GLOBAL EXISTENCE OF SOLUTIONS OF SEMILINEAR WAVE EQUATIONS WITH DATA OF NON COMPACT SUPPORT IN ODD SPACE DIMENSIONS

Hideo Kubo

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Hideo Kubo

Department of Mathematics
Hokkaido University
Sapporo 060, Japan

§1 Introduction and statement of the results

This paper studies the existence of global solutions of the Cauchy problem of the form:

$$(E) \qquad \begin{cases} u_{tt} - \Delta_x u = a u_t^p + b|\nabla_x u|^{2q} & \text{in} \quad \mathbb{R}^n \times [0, \infty), \\ u(x,0) = f_0(x), \quad u_t(x,0) = g_0(x) & \text{for} \quad x \in \mathbb{R}^n, \end{cases}$$

where a and b are real constants, p and q are positive integers.

For $n > 3$, S.Klainerman [7] showed the existence of unique global C^∞-solutions of (E) with $p \geq 2$ for the sufficiently small and smooth data which are compactly supported. When $n = 3$ and $p \geq 2$, he obtained the lower bounds for the so-called life span, namely, the maximal time interval where C^∞-solutions of (E) exist. When the initial data may have singularity, P.Godin [4] and [5] obtained the global and long time existence results of piecewise smooth progressing waves of (E) with $p \geq 2$ or second order quasilinear equations.

On the other hand, when $n = 3$, F.John [6] has obtained the same life span estimate as [7] for radial C^2-solutions of (E) with $a = 1$, $b = 0$, and $p = 2$, even if the data do not have compact support. When $a = 1$, $b = 0$, and $p \geq 3$, H.Takamura [8] showed the global existence of unique radial C^2-solutions of (E) with $n = 3$.

Furthermore, F.Asakura [2] applied the method proposed in [6] to the case $n = 5$. But unfortunately, it seems that his estimates for the nonlinear terms would need more explanations. Although it is possible to complete the proof following [2], the real proof would make the computations very involved and technical. In fact, there arise undesirable unbounded terms which require nice cancellations among themselves.

One of the aims of this paper is to present a new formulation of the problem. This enables us to avoid such technicalities as well as the previous proof much simpler. Further details will be stated in Remark 5.7 below. We next point out the assumption of the regularity for the initial data. It is assumed that $f \in C^5(\mathbb{R})$, $g \in C^4(\mathbb{R})$ in [2]. By virtue of Proposition 2.1 below, however, it is possible to weaken these assumptions as $f \in C^4(\mathbb{R})$, $g \in C^3(\mathbb{R})$. Moreover, our formulation leads to a natural regularity result as far as radial solutions are concerned.

Let $u(x,t)$ be a C^2-solution of (E) which depends only on $r = |x|$ and t. Then, f_0 and g_0 are radially symmetric. Note that for any radially symmetric function on $\mathbb{R}^n$, the associated function of $r = |x|$ on the half line can be extended to the whole line as an even function with the same regularity. Indeed, we use Taylor's formula with respect to r at the origin to find that the odd order derivatives vanish at the origin. Let f and g be the extensions of f_0 and g_0 as above, respectively. We consider the radially symmetric version of (E) in $\mathbb{R} \times [0,\infty)$:

$$(E^*) \qquad \begin{cases} u_{tt} - u_{rr} - \dfrac{n-1}{r} u_r = G(u_t, u_r) & \text{for} \quad (r,t) \in \mathbb{R} \times [0,\infty), \\ u(r,0) = f(r), \quad u_t(r,0) = g(r) & \text{for} \quad r \in \mathbb{R}, \end{cases}$$

where $G(u_t, u_r) = a u_t^p + b u_r^{2q}$.

We set for $f \in C^{m+k+3}(\mathbb{R})$ and $g \in C^{m+k+2}(\mathbb{R})$,

$$H_f(\rho) = \int_0^\rho (\frac{1}{2\sigma}\frac{d}{d\sigma})^m (\sigma^{2m+1} f(\sigma)) d\sigma,$$
$$\eta_k = \max_{0 \le j \le k+3} \sup_{\rho \in \mathbb{R}} \{|H_f^{(j+1)}(\rho)| + |H_g^{(j)}(\rho)|\},$$

where $m = \dfrac{n-3}{2}$ and $k \in \mathbb{Z}_+$, the set of nonnegative integers.

In addition, we consider the following condition:

$$\text{(A.1)} \qquad \begin{aligned} p,\ 2q &\ge \frac{n+3}{4} \quad \text{if} \quad n \ge 5, \\ p,\ 2q &\ge 3 \quad \text{if} \quad n = 3. \end{aligned}$$

Now we state the main result of this paper.

MAIN THEOREM. *Let n be an odd integer with $n \ge 3$. Suppose that $f \in C^{m+k+3}(\mathbb{R})$ and $g \in C^{m+k+2}(\mathbb{R})$ are even functions. If η_k is sufficiently small and $(A.1)$ holds, the problem (E^*) admits a unique global C^{k+2}-solution which is even in r.*

We notice that this theorem refines the results of [2] and [8]. But, for $n \ge 7$, the assumption $(A.1)$ is stronger than the condition $p,\ 2q > \dfrac{n+1}{n-1}$ which was conjectured in [8]. It is remarkable that if $a + b = 0$, $p = 2$, and $q = 1$, the problem (E) has a unique global solution in general space dimensions. Indeed, setting $v(x,t) = exp\{-au(x,t)\}$, we can rewrite the equation as $v_{tt} - \Delta_x v = 0$.

The proof of the Main Theorem for $k \ge 1$ has the same line as that of the case $k = 0$. So we concentrate the latter case, but a necessary modification will be discussed in Remark 2.2.

This paper is organized as follows. In Section 2, we convert the problem (E^*) into a system of integral equations (H) and fix some notations. The proof of the Main Theorem, which is given in Section 3, is completely classical and independent of energy estimates and of formation of support of data. In Sections 4 and 5, we carry out the estimates for solutions of the linear problem and for the nonlinear terms, which are used in Section 3. In Appendix, we prove the assertion of the Main Theorem for $n = 3$. It is a simple application of the arguments done for the case $n \geq 5$.

§2 Formulation of the Problem

Since the uniqueness of solutions to (E) is valid (see e.g. [6], Theorem 4 or R.Agemi [8], p.156), in order to prove the Main Theorem for $k = 0$, it suffices to construct a C^2-solution of (E^*). To do this, we convert the problem (E^*) into a system of integral equations (H).

As is well known, a solution of (E^*) is furnished by a solution of the following integral equation:

$$u(r,t) = u^0(r,t) + L(u_t, u_r)(r,t) \qquad \text{for} \quad r \neq 0, \tag{2.1}$$

where $u^0(r,t)$ is the solution of the linear wave equation, (3.2) below, which will be studied in Section 4 and

$$\begin{aligned} L(u_1,u_2)(r,t) =& \frac{(-1)^m}{2m!\, r^{2m+1}} \int_0^t d\tau \int_{t-\tau-r}^{t-\tau+r} (\rho^{2m+1}\{au_1^p(\rho,\tau) + bu_2^{2q}(\rho,\tau)\}) \\ & (\frac{\partial}{\partial\rho}\frac{1}{2\rho})^m (\phi^m(\rho,r,t-\tau))d\rho, \\ \phi(\rho,r,t) =& r^2 - (t-\rho)^2, \quad m = \frac{n-3}{2}. \end{aligned} \tag{2.2}$$

(See e.g. Courant-Hilbert [3], Chap.VI, §13). Suppose that $u(r,t)$ is a C^2-solution of (2.1), then $u_t(r,t)$ and $u_r(r,t)$ satisfy the following system with $u_1 = u_t$, $u_2 = u_r$:

$$\begin{cases} u_1(r,t) = u_t^0(r,t) + D_t L(u_1,u_2)(r,t), \\ u_2(r,t) = u_r^0(r,t) + D_r L(u_1,u_2)(r,t), \end{cases} \tag{2.3}$$

for $r \neq 0$.

Conversely, we can construct $u \in C^2(\mathbb{R} \times [0,\infty))$ satisfying (2.1) from the C^1-solution of (2.3). In fact:

PROPOSITION 2.1. *Let u_1, $u_2 \in C^1(\mathbb{R} \times [0,\infty))$ satisfy (2.3). Set*

$$u(r,t) = f(r) + \int_0^t u_1(r,\tau)d\tau. \tag{2.4}$$

Then $u(r,t)$ is a C^2-solution of (2.1) in $\mathbb{R} \times [0,\infty)$.

PROOF: Since $L(u_1,u_2)(r,t)$ is a C^2-function for $r \neq 0$ and $u^0(r,t)$ is a C^2-solution of the linear problem, it follows from (2.3) that

$$D_r u_1(r,t) = D_t u_2(r,t) \qquad \text{for} \quad r \neq 0. \tag{2.5}$$

Moreover, since u_1, $u_2 \in C^1(\mathbb{R} \times [0,\infty))$, (2.5) holds for any $(r,t) \in \mathbb{R} \times [0,\infty)$. We next deduce the following relation:

$$f(r) + \int_0^t u_1(r,\tau)d\tau = f(0) + \int_0^t u_1(0,\tau)d\tau + \int_0^r u_2(\rho,t)d\rho. \tag{2.6}$$

Indeed, by (2.5), we have

$$u_2(r,t) = u_2(r,0) + \int_0^t D_r u_1(r,\tau)d\tau. \tag{2.7}$$

Noting that $u_2(r,0) = f'(r)$ and integrating (2.7) with respect to r, we obtain (2.6).

Combining (2.4) and (2.6), we have $u_t(r,t) = u_1(r,t)$, $u_r(r,t) = u_2(r,t)$. This implies that $u(r,t) \in C^2(\mathbb{R} \times [0,\infty))$ and that

$$u_1(r,t) = u_t^0(r,t) + D_t L(u_t,u_r)(r,t) \qquad \text{for} \quad r \neq 0. \tag{2.8}$$

Integrating (2.8) with respect to t and using(2.4), we have (2.1).

From now on, let $m \geq 1$. Making use of the decay property of the solutions to the wave equations, we convert the problem (2.3) into the following:

$$(H) \qquad \begin{cases} U(r,t) = U^0(r,t) + \dfrac{r^{m+1}}{< r >^{m-1}} D_t L(\dfrac{< r >^{m-1}}{r^{m+1}} U, \dfrac{< r >^{m-1}}{r^{m+1}} V)(r,t), \\ V(r,t) = V^0(r,t) + \dfrac{r^{m+1}}{< r >^{m-1}} D_t L(\dfrac{< r >^{m-1}}{r^{m+1}} U, \dfrac{< r >^{m-1}}{r^{m+1}} V)(r,t), \end{cases}$$

where

$$U^0(r,t) = \frac{r^{m+1}}{< r >^{m-1}} u_t^0(r,t), \quad V^0(r,t) = \frac{r^{m+1}}{< r >^{m-1}} u_r^0(r,t),$$
$$< r > = \sqrt{1+r^2}.$$

Note that r^{m+1} is the usual decay rate and the factor $< r >^{m-1}$ is necessary to control the behavior of functions as r tends to ∞.

We introduce some function spaces on which we intend to study the problem (H). Let $B^k(\mathbb{R} \times [0,\infty))$ denote the space of functions $u(r,t) \in C^k(\mathbb{R} \times [0,\infty))$ such that

$$\|u\|_k = \max_{|\alpha| \leq k} \sup\{|D^\alpha u(r,t)| : (r,t) \in \mathbb{R} \times [0,\infty)\} < \infty.$$

We agree that Γ_1 and Γ_2 defined by the following are the closed subspaces of $B^k(\mathbb{R} \times [0,\infty))$:

$$\Gamma_1 = \{U \in B^{m+2}(\mathbb{R} \times [0,\infty)) : D_1^i U(0,t) = 0 \ (0 \leq i \leq m),$$
$$U(-r,t) = (-1)^{m+1} U(r,t)\},$$
$$\Gamma_2 = \{V \in B^{m+2}(\mathbb{R} \times [0,\infty)) : D_1^i V(0,t) = 0 \ (0 \leq i \leq m),$$
$$V(-r,t) = (-1)^m V(r,t)\}.$$

$\Gamma = \Gamma_1 \times \Gamma_2$ is a Banach space with respect to the norm:

$$\|(U,V)\|_\Gamma = \|U\|_{m+2} + \|V\|_{m+2}.$$

Note that if we find $(U,V) \in \Gamma$ satisfying (H), we construct a solution of the original problem (E^*). In fact, setting

$$u_1(r,t) = \frac{< r >^{m-1}}{r^{m+1}} U(r,t), \quad u_2(r,t) = \frac{< r >^{m-1}}{r^{m+1}} V(r,t),$$

we find that u_1, u_2 satisfy (2.3) and belong to $C^1(\mathbb{R} \times [0,\infty))$.

REMARK 2.2. *The regularity of the solution of (E^*) is closely related to the definition of the function space Γ. Let $k \in \mathbb{Z}_+$. If we intend to get the C^{k+2}-solution* *of* (E^*), *it is necessary to consider the problem* (H) *on the following function space* $\Gamma = \Gamma_{1,k} \times \Gamma_{2,k}$, *where*

$$\Gamma_{1,k} = \{U \in B^{m+k+2}(\mathbb{R} \times [0,\infty)) : D_1^i U(0,t) = 0 \ (0 \le i \le m), \\ U(-r,t) = (-1)^{m+1} U(r,t)\},$$

$$\Gamma_{2,k} = \{V \in B^{m+k+2}(\mathbb{R} \times [0,\infty)) : D_1^i V(0,t) = 0 \ (0 \le i \le m), \\ V(-r,t) = (-1)^{m} V(r,t)\}.$$

Accordingly, we have to change the statements of the propositions and lemmas below. Although it seems an easy task, we shall point out the essential point in Remark 5.3.

§3 Proof of the Main Theorem

In this section, we shall prove the Main Theorem with $m \ge 1$ by assuming the following two propositions. One is concerned with a priori estimates for the integral terms of the system of integral equations (H) which will be proved in Section 5. To state it, we set

$$\text{(3.1)} \qquad \begin{aligned} \Xi_1(U,V) &= \frac{r^{m+1}}{< r >^{m-1}} D_t L\Big(\frac{< r >^{m-1}}{r^{m+1}} U, \frac{< r >^{m-1}}{r^{m+1}} V\Big), \\ \Xi_2(U,V) &= \frac{r^{m+1}}{< r >^{m-1}} D_r L\Big(\frac{< r >^{m-1}}{r^{m+1}} U, \frac{< r >^{m-1}}{r^{m+1}} V\Big), \end{aligned}$$

where $m = \dfrac{n-3}{2}$. Throughout this paper we denote $A = O(B)$ when there exists a universal constant κ such that $|A| \le \kappa |B|$ for all A, B in question.

PROPOSITION 3.1. *Let n be an odd integer with $n \ge 5$. Suppose that* $(A.1)$ *holds. Then for* $W = (U,V) \in \Gamma$, $\Xi_i(W) \in \Gamma_i$ *and*

$$||\Xi_i(W)||_{m+2} = O(||U||_{m+2}^p + ||V||_{m+2}^{2q}) \quad (i = 1,2).$$

The other is concerned with the properties of the solutions of the linear problem of the form:

$$
(3.2) \qquad \begin{cases} u_{tt} - u_{rr} - \dfrac{n-1}{r} u_r = 0 & \text{for} \quad (r,t) \in \mathbb{R} \times [0,\infty), \\ u(r,0) = f(r), \quad u_t(r,0) = g(r) & \text{for} \quad r \in \mathbb{R}, \end{cases}
$$

where f and g are the same functions as in (E^*). The following proposition below will be proved in Section 4.

PROPOSITION 3.2. *Let n be an odd integer with $n \geq 5$. Suppose that $f \in C^{m+3}(\mathbb{R})$ and $g \in C^{m+2}(\mathbb{R})$ are even functions. Then the problem (3.2) admits a unique global C^2-solution $u^0(r,t)$. Set*

$$
U^0(r,t) = \frac{r^{m+1}}{< r >^{m-1}} u_t^0(r,t), \quad V^0(r,t) = \frac{r^{m+1}}{< r >^{m-1}} u_r^0(r,t).
$$

In addition, suppose that $\eta_0 \leq \infty$. Then $U^0 \in \Gamma_1$, $V^0 \in \Gamma_2$ and

$$
||U^0||_{m+2} = O(\eta_0), \quad ||V^0||_{m+2} = O(\eta_0),
$$

where η_0 is the same number as in the Main Theorem.

As we have already pointed out in Section 2, proving the Main Theorem is equivalent to solving the problem (H):

$$
(H) \qquad \begin{cases} U(r,t) = U^0(r,t) + \Xi_1(U,V)(r,t), \\ V(r,t) = V^0(r,t) + \Xi_2(U,V)(r,t). \end{cases}
$$

This is done by the contraction mapping principle. For $W = (U,V) \in \Gamma$, define an operator Ξ on Γ by

$$
\Xi(W)(r,t) = (U^0(r,t) + \Xi_1(W)(r,t), \quad V^0(r,t) + \Xi_2(W)(r,t)).
$$

For $\delta \in (0,1)$, set $B_\delta = \{W \in \Gamma : ||W||_\Gamma \leq \delta\}$. Now we show that Ξ maps B_δ into itself and is locally Lipschitz continuous. Combining the results of Propositions

3.1 and 3.2, we have

$$\begin{aligned}||\Xi(W)||_\Gamma &= O(||(U^0,V^0)||_\Gamma + ||(\Xi_1(W),\Xi_2(W))||_\Gamma) \\ &= O(\eta_0) + O(||U||^p_{m+2} + ||V||^{2q}_{m+2}) \\ &= O(\eta_0) + O(||W||^l_\Gamma),\end{aligned}$$

where $l=\min(p,2q)$.

Similarly, it is shown that Ξ is locally Lipschitz continuous: For W, $W^* \in \Gamma$,

$$||\Xi(W) - \Xi(W^*)||_\Gamma = O(\{||W||_\Gamma + ||W^*||_\Gamma\}^{l-1}||W - W^*||_\Gamma).$$

These imply that Ξ is a contraction mapping in B_δ, provided that η_0 and δ are sufficiently small. This completes the proof of the Main Theorem for $n \geq 5$ and $k = 0$.

§4 The linear wave equation in odd space dimension

This section is devoted to the study of the problem (3.2). Let $u^0(r,t)$ be a solution of (3.2). When n is an odd integer with $n \geq 3$, $u^0(r,t)$ is represented as

$$\begin{aligned}u^0(r,t) = &\frac{1}{2m!\, r^{2m+1}}\frac{\partial}{\partial t}\int_{t-r}^{t+r} D_\rho H_f(\rho)\cdot\phi^m(\rho,r,t)d\rho \\ &+\frac{1}{2m!\, r^{2m+1}}\int_{t-r}^{t+r} D_\rho H_g(\rho)\cdot\phi^m(\rho,r,t)d\rho,\end{aligned}\tag{4.1}$$

where $H_f(\rho) = \int_0^\rho (\frac{1}{2\sigma}\frac{d}{d\sigma})^m(\sigma^{2m+1}f(\sigma))d\sigma$ and $m = \frac{n-3}{2}$. (See e.g. Courant-Hilbert [3], Chap.VI, §13).

First, we shall study the function $\phi(\rho,r,t) = r^2 - (t-\rho)^2$.

LEMMA 4.1. *Let m be a positive integer. For all $\alpha \in \mathbb{Z}^2_+$ with $|\alpha| \leq m$, there exists a polynomial ψ_α of degree $2m - |\alpha|$ such that*

(i) $D_r^{\alpha_1}D_t^{\alpha_2}\phi^m(\rho,r,t) = r^{2m-|\alpha|}\psi_\alpha(\frac{t-\rho}{r})$;

(ii) $\psi_\alpha(\pm 1) = 0 \qquad \text{for} \quad |\alpha| \leq m-1.$

PROOF: Since $\phi^m(\rho, r, t) = r^{2m}(1-(\frac{t-\rho}{r})^2)^m$, by setting $\psi_0(\sigma) = (1-\sigma^2)^m$, the assertions are valid for $|\alpha| = 0$. From now on, let $|\alpha| \geq 1$. We write

$$\phi^m(\rho, r, t) = \{r - (t-\rho)\}^m \{r + t - \rho\}^m.$$

By Leibniz' rule, we have

$$D^\alpha \phi^m(\rho, r, t) = \sum_{\beta \leq \alpha} C_\beta \{r - (t-\rho)\}^{m-|\beta|}\{r + t - \rho\}^{m-|\alpha-\beta|}.$$

where C_β is a universal constant. So, setting
$\psi_\alpha(\sigma) = \sum_{\beta \leq \alpha} C_\beta (1-\sigma)^{m-|\beta|}(1+\sigma)^{m-|\alpha-\beta|}$, we obtain the desired results.

Next we shall show the following proposition from which we can derive Proposition 3.2.

PROPOSITION 4.2. *Let m be a positive integer. For $h \in B^3(\mathbb{R})$, set*

$$w(r,t) = \frac{1}{r^{2m+1}} \int_{t-r}^{t+r} h(\rho) D_\rho \phi^m(\rho, r, t) d\rho. \tag{4.2}$$

Put $W_1(r,t) = \dfrac{r^{m+1}}{< r >^{m-1}} w_t(r,t), \quad W_2(r,t) = \dfrac{r^{m+1}}{< r >^{m-1}} w_r(r,t).$
Then $W_i \in \Gamma_i$ *and* $||W_i||_{m+2} = O(||h||_3) \quad (i = 1, 2).$

PROOF: Since $\phi(\rho, r, t)$ is even in r, $w(r,t)$ is also even in r. Hence, by the definition of $W_i(r,t)$, we find that
$W_1(r,t) = (-1)^{m+1} W_1(-r,t) \quad \text{and} \quad W_2(r,t) = (-1)^m W_2(-r,t).$

From now on, let $m \geq 2$. By virtue of Lemma 4.1,

$$D^\alpha \phi^m(t \pm r, r, t) = 0 \qquad \text{for} \quad |\alpha| \leq m-1. \tag{4.3}$$

So we have from (4.2),

$$W_1(r,t) = \frac{1}{< r >^{m-1}\, r^m} \int_{t-r}^{t+r} h(\rho) D_t D_\rho \phi^m(\rho, r, t) d\rho \tag{4.4}$$

and

$$
(4.5)\qquad \begin{aligned} W_2(r,t) = & \frac{1}{<r>^{m-1} r^m} \int_{t-r}^{t+r} h(\rho) D_r D_\rho \phi^m(\rho,r,t) d\rho \\ & - \frac{2m+1}{<r>^{m-1} r^{m+1}} \int_{t-r}^{t+r} h(\rho) D_\rho \phi^m(\rho,r,t) d\rho. \end{aligned}
$$

First, we shall carry out the estimate for $W_1(r,t)$. Let $|\alpha| \leq m-2$. By noticing (4.3) and using Leibniz' rule, it follows from (4.4) that

$$
(4.6)\qquad \begin{aligned} & D^\alpha W_1(r,t) \\ & = \sum_{k=0}^{\alpha_1} C_k D_r^k \left(\frac{1}{<r>^{m-1} r^m}\right) \int_{t-r}^{t+r} h(\rho) D_r^{\alpha_1-k} D_t^{\alpha_2+1} D_\rho \phi^m(\rho,r,t) d\rho. \end{aligned}
$$

By Lemma 4.1, for each k, there exists a polynomial ψ_k of degree $2m-|\alpha|+k-2$ such that

$$
D_r^{\alpha_1-k} D_t^{\alpha_2+2} \phi^m(\rho,r,t) = r^{2m-|\alpha|+k-2} \psi_k\left(\frac{t-\rho}{r}\right).
$$

Hence, changing the variable as $\sigma = \dfrac{t-\rho}{r}$ in (4.6), we have

$$
(4.7)\qquad \begin{aligned} & D^\alpha W_1(r,t) \\ & = \sum_{k=0}^{\alpha_1} C_k r^{2m-|\alpha|+k-1} D_r^k \left(\frac{1}{<r>^{m-1} r^m}\right) \int_{-1}^{1} h(t-r\sigma) \psi_k(\sigma) d\sigma. \end{aligned}
$$

Since $D_r^l(<r>^{1-m}) = O(<r>^{1-m})$ for any nonnegative integer l, it follows from (4.7) that $D^\alpha W_1(r,t) = O(||h||_0)$.

From now on, let $|\alpha| = m-1$ and $\alpha_1 \geq 1$. In a similar fashion, we have

$$
(4.8)\qquad \begin{aligned} & D^\alpha W_1(r,t) \\ = & \sum_{k=0}^{\alpha_1} C_k D_r^k \left(\frac{1}{<r>^{m-1} r^m}\right) \int_{t-r}^{t+r} h(\rho) D_r^{\alpha_1-k} D_t^{\alpha_2+1} D_\rho \phi^m(\rho,r,t) d\rho \\ & + \frac{C_0}{<r>^{m-1} r^m} \{ h(t+r) D_r^{\alpha_1-1} D_t^{\alpha_2+1} D_\rho \phi^m(t+r,r,t) \\ & + h(t-r) D_r^{\alpha_1-1} D_t^{\alpha_2+1} D_\rho \phi^m(t-r,r,t) \} \\ = & \sum_{k=0}^{\alpha_1} C_k r^{m+k} D_r^k \left(\frac{1}{<r>^{m-1} r^m}\right) \int_{-1}^{1} h(t-r\sigma) \psi_k(\sigma) d\sigma \\ & + \frac{C_{\alpha_1}}{<r>^{m-1}} \{ h(t+r)\psi_1(-1) + h(t-r)\psi_1(1) \}, \end{aligned}
$$

where ψ_k are polynomials of degree $m+k-1$ $(0 \leq k \leq \alpha_1)$. Hence, for all β with $|\beta| \leq 3$, we conclude from (4.8) that $D^\beta D^\alpha W_1(r,t) = O(\|h\|_3)$. When $|\alpha| = m-1$ and $\alpha_1 = 0$, we easily get the same estimates as above. This completes the estimate for $W_1(r,t)$. Similarly, we get the same bound for $W_2(r,t)$ from (4.5).

We next show that $D_r^j W_1(0,t) = 0$ for $0 \leq j \leq m$. Notice that $D_t\phi = -D_\rho\phi$. Integrating by parts in (4.4), we have

$$
\begin{aligned}
W_1(r,t) &= \frac{-1}{<r>^{m-1} r^m} \int_{t-r}^{t+r} h(\rho) D_\rho^2 \phi^m(\rho,r,t) d\rho \\
&= \frac{-1}{<r>^{m-1} r^m} \int_{t-r}^{t+r} h''(\rho) \phi^m(\rho,r,t) d\rho.
\end{aligned} \tag{4.9}
$$

Differentiating (4.9) i times, we get

$$
D_r^i W_1(r,t) = \sum_{l=0}^{i} C_l D_r^{i-l}\left(\frac{1}{<r>^{m-1} r^m}\right) \int_{t-r}^{t+r} h''(\rho) D_r^l \phi^m(\rho,r,t) d\rho. \tag{4.10}
$$

By virtue of Lemma 4.1, we have therefore,

$$
D_r^i W_1(r,t) = O(r^{m-i+1}) \qquad \text{for} \quad 0 \leq i \leq m.
$$

Letting $r \to 0$, we get the desired results. Similarly, we obtain $D_r^i W_2(0,t) = 0$ for $0 \leq i \leq m-1$, which implies that

$$
W_2(r,t) = \frac{r^m}{m!} D_1^m W_2(0,t) + \frac{r^{m+1}}{m!} \int_0^1 (1-\lambda)^m D_1^{m+1} W_2(r\lambda, t) d\lambda. \tag{4.11}
$$

Using the definition of $W_2(r,t)$, we have from (4.11) that

$$
D_r^m W_2(0,t) = \frac{m!\, r}{<r>^{m-1}} w_r(r,t) - r \int_{-1}^1 (1-\lambda)^m D_1^{m+1} W_2(r\lambda, t) d\lambda. \tag{4.12}
$$

On the other hand, from (4.2), we have

$$
\begin{aligned}
w(r,t) &= -\frac{1}{r^{2m+1}} \int_{t-r}^{t+r} h'(\rho) \phi^m(\rho,r,t) d\rho \\
&= -\int_{-1}^1 h'(t-r\sigma)(1-\sigma^2)^m d\sigma.
\end{aligned}
$$

Since $h \in B^3(\mathbb{R})$, we find that $w_r \in C^1(\mathbb{R} \times [0,\infty))$. And also $D_r^{m+1} W_2 \in C^1(\mathbb{R} \times [0,\infty))$. Therefore, it follows from (4.12) that $D_1^m W_2(0,t) = 0$. This completes the proof for $m \geq 2$. Similarly, we can derive the assertions for $m = 1$. We omit the further details.

We are now in a position to prove Proposition 3.2.

PROOF OF PROPOSITION 3.2: We recall that $u^0(r,t)$ is represented as (4.1). Noticing that $D_t\phi = -D_\rho\phi$ in the first term and integrating by parts in the second term, we have

$$
\begin{aligned}
u^0(r,t) = &- \frac{1}{2m!\, r^{2m+1}} \int_{t-r}^{t+r} D_\rho H_f(\rho) D_\rho \phi^m(\rho,r,t)\,d\rho \\
&- \frac{1}{2m!\, r^{2m+1}} \int_{t-r}^{t+r} H_g(\rho) D_\rho \phi^m(\rho,r,t)\,d\rho.
\end{aligned}
\tag{4.13}
$$

By the assumptions on f and g, we find that $D_\rho H_f$, $H_g \in B^3(\mathbb{R})$. Therefore, we apply Proposition 4.2 to each term of (4.13) to get the desired results.

§5 Proof of Proposition 3.1

Our goal in this section is to prove the following which are equivalent to the assertions of Proposition 3.1: For $(U,V) \in \Gamma$,

$$
\begin{aligned}
\Xi_1(U,V)(-r,t) &= (-1)^{m+1}\Xi_1(U,V)(r,t), \\
\Xi_2(U,V)(-r,t) &= (-1)^{m}\Xi_2(U,V)(r,t);
\end{aligned}
\tag{5.1}
$$

$$
||\Xi_i(U,V)||_{m+2} = O(||U||^p_{m+2} + ||V||^{2q}_{m+2}) \quad (i = 1,2);
\tag{5.2}
$$

$$
D_r^k \Xi_i(U,V)(0,t) = 0 \quad (i = 1,2 \quad \text{and} \quad 0 \leq k \leq m).
\tag{5.3}
$$

For convenience, we set

$$
\begin{cases}
u_1(r,t) &= \dfrac{<r>^{m-1}}{r^{m+1}} U(r,t), \\
u_2(r,t) &= \dfrac{<r>^{m-1}}{r^{m+1}} V(r,t),
\end{cases}
$$

where $(U, V) \in \Gamma$. And we rewrite $L(u_1, u_2)(r, t)$ by making use of following properties: $\rho^{2m+1}G(u_1, u_2)(\rho, \tau)$ has appropriate regularity which will be shown in the proof of Proposition 5.5, STEP II. By virtue of Lemma 4.1 (*ii*), integrating by parts m times in (2.2), we have

$$
\begin{aligned}
&L(u_1, u_2)(r, t) \\
(5.4) \quad &= \frac{(-1)^m}{2m!\, r^{2m+1}} \int_0^t d\tau \int_{\gamma_1}^{\gamma_2} \left(\frac{1}{2\rho}\frac{\partial}{\partial \rho}\right)^m (\rho^{2m+1} G(u_1, u_2)(\rho, \tau)) \phi^m(\rho, r, t-\tau) d\rho,
\end{aligned}
$$

where $\quad \gamma_1 = t - \tau - r, \quad \gamma_2 = t - \tau + r.$

We shall split this section into four subsections.

5.1 Proof of (5.1)

Since $\phi(\rho, r, t)$ is even in r, it follows from (5.4) that $L(u_1, u_2)(r, t)$ is also even in r. Then it follows from (3.1) that

$$
\begin{aligned}
\Xi_1(U, V)(-r, t) &= \frac{(-r)^{m+1}}{< -r >^{m-1}} D_t L(u_1, u_2)(-r, t) \\
&= (-1)^{m+1} \Xi_1(U, V)(r, t)
\end{aligned}
$$

and

$$
\begin{aligned}
\Xi_2(U, V)(-r, t) &= \frac{(-r)^{m+1}}{< -r >^{m-1}} D_r L(u_1, u_2)(-r, t) \\
&= (-1)^m \Xi_2(U, V)(r, t).
\end{aligned}
$$

This completes the proof of (5.1).

5.2 Formulation of the Integral Operator and Preliminaries

First of all, we shall prove that the integrand of $L(u_1, u_2)(r, t)$ is odd in ρ at $\tau = t$. Since $u_1(\rho, \tau)$ is even in ρ, $G(u_1, u_2)(\rho, \tau)$ is also even in ρ. Hence, $\left(\frac{1}{2\rho}\frac{\partial}{\partial \rho}\right)^m (\rho^{2m+1} G(u_1, u_2)(\rho, \tau))$ is odd in ρ. On the other hand, $\phi(\rho, r, 0)$ is even in ρ, so the assertion follows. Now we compute the derivatives of $L(u_1, u_2)(r, t)$:

$$
\begin{aligned}
&D_t L(u_1, u_2)(r, t) \\
(5.5) \quad &= \frac{1}{2m!\, r^{2m+1}} \int_0^t d\tau \int_{\gamma_1}^{\gamma_2} \left(\frac{1}{2\rho}\frac{\partial}{\partial \rho}\right)^m (\rho^{2m+1} G(\rho, \tau)) D_t(\phi^m(\rho, r, t-\tau)) d\rho,
\end{aligned}
$$

$$
\begin{aligned}
&D_r L(u_1,u_2)(r,t)\\
(5.6)\quad &= \frac{1}{2m!\,r^{2m+1}}\int_0^t d\tau\int_{\gamma_1}^{\gamma_2}\left(\frac{1}{2\rho}\frac{\partial}{\partial\rho}\right)^m(\rho^{2m+1}G(\rho,\tau))D_r(\phi^m(\rho,r,t-\tau))d\rho\\
&\quad -\frac{2m+1}{2m!\,r^{2m+2}}\int_0^t d\tau\int_{\gamma_1}^{\gamma_2}\left(\frac{1}{2\rho}\frac{\partial}{\partial\rho}\right)^m(\rho^{2m+1}G(\rho,\tau))\phi^m(\rho,r,t-\tau)d\rho,
\end{aligned}
$$

where we have used $\phi(\gamma_i,r,t-\tau)=0 \quad (i=1,2)$.

Here we introduce some integral operators as follows:
For $X:\mathbb{R}\times[0,\infty)\longrightarrow\mathbb{R}$ and a positive integer s,

$$
\begin{aligned}
&\Phi_s X(r,t)\\
(5.7)\quad &= \frac{1}{2m!<r>^{m-1}\,r^m}\int_0^t d\tau\int_{\gamma_1}^{\gamma_2}\left(\frac{1}{2\rho}\frac{\partial}{\partial\rho}\right)^m\left(\rho^{2m+1}\left(\frac{<\rho>^{m-1}}{\rho^{m+1}}X(\rho,\tau)\right)^s\right)\\
&\qquad\times D_t(\phi^m(\rho,r,t-\tau))d\rho,
\end{aligned}
$$

$$
\begin{aligned}
&\Omega_s X(r,t)\\
(5.8)\quad &= \frac{1}{2m!<r>^{m-1}\,r^m}\int_0^t d\tau\int_{\gamma_1}^{\gamma_2}\left(\frac{1}{2\rho}\frac{\partial}{\partial\rho}\right)^m\left(\rho^{2m+1}\left(\frac{<\rho>^{m-1}}{\rho^{m+1}}X(\rho,\tau)\right)^s\right)\\
&\qquad\times D_r(\phi^m(\rho,r,t-\tau))d\rho,
\end{aligned}
$$

$$
\begin{aligned}
(5.9)\quad &\\
&\Psi_s X(r,t)\\
&= -\frac{2m+1}{2m!<r>^{m-1}\,r^{m+1}}\int_0^t d\tau\int_{\gamma_1}^{\gamma_2}\left(\frac{1}{2\rho}\frac{\partial}{\partial\rho}\right)^m\left(\rho^{2m+1}\left(\frac{<\rho>^{m-1}}{\rho^{m+1}}X(\rho,\tau)\right)^s\right)\\
&\qquad\times D_r(\phi^m(\rho,r,t-\tau))d\rho.
\end{aligned}
$$

Then it follows from (3.1), (5.5) and (5.6) that

$$
\begin{cases}
\Xi_1(U,V)(r,t)= & a\Phi_p U(r,t)+b\Phi_{2q}V(r,t),\\
\Xi_2(U,V)(r,t)= & a\Omega_p U(r,t)+b\Omega_{2q}V(r,t)+a\Psi_p U(r,t)+b\Psi_{2q}V(r,t),
\end{cases}
$$

which imply that proving (5.2) is equivalent to carrying out the estimates for the derivatives of $\Phi_s X(r,t)$, $\Omega_s X(r,t)$, and $\Psi_s X(r,t)$ up to order $m+2$.

To do this, in the rest of this subsection, we shall study the properties of functions $X(r,t)$ which belong to the following function space:

$$
\Lambda=\{X\in B^{m+2}(\mathbb{R}\times[0,\infty)):\ D_1^i X(0,t)=0 \quad (0\le i\le m)\}.
$$

Note that if $(U,V) \in \Gamma$, then U, $V \in \Lambda$. Using Taylor's formula with respect to r at the origin, we have

$$D^\alpha X(r,t) = \frac{r^{k+1}}{k!}\int_0^1 (1-\sigma)^k D_r^{k+1} D^\alpha X(r\sigma,t)d\sigma, \tag{5.10}$$

where $0 \le k \le m-|\alpha|$, $0 \le |\alpha| \le m$. This implies that the lemma below holds.

LEMMA 5.1. *For all $X \in \Lambda$, there exist $P_i \in B^{m+2-i}(\mathbb{R}\times[0,\infty))$ $(1 \le i \le m+1)$ and $R_\alpha \in B^{m+2-|\alpha|}(\mathbb{R}\times[0,\infty))$ $(1 \le |\alpha| \le m)$ such that*

(i) $X(r,t) = r^i P_i(r,t) \quad (1 \le i \le m+1)$,
$D^\alpha X(r,t) = r^{m+1-|\alpha|}R_\alpha \quad (1 \le |\alpha| \le m)$;

(ii) $D^\beta P_i(r,t) = O(<r>^{-i}\, ||X||_{|\beta|+i}) \quad (|\beta| \le m+2-i)$,
$D^\beta R_\alpha(r,t) = O(<r>^{-(m+1-|\alpha|)}\, ||X||_{|\alpha|+|\beta|}) \quad (|\beta| \le m+2-|\alpha|)$.

Next we shall carry out the estimates for functions of the type

$$Y_i(r,t) = r^{1+i-(m+1)s}X^s(r,t) \qquad 0 \le i \le m,$$

which is appeared as a part of the integrand in the integral operators.

LEMMA 5.2. *For all $X(r,t) \in \Lambda$,*

$$D^\alpha Y_i(r,t) = O(<r>^{-(m-i)-(m+1)(s-1)}\, ||X||_{m+2}^s) \qquad (|\alpha| \le i+2). \tag{5.11}$$

PROOF: When $r \ge 1$, it is easy to see that (5.10) holds. So, let $0 \le r \le 1$. In this case, it suffices to show

$$D^\alpha Y_i(r,t) = O(||X||_{m+2}^s) \qquad (|\alpha| \le i+2). \tag{5.12}$$

We divide the proof into three steps.

STEP I: $|\alpha| = 0$.

By Lemma 5.1 (i), we have

$$\begin{aligned} Y_i(r,t) =& r^{-(m-i)}X(r,t)(r^{-(m+1)}X(r,t))^{s-1} \\ =& P_{m-i}(r,t)(P_{m+1}(r,t))^{s-1}, \end{aligned}$$

where $P_0(r,t) = X(r,t)$. Hence we use Lemma 5.1 (ii) to obtain (5.12).

STEP II: $1 \leq |\alpha| \leq i+1$.

By Leibniz' rule, we have

$$D_r^{\alpha_1} D_t^{\alpha_2} Y_i(r,t) = \sum_{l=0}^{\alpha_1} C_l D_r^{\alpha_1 - l}(r^{1+i-(m+1)s}) D_r^l D_t^{\alpha_2} X^s(r,t). \tag{5.13}$$

Now we divide this sum into three parts to estimate them.

CASE I: $l + \alpha_2 = 0$.

In this case, we have

$$\begin{aligned} C_0 D^{\alpha_1}(r^{1+i-(m+1)s} X^s(r,t) &= r^{1+i-(m+1)s-\alpha_1} X^s(r,t) \\ &= P_{m-i+\alpha_1}(r,t)(P_{m+1}(r,t))^{s-1} \\ &= O(||X||_{m+2}). \end{aligned}$$

CASE II: $1 \leq l + \alpha_2 \leq i$.

By the chain rule,we have

$$D_r^l D_t^{\alpha_2} X^s(r,t) = \sum_{k=1}^{l+\alpha_2} C_k X^{s-k}(r,t) \sum_{\sigma \in \Upsilon_{k,\beta}} C_\sigma \prod_{j=1}^{k} D^{\sigma_j} X(r,t),$$

where $\beta = (l, \alpha_2)$, C_σ is a universal constant and

$$\Upsilon_{k,\beta} = \{\sigma \in (\mathbb{Z}_+^2)^k : \sum_{j=1}^{k} \sigma_j = \beta, \ \sigma_j \neq 0\}.$$

So, each term of sum in (5.13) satisfying $1 \leq l + \alpha_2 \leq i$ has the following estimate:

$$\begin{aligned} & r^{1+i-(m+1)s-(\alpha_1-l)} X^{s-k}(r,t) \prod_{j=1}^{k} D^{\sigma_j} X(r,t) \\ =& P_{m-i+|\alpha|}(r,t) P_{m+1}^{s-1-k}(r,t) \prod_{j=1}^{k} R_{\sigma_j}(r,t) \\ =& O(||X||_{m+2}^s). \end{aligned}$$

CASE III: $l+\alpha_2 = i+1$.

In this case, we have $|\alpha| = i+1$ and

$$
\begin{aligned}
& C_{\alpha_1} r^{1+i-(m+1)s} D_r^{\alpha_1} D_t^{\alpha_2} X^s(r,t) \\
=& C_{\alpha_1} r^{1+i-(m+1)s} \sum_{k=1}^{|\alpha|} C_k X^{s-k}(r,t) \sum_{\sigma \in \Upsilon_{k,\alpha}} C_\sigma \prod_{j=1}^{k} D^{\sigma_j} X(r,t).
\end{aligned} \tag{5.14}
$$

When $i=m$ and $k=1$, we have

$$
\begin{aligned}
& r^{(m+1)(1-s)} X^{s-1}(r,t) D^{\sigma_1} X(r,t) \\
=& D^\alpha X(r,t) (P_{m+1}(r,t))^{s-1} \\
=& O(||X||_{m+2}^s).
\end{aligned}
$$

Proceeding as in Case II, we easily deal with the other case. Hence, from (5.14), we get

$$r^{1+i-(m+1)s} D^\alpha X^s(r,t) = O(||X||_{m+2}^s).$$

Combining these estimates, it follows from (5.13) that (5.12) also holds in this step.

STEP III: $|\alpha| = i+2$.

we note that $D^\beta Y_i(r,t)$ with $|\beta| = i+1$ are represented by $P_j(r,t)$ $(1 \le j \le m+1)$ and $R_\sigma(r,t)$ $(|\sigma| \le i+1)$. By virtue of Lemma 5.1 (*ii*), it follows from the representations of $D^\beta Y_i(r,t)$ that (5.12) holds for any $\alpha \in \mathbb{Z}_+$ with $|\alpha| = i+2$. This completes the proof of Proposition 5.2.

REMARK 5.3. *We set*

$$\Lambda_k = \{X \in B^{m+k+2}(\mathbb{R} \times [0,\infty)) : \; D_1^i X(0,t) = 0 \quad (0 \le i \le m)\},$$

where $k \in \mathbb{Z}_+$. Then we easily obtain an analogue to Lemma 5.2:
For all $X(r,t) \in \Lambda_k$,

$$D^\alpha Y_i(r,t) = O(< r >^{-(m-i)-(m+1)(s-1)} ||X||_{m+k+2}^s) \qquad |\alpha| \le i+k+2.$$

Indeed, it follows from (5.10) that $P_i \in B^{m+k+2-i}(\mathbb{R} \times [0,\infty))$ $(1 \le i \le m+1)$ *and* $R_\alpha \in B^{m+k+2-|\alpha|}(\mathbb{R} \times [0,\infty))$ $(1 \le |\alpha| \le m)$ *for all* $X \in \Lambda_k$. *Hence, we obtain the estimates for* $Y_i(\rho, \tau)$ *up to order* $i + k + 2$.

For $X(r,t) \in \Lambda$, set

$$Q_i(r,t) = < r >^{(m-1)s}\, Y_i(r,t) \qquad (0 \le i \le m).$$

Corresponding to $(A.1)$, we consider the following assumption:

$$\text{(A.2)} \qquad s \ge \frac{m+3}{2}.$$

Corollary 5.4. *Let* $(A.2)$ *hold. Then for all* $X(r,t) \in \Lambda$,

$$D^\alpha Q_i(r,t) = O(< r >^{-2}\, \|X\|^s_{m+2}) \qquad (|\alpha| \le i+2).$$

In particular, $\quad Q_0(r,t) = O(< r >^{-3}\, \|X\|^s_{m+2})$.

Proof: Notice that $D^k(< r >^{(m-1)s}) = O(< r >^{(m-1)s})$ for any positive integer k. It follows from Lemma 5.2 that

$$\begin{aligned} D^\alpha Q_i(r,t) &= O(< r >^{(m-1)s} < r >^{-(m-i)-(m+1)(s-1)}\, \|X\|^s_{m+2}) \\ &= O(< r >^{-2s+m+1-(m-i)}\, \|X\|^s_{m+2}). \end{aligned}$$

Since $(A.2)$ implies that $-2s + m + 1 \le -2$ and $s \ge 2$, the desired result follows.

5.3 Proof of (5.2)

The aim of this section is to show the following proposition. As we have already pointed out in Subsection 5.2, (5.2) follows from this proposition.

Proposition 5.5. *Let* $(A.2)$ *hold. Then for all* $X(r,t) \in \Lambda$,

(i) $D^\alpha \Phi_s X(r,t) = O(\|X\|^s_{m+2})$;

(ii) $D^\alpha \Omega_s X(r,t) = O(\|X\|^s_{m+2})$;

(iii) $D^\alpha \Psi_s X(r,t) = O(\|X\|^s_{m+2})$,

where $|\alpha| \le m+2$.

PROOF: Here we shall prove only (i). Because one can get (ii) and (iii) from (5.8) and (5.9) in a similar fashion. By (5.1), it suffices to show (i) for $r \ge 0$.

STEP I: $|\alpha| = 0$.

First, for $X(r,t) \in \Lambda$, set

$$\Theta_s X(r,t) = \frac{1}{2 < r >^{(m-1)s}} \int_0^t Q_m(\gamma_1, \tau) d\tau.$$

It follows from Corollary 5.4 that $Q_m(\gamma_i, \tau) = O(< r >^{-2} \|X\|^s_{m+2})$. Hence, $\Theta(\pm r, t) = O(\|X\|^s_{m+2})$.

Notice that $D^j_\rho D_t \phi^m(\gamma_i, r, t-\tau) = 0 \quad (0 \le j \le m-2)$ and that

$$D^{m-1}_\rho D_t \phi^m(\gamma_i, r, t-\tau) = \begin{cases} -m!(-2r)^m, & i=1, \\ -m!(2r)^m, & i=2. \end{cases}$$

Integrating by parts m times in (5.7), we have

$$\begin{aligned} \Phi_s X(r,t) = {} & \Theta_s X(r,t) - \Theta_s X(-r,t) \\ & + \frac{(-1)^m}{2m! < r >^{m-1} r^m} \int_0^t d\tau \int_{\gamma_1}^{\gamma_2} \rho^{2m+1} \left(\frac{< \rho >^{m-1}}{\rho^{m+1}} X(\rho,\tau)\right)^s \\ & \times \left(\frac{\partial}{\partial \rho} \frac{1}{2\rho}\right)^m D_t \phi^m(\rho, r, t-\tau) d\rho. \end{aligned} \tag{5.15}$$

Next we shall carry out the estimate for the last term. Since $\phi(\rho, r, \tau)$ is a polynomial of degree $2m$ in ρ, we find that $(\frac{1}{2\rho}\frac{\partial}{\partial \rho})^m \phi^m(\rho, r, t-\tau)$ is even in ρ. Hence, the region of the integration in the last term becomes
$\{(\rho,\tau) \in \mathbb{R} \times [0,\infty) : \ 0 \le \tau \le t, \ |\gamma_1| \le \rho \le \gamma_2\}$. By writing

$$\left(\frac{\partial}{\partial \rho} \frac{1}{2\rho}\right)^m \phi^m(\rho, r, t-\tau) = \sum_{j=0}^m C_j \rho^{-2m+j} D^j_\rho \phi^m(\rho, r, t-\tau)$$

and setting

$$I_j(r,t) = \frac{1}{< r >^{m-1} r^m} \int_0^t d\tau \int_{|\gamma_1|}^{\gamma_2} Q_j(\rho,\tau) D_t D^j_\rho \phi^m(\rho, r, t-\tau) d\rho,$$

it follows from (5.15) that

$$\Phi_s X(r,t) - \{\Theta_s X(r,t) - \Theta_s X(-r,t)\} = \sum_{j=0}^{m} C_j I_j(r,t). \tag{5.16}$$

By Lemma 4.1 (i), there exists a polynomial $\psi_j(\sigma)$ such that

$$D_t D_\rho^j \phi^m(\rho, r, t-\tau) = r^{2m-j-1}\psi_j\left(\frac{t-\tau-\rho}{r}\right),$$

so

$$I_j(r,t) = \frac{r^{m-j-1}}{<r>^{m-1}} \int_0^t d\tau \int_{|\gamma_1|}^{\gamma_2} Q_j(\rho,\tau)\psi_j\left(\frac{t-\tau-\rho}{r}\right) d\rho.$$

Notice that $\dfrac{t-\tau-\rho}{r} = O(1)$ for $|\gamma_1| \le \rho \le \gamma_2$. Since ψ_j is a polynomial, we have

$$I_j(r,t) = \frac{r^{m-j-1}}{<r>^{m-1}} \int_0^t d\tau \int_{|\gamma_1|}^{\gamma_2} Q_j(\rho,\tau) d\rho. \tag{5.17}$$

It follows from Corollary 5.4 that

$$Q_j(\rho,\tau) = \begin{cases} O(<\rho>^{-2} \|X\|_{m+2}^s), & 1 \le j \le m, \\ O(<\rho>^{-3} \|X\|_{m+2}^s), & j = 0. \end{cases}$$

Hence, from (5.17), we have

$$I_j(r,t) = \begin{cases} O\left(\dfrac{\|X\|_{m+2}^s}{r} \displaystyle\int_0^t d\tau \int_{|\gamma_1|}^{\gamma_2} <\rho>^{-2} d\rho\right), & 1 \le j \le m, \\ O\left(\dfrac{\|X\|_{m+2}^s}{r} \displaystyle\int_0^t d\tau \int_{|\gamma_1|}^{\gamma_2} <\rho>^{-3} d\rho\right), & j = 0. \end{cases}$$

To continue the argument, we need the lemma below which is easily proved if we change the variables as $\xi = \tau + \rho, \quad \eta = \tau - \rho$.

LEMMA 5.6.
(i) $\displaystyle\int_0^t d\tau \int_{\gamma_1}^{\gamma_2} <\rho>^{-2} d\rho = O(r);$
(ii) $\displaystyle\int_0^t d\tau \int_{|\gamma_1|}^{\gamma_2} <\rho>^{-3} d\rho = O(1).$

By this lemma, we conclude $I_j(r,t) = O(\|X\|_{m+2}^s)$. This completes the proof of (i) for $|\alpha| = 0$.

STEP II: $1 \leq |\alpha| \leq m+2$.

Since

$$(\frac{1}{2\rho}\frac{\partial}{\partial\rho})^m(\rho^{2m+1}(\frac{<\rho>^{m-1}}{\rho^{m+1}}X(\rho,\tau))') = \sum_{j=0}^{m} C_j D_\rho^j Q_j(\rho,\tau),$$

if we set

$$K_j(r,t) = \frac{1}{<r>^{m-1}\,r^m}\int_0^t d\tau \int_{\gamma_1}^{\gamma_2} D_\rho^j Q_j(\rho,\tau) D_t\phi^m(\rho,r,t-\tau)d\rho,$$

then it follows from (5.6) that

$$\Phi_s X(r,t) = \sum_{j=0}^{m} C_j K_j(r,t). \tag{5.18}$$

Notice that for $X(r,t) \in \Lambda$, $D_\rho^j Q_j(\rho,\tau)$ is even in ρ. By virtue of Corollary 5.4, $D_\rho^j Q_j(\rho,\tau) \in B^2(\mathbb{R}\times[0,\infty)$. Therefore, proceeding as in the proof of Proposition 4.2, we obtain

$$D^\alpha K_j(r,t) = O(\|X\|_{m+2}^s) \qquad \text{for} \quad 1 \leq |\alpha| \leq m+2.$$

This completes the proof of (i).

REMARK 5.7. *In the proof above, we use two representations of $\Phi_s X(r,t)$: (5.16) and (5.18). A crucial difference between them consists in the region of the integration. Although it seems that the case $\gamma_1 \leq 0$ is not discussed explicitly, the integral operators introduced in [2] correspond to (5.16) from this point of view. (See [2], p.368, (3.9).) When we intend to estimate $\Phi_s X(r,t)$ itself, we have to use (5.16) in order to apply Lemma 5.6 (ii). But (5.16) is inadequate to estimate its derivatives. In fact, if we carry it out following [2], some terms of the fourth order derivatives may be unbounded when $\gamma_1 \leq 0$. However, the sum of such unbounded terms is bounded. But this method requires to compute the coefficients of such* *terms and to cancel them each other. To avoid such computations, we used (5.18) in Step II in the proof of Proposition 5.5.*

REMARK 5.8. *Even if we do not assume (A.1), we can get the estimate of $\Phi_s X(r,t)$ with $s \geq 2$ in the region $R = \{(\rho, \tau) : 0 \leq \tau \leq t, |\gamma_1| \leq \rho \leq r\}$.*
Indeed, (5.16) can be rewritten as

$$I_j(r,t) = \frac{r^{m-j-1}}{< r >^{m-1}} \iint_R \rho^j Q_0(\rho,\tau) d\rho d\tau$$
$$= O(\frac{r^{m-1}}{< r >^{m-1}} \iint_R Q_0(\rho,\tau) d\rho d\tau).$$

Hence, it follows from Corollary 5.4 and Lemma 5.6 that $I_j(r,t) = O(||X||^s_{m+2})$.

5.4 Proof of (5.3)

By writing

$$(\frac{1}{2\rho}\frac{\partial}{\partial \rho})^m (\rho^{2m+1} G(\rho,\tau)) = \sum_{j=0}^{m} C_j D^j_\rho (\rho^{1+j} G(\rho,\tau)),$$

it follows from (5.4) that

$$L(u_1,u_2)(r,t) = \sum_{j=0}^{m} \frac{C_j}{r^{2m+1}} \int_0^t d\tau \int_{\gamma_1}^{\gamma_2} D^j_\rho (\rho^{1+j} G(\rho,\tau)) \phi^m(\rho,r,t-\tau) d\rho.$$

We recall the proof of Proposition 3.2 in order to show (5.3). Then we find that it suffices to prove the following:

$$D^j_\rho(\rho^{1+j} G(\rho,\tau)) \in C^2(\mathbb{R} \times [0,\infty)), \tag{5.19}$$

$$D_r L(u_1,u_2)(r,t) \in C^0(\mathbb{R} \times [0,\infty)), \tag{5.20}$$

$$D^{m+1}_r \Xi(U,V)(r,t) \in C^0(\mathbb{R} \times [0,\infty)). \tag{5.21}$$

Since U, $V \in \Lambda$ for $(U,V) \in \Gamma$, we can apply Corollary 5.4 to $\rho^{1+j} G(\rho,\tau)$ to get (5.19). It follows from (5.19) that $L(u_1,u_2)(r,t) \in C^2(\mathbb{R} \times [0,\infty))$. This implies (5.20). By (5.2), (5.21) holds. This completes the proof of Proposition 3.1.

APPENDIX

In this Appendix, we shall study the problem (E^*) in three space dimensions. Instead of (H), we consider the following system of integral equations (H^*):

$$(H^*) \qquad \begin{cases} U(r,t) = U^0(r,t) + rD_tL(\frac{U}{r}, \frac{U}{r})(r,t), \\ V(r,t) = V^0(r,t) + rD_rL(\frac{U}{r}, \frac{U}{r})(r,t), \end{cases}$$

where $U^0(r,t) = ru_t^0(r,t)$, $V^0(r,t) = ru_r^0(r,t)$, and $L(u_1,u_2)(r,t)$ is given in (2.2) with $m = 0$. Solving the problem (E^*) with $n = 3$ is equivalent to finding a solution of (H^*) belonging to Γ with $m = 0$. So, the propositions below are essential.

PROPOSITION A.1. *Suppose that $f \in C^3(\mathbb{R})$ and $g \in C^2(\mathbb{R})$ are even functions. Then the problem below admits a unique global C^2-solution $u^0(r,t)$:*

$$\begin{cases} u_{tt} - u_{rr} - \frac{2}{r}u_r = 0 & \text{for} \quad (r,t) \in \mathbb{R} \times [0,\infty), \\ u(r,0) = f(r), \quad u_t(r,0) = g(r) & \text{for} \quad r \in \mathbb{R}, \end{cases}$$

where f and g are the same functions as in (E^*).
In addition, if $\eta_0 < \infty$, then $U^0 \in \Gamma_1$, $V^0 \in \Gamma_2$ and

$$||U^0||_2 = O(\eta_0), \quad ||V^0||_2 = O(\eta_0),$$

where η_0 is the same number as in the Main Theorem.

PROPOSITION A.2. *Suppose that p, $2q \geq 3$. Then for $W = (U,V) \in \Gamma$, $\Xi_i(W) \in \Gamma_i$ and*

$$||\Xi_i(W)||_2 = O(||U||_2^p + ||V||_2^{2q}) \quad (i = 1,2),$$

where $\Xi_1(U,V) = rD_tL(\frac{U}{r}, \frac{V}{r})$, $\Xi_2(U,V) = rD_rL(\frac{U}{r}, \frac{V}{r})$.

Proceeding as in Section 3, we obtain the solution of (H^*) which belongs to Γ. This implies that the assertion of the Main Theorem is valid,if $n = 3$ and $k = 0$.

ACKNOWLEDGMENT

The author would like to express his gratitude to Professor Kôji Kubota for his stimulating discussions and constant encouragement.

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