Developable of a Curve and Determinacy Relative to Osculation-Type

Goo Ishikawa

Series ♯151. May 1992

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

♯ 123: A. Arai, Fock-space representations of the relativistic supersymmetry algebra in the two-dimensional space-time, 13 pages. 1991.

♯ 124: S. Izumiya, The theory of Legendrian unfoldings and first order differential equations, 16 pages. 1991.

♯ 125: T. Hibi, Face number inequalities for matroid complexes and Cohen-Macaulay types of Stanley-Reisner rings of distributive lattices, 17 pages. 1991.

♯ 126: S. Izumiya, Completely integrable holonomic systems of first order differential equations, 35 pages. 1991.

♯ 127: G. Ishikawa, S. Izumiya and K. Watanabe, Vector fields near a generic submanifold, 9 pages. 1991.

♯ 128: A. Arai, I. Mitoma, Comparison and nuclearity of spaces of differential forms on topological vector spaces, 27 pages. 1991.

♯ 129: K. Kubota, Existence of a global solution to a semi-linear wave equation with initial data of non-compact support in low space dimensions, 53 pages. 1991.

♯ 130: S. Altschuler, S. Angenent and Y. Giga, Mean curvature flow through singularities for surfaces of rotation, 62 pages. 1991.

♯ 131: M. Giga, Y. Giga and H. Sohr, L^p estimates for the Stokes system, 13 pages. 1991.

♯ 132: Y. Okabe, T. Ootsuka, Applications of the theory of KM_2O-Langevin equations to the non-linear prediction problem for the one-dimensional strictly stationary time series, 27 pages. 1992.

♯ 133: Y. Okabe, Applications of the theory of KM_2O-Langevin equations to the linear prediction problem for the multi-dimensional weakly stationary time series, 22 pages. 1992.

♯ 134: P. Aviles, Y. Giga and N. Komuro, Duality formulas and variational integrals, 22 pages. 1992.

♯ 135: S. Izumiya, The Clairaut type equation, 6 pages. 1992.

♯ 136: S. Izumiya, Singular solutions of first order differential equations, 6 pages. 1992.

♯ 137: S. Izumiya, W.L. Marar, The Euler characteristic of a generic wave front in a 3-manifold, 6 pages. 1992.

♯ 138: S. Izumiya, W.L. Marar, The Euler characteristic of the image of a stable mapping from a closed n-manifold to a $(2n-1)$-manifold, 5 pages. 1992.

♯ 139: Y. Giga, Z. Yoshida, A bound for the pressure integral in a plasma equilibrium, 20 pages. 1992.

♯ 140: S. Izumiya, What is the Clairaut equation ?, 13 pages. 1992.

♯ 141: H. Takamura, Weighted deformation theorem for normal currents, 27 pages. 1992.

♯ 142: T. Morimoto, Geometric structures on filtered manifolds, 104 pages. 1992.

♯ 143: G. Ishikawa, T. Ohmoto, Local invariants of singular surfaces in an almost complex four-manifold, 9 pages. 1992.

♯ 144: K. Kubota, K. Mochizuki, On small data scattering for 2-dimensional semilinear wave equations, 22 pages. 1992.

♯ 145: T. Nakazi, K. Takahashi, Hyponormal Toeplitz operators and extremal problems of Hardy spaces, 30 pages. 1992.

♯ 146: N. Hayashi, T. Ozawa, Remarks on nonlinear Schrödinger equations in one space dimension, 10 pages. 1992.

♯ 147: M. Sato, Interface evolution with Neumann boundary condition, 16 pages. 1992.

♯ 148: Y. Okabe, Langevin equations and causal analysis, 49 pages. 1992.

♯ 149: Y. Giga, S. Takahashi, On global weak solutions of the nonstationary two-phase Stokes flow, 25 pages. 1992.

♯ 150: G. Ishikawa, Determinacy of envelope of the osculating hyperplanes to a curve, 9 pages. 1992.

Developable of a Curve and Determinacy Relative to Osculation-Type

Goo ISHIKAWA

Department of Mathematics, Hokkaido University, Sapporo 060, JAPAN

0. Introduction

The ruled surface by the tangent lines to a space curve is called the developable surface of the curve. In general the developable of a curve in $(n+1)$-dimensional projective space is defined as the hypersurface "ruled" by osculating $(n-1)$-subspaces to the curve.

In this paper we study the local C^∞ classification of singularities appearing in developables. First we recall basic definitions and known facts needed to formulate our results ([S1],[I4]).

Let $\gamma : M \longrightarrow \mathbf{R}P^{n+1}$ be a C^∞ parametrized curve, where M is a one-dimensional manifold and $n \geqq 1$. We call the germ γ_p at a point $p \in M$ of finite osculation-type (or simply, of finite type) if γ is represented by $x_i = t^{a_i} + o(t^{a_i}), 1 \leqq i \leqq n+1$, for a C^∞ coordinate t of (M,p) and an affine coordinate $(x_1, \dots, x_{n+1})$ of $\mathbf{R}P^{n+1}$ centered at $\gamma(p)$, where each a_i is a natural number and $1 \leqq a_1 < \cdots < a_{n+1}$. Then $\mathcal{A} = (a_1, a_2, \dots, a_{n+1})$ is a local projective invariant of the germ γ_p and we call $\mathcal{A}$ the type of γ_p; $\mathrm{type}(\gamma_p) = \mathcal{A}$.

A point $p \in M$ is called an ordinary point if $\mathrm{type}(\gamma_p) = (1, 2, \dots, n, n+1)$, and, otherwise, it is called a special point. Special points of finite type are isolated in M.

For each $p \in M$ where γ_p is of finite type and for each i, $(0 \leqq i \leqq n+1)$, we set $O_i(\gamma, p) = \{x_{i+1} = \cdots = x_{n+1} = 0\} \subset T_{\gamma(p)}\mathbf{R}P^{n+1}$ under the above affine representation of γ_p. The corresponding projective subspace of $\mathbf{R}P^{n+1}$ through $\gamma(p)$ of dimension i is also denoted by $O_i(\gamma, p)$. Further we define the osculating i-bundle $O_i(\gamma) = \bigcup_{p \in M} O_i(\gamma, p)$ in the pullback bundle $\gamma^{-1}T\mathbf{R}P^{n+1}$. The natural parametrization $\mathrm{dev}(\gamma) : O_{n-1}(\gamma) \longrightarrow \mathbf{R}P^{n+1}$ defined by $(p, q) \mapsto q$, where $q \in O_{n-1}(\gamma, p) (\subset \mathbf{R}P^{n+1})$, is called a developable of γ.

The germ $\mathrm{dev}(\gamma)_p$ of $\mathrm{dev}(\gamma)$ at $(p, 0)$ is determined up to the projective equivalence by the projective class of γ_p. In this paper we consider a weaker equivalence, that is, C^∞ right-left equivalence. By the C^∞ class of developable, we mean the C^∞ right-left equivalence class of $\mathrm{dev}(\gamma)_p$.

Up to the present, several results are known on the C^∞ classification of developables.

It is classically known that, for a space curve-germ γ, $n = 2$, and at each ordinary point p, the developable has cuspidal singularities along γ and $\mathrm{dev}(\gamma)_p$ is C^∞ equivalent to the map-germ $(x, t) \mapsto (x, t^2, t^3)$ at the origin.

Cleave [Cl], Gaffney, du Plessis [GP] and Shcherbak [S1] prove recently that, if γ_p is of type $(1, 2, 4)$, $\mathrm{dev}(\gamma)_p$ is C^∞ equivalent to $(x, t) \mapsto (x, t^2, xt^3)$, which is called the open umbrella [A2].

Further, Mond proceeds to give C^∞ normal forms of developable of curves of type $(1,2,2+r)$, $r \leqq 5$, and of type $(1,3,4)$ in [M1],[M2].

In the case of arbitrary dimension, Shcherbak [S1] shows that the developable of a curve of type $(2,3,\dots,n+1,n+2)$ is C^∞ equivalent to the (parametrization of) n-dimensional swallowtail, generalizing the observation of Arnol'd [A1] for a curve of type $(2,3,4)$. These follows from the Legendre singularity theory [AGV].

In connection with the study of projections of wave front sets ([G]), Shcherbak, further in [S2], gives the C^∞ normal form of the union of the developable of a curve-germ γ_p of type $(1,2,\dots,n,n+2)$ and the osculating hyperplane $O_n(\gamma,p)$. In that paper, it is also observed that, in the complex analytic category, the developable of a curve of type $(1,3,5)$ is equivalent to the variety of irregular orbits of the finite reflection group H_3 in $\mathbf{C}^3$.

We notice that the type of a curve determines the C^∞ class of developable of the curve in the above mentioned cases.

Inspired with these previous results, we are led to the natural problem that whether or not a type of a curve-germ γ_p determines the C^∞ class of map-germ $\mathrm{dev}(\gamma)_p$: We call $\mathsf{A} = (a_1, a_2, \dots, a_{n+1})$ determinative if $\mathrm{type}(\gamma_p) = \mathrm{type}(\gamma'_{p'}) = \mathsf{A}$ implies that $\mathrm{dev}(\gamma)_p$ is C^∞ right-left equivalent to $\mathrm{dev}(\gamma')_{p'}$.

The purpose of this paper is to give the complete solution of this determinacy problem.

THEOREM 1. *A type A of a curve-germ in $\mathbf{R}P^{n+1}$ is determinative if and only if A is one of following types:*

$(I)_{n,r}$ $\mathsf{A} = (1,2,\dots,n,n+r), \quad r = 1,2,\dots,$

$(II)_{n,i}$ $\mathsf{A} = (1,2,\dots,i,i+2,\dots,n+1,n+2), \quad 0 \leqq i \leqq n-1,$

$(III)_n$ $\mathsf{A} = (3,4,\dots,n+2,n+3),$

$(IV)_m$ $\mathsf{A} = (2,2m+1), m = 1,2,\dots,$ $\quad (V)$ $\mathsf{A} = (3,5),$ $\quad (VI)$ $\mathsf{A} = (1,3,5).$

Further, in this case, if $\mathrm{type}(\gamma_p) = \mathsf{A}$, then the map-germ $\mathrm{dev}(\gamma)_p$ is C^∞ right-left equivalent to $(x', U(x',t), U_r(x',t)) : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^{n+1}, 0$, where $(x',t) = (x_1,\dots,x_{n-1},t)$ is a coordinate of $(\mathbf{R}^n, 0)$,

$$U(x',t) = \frac{t^{a_n}}{a_n!} + x_1 \frac{t^{a_n - a_1}}{(a_n - a_1)!} + \cdots + x_{n-1} \frac{t^{a_n - a_{n-1}}}{(a_n - a_{n-1})!},$$

$r = a_{n+1} - a_n$ and

$$U_r(x',t) = \int_0^t \frac{u^r}{r!} \frac{\partial U}{\partial t}(x',u)du.$$

If such a determinacy for a type A is established once, then to have the normal form of developables of curves of type A is reduced to just a calculation of an example. The above normal form in each case is in fact a parametrization of the developable of a typical curve of the given type. For example, if $\mathsf{A} = (1,3,5)$, then the normal form of developable is given by $(x_1, t) \mapsto (x_1, \frac{t^3}{6} + x_1 \frac{t^2}{2}, \frac{t^5}{20} + \frac{x_1 t^4}{8})$.

The figures of the developable surfaces of space curves of type $(1,2,4)$, $(1,3,4)$, $(2,3,4)$ and $(1,3,5)$ can be seen, for instance, in [Cl], [M2], [AGV] and [S2] respectively. Here we draw the figure of the developable surface of a curve of type $(3,4,5)$, which seems to have not appeared in any literature:

(Please insert here the figure 1.)

We remark, in this stage, that the developable has a geometric meaning as a component of the envelope of the one-parameter family of osculating hyperplanes to a curve-germ γ_p ([I4]). In the case $a_{n+1} - a_n > 1$, the envelope also has a component $O_n(\gamma, p)$ itself. In this case therefore it is natural to classify developables by diffeomorphisms preserving $O_n(\gamma, p)$. By the C^∞ class of "full developable" of a curve-germ, we mean the class of developable under this equivalence. Then we have the following result:

THEOREM 2. *A type A of a curve-germ determines the C^∞ class of full developable if and only if A is one of types $(I)_{n,r}$ and $(II)_{n,i}$ in Theorem 1.*

There are two aspects to investigate the singularities of developables: One is the Legendre singularity theory [AGV] as in [A1], [S1], [S2], and the other is analizing directly the parametrization as in [GP], [M1], [M2]. We utilize the first aspect in [I4], and prove in that paper the types $(\mathrm{II})_{n,i}$ and $(\mathrm{III})_n$ determine the C^∞ classes of developables respectively. Furthermore we prove the types $(\mathrm{II})_{n,i}$ determine the C^∞ classes of full envelope, therefore, the C^∞ classes of full envelopes (Corollaries 1 and 2 of [I4]).

In this paper we pursue the second aspect to solve the problem completely.

The methods used in this paper owe heavily to the infinitesimal computations, which are standard in the singularity theory. In fact the arguments in §§3 and 4 are inspired by the paper [M2] of Mond. The special feature of this paper, however, is picking up the algebraic objects H_k in §2. Without these footholds, the computations in §§3 and 4 would become ungovernable.

In §1, after recalling the notion of dual curves, we give the concrete parametrization of a developable. After a preliminary in §2, we examine a necessary condition for a type to determine the C^∞ class of developable in §3. In §4 we show the determinacy of the parametization of developables for curves of types $(\mathrm{I})_{n,r}$, $(\mathrm{IV})_m$, (V) and (VI), and thus we prove Theorems 1 and 2. As an application of Theorem 1, we reprove the Mond's result on the developables of curves of type $(1,2,2+r)$ in §5.

Theorem 1 indicates that it is need other invariants than the osculation-type to determine the C^∞ class of developable of a curve of general type: It is an interesting problem to classify developables of curves of type $(1,3,6)$, for instance.

In this paper we do not treat the bifurcation problem of curves and their developables. For this interesting problem, see [K],[M2]. Besides, developables of curves can be regarded as (singular) solutions of homogeneous Monge-Ampère equations [Mo],[D]. The study from this aspect will be given in a forthcoming paper.

Hereafter all mappings, diffeomorphisms and vector fields are assumed of class C^∞.

1. Dual curves and generating families

Denote the dual projective space of $\mathbf{R}P^{n+1}$ by $\mathbf{R}P^{n+1*}$, which is the space of all hyperplanes in $\mathbf{R}P^{n+1}$. Then we naturally identify $\mathbf{R}P^{n+1**}$ with $\mathbf{R}P^{n+1}$. For a curve $\gamma : M \longrightarrow \mathbf{R}P^{n+1}$, γ_p being of finite type at each point $p \in M$, we define the dual $\gamma^* : M \longrightarrow \mathbf{R}P^{n+1*}$ of γ by $p \mapsto O_n(\gamma, p)$. We describe the developable of γ by the dual curve γ^* in $\mathbf{R}P^{n+1*}$:

LEMMA 1.1. ([S1]).

(0) *γ^* is a C^∞ map.*

(1) *If γ_p is of type $\mathsf{A} = (a_1, \dots, a_{n+1})$, then γ^*_p is also of finite type $\mathsf{A}^* = (a_{n+1} - a_n, a_{n+1} - a_{n-1}, \dots, a_{n+1} - a_1, a_{n+1})$.*

(2) *$O_i(\gamma^*, p) = O_{n-i}(\gamma, p)^*$, the dual of $O_{n-i}(\gamma, p)$, $0 \leqq i \leqq n$.*

(3) *$\gamma^{**} = \gamma$.*

(4) *$dev(\gamma)$ is identified with $front(\gamma^*) : O_{n-1}(\gamma) = O_1(\gamma^*)^* \longrightarrow \mathbf{R}P^{n+1}$ defined by $(p, q) \mapsto q$, where $q \in O_1(\gamma^*, p)^* \subset \mathbf{R}P^{n+1**} = \mathbf{R}P^{n+1}$.*

Set $\mathsf{A}^* = \mathsf{B} = (b_1, \dots, b_{n+1})$ and $a_0 = 0$. Then $b_i = a_{n+1} - a_{n+1-j}, 1 \leqq i \leqq n+1$.

Set $Q = \{(p, q) | p \in q^*\} \subset \mathbf{R}P^{n+1} \times \mathbf{R}P^{n+1*}$. The both natural identifications $Q \cong PT^*\mathbf{R}P^{n+1}$ and $Q \cong PT^*\mathbf{R}P^{n+1*}$ induce the same contact structure on Q, [S1]. Then front(γ^*) lifts to an isotropic mapping $O_1(\gamma^*)^* \longrightarrow Q$ naturally. Therefore the developable is regarded to be a wave front set of a Legendre variety, which in general has singular points.

We take an affine representaive of $\gamma^*_p : y_i = y_i(t)$ with the order of $y_i(t) = b_i$. We define the affine coordinate $x = (x_1, \dots, x_{n+1})$ of $\mathbf{R}P^{n+1}$ correspondingly such that the projective duality is described by $\sum_{j=0}^{n+1} x_j y_{n+1-j} = 0$, with $x_0 = y_0 = 1$. Set

$$F(x,t) = y_{n+1}(t) + x_1 y_n(t) + \cdots + x_n y_1(t) + x_{n+1} = \sum_{j=0}^{n+1} x_j y_{n+1-j}(t).$$

The one-parameter family of osculating hyperplanes of γ near p is defined by $F = 0$. Then $O_1(\gamma^*)^*$ is obtained when we solve the system of equations $F = \partial F/\partial t = 0$ first for $t \neq 0$, and then extend to $t = 0$. Thus we have

$$x_n(x', t) = -(1/\dot{y}_1)(\dot{y}_{n+1} + x_1 \dot{y}_n + \cdots + x_{n-1}\dot{y}_2),$$

where $x' = (x_1, \dots, x_{n-1})$, and $x_{n+1}(x', t)$ is determined by

$$\partial x_{n+1}/\partial t = -y_1 \partial x_n / \partial t, \quad x_{n+1} \in t^r E_{x',t},$$

where $r = b_1 = a_{n+1} - a_n$. The developable is then parametrized by the germ $f : \mathbf{R}^{n-1} \times \mathbf{R}, 0 \longrightarrow \mathbf{R}P^{n+1}, \gamma(p)$ defined by $(x', t) \mapsto (x', x_n(x', t), x_{n+1}(x', t))$. Remark that the singular locus $\Sigma(f) \subset \mathbf{R}^{n-1} \times \mathbf{R}, (0,0)$ of f is equal to $\{\partial x_n/\partial t = 0\}$. Therefore $\Sigma(f)$ contains the component $\{t = 0\}$ if and only if $a_n - a_{n-1} > 1$.

In the following sections we fix the weight of each variable as follows: $w(x_j) = a_j, 1 \leqq j \leqq n+1$ and $w(t) = 1$. Then the map-germ f is weighted semi-homogeneous, the initial part of which is weighted-homogeneous of weight A. We use the notion of order with respect to these weights.

We denote by E_x the $\mathbf{R}$-algebra of function-germs of variable x at a point fixed in a context and denote by m_x the unique maximal ideal of E_x. For instance, in the next section, $E_{x',t,\lambda}$ denotes the totality of function-germs on the (x', t, λ)-space $\mathbf{R}^{n-1} \times \mathbf{R} \times \mathbf{R}$ at $(0, 0, \lambda_0)$.

We set $e(m) = t^m/m!$ for brevity.

2. Algebraic preliminary.

Let m and s be natural numbers with $m > s \geqq 1$, and $\alpha_1, \dots, \alpha_{m-s} : \mathbf{R}^{n-1}, 0 \longrightarrow \mathbf{R}, 0$ be analytic map-germs with $\alpha_{m-s} \neq 0$. We set $U = e(m) + \alpha_1(x')e(m-1) + \cdots + \alpha_{m-s}(x')e(s)$.

We will apply the results in this section to the proof of Theorems 1 and 2, setting $m = a_n, s = a_n - a_{n-1}$ and $\alpha_{a_i} = x_i, 1 \leqq i \leqq n-1$, otherwise, $\alpha_j = 0$.

Consider a parameter $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathbf{R}^\ell$ for some $\ell \geqq 0$, and fix $\lambda_0 \in \mathbf{R}^\ell$. Set

$$H_k = \{h \in t^k E_{x',t,\lambda} \mid \partial h/\partial t \in (t^k \partial U/\partial t) E_{x',t,\lambda}\} \subset E_{x',t,\lambda},$$

for each $k = 0, 1, 2, \dots$. Then we first observe the following:

LEMMA 2.1. *$H_k = H_0 \cap t^{k+s} E_{x',t,\lambda}$, for $k \geqq 1$.*

PROOF: Let $h \in H_k$. Then $\partial h/\partial t \in (t^k \partial U/\partial t) E_{x',t,\lambda} \subset t^{k+s-1} E_{x',t,\lambda}$, since $\partial U/\partial t \in t^{s-1} E_{x',t}$. Since $h \in t E_{x',t,\lambda}$, we have $h \in t^{k+s} E_{x',t,\lambda}$.

Let $h \in H_0 \cap t^{k+s} E_{x',t,\lambda}$. Then $\partial h/\partial t = t^{k+s-1} K = (\partial U/\partial t) L$ for some $K, L \in E_{x',t,\lambda}$. Write $\partial U/\partial t = t^{s-1} P$, where $P|_{t=0}$ is not a zero divisor in $E_{x',\lambda}$. Then we have $t^r K = PL$. Hence $\partial^j L/\partial t^j|_{t=0} = 0$, $0 \leqq j \leqq k-1$, and therefore $L \in t^k E_{x',t,\lambda}$. This completes the proof.

In general we consider, for a map-germ $G : \mathbf{R}^M, y_0 \longrightarrow \mathbf{R}^N, x_0$, with the induced ring homomorphism $G^* : E_x \longrightarrow E_y$. Then we set as in [I2],

$$H_G = \{h \in E_y \mid dh \in \sum_{i=1}^{N} E_y \cdot dG_i\}.$$

Then, as easily verified, $H_G \supset G^* E_x$ and thus H_G is a $G^* E_x$ module. Moreover, for any $h_1, \dots, h_m \in H_G$ and for a C^∞ function $\tau : \mathbf{R}^m \longrightarrow \mathbf{R}$, the composition $\tau \circ (h_1, \dots, h_m)$ also belongs to H_G.

Let X be a vector field germ over the source $\mathbf{R}^M, y_0$ of G. Denote also by $X : E_y \longrightarrow E_y$ the induced derivation. Then we have

LEMMA 2.2. *If $X(G^*E_x) \subset H_G$, then $X(H_G) \subset H_G$.*

PROOF: Let $h \in H_G$. Set $dh = \sum_{i=1}^N A_i dG_i$. Then we have

$$d(Xh) = L_X(dh) = L_X(\sum_{i=1}^N A_i dg_i) = \sum_{i=1}^N (XA_i)dG_i + \sum_{i=1}^N A_i d(XG_i),$$

where L_X denotes the Lie derivative. Since each XG_i belongs to H_G, we see $d(Xh) \in \sum_{i=1}^N E_y dG_i$, therefore $Xh \in H_g$.

Returning the original situation, we set $G = (x', U, \lambda) : \mathbb{R}^{n+\ell}, (0, \lambda_0) \longrightarrow \mathbb{R}^{n+\ell}$, which is a map-germ from the (x', t, λ)-space to the (x', x_n, λ)-space. Then we see $H_G = H_0$ and $H_k, k \geqq 0$, is a $G^*E_{x',x_n,\lambda}$-module.

LEMMA 2.3. *Let X be a vector field germ over the source $\mathbb{R}^{n+\ell}, (0, \lambda_0)$. If $Xx_j, 1 \leqq j \leqq n-1, X\lambda_j, 1 \leqq i \leqq \ell$, and XU belong to H_0 and if $Xt \in tE_{x',t,\lambda}$, then $X(H_k) \subset H_k, k \geqq 0$.*

PROOF: By the assumption we have $X(t^{k+s}E_{x',t,\lambda}) \subset t^{k+s}E_{x',t,\lambda}$. By Lemma 2.2, we have $X(H_0) \subset H_0$. Then Lemma 2.1 implies Lemma 2.3.

Define U_j, $j \geqq 0$, as in Theorem 1, by $\partial U_j/\partial t = e(j)\partial U/\partial t, U_j|_{t=0} = 0$. Then $U_0 = U$. If $j \geqq k$, then $U_j \in H_k$.

We utilize in §4 the following lemma, which describe the structure of $H_k, k \geqq 0$, as $G^*E_{x',x_n,\lambda}$-module.

LEMMA 2.4. (1) *Let $S \subset H_k, k \geqq 1$, be a finite subset. Then S generates H_k as $G^*E_{x',x_n,\lambda}$-module if and only if i^*S generate*

$$t^{k+a_n}E_t / t^{k+2a_n}E_t \cong \mathbb{R}^{a_n},$$

where $i^* : H_k \longrightarrow t^{k+a_n}E_t$ is induced by $i(t) = (0, t, \lambda_0)$.

(1') *$H_k, k \geqq 1$, is generated by $S' = \{U_k, U_{k+1}, \ldots, U_{k+a_n-1}\}$ as $G^*E_{x',x_n,\lambda}$-module.*

(2) *H_0 is generated by $S'' = \{1, U_1, \ldots, U_{a_n-1}\}$ as $G^*E_{x',x_n,\lambda}$-module.*

PROOF: (2) By [I1], (cf. [I3]), we see H_0 is a differentiable algebra in the sense of Malgrange [M] generated by the compoments of x' and λ and $U, U_1, \ldots, U_{a_n-1}$. This means that any $h \in H_0$ can be written in the form $h = \tau(x', U, \lambda, U_1, \ldots, U_{a_n-1})$, for a function-germ $\tau : \mathbb{R}^{n-1+\ell+a_n-1}, 0 \longrightarrow \mathbb{R}$. Therefore, as easily verified, we see

$$H_0 = G^*m_{x',x_n,\lambda}H_0 + G^*E_{x',x_n,\lambda} + \sum_{j=1}^{a_n-1} G^*E_{x',x_n,\lambda}U_j.$$

Thus (2) follows from Malgrange's preparation theorem for differentiable algebras [M].

(1') By Lemma 2.1, $H_k = H_0 \cap t^{k+s}E_{x',t,\lambda}$. Let $h \in H_k$. Then, by (2), we can write $h = c_0 \circ G + \sum_{j=1}^{a_n-1} c_j \circ G\ U_j$, where $c_j \in E_{x',x_n,\lambda}, 0 \leqq j \leqq a_n - 1$. Set $c_j(x', x_n, \lambda) =$

$c_{j0}(x',\lambda)+c_{j1}(x',x_n,\lambda)x_n$, $0 \leqq j \leqq a_n-1$, $c_{01}(x',x_n,\lambda)=c_{010}(x',\lambda)+c_{011}(x',x_n,\lambda)x_n$, $h_0 = c_{00}+c_{010}U+\sum_{j=1}^{a_n-1} c_{j0}U_j$ and $h_1 = c_{011}\circ G\ UU+\sum_{j=1}^{a_n-1} c_{j1}\circ G\ UU_j$. Then $h=h_0+h_1$. Remark that $UU_j, j \geqq 0$, is an $E_{x'}$-linear combination of $U_{j+s},\ldots,U_{j+a_n}$. In fact, this is verified from the equality $\partial(UU_j)/\partial t=(U_j+Ue(j))\partial U/\partial t$, and that $U_j+Ue(j)$ is an $E_{x'}$-linear combination of $e(j+s),\ldots,e(j+a_n)$. Substituting UU_j, $0\leqq j\leqq k-1$, appearing in h_1, by such a linear combination iteratively, we have

$$h \equiv \beta_0+\beta_1 U+\beta_2 U_1+\cdots+\beta_k U_{k-1},$$

modulo $G^*E_{x',x_n,\lambda}\cdot S' \subset H_k \subset t^{k+s}E_{x',t,\lambda}$, where $\beta_j \in E_{x',\lambda}, 0\leqq j\leqq k$. Comparing terms of $t^0,t^s,\ldots,t^{s+k-1}$, we see $\beta_0=\beta_1=\cdots=\beta_k=0$. This shows (1').

(1) By (1'), we see that H_k is a finite $G^*E_{x',x_n,\lambda}$-module. Since i induces an isomorphism of $\mathbb{R}$ vector spaces $H_k/G^*m_{x',x_n,\lambda}H_k$ and $t^{k+a_n}E_t/t^{k+2a_n}E_t$, we have the required result, by the Malgrange-Mather preparation theorem [B].

3. Necessary condition

Take γ^* particularly such that $y_1 = -e(a_{n+1}-a_n)$ and $e(a_n-a_{n-j}) = -\dot{y}_{j+1}/\dot{y}_1$, $1\leqq j\leqq n$, $a_0=0$. It sufficies to set precisely

$$y_{j+1}=\binom{a_{n+1}-a_{n-j}-1}{a_n-a_{n-j}}e(a_{n+1}-a_{n-j}),\quad 1\leqq j\leqq n.$$

We set as in Theorem 1,

$$U=e(a_n)+x_1e(a_n-a_1)+\cdots+x_{n-1}e(a_n-a_{n-1}),$$

$$U_j=\int_0^t e(j)\frac{\partial U}{\partial t}dt,\quad j=0,1,2,\ldots,$$

$r=a_{n+1}-a_n$, $s=a_n-a_{n-1}$, $x'=(x_1,\ldots,x_{n-1})$ and

$$f=(x',U,U_r):\mathbb{R}^{n-1}\times\mathbb{R},0\to\mathbb{R}^{n+1},0.$$

Then U_j and f are weighted homogeneous of weight a_n+j and A respectively and f parametrizes the developable of the curve $\gamma=(\gamma^*)^*$.

The following necessary condition for determinacy is the base to pass various types through a sieve:

LEMMA 3.1. *If A determines C^∞ class of developable, then, for any $\ell\in\mathbb{N}$,*

$$U_{r+\ell}=XU_r+Z\circ f,$$

for a weighted homogeneous vector field germ X over $\mathbb{R}^{n-1}\times\mathbb{R},0$ and a weighted homogeneous function-germ Z on $\mathbb{R}^{n+1},0$ with $Xx_j\in f^*m_x, w(Xx_j)=a_j+\ell, 1\leqq j\leqq n-1, w(Xt)=1+\ell, XU\in f^*m_x$, $Z\in m_x$ and $w(Z)=a_{n+1}+\ell$.

If A determines the C^∞ class of full developable, then moreover Z can be taken from $x_{n+1}E_x$.

PROOF: Consider everything in the jet space $J^N(n, n+1)$ for sufficiently large N.

Let γ^*_λ be a type-preserving deformation of γ^* defined by

$$y_{1,\lambda} = y_1 - \lambda e(r+j), \text{ and, } -\dot{y}_{i+1,\lambda}/\dot{y}_{1,\lambda} = e(a_n - a_{n-i}), \ 1 \leqq i \leqq n.$$

Corresponding parametrization of developable is then given by $f_\lambda = (x', U, U_r + \lambda U_{r+\ell})$. Using the notations in [Ma], we have a necessary condition:

$$\partial f_\lambda/\partial\lambda|_{\lambda=0} = tf(X) - wf(Y), \text{ mod. } m_{x',t}^{N+1},$$

for some vector field germs X and Y over $\mathbb{R}^{n-1}\times\mathbb{R}, 0$ and $\mathbb{R}^{n+1}, 0$ with $X(0) = 0$ and $Y(0) = 0$ respectively.

Writing $X = \sum_{j=1}^{n-1} X_j\partial/\partial x_j + X_n\partial/\partial t$ and $Y = \sum_{k=1}^{n+1} Y_k\partial/\partial x_k$, we have $X_j = Y_j \circ f$, $1 \leqq j \leqq n-1$, $XU = Y_n \circ f$, and,

$$U_{r+\ell} = XU_r - Y_{n+1}\circ f, \text{ mod. } m_{x',t}^{n+1}.$$

Remark that the condition $Yx_{n+1} \in x_{n+1}E_x$ is equivalent to that $Y_{n+1} \in x_{n+1}E_x$.

From the weighted homogenuity of f and $U_{r+\ell}$, we can choose X and $Z = -Y_{n+1}$ satisfying the required conditions.

This shows Lemma 3.1.

Now let $n \geqq 2$. We cover the domain of types A except $(\mathrm{I})_{n,r}$, $(\mathrm{II})_{n,i}$ and $(\mathrm{III})_n$ by the following not necessarily disjoint nine classes:

(i) $r \geqq 2$ and $s \geqq 2$.

(ii) $s = 1, r \geqq 2$ and $a_{j+1} - a_j \geqq 2$ for some $j, 1 \leqq j \leqq n-2$.

(iii) $s \geqq 2, r = 1$ and $n \geqq 3$.

(iv) $s \geqq 2, r = 1$ and $a_1 \geqq 2$.

(v) $a_{j+1} - a_j \geqq 2, a_{j'+1} - a_{j'} \geqq 2$, for some j, j' with $1 \leqq j < j' \leqq n-2$.

(vi) $a_{j+1} - a_j \geqq 3$ for some $j, 1 \leqq j \leqq n-2$.

(vii) $a_1 \geqq 2$ and $a_{j+1} - a_j \geqq 2$ for some $j, 1 \leqq j \leqq n-2$.

(viii) $a_1 \geqq 4$ and $r = 1$.

(ix) $(1, s+1, s+2)$ or $(1, s+1, 2s+1)$, for $s \geqq 3$.

Then we have

LEMMA 3.2. *Assume $n \geqq 2$. Let $\ell \in \mathbb{N}$. Let $X = \sum_{j=1}^{n-1} X_j\partial/\partial x_j + X_n\partial/\partial t$ be a weighted homogeneous vector field germ over $\mathbb{R}^{n-1}\times\mathbb{R}, 0$ with $X_j \in f^*m_x, w(X_j) = a_j + \ell, 1 \leqq j \leqq n-1, w(X_n) = 1 + \ell$ and $XU \in f^*m_x$.*

If $\ell = 1$ and A is in one of classes from (i) *to* (vii), (*resp.* $\ell = 2$ *and* A *is in one of classes* (viii) *and* (ix)), *then necessarily* X *satisfies*

$$X_j \in m_{x'}E_{x',t}, \ 1 \leqq j \leqq n-1, \quad \text{and} \quad Xt \in x_1tE_{x',t} \subset tE_{x',t'} \cap m_{x'}E_{x',t}.$$

PROOF: Write $X_j = Y_j(x', U, U_r), 1 \leqq j \leqq n-1$, and

$$(*) \qquad XU = \sum_{j=1}^{n-1} X_j e(a_n - a_j) + X_n \partial U/\partial t = Y_n(x', U, U_r).$$

Let $\ell = 1$. Since $w(X_j) = a_j + 1 \leqq a_n < a_{n+1}$, we see $X_j = Y_j(x') \in m_{x'}^2, 1 \leqq j \leqq n-2$, and $X_{n-1} = \alpha U + \beta(x'), \alpha \in \mathsf{R}, \beta \in m_{x'}^2$. Further, if $s = a_n - a_{n-1} \geqq 2$, then $\alpha = 0$. Set $X_n = At^2 + Bx_1t + C(x_1, x_2)$ wit $A, B \in \mathsf{R}, C \in m_{x'}$.

If $s \geqq 2$, then we have $C = 0$, comparing the terms in (*) of order $s-1$ with respect to t. Therefore, in the case (i), we see $X_j \in m_{x'}^2 \subset m_{x'}E_{x',t}$. Moreover, we have $A = 0$, comparing the terms of t^{a_n+1} in (*). Hence, $X_n = Bx_1t \in x_1tE_{x',t}$.

In the case (ii), $a_n - a_j > a_n - a_{j+1} + 1$. We compare the terms of order $a_n - a_{j+1} + 1$ with respect to t in (*). Then we have $C = 0$. Further, set $x_1 = \cdots = x_j = x_{j+2} = \cdots = x_{n-1} = 0$. Then, from (*), we see $\alpha tU(x_{j+1}, t) + At^2\partial U/\partial t(x_{j+1}, t) = 0$. Since two polynomials $tU(x_{j+1}, t)$ and $t^2\partial U/\partial t(x_{j+1}, t)$ are linearly independent over R, we have $\alpha = A = 0$.

Also for the cases from (iii) to (vii), the proofs are similar, and we omit them.

Let $\ell = 2$.

In the case (viii), we see that $X_n = At^3$, for some $A \in \mathsf{R}$, and that $XU \in m_{x'}^2$, since $w(X_n) = 3$ and $w(XU) = a_n + 2$. If $s \geqq 3$, then $X_j \in m_{x'}^2, 1 \leqq j \leqq n-1$. Hence $A = 0$. In the case $s = 1$, (resp. 2) and $n = 2$, we set $X_1 = \alpha U_1 + \beta(x')$, (resp. $X_1 = \alpha U + \beta(x')$), $\alpha \in \mathsf{R}, \beta \in m_{x'}^2$. Then, by (*), $\alpha U_1 t + At^3\partial U/\partial t = 0$ (resp. $\alpha U_1 e(2) + At^3\partial U/\partial t = 0$). In both cases we have $\alpha = A = 0$. Also for the case $n \geqq 3$, the proof is similar.

In the case (ix), we have $X_1 \in m_{x'}^2$ and $XU \in m_{x'}E_{x',t}$, since $w(X_1) = 3, w(XU) = s+3$. Comparing the terms of order $s-1$ and $s+3$ with respect to t in (*), respectively, we see $X_n \in x_1tE_{x',t}$.

Q.E.D.

Now assume the equality $U_{r+\ell} = XU_r + Z \circ f$ holds as in Lemma 3.1 and assume further X satisfies $Xx_j \in m_{x'}E_{x',t}, 1 \leqq j \leqq n-1$, and $Xt \in tE_{x',t} \cap m_{x'}E_{x',t}$.

We recall object H_k introduced in §2 for the case without parameter ($\ell = 0$). Remark that all components of f belong to H_0, and therefore $f_x^E \subset H_0$. Since $Xx_j, 1 \leqq j \leqq n-1$ and XU belong to $f^*m_x \subset H_0$ and $Xt \in tE_{x',t}$ from the assumption, and $U_r \in H_r$, we see XU_r belongs to H_r, by Lemma 2.3. Since $U_{r+\ell}$ also belongs to H_r, the weighted homogeneous polynomial $Z \circ f = Z(x', U, U_r)$ must be an element of H_r of weight $a_{n+1} + \ell$. Moreover, since $Xx_j, 1 \leqq j \leqq n-1$ and Xt belong to $m_{x'}E_{x',t}$, we have $XU_r \in m_{x'}E_{x',t}$. Hence $U_{r+\ell}(0, t) = Z \circ f(0, t)$.

If $\ell < a_{n+1} = w(U_r)$, then we can write $Z \circ f$ as the following form: $Z \circ f = A(x')U_r + \alpha P$, with $\alpha \in \mathbf{R} - 0$ and

$$P = U^m + p_1(x')U^{m-1} + \cdots + p_{m-1}(x')U + p_m(x'),$$

with $m \in \mathbf{N}, m > 1$ and $p_i(0) = 0, 1 \leqq i \leqq m$. Then we see that P belongs to H_r and P is of weight $a_{n+1} + \ell$.

By Lemma 2.1, we have $ma_n = a_{n+1} + \ell$, and $ms \geqq r + s$.

Then we easily see that $1 < m \leqq 1 + (\ell/a_{n-1})$ and that the last equality holds only if $s = (ra_{n-1}/\ell)$. Therefore, $a_{n-1} \leqq \ell$ and further if $a_{n-1} = \ell$ then $s = r$.

Now first we apply the above argument to the classes from (i) to (vii) for $\ell = 1$. Then we have $n = 2$, $a_1 = 1$ and $s = r$. Therefore only the types $(1, s+1, 2s+1)$ survive in this case.

Next apply to the classes (viii) and (ix) for $\ell = 2$. Then we have $a_1 \leqq a_{n-1} \leqq 2$. Thus the class (viii) drops out. For the types $(1, s+1, 2s+1), s \geqq 3$, we have $m = (a_{n+1}+2)/a_n = 2 + 1/(s+1)$, which leads a contradiction. Also for the types $(1, s+1, s+2), s \geqq 3$, we have $m = 1 + 3/(s+1)$, which leads a contradiction.

By Lemma 3.2, only the type $(1, 3, 5)$ satisfies the necesary condition in Lemma 3.1 except for the types $(\mathrm{I})_{n,r}$, $(\mathrm{II})_{n,i}$ and $(\mathrm{III})_n$ in the case $n \geqq 2$.

In the case $n = 1$, set $A = (a_1, a_2) = (s, s+r)$. The necessary condition of Lemma 3.1 is reduced to that, for any $\ell \in \mathbf{N}$, there exist $A(t)$ of weight $1 + \ell$, $Y_1(x_1, x_2)$ of weight $s + \ell$, $Y_2(x_1, x_2)$ of weight $s + r + \ell$ and $c \neq 0$ such that

$$A(t)t^{s-1} = Y_1(t^s, t^{r+s}), \qquad t^{s+r+\ell} = cA(t)t^{s+r-1} + Y_2(t^s, t^{s+r}).$$

Then it is easy to see that only the types $(\mathrm{IV})_m$ and (V) satisfy this condition.

Summarizing these arguments, we have

PROPOSITION 3.3. *If A is not one of types in Theorem 1, then A does not determine the C^∞ class of developable.*

REMARK 3.4: In the case $A = (2, 2m+1), (3, 5), (1, 3, 5)$ or $(3, 4, \ldots, n+1, n+2)$, we see $Z \notin x_{n+1}E_x$ in the above argument. By Lemma 3.1, we see the type $(2, 2m+1)$ (resp. $(3, 5)$, $(1, 3, 5)$, $(3, 4, \ldots, n+1, n+2)$) does not determine the C^∞ class of full developable.

4. Developables of curves of type (1, 2, ..., n–1, n–1+s, n–1+s+r)

To clarify the determinacy of developables of curves of types $(\mathrm{I})_{n,r}$, $(\mathrm{IV})_m$, (V) and (VI), we study the triviality of deformations of the parametrization f in spacial cases.

PROPOSITION 4.1. *Let s and r be positive integers. If $\mathsf{A} = (1, 2, \ldots, n-1, n-1+s, n-1+s+r)$, then the map-germ $f : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^{n+1}, 0$ is C^∞ right left equivalent to (x', U, V), such that $U = e(n-1+s) + x_1 e(n-2+s) + \cdots + x_{n-1} e(s)$, V belongs to H_r introduced in §2 ($\ell = 0$), and that the initial part of V is equal to U_r.*

PROOF: Since $\mathsf{B} = \mathsf{A}^* = (r, r+s, r+s+1, \ldots, r+s+n-2, r+s+n-1)$, the dual curve can be represented by

$$y_1 = -e(r) + o(t^r),$$
$$y_{j+1} = \binom{r+s+j-2}{s+j-1} e(r+s+j-1) + o(t^{r+s+j-1}), \qquad 1 \leqq j \leqq n.$$

(See the beginning of §3.)

Then we see that the initial part of the component x_n (resp. x_{n+1}) of f is equal to U (resp. U_r) and that $\partial x_{n+1}/\partial t \in (t^r \partial x_n/\partial t)E_{x',t}$ and $x_{n+1} \in t^r E_{x',t}$.

We intend to find a diffeomorphism-germ $\sigma : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$ of the (x', t) space such that σ is of form $\sigma(x', t) = (X'(x'), tT(x', t))$ and that $f \circ \sigma^{-1} = (x', U, V)$ for some $V \in m_{x',t}$. Then we see automatically that $\partial V/\partial t \in (t^r \partial U/\partial t)E_{x',t}$, and $V \in t^r E_{x',t}$, that is, $V \in H_r$. After a coordinate change $x_{n+1} \mapsto a x_{n+1}, a \in \mathbf{R} - 0$, we may assume the initial part of V is equal to U_r.

To obtain such a σ, we set $U' = x_n - U$ and set

$$\mathcal{U}(x', t, \lambda) = U(x', t) + \lambda U'(x', t), \qquad \lambda \in \mathbf{R}.$$

By the homotopy method, it sufficies to check, for each $\lambda_0 \in \mathbf{R}$, that $U' = \xi \mathcal{U}$ for some vector field germ ξ over $\mathbf{R}^n \times \mathbf{R}, (0, \lambda_0)$ of form

$$\xi = \sum_{i=1}^{n-1} R_i(x', \lambda) \frac{\partial}{\partial x_i} + tQ(x', t, \lambda) \frac{\partial}{\partial t},$$

with $R_i(0, \lambda) = 0, 1 \leqq i \leqq n-1$, the existence of which is easily verified by the Malgrange-Mather division theorem [B].

Q.E.D.

Next we set $W(x', t, \lambda) = U_r + \lambda(V - U_r)$, for $\lambda \in \mathbf{R}$, and set

$$F = (x', U, W, \lambda) : \mathbf{R}^n \times \mathbf{R}, 0 \times \mathbf{R} \longrightarrow \mathbf{R}^{n+1} \times \mathbf{R}, 0 \times \mathbf{R}.$$

Then, to trivialize this family, it sufficies to solve, for each $\lambda_0 \in \mathbf{R}$, the equation

$$\partial F/\partial \lambda = tF(\tilde{X}) - \omega F(\tilde{Y}),$$

at $(0,0,\lambda_0)$, for some vector field germ $\tilde{X}$ over $\mathbf{R}^n \times \mathbf{R}, (0,\lambda_0)$ and $\tilde{Y}$ over $\mathbf{R}^{n+1} \times \mathbf{R}, (0,\lambda_0)$ respectively. Set $\tilde{X} = X + \partial/\partial\lambda$. Then it sufficies to solve

$$(**) \qquad V - U_r = XW + Z \circ F,$$

for a family X of vector field germ over $\mathbf{R}^n, 0$ and a family Z of function germs on $\mathbf{R}^{n+1}, 0$ near λ_0, respectively, such that Xx_j, $1 \leqq n-1$, and XU belong to $F^*E_{x,\lambda}$.

Recall H_k introduced in §1 in the one-parameter case ($\ell = 1$). Then $V - U_r$ and W belong to H_r.

Denote by T the totality of function germs in $E_{x',t,\lambda}$ of form of the right hand side of (**), and by T' the set of function germs in T with the condition $Z \in x_{n+1}E_{x,\lambda}$.

We set, as in §2,

$$G = (x', U, \lambda) : \mathbf{R}^n \times \mathbf{R}, 0 \times \mathbf{R} \longrightarrow \mathbf{R}^n \times \mathbf{R}, 0 \times \mathbf{R}.$$

Then H_r, T and T' are $G^*E_{x',x_n,\lambda}$-modules.

If $H_r \subset T$, for each $\lambda_0 \in \mathbf{R}$, then A determines the C^∞ class of developable. If $H_r \subset T'$, for each $\lambda_0 \in \mathbf{R}$, then A determines the C^∞ class of full developable. Lemma 2.3 makes easy verifying the inclusion $H_r \subset T$ or $H_r \subset T'$. Remark that $a_n = n - 1 + s$. Now we examine, that there exists a finite system $S = \{h_1, \ldots, h_{n-1+s}\}$ of elements in $H_r \cap T$ or $H_r \cap T'$ generating H_r as $G^*E_{x',x_n,\lambda}$-module.

LEMMA 4.2. (1) *For $(I)_{n,r}$, $(s = 1)$, there exist vector fields $\xi_1, \ldots, \xi_n$ over $\mathbf{R}^n \times \mathbf{R}, (0,\lambda_0)$ such that $\xi_j x_i \in G^*E_{x',x_n,\lambda}, \xi_j U \in G^*E_{x',x_n,\lambda}, \xi_j t \in tE_{x',t,\lambda}, 1 \leqq j \leqq n, 1 \leqq i \leqq n-1$, and that the set of elements $h_1 = \xi_1 W, \ldots, h_n = \xi_n W$ of $H_r \cap T'$ generates H_r.*

(2) *For $(1,3,5), (n = 2, s = 2, r = 2)$, set*

$$\xi = -(t^3/3! + xt^2/2! - x^3)\partial/\partial x + (t^3/3! + (1/3)xt^2/2!)\partial/\partial t.$$

Then the set of elements $h_1 = W, h_2 = U^2$ and $h_3 = \xi W$ of $H_2 \cap T$ generates H_2.

(3) *For $(3,5), (n = 1, s = 3, r = 2)$, we define $a \in E_{t,\lambda}$ by $a(t,\lambda)\partial U/\partial t = W$. Then the set of elements $h_1 = W, h_2 = U^2$ and $h_3 = a\partial W/\partial t$ of $H_2 \cap T$ generates H_2.*

(4) *For $(2, 2m+1), (n = 1, s = 2, r = 2m-1)$, the set of elements $h_1 = W$ and $h_2 = U^{m+1}$ of $H_{2m-1} \cap T$ generates H_{2m-1}.*

PROOF: (1) Remark that, in this case, $U = e(n) + x_1 e(n-1) + \cdots + x_{n-1}e(1)$. Since $1, \partial U/\partial x_1, \ldots, \partial U/\partial x_{n-1}$ generate $E_{x',t,\lambda}$ over $G^*E_{x',x_n,\lambda}$, we can write, for $1 \leqq j \leqq n$,

$$t^j \partial U/\partial t = \sum_{i=1}^{n-1} a_{ij} \circ G\ \partial U/\partial x_i + a_{0j} \circ G, \qquad a_{ij} \in E_{x',x_n,\lambda}.$$

Then we set $\xi_j = -\sum_{i=1}^{n-1} a_{ij} \circ G\partial/\partial x_i + t^j\partial/\partial t$. Then we see $\xi_j W$ belongs to H_r and T', by Lemma 2.3. The order of $(\xi_j W)(0,t,\lambda_0)$ with respect to t is equal to $j + n + r - 1$. Thus

$\{i^*h_1, \dots, i^*h_n\}$ generate $t^{n+r}E_t/t^{2n+r-1}E_t$. Hence, by Lemma 2.4, we have the required result.

Similarly also (2), (3) and (4) follow from Lemma 2.4.

PROOF OF THEOREM 1: By Lemma 4.2, we see the types $(\mathrm{I})_{n,r}$, $(\mathrm{IV})_m$, (V) and (VI) determine the C^∞ classes of developables respectively. In [I4], we prove so the types $(\mathrm{II})_{n,i}$ and $(\mathrm{III})_n$. By Proposition 3.3, other classes do not determine the C^∞ classes of developables. These yields Theorem 1.

PROOF OF THEOREM 2: By Lemma 4.2.(1), we see the types $(\mathrm{I})_{n,r}$ determines the C^∞ class of full developables. We prove so for the types $(\mathrm{II})_{n,i}$ in [I4]. Combined with Remark 3.4, we have Theorem 2.

5. Mond's theorem

Based on Theorem 1, we reprove the following result obtained by Mond [M1], [M2, Corollary 0.2]:

THEOREM 5.1 (MOND). *Let $\gamma : \mathbf{R}, 0 \longrightarrow \mathbf{R}P^3$ be a curve-germ of type $(1, 2, 2+r)$. Then dev(γ) : $\mathbf{R}^2, 0 \longrightarrow \mathbf{R}P^3$ is a topological embedding if r is odd, and dev(γ) has a single curve of selfintersection if r is even.*

PROOF: By Theorem 1, dev(γ) is C^∞ equivalent to the germ at 0 of

$$f(x,t) = (x, \frac{t^2}{2} + xt, \int_0^t \frac{s^r}{r!}(s+x)ds) : \mathbf{R}^2 \longrightarrow \mathbf{R}^3.$$

Now, assume $f(x_1, t_1) = f(x_2, t_2), (x_i, t_i) \in \mathbf{R}^2, i = 1, 2$. Then we see $x_1 = x_2, x_1 = -(1/2)(t_1 + t_2)$ and $\int_{t_1}^{t_2} s^r(s + x_1)ds = 0$. Thus, setting $\sigma = s + x_1$, we have

$$\int_{-a}^{a} (\sigma - x_1)^r \sigma d\sigma = 0 \qquad \cdots (*),$$

where $a = (1/2)(t_2 - t_1)$.

If r is odd, then the left hand side of (*) is equal to an integral from $-a$ to a with a polynomial integrand with positive monomials of even degrees. Hence we have $a = 0$. This means that $(x_1, t_1) = (x_2, t_2)$ and that f is injective. Since f is a proper mapping, we see f is a topological embedding.

By a similar argument, if r is even, then we have $x_1 = 0$ or $(x_1, t_1) = (x_2, t_2)$. Further $f|\{x = 0\} = (0, t^2/2, (r+1)\{t^{r+2}/(r+2)!\})$ is 2 to 1. Therefore in this case we see f is an embedding in the complement of a double point curve $\{x = 0\}$.

REFERENCES

[A1] V.I. Arnol'd, *Lagrangian manifolds with singularities, asymptotic rays, and the open swallowtail*, Funct. Anal. Appl. **15-4** (1981), 235–246.

[A2] V.I. Arnol'd, "Singularities of Caustics and Wave Fronts," Kluwer Academic Publishers, 1990.

[AGV] V.I. Arnol'd, S.M. Gusein-Zade, A.N. Varchenko, "Singularities of Differentiable Maps I," Birkhäuser, 1985.

[B] Th. Bröcker, "Differentiable Germs and Catastrophes," London Math. Soc. Lect. Note Series 17, Cambridge Univ. Press, 1975.

[Cl] J.P. Cleave, *The form of the tangent developable at points of zero torsion on space curves*, Math. Proc. Camb. Phil. Soc. **88** (1980), 403–407.

[D] Ph. Delanoë, *L'opérateur de Monge-Ampère réel et la géométrie des sous-variétés*, in "in Geometry and Topology of Submanifolds," ed. by J.-M. Morvan and L. Verstraelen, World Scientific, 1989, pp. 49–72.

[GP] T. Gaffney, A. du Plessis, *More on the determinacy of smooth map-germs*, Invent. Math. **66** (1982), 137–163.

[G] V.V. Golunov, *Singularities of projections of complete intersections*, Current Problems in Math., VITINI, Akad. Nauk. SSSR **22** (1983), 167–206.

[I1] G.Ishikawa, *Families of functions dominated by distributions of C-classes of map-germs*, Ann. Inst. Fourier **33–2** (1983), 199–217.

[I2] G. Ishikawa, *The local model of an isotropic map-germ arising from one dimensional symplectic reduction*, Math. Proc. Camb. Phil. Soc. **111–1** (1992), 103–112.

[I3] G. Ishikawa, *Parametrization of a singular Lagrangian variety*, to appear, Trans. Amer. Math. Soc..

[I4] G. Ishikawa, *Determinacy of envelope of the osculating hyperplanes to a curve*, Prepint, Hokkaido University.

[K] M.É. Kazaryan, *Singularities of the boundary of fundamental systems, flat points of projective curves, and Schubert cells*, in "Itogi Nauki Tekh., Ser. Sorrem. Probl. Mat. (Comtemporary Problems of Mathematics) 33," VITINI, 1988, pp. 215–232.

[M] B. Malgrange, "Ideals of Differentiable Functions," Oxford Univ. Press, 1966.

[Ma] J.N. Mather, *Stability of C^∞ mappings III: Finitely determined map-germs*, Publ. Math. I.H.E.S. **35** (1968), 127–156.

[M1] D. Mond, *On the tangent developable of a space curve*, Math. Proc. Camb. Phil. Soc. **91** (1982), 351–355.

[M2] D. Mond, *Singularities of the tangent developable surface of a space curve*, Quart. J. Math. Oxford **40** (1989), 79–91.

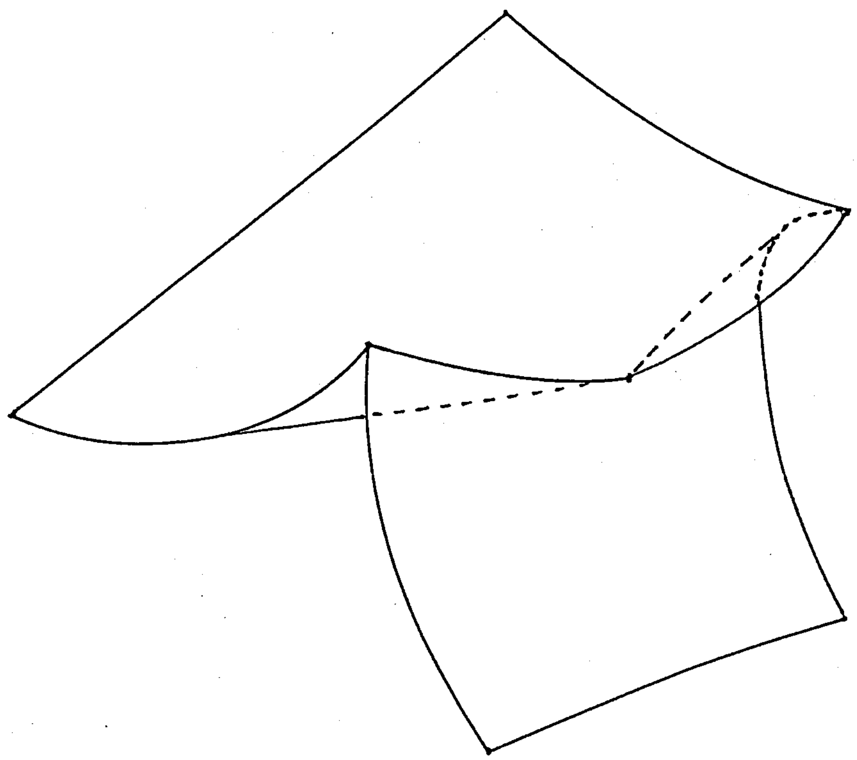
[Mo] T. Morimoto, *La géométrie des équations de Monge-Ampère*, C. R. Acad. Sc. Paris **289** (1979), 25–28.

[P] R. Piene, *Numerical characters of a curve in projective n-space*, in "Real and Complex Singularities," ed. by P. Holm, Sijthoff & Noordhoff International Publishers, 1977, pp. 475–495.

[S1] O.P. Shcherbak, *Projectively dual space curves and Legendre singularities*, Trudy Tbiliss. Univ. **232-233** (1982), 280–336.

[S2] O.P. Shcherbak, *Wavefront and reflection groups*, Russian Math. Surveys **43-3** (1988), 149–194.

Figure 1 : The developable of a curve of type (3,4,5).



(Please insert this in p.   )