the dual $\gamma^* : M \rightarrow \mathbb{R}P^{n+1*}$ of γ by $p \mapsto O_n(\gamma, p)$, and describe the developable of γ by the dual curve γ^* in $\mathbb{R}P^{n+1*}$.

Then it is known the followings ([S1]):

(0) γ^* is a C^∞ map.

(1) Let $p \in M$. If γ_p is of type $A = (a_1, \dots, a_{n+1})$, then γ_p^* is also of finite type $A^* = (a_{n+1} - a_n, a_{n+1} - a_{n-1}, \dots, a_{n+1} - a_1, a_{n+1})$.

(2) Let $p \in M$. Then $O_i(\gamma^*, p) = O_{n-i}(\gamma, p)^*$, the dual of $O_{n-i}(\gamma, p)$, $0 \leq i \leq n$.

(3) $\gamma^{**} = \gamma$.

Set $A^* = B = (b_1, \dots, b_{n+1})$ and $a_0 = 0$. Then $b_i = a_{n+1} - a_{n+1-j}$, $1 \leq i \leq n+1$.

Take an affine representative of $\gamma_p^* : y_i = y_i(t)$ with the order of $y_i(t) = b_i$. We define the affine coordinate $x = (x_1, \dots, x_{n+1})$ of $\mathbb{R}P^{n+1}$ at $\gamma(p)$ correspondingly such that the projective duality is described by $\sum_{j=0}^{n+1} x_j y_{n+1-j} = 0$, with $x_0 = y_0 = 1$. Set

$$F(x, t) = y_{n+1}(t) + x_1 y_n(t) + \dots + x_n y_1(t) + x_{n+1} = \sum_{j=0}^{n+1} x_j y_{n+1-j}(t).$$

The one-parameter family of osculating hyperplanes of γ near p is defined by $F = 0$. The developable of γ_p is obtained when we solve the system of equations $F = \partial F / \partial t = 0$ first for $t \neq 0$, and then extend to $t = 0$. Thus we have a natural parametrization $f : \mathbb{R}^n, 0 \rightarrow \mathbb{R}P^{n+1}, \gamma(p)$ of the form $f(x', t) = (x', x_n(x', t), x_{n+1}(x', t))$ where $x' = (x_1, \dots, x_{n-1})$.

Simply solving the system of equations $F = \partial F / \partial t = 0$ we get the envelope of γ_p . If $b_1 = a_{n+1} - a_n > 1$, then the envelope contains the component $\{x_{n+1} = 0\} = O_n(\gamma, p)$ in addition to the developable.

According to the framework of the Legendre singularity theory [AGV], we consider the diagram

$$\{F = 0\} \xrightarrow{j} \mathbb{R}^{n+1} \times \mathbb{R}, 0 \xrightarrow{\pi} \mathbb{R}^{n+1}, 0,$$

where j is the inclusion and π is the projection to the first component. We call this diagram or simply F itself the generating family of γ^* . The envelope is regarded as the set of critical values of $\pi \circ j$, which can be seen the wave front sets of (singular) Legendre varieties [A2]. The determinacy of generating families under the \mathcal{K} -equivalence [AGV] relative to an osculation type leads that of envelopes.

We achieve to trivialize the type preserving deformations of generating families for the cases (I) $_{n,r}$, (II) $_{n,i}$, (III) $_n$, in the following section, and prove Theorems 1 and 2.

We notice that the generating family F is a Morse family, that is, $\{F = \partial F / \partial t = 0\}$ is non-singular in $\mathbb{R}^{n+1} \times \mathbb{R}, (0, 0)$ if and only if $b_1 = 1$. From the general results on the classifications of wave front sets [AGV], if $b_1 = 1$, then the classifications of envelopes under the C^∞ equivalence, them of parametrizations of developables under the C^∞ right left equivalence and them of generating families under the \mathcal{K} -equivalence coincide. Hence, by Theorem 1, we have

COROLLARY 1. *If A is one of types (II) $_{n,i}$ and (III) $_n$ in Theorem 1, then A determines the C^∞ class of the parametrization of developable.*

COROLLARY 2. *If A is one of types (II) $_{n,i}$ in Theorem 1, then A determines the C^∞ class of full developable.*

Here we mean by C^∞ class of full developable the class of developable under the equivalence preserving $O_n(\gamma, p)$. Corollary 2 generalizes Mond's result [M2] on space curves of type (1, 3, 4).

It seems not so clear that merely the determinacy of generating families relative to osculation type implies that of parametrizations of developables, in the case $b_1 > 1$, for instance, in the case (I) $_{n,r}$, $r > 1$. In [I], we analyze directly the parametrizations as in [M1], [M2], and establish the characterization of types determining C^∞ class of developables.

For the general theory of envelope, see [T], [Bru], [Ca]. Discussions on not necessarily type preserving deformations can be seen in [M2], [K]. We also remark that the family of osculating hyperplanes defines "web structure" on the complement of envelope. For this subject, see [N].

In the next section, we fix the weight of each variable as follows: $w(x_j) = a_j, 1 \leq j \leq n+1$ and $w(t) = 1$. Then the function-germ F is weighted-semi-homogeneous, the initial part

of which is weighted-homogeneous of weight a_{n+1} . This fact simplifies the subsequent discussions.

We denote by E_x the \mathbb{R} -algebra of function-germs of variable x at a point fixed in a context and denote by m_x the unique maximal ideal of E_x . For instance, in the next section, $E_{x,t,\lambda}$ denotes the totality of function-germs on the (x,t,λ) -space $\mathbb{R}^{n+1} \times \mathbb{R} \times \mathbb{R}$ at $(0,0,\lambda_0)$.

We set $e(\ell) = t^\ell / \ell!$.

Hereafter all mappings, diffeomorphisms and vector fields are assumed of class C^∞ .

1. Determinacy of a generating family

Let γ be a curve-germ of type A and γ^* the dual to γ of type $A^* = B$. Then there exists an affine representation of γ^* : $y_j = e(b_j) + v_j(t)$, $1 \leq j \leq n+1$, with $v_j \in m_t^{b_j+1}$.

Consider an initial part preserving deformation

$$F_\lambda = \sum_{j=0}^{n+1} x_j (e(b_{n+1-j}) + \lambda v_{n+1-j}),$$

$\lambda \in \mathbb{R}$, of the generating family F of γ^* , where $b_0 = 0$ and $v_0 = 0$.

For each λ_0 , we seek to solve the equation for A_λ and Φ_λ :

$$F_{\lambda_0} = A_\lambda \cdot F_\lambda \circ \Phi_\lambda,$$

where A_λ (resp. Φ_λ) is a family of function-germs (resp. a family of π fiber preserving diffeomorphism germs) of $\mathbb{R}^{n+1} \times \mathbb{R}$ at the origin with $A_{\lambda_0} = 1$ and Φ_{λ_0} the identity. Further we examine if Φ_λ preserves $\{t = 0\}$ in the context of Theorem 2.

Now we introduce, as tangent spaces in our problem, an $E_{x,\lambda}$ -module T (resp. T') consisting of function-germs in $E_{x,t,\lambda}$ of the form $h_\lambda \cdot F_\lambda + X_\lambda F_\lambda$, where $h_\lambda = h_\lambda(x,t)$ and X_λ is a π fiber preserving vector field germ over $\mathbb{R}^{n+1} \times \mathbb{R}, (0,0)$ with $X_\lambda(0,0) = 0$ (resp. and $X_\lambda t \subset tE_{x,t,\lambda}$).

As the corresponding infinitesimal condition, we need to check that $\partial F_\lambda / \partial \lambda \in T$ or T' , for any $\lambda_0 \in \mathbb{R}$. Remark that the spaces T and T' depend on $\lambda_0 \in \mathbb{R}$.

The space of variations in our problem is

$$V = \sum_{j=0}^{n+1} x_j m_t^{b_{n+1-j}+1} = \sum_{j=0}^{n+1} x_j m_t^{a_{n+1}-a_j+1} \subset m_t^{b_1+1} E_{x,t},$$

where we set $a_0 = 0$. In fact we see $\partial F_\lambda / \partial \lambda = \sum_{j=0}^{n+1} x_j v_{n+1-j} \in V$. Thus we have

PROPOSITION 1.1.

(1) *If $V \subset T$, for any $\lambda_0 \in \mathbb{R}$, then the type A determines the C^∞ class of envelope.*

(2) If $V \subset T'$, for any $\lambda_0 \in \mathbb{R}$, then the type A determines the C^∞ class of full envelope.

We now examine the condition $V \subset T$ or $V \subset T'$. We set $V_m = \{h \in V \mid \text{ord } h \geq a_{n+1} + m\}$, $m \geq 1$, and $V_0 = \sum_{j=0}^{n+1} x_j m_i^{a_{n+1}-a_j}$. Then we have the order filtration $V_0 \supset V = V_1 \supset V_2 \supset \dots$.

Remark that multiplying t maps V_m to V_{m+1} isomorphically for $m \geq 0$.

We need the following in the proof of Proposition 1.3 below:

LEMMA 1.2. (1) For any A ,

$$\begin{aligned} F_\lambda - F_0 &= \lambda \sum_{j=0}^{n+1} x_j v_{n+1-j} \in E_{x,\lambda} V, \\ F_\lambda - e(a_{n+1}) &= \lambda v_{n+1} + \sum_{j=1}^{n+1} x_j (\partial F_\lambda / \partial x_j) \in E_{x,\lambda} V + T', \\ t \partial F_\lambda / \partial t - t \partial F_0 / \partial t &= \lambda \sum_{j=0}^{n+1} x_j t (dv_{n+1-j} / dt) \in E_{x,\lambda} V, \end{aligned}$$

and

$$x_\ell \partial F_\lambda / \partial x_j - x_\ell e(b_{n+1-j}) = \lambda x_\ell v_{n+1-j} \in E_{x,\lambda} V_{a_\ell - a_j + 1}, \quad \text{if } \ell \geq j.$$

(2) If $A^* = B = (1, 2, \dots, n, n+r)$, then $\partial F_\lambda / \partial t - e(n+r-1) \in T' + E_{x,\lambda} V$.

PROOF: (1) is clear.

To see (2), we remark that $F_\lambda = e(n+r) + \sum_{j=1}^{n+1} x_j (e(n+1-j) + \lambda v_{n+1-j})$. Then $x_j \partial F_\lambda / \partial x_{j+1} = x_j e(n-j) + \lambda x_j v_{n-j}$, $0 \leq j \leq n$. Therefore,

$$\begin{aligned} \partial F_\lambda / \partial t - e(n+r-1) &= \sum_{j=1}^n x_j (e(n-j) + \lambda dv_{n+1-j} / dt) \\ &\equiv \lambda \sum_{j=1}^n x_j ((dv_{n+1-j} / dt) - v_{n-j}) \pmod{T'}. \end{aligned}$$

Write $(dv_{n+1-j} / dt) - v_{n-j} = c_j e(n+1-j) + w_{n+1-j}$, with $w_{n+1-j} \in m_i^{n+2-j}$, $c_j \in \mathbb{R}$, $1 \leq j \leq n$. Then we see

$$\partial F_\lambda / \partial t - e(n+r-1) \equiv \lambda \sum_{j=1}^n (c_j x_j \partial F_\lambda / \partial x_j - \lambda c_j x_j v_{n+1-j} + x_j w_{n+1-j}) \equiv 0,$$

modulo $T' + E_{x,\lambda} V$.

Q.E.D.

We define an $E_{x,\lambda}$ -module Γ (resp. Γ') by the quotient $\Gamma = (E_{x,\lambda} V + T) / T$ (resp. $\Gamma' = (E_{x,\lambda} V + T') / T'$), and an \mathbb{R} -module Λ (resp. Λ') by $\Lambda = \Gamma / m_{x,\lambda} \Gamma$ (resp. $\Lambda' = \Gamma' / m_{x,\lambda} \Gamma'$). Let $\rho : V \rightarrow \Lambda$ (resp. $\rho' : V \rightarrow \Lambda'$) denotes the canonical projection. Set $\Lambda_j = \mathbb{R} \cdot \rho(V_j) \subset \Lambda$ (resp. $\Lambda'_j = \mathbb{R} \cdot \rho'(V'_j) \subset \Lambda'$). Then the following is the key to show the determinacy relative to a type:

PROPOSITION 1.3. The $E_{x,\lambda}$ -module Γ (resp. Γ') has a finite system of generators $H \subset V$ satisfying that

- (1) any element $h \in H$ is of finite order, and
- (2) $\rho(H \cap V_m)$ (resp. $\rho'(H \cap V_m)$) generates Λ_m (resp. Λ'_m) over \mathbf{R} , for $m \geq 1$.

Further, if \mathbf{A} is one of types $(I)_{n,r}$, $(II)_{n,i}$ and $(III)_n$ in Theorem 1, then H can be chosen to satisfy also

- (3) $\rho(H \cap V_m) \subset \Lambda_{m+1}$, for $m \geq 1$.

If \mathbf{A} is one of types $(I)_{n,r}$ and $(II)_{n,i}$, then H can be chosen to satisfy also

- (4) $\rho'(H \cap V_m) \subset \Lambda'_{m+1}$, for $m \geq 1$.

PROOF: Define functions $Y_0 = 1, Y_1, \dots, Y_{a_{n+1}-1}$ by

$$Y_{b_j} = \partial F_\lambda / \partial x_j = e(n+1-j) + \lambda v_{n+1-j}, \quad 1 \leq j \leq n+1$$

and otherwise $Y_k = e(k)$. Then, by the Malgrange-Mather division theorem [Brö], they generate the $E_{x,\lambda}$ -module $E_{x,t,\lambda} / (t \partial F_\lambda / \partial t) E_{x,t,\lambda}$. Therefore, each $h \in V$ can be written as

$$h = Q(x, t, \lambda) \cdot t \frac{\partial F_\lambda}{\partial t} + \sum_{j=0}^{a_{n+1}-1} R_j(x, \lambda) Y_j, \quad \dots (*),$$

where $Q \in E_{x,t,\lambda}$, $R_j \in E_{x,\lambda}$, $1 \leq j \leq a_{n+1} - 1$.

Since $h \in m_t^{b_1+1} E_{x,t,\lambda}$ and $t \partial F_\lambda / \partial t \in m_t^{b_1} E_{x,t,\lambda}$, we see $R_j = 0$ for $j < b_1$, comparing the order of each term with respect to t . In general we have

$$R_k \in \langle x_n, x_{n-1}, \dots, x_{n+1-j} \rangle E_{x,\lambda}, \quad \text{for } k < b_{j+1}.$$

To see this we define an $E_{x,\lambda}$ -module homomorphism $\Phi_k : E_{x,t,\lambda} \rightarrow E_{x,\lambda}$ by $\Phi_k(\rho) = \partial^k \rho / \partial t^k |_{t=0}$. Then $\Phi_k(h)$, $\Phi_k(t \partial F_\lambda / \partial t)$ and $\Phi_k(Y_j)$, $j < k$, are in $\sum_{\ell=n+1-j}^n x_\ell E_{x,\lambda}$, $\Phi_k(Y_j) = 0$, $j > k$, and $\Phi_k(Y_k) = 1$. Thus $\Phi_k(R_k) = R_k$ is also in $\sum_{\ell=n+1-j}^n x_\ell E_{x,\lambda}$.

Since $Q t \partial F_\lambda / \partial t$ and $R_{b_j} \cdot \partial F_\lambda / \partial x_j$, $1 \leq j \leq n+1$ belong to $T' \subset T$ we can take

$$H = \{x_\ell e(k) \mid b_j < k < b_{j+1}, n+1-j \leq \ell \leq n, 1 \leq j \leq n\},$$

as a finite system of generators of Γ and Γ' . Remark that H consists of monomials and satisfies (1).

To see (2), we remark that $\text{ord} R_j \geq \text{ord} h - \text{ord} Y_j$, $b_1 \leq j \leq a_{n+1} - 1$, which follows from the formal uniqueness of the expression (*).

It remains to see (3) and (4). Remark that the number of elements of H is equal to $\sum_{j=1}^n (a_j - j)$.

If \mathbf{A} is of type $(I)_{n,r}$, then $H = \emptyset$. Thus (3) and (4) are clear.

Let \mathbf{A} be of type $(II)_{n,i}$. Then \mathbf{B} is of type $(II)_{n,n-i}$ and therefore

$$H = \{x_\ell e(n-i+1) \mid i+1 \leq \ell \leq n\}.$$

Let A be of type $(III)_n$. Then

$$H = \{x_1 e(n+1), \dots, x_n e(n+1), x_1 e(n+2), \dots, x_n e(n+2)\}.$$

We denote by $V^{(m)}$ the \mathbb{R} -vector space of weighted homogeneous polynomials in V of weight $a_{n+1} + m$. We see $V^{(m)} \cong V_m/V_{m+1}$ and $\dim V^{(m)} = n+2$, $m \geq 1$.

Then the following simple facts are essential:

LEMMA 1.4. (i) If A is $(II)_{n,i}$, then the system of elements in $V^{(m)}$

$$S : t^m F_0, t^{m+1} \partial F_0 / \partial t, x_1 e(n+m+1), \dots, x_i e(n+m+2-i),$$

$$x_{i+1} e(n+m-i), \dots, x_{i+m-1} e(n-i+2), x_{i+m+1} e(n-i), \dots, x_{n+1} e(m),$$

forms a basis of the vector space $V^{(m)}$, $1 \leq m \leq n-i$.

(ii) If A is $(III)_n$, then the system of elements in $V^{(m)}$

$$S : t^m F_0, t^{m+1} \partial F_0 / \partial t, x_1 e(n+m), \dots, x_{m-1} e(n+2), x_{m+1} e(n), \dots, x_{n+1} e(m)$$

forms a basis of the vector space $V^{(m)}$, $1 \leq m \leq n+1$.

Now let $h \in V^{(m)}$. Then, by Lemma 1.4, h is a linear combination of elements of S over \mathbb{R} .

If each element of S belongs $E_{x,\lambda} V_{m+1} + m_{x,\lambda} V + T$ (resp. $E_{x,\lambda} V_{m+1} + m_{x,\lambda} V + T'$), then so is h . This means that $\rho(h) \in \Lambda_{m+1}$ (resp. $\rho'(h) \in \Lambda'_{m+1}$).

Let A be of type $(II)_{n,i}$. Then

$$t^m F_0 = t^m F_\lambda - t^m (F_\lambda - F_0) \subset T' + E_{x,\lambda} V_{m+1},$$

and

$$t^{m+1} \partial F_0 / \partial t = t^m \cdot t \partial F_\lambda / \partial t - t^m (t \partial F_\lambda / \partial t - t \partial F_0 / \partial t) \subset T' + E_{x,\lambda} V_{m+1}.$$

Further, consider an element of type $x_k e(\ell)$ in S . Then $a_k + \ell = a_{n+1} + m$. If $\ell > n+2 = a_{n+1}$, then $e(\ell) \in V$ and thus $x_k e(\ell) \in m_{x,\lambda} V$. If $\ell = n+2$, then $x_k e(n+2) = x_k F_\lambda - x_k (F_\lambda - e(n+2))$, which belongs to $T' + m_{x,\lambda} V$. If $\ell < n+2$, then $\ell \neq n-i+1$ and $e(\ell) = \partial F_0 / \partial x_\alpha$ for some $\alpha \leq k$. Then $\ell = b_{n+1-\alpha}$, therefore, $a_k - a_\alpha = m$. Therefore, $x_k e(\ell) = x_k \partial F_\lambda / \partial x_\alpha - x_k (\partial F_\lambda / \partial x_\alpha - \partial F_0 / \partial x_\alpha) \in E_{x,\lambda} V_{m+1} + m_{x,\lambda} V + T'$.

From this we see $\rho'(H \cap V_m) \subset \Lambda'_{m+1}$.

Let A be of type $(III)_n$. Then we have $t^m F_0$ and $t^{m+1} \partial F_0 / \partial t$ belong to $T' + E_{x,\lambda} V_{m+1}$ similarly as above. If $\ell > n+3 = a_{n+1}$, then $e(\ell) \in V$ and $x_k e(\ell) \in m_{x,\lambda} V$. If $\ell = n+3$, then $x_k e(n+3) = x_k F_\lambda - x_k (F_\lambda - e(n+3)) \in T' + m_{x,\lambda} V$. If $\ell = n+2$, then $x_k e(n+2) = x_k \partial F_\lambda / \partial t - x_k (\partial F_\lambda / \partial t - e(n+2))$. The first term belongs to T , and the second term belongs to $T' + m_{x,\lambda} V$ by Lemma 1.2.(2). If $\ell \leq n$, then $e(\ell) = \partial F_0 / \partial x_\alpha$ for some $\alpha \leq k$. Thus $x_k e(\ell) \in E_{x,\lambda} V_{m+1} + m_{x,\lambda} V + T$.

From this we know that $\rho'(H \cap V_m) \subset \Lambda_{m+1}$.

Thus Proposition 1.3 is proved.

PROOF OF LEMMA 1.4: Set $P = e(n+2) + x_{i+m}e(n+1-i-m)$ (resp. $P = e(n+3) + x_m e(n+1)$), which is equal to F_0 restricted to $\{x_1 = \cdots = x_{i+m-1} = x_{i+m+1} = \cdots = x_{n+1} = 0\}$ (resp. $\{x_1 = \cdots = x_{m-1} = x_{m+1} = \cdots = x_{n+1} = 0\}$). Then Lemma follows from the simple fact that $t^m P$ and $t^{m+1} \partial P / \partial t$ are linearly independent over \mathbb{R} .

PROOF OF THEOREM 1: By (1) and (2) of Proposition 1.3, we see $\Lambda_j = 0$, for sufficiently large j . By (3), we have $\Lambda_1 = \Lambda_2 = \cdots$, therefore, $\Lambda = 0$. Hence $\Gamma = m_{x,\lambda} \Gamma$. Since Γ is finitely generated, Nakayama's lemma implies that $\Gamma = 0$. This means $V \subset T$. Thus Proposition 1.1.(1) implies Theorem 1. The normal forms are obtained when we choose γ_p and affine coordinates such that γ_p^* is represented by $y_1 = t^{a_{n+1}-a_n}, \dots, y_{n+1} = t^{a_{n+1}}$.

PROOF OF THEOREM 2: Similarly as above, by (1), (2) and (4) of Proposition 1.3, we have $\Gamma' = 0$ and therefore $V \subset T'$. Then, by Proposition 2.1.(2), we have Theorem 2.

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