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LATTICES OF INTERMEDIATE SUBFACTORS

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LATTICES OF INTERMEDIATE SUBFACTORS

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Abstract. Let N be a irreducible subfactor of a type II_1 factor M. If Jones index [M:N] is finite, then the set $\mathcal{L}at(N \subset M)$ of the intermediate subfactors for the inclusion $N \subset M$ form a *finite* lattice. The (co-) commuting square conditions for intermediate subfactors are related with the modular identity in the lattice $\mathcal{L}at(N \subset N)$. In particular simplicity of a finite group G is characterized in terms of (co-) commuting square conditions of intermediate subfactors for $N \subset M = N \rtimes G$.

1. Introduction.

The study of lattice structure of von Neumann subalgebras started with the fundamental paper "On rings of operators" [MN] by Murray and von Neumann in 1936. Let H be a Hilbert space and B(H) the algebras of bounded linear operators on H. For von Nuemann subalgebras M and N of B(H), let $M \vee N = (M \cup N)''$ and $M \wedge N = M \cap N$. Then the set of von Neumann subalgebras of B(H) forms a lattice. In the introduction of the paper [MN], they considered the lattice to motivate the definition of factors. They also considered in [MN] the lattice of the projections of a factor, which had become much more important than the lattice of von Neumann subalgebras.

But there exist a few important contribution to the lattice structure of von Nuemann subalgebras after them. For example, in 1963 H.Araki[Ar] established a lattice isomorphism from a lattice of subspaces of a Hilbert space into a lattice of von Neumann subalgebras in the quantum field theory. See also [HK] and [DHR]. A Golois correspondence between intermediate subfactor lattices of type II_1 factors and subgroup lattices was initiated by Nakamura and Takeda [NT1], [NT2] in 1960 already. More generally, the structure of von Neumann subalgebras were investigated by Skau [Sk] and Christensen [Chr].

Since the work [Jo] of Jones on index for subfactors, the classification of subfactors has been studied by many people ([BN], [EK],[GDJ],[HKo],[HS],[I1],[I2],[I3],[IK],[Ka],[Ko],[KY],[Lo],[O1], [PP1],[P1],[P2],[SV],[We1],[We2],[Yo]...)

In this note we bigin to investigate the lattice structure of intermediate subfactors. We shall show that (co-) commuting square condition is related with the modular identity in the intermediate subfactor lattice. As a bonus we can characterize the simplicity of groups in terms of (co-) commuting square condition.

A nice characterization of intermediate subfactors has been obtained by Bisch [Bis] and Ocneanu [O2], although I was prepairing this note without their result and did not use it.

I would like to thank S.Sakai and F.Goodman who remind me the work of continuous geometry [Ne] by von Neumann when I was working on the relation among subfactors, Latin squares and finite geometry [MW]. I also thank Y.Kato for a comment on lattice theory. I am indebted to H.Kosaki for hastening me to write this note by letting me know the work of Bisch [Bis].

We refer to a book [Bir] by Birkhoff for Lattice theory and a book [Su] by Suzuki for subgroup lattices.

2. Finiteness of intermediate subfactor lattices

Let M be a factor and N a subfactor of M such that $N' \cap M = \mathbb{C}$. Let K be an intermediate von Neumann subalgebra for the inclusion $N \subset M$. Since $K' \cap K \subset N' \cap M = \mathbb{C}$, K is a factor automatically. Therefore the set $\mathcal{L}at(N\subset M)$ of all intermediate subfactors for $N \subset M$ forms a lattice under the two operations $\wedge$ and $\vee$ defined by

$$K_1 \wedge K_2 = K_1 \cap K_2 \qquad \text{and} \qquad K_1 \vee K_2 = (K_1 \cup K_2)''.$$

The lattice $\mathcal{L}at(N\subset M)$ clearly have a least element N and a greatest element M. Galois theory suggests us that many properties of the inclusion $N \subset M$ must be analized through the study of $\mathcal{L}at(N\subset M)$.

For $K \in \mathcal{L}at(N\subset M)$, the Jones projection e_K^M is defined as the projection of $L^2(M)$ onto $L^2(K)$ and we have $e_K^M \in N' \cap \langle M, e_N^M \rangle$. Furthermore for $A, B \in \mathcal{L}at(N\subset M)$, $A \subset B$ if and only if $e_A^M \le e_B^M$, because if A is not containded in B, then there exist $a \in A$ with $a \notin B$. Since $(I - E_B^M)(a) \neq 0$, we have $(I - e_B^M)e_A^M \neq 0$.

We also note that for $A,B \in \mathcal{L}at(N\subset M)$, we have $e_{A\wedge B} = e_A \wedge e_B$. But $e_{A\vee B} \neq e_A \vee e_B$ in general, see [SW].

Example 2.1. Let P be a type II_1 factor, G a finite group and α : $G \to \mathrm{Aut}\, P$ an outer action. Then the crossed product $M = P \rtimes_\alpha G \supset N = P$ and the fixed point algebra $N = P^G \subset M = P$ give two kinds of inclusions which are dual in a certain sense. By Nakamura-Takeda [NT1] and [NT2], the intermediate subfactor lattice $\mathcal{L}at(P \subset P\rtimes_\alpha G)$ is isomorphic to the subgroup lattice $\mathcal{L}(G)$ of G and the intermediate subfactor lattice $\mathcal{L}at(P^G \subset P)$ is isomorphic to the dual lattice of

$\mathcal{L}(G)$. In particular, if G is abelian, $\mathcal{L}at(P \subset P \rtimes_\alpha G)$ and $\mathcal{L}at(P^G \subset P)$ are isomorphic.

It is a fruitful idea to regard the intermediate subfactor lattice $\mathcal{L}at(N \subset M)$ as a generalization of the subgoup lattice $\mathcal{L}(G)$.

If G is a finite group, then the subgroup lattice $\mathcal{L}(G)$ is clearly a finite set. Then we may expect that the finiteness of the Jones index [M:N] implies that $\mathcal{L}at(N \subset M)$ is a finite set. But it is easy to see that it fails. In fact for example, let $M = N \otimes M_4(\mathbb{C})$. Then [M:N] is finite but $\mathcal{L}at(N \subset M)$ is an infinite set. But Example 2.1. also suggest that we may suppose that $N' \cap M = \mathbb{C}$. Then the above analogy works well. But we should note that the perturbation theory of von Neumann subalgebras due to E. Christensen [Chr], Pimsner and Popa [PP2] or B. Mashhood[Ma] is essentially used to prove it.

Theorem 2.2. Let M be a type II_1 factor and N a subfactor of M with $N' \cap M = \mathbb{C}$. If Jones index [M:N] is finite, then the intermediate subfactor lattice $\mathcal{L}at(N \subset M)$ is a finite set.

Proof. Let $M_1 = \langle M, e_N^M \rangle$. Since [M:N] is finite, $N' \cap M_1$ is finite dimensional. Hence, we have

$$\begin{aligned} &{}^{\#}\{[M:K] \mid K \in \mathcal{L}at(N \subset M)\} \\ &= {}^{\#}\{(tr_{M_1}(e_K^M))^{-1} \mid K \in \mathcal{L}at(N \subset M)\} \\ &\leq {}^{\#}\{(tr_{M_1}(p))^{-1} \mid p \text{ is a non-zero projection in } N' \cap M_1\} \\ &\leq 2^{\dim(N' \cap M_1)} < \infty . \end{aligned}$$

Therefore it suffices to show that for a fixed constant $c > 1$

$\{ K \in \mathcal{L}at(N \subset M) \mid [M:K] = c \}$ is a finite set. On the contrary suppose that the set was an infinite set. Since $N' \cap M_1$ is finite dimensional, $\{ p \in N' \cap M_1 \mid p$ is a projection with $tr_{M_1}(p) = c \}$ is a compact Housdorff space in uniform topology. Therefore there exists a sequence $(K_n)_n$ in $\mathcal{L}at(N \subset M)$ and a projection $p \in N' \cap M_1$ such that $\| e^M_{K_n} - p \| \to 0 \quad (n \to \infty)$, $tr_{M_1}(e^M_{K_1}) = tr_{M_1}(p) = c$ and $K_n \neq K_m$ $(n \neq m)$. By Pimsner and Popa [PP2;proposition] or Mashhood [Ma;6.2 Theorem] for a constant $c > 1$, there exists $\delta > 0$ such that if two subfactors $A, B \subset M$ satisfy $[M:A] = [M:B] = c$ and $\| E^M_B(a) - a \|_2 < \delta$ for $a \in A$ with $\|a\| \leq 1$, then there exists a unitary $u \in M$ such that $B = uAu^*$. Since we have that

$$\| E^M_B(a) - a \|_2 \leq \| E^M_B(a) - E^M_A(a) \|_2 \leq \| e^M_B - e^M_A \|$$

for $a \in A$ with $\|a\| \leq 1$, choosing a subsequence, we may assume that there exists a sequence $(K_n)_n$ in $\mathcal{L}at(N \subset M)$ and unitaries $u_n \in M$ such that $K_n = u_n K_1 u_n^*$ and $K_n \neq K_m$ (if $n \neq m$).

Fix natural numbers n and m with $n \neq m$. Define an onto $*$-isomorphism $\varphi : K_n \to K_m$ by

$$\varphi(x) = u_m u_n^* x (u_m u_n^*)^* \qquad \text{for } x \in K_n .$$

Then for any $z \in N$ $(\subset K_n)$, we have that $\varphi(z) u_m u_n^* = u_m u_n^* z$ and

$$E^M_{K_m}(\varphi(z) u_m u_n^*) = E^M_{K_m}(u_m u_n^* z) .$$

Since $z \in N \subset K_m$ and $\varphi(z) \in \varphi(N) \subset \varphi(K_n) \subset K_m$, we have

$$\varphi(z) E^M_{K_m}(u_m u_n^*) = E^M_{K_m}(u_m u_n^*) z ,$$

that is,

$$u_m u_n^* z (u_m u_n^*)^* E^M_{K_m}(u_m u_n^*) = E^M_{K_m}(u_m u_n^*) z .$$

Hence we have that $u_n u_m^* E^M_{K_m}(u_m u_n^*) \in N' \cap M = \mathbb{C}$ and there exists a scalar λ such that $E^M_{K_m}(u_m u_n^*) = \lambda u_m u_n^*$. Suppose that $\lambda \neq 0$. Then $u_m u_n^* \in K_m$ and $K_n = \varphi^{-1}(K_m) = (u_m u_n^*)^* K_m (u_m u_n^*) = K_m$. This is a contradiction. Therefore $\lambda = 0$ and $E^M_{K_m}(u_m u_n^*) = 0$.

Hence we have $E^M_N(u_m u_n^*) = E^{K_m}_N E^M_{K_m}(u_m u_n^*) = 0$.

Let H_n be the $\| \|_2$ closure of $\eta(Nu_n)$ in $L^2(M)$ for $n = 1,2,3,...$ If $n \neq m$, then H_n and H_m are orthogonal each other. In fact For $x \in N$ and $y \in N$,

$$(\eta(xu_m)|\eta(yu_n)) = tr(u_n^* y^* x u_m) = tr(u_m u_n^* y^* x) = tr(E^M_N(u_m u_n^*) y^* x) = 0$$

Since $\dim_N H_n = 1$, $[M:N] = \dim_N L^2(M) \geq \dim_N(\oplus_n H_n) = \infty$. This contradicts to the assumption that [M:N] is finite. Therefore $\mathcal{L}at(N \subset M)$ is a finite set. qed.

Remark. By the above theorem, we can draw pictures of intermediate subfactor lattices $\mathcal{L}at(N \subset M)$ by their Hasse diagrams.

Remark. If we drop the condition that $N' \cap M = \mathbb{C}$, then the lattice $\mathcal{L}at(N \subset M)$ is not a finite set in general. But clearly the lattice $\mathcal{L}at(M \subset N)$ is of finite hight, that is , the lengths of the chains in $\mathcal{L}at(N \subset M)$ is bounded.

The following duality on basic construction is evident to prove but useful to note.

Proposition 2.3 Let M be a factor and N a subfactor of M with $N' \cap M = \mathbb{C}$. Then $\mathcal{L}at(M \subset \langle M, e_N \rangle)$ is the dual lattice of $\mathcal{L}at(N \subset M)$.

Proof. It immedietly follows from the fact that

$$M = J_M M' J_M \subset J_M N' J_M = \langle M, e_N \rangle .$$

In the rest of the section we show the the class of the intermediate subfactor lattices is really larger than the class of the subgroup lattices and their duals.

Definition 2.4 Let $\mathcal{L}$(Groups) be the class of finite lattices which are isomorphic to $\mathcal{L}at(N \subset N \rtimes G)$ for a certatin II_1 factor N and an outer action $\alpha : G \to \mathrm{Aut}\, N$ of a finite group G. Similarly let $\mathcal{L}$(Group duals) be the class of finite lattices which are isomorphic to $\mathcal{L}at(M^G \subset M)$ for a certain II_1 factor M and an outer action α of a finite group G on M. Let $\mathcal{L}$(Subfactors) be the class of finite lattices which are isomorphic to $\mathcal{L}at(N \subset M)$ for a certain subfactor $N \subset M$ of type II_1 factor M such that $[M:N] < \infty$ and $N' \cap M = \mathbb{C}$. We note that $\mathcal{L}$(Groups) is in fact the class of finite lattices which are isomorphic to subgroup lattices and $\mathcal{L}$(Group duals) is in fact the class of finite lattices which are dually isomorphic to subgroup lattices. And $\mathcal{L}$(Subfactors) clearly contains both $\mathcal{L}$(Groups) and $\mathcal{L}$(Group duals).

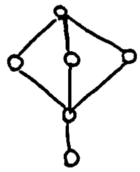
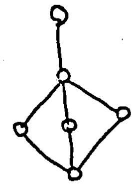
Proposition 2.5 The following hold:

(1) There exists a lattice L such that $L \in \mathcal{L}$(Groups) and $L \notin \mathcal{L}$(Group duals).

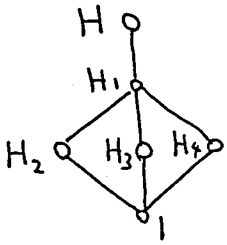
(2) There exists a lattice L such that $L \in \mathcal{L}$(Group duals) and $L \notin \mathcal{L}$(Groups) .

(3) There exists a lattice L such that $L \in \mathcal{L}$(Subfactors) , $L \notin \mathcal{L}$(Groups) and $L \notin \mathcal{L}$(Group duals) .

Proof.

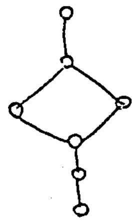
(1) Consider a lattice L :  and its dual lattice $\hat{L}$: 

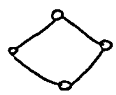
Let $G = \langle\langle x, y \mid x^4 = 1,\ x^2y^{-2} = 1,\ y^{-1}xyx = 1 \rangle\rangle$ be the quaternion group. Then the subgroup lattice $\mathcal{L}(G)$ is isomorphic to L by [Wei; Result 4.4.2] . Since $L \simeq \mathcal{L}(G) \simeq \mathcal{L}at(R \subset R \rtimes G)$, we have $L \in \mathcal{L}$(Groups). We shall show that $L \notin \mathcal{L}$(Group duals). On the contrary assume that L $\in \mathcal{L}$(Group duals). Then there exists a finite group H such that $\hat{L} \simeq \mathcal{L}(H)$. Consider subgroups H_1, H_2, H_3 and H_4 of H as follows:



Take $a \in H \setminus H_1$. Let $A = \langle\langle a \rangle\rangle$ be the subgroup of G generated by a. Since A is not contained in any H_1, H_2, H_3, H_4 , {1}, we have A = H. This shows that H is a cyclic group. Since H is abelian, $\hat{L} \simeq \mathcal{L}(H)$ must be self dual. This is a contradiction. Thus $L \notin \mathcal{L}$(Group duals)

(2)By the above argument and duality, we have that $\hat{L} \in \mathcal{L}$(Group duals) and $\hat{L} \notin \mathcal{L}$(Groups).

(3) Considedr a lattice K :  . We can use the same argument as above to show that $K \notin \mathcal{L}$(Groups) and $K \notin \mathcal{L}$(Group duals).

We note that the lattice  is isomorphic to $\mathcal{L}(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z})$.

Since we can add any chains on the top or bottom in the class

$\mathcal{L}$(Subfactors), which will be shown later in Theorem 4.6, we have that $K \in \mathcal{L}$(Subfactors) .

3. Commuting squares and modular identity

Recall that continuous geometry was invented by von Neumann [Ne] as a continuous analogue of projective geometry and the first example was given by a projecton lattice of a type II_1 factor. Continuous geometry is a continuous complemented modular lattice. Since we can regard intermediate subfactor lattice as a quantization of continuous geomety , it seems to be important to study modular identity in in intermediate subfactor lattices first. We shall show that modular identity is connected with commuting (and co-commuting) square conditions. As a bonus we can characterize simplicity of groups in terms of (co-)commuting square conditions of intermediate subfactors for inclusions of crossed products.

Recall that a lattice L is said to be *modular* if L satisfies the following modular identity for any $x, y, z \in L$:

$$\text{If } x \leq z, \quad \text{then } x \vee (y \wedge z) = (x \vee y) \wedge z \quad .$$

Dedekind showed that the set of normal subgroups of a group forms a modular lattice. The modular identity in the subgroup lattice $\mathcal{L}(G)$ of a group G is closely connected with the notion of permutable subgroups. Two subgroup H and K of a group G are said to be *permutable* if HK = KH, so that $H \vee K = HK = KH$. It is also known that H and K are permutable if and only if $[H : H \cap K] = [H \vee K : K]$. If H and K are permutable, then modular identity

$$(H \vee K) \wedge C = H \vee (K \wedge C) \qquad \text{for} \quad C \supset H$$

is satisfied. A subgroup H of a group G is called *quasi-normal* if H is permutable with all subgroups of G. Inparticular normal subgroups are quasi-normal. See a book [Su] for details.

Sano and Watatani [SW] noticed that the permutablity of subgroups is connected with commuting square condition for the commutants. In the below we shall introduce the notion of co-commuting squares to clarify the relation. Co-commuting square condition is also related with a formula of relative entropy as in Wierzbicki and Watatani[WW].

Definition 3.1 Let N be a von Neumann algebra on a Hilbert space H with a fixed (finte faithful normal) trace τ' on N'. A diagram $\begin{matrix} A & \subset & M \\ \cup & & \cup \\ N & \subset & B \end{matrix}$ of von Neumann algebras on H is called a *co-commuting square* if the diagram $\begin{matrix} A' & \subset & N' \\ \cup & & \cup \\ M' & \subset & B' \end{matrix}$ of their commutants is a commuting square. In particular we have $M = A \vee B$ because $M' = A' \cap B'$.

In the below we consider only the case that N' is a finite factor, therefore we do not worry about the choice of traces. Furthermore if M is a type II_1 factor and N a subfactor of M with $[M:N] < \infty$, then co-commuting square condition does not depend on the choice of Hilbert space H on which M acts if N' is finite as shown in [SW;Proposition 4.1] or [WW]. In the terminology of angles between two subfactors in [SW], the above co-commuting square condition is written as $\text{Op-ang}_M(A,B) = \{ \frac{\pi}{2} \}$ by definition. The following proposition is essentially rewriting of [SW;Theorem 7.8].

Proposition 3.2. Let M be a type II_1 factor and N a subfactor with $[M:N] < \infty$. Let A and B be intermediate subfactors for $N \subset M$. Suppose that a diagram $\begin{matrix} A & \subset & A\vee B \\ \cup & & \cup \\ A\wedge B & \subset & B \end{matrix}$ is a commuting square. Then the following conditions are equivalent:

(1) the diagram $\begin{matrix} A & \subset & A\vee B \\ \cup & & \cup \\ A\wedge B & \subset & B \end{matrix}$ is a co-commuting square.

(2) $A \vee B = AB$

(3) $AB = BA$

(4) $[A : A\wedge B] = [A\vee B : B]$, i.e. $(A\vee B, A, B, A\wedge B)$ is a parallelogram.

(5) $A \vee B = \overline{AB}^{\sigma\text{strong}}$

where $AB = \{ \sum_{\text{finite}} a_i b_i \mid a_i \in A$ and $b_i \in B \}$ and we may replace the role of A and B in the above conditions.

Definition 3.3 Let M be a II_1 factor and N a subfactor of N with $[M:N] < \infty$. Let A be an intermediate subfactor for $N \subset M$. Then we call that A is *double commuting* if for any intermediate subfactor B for $N \subset M$, a diagram $\begin{matrix} A & \subset & A\vee B \\ \cup & & \cup \\ A\wedge B & \subset & B \end{matrix}$ is a commuting and co-commuting square.

For example it is trivial that M and N are double commuting intermediate subfactors for $N \subset M$. More non-trirvial examples are given by the crossed products by normal subgroups as follows:

Proposition 3.4 Let N be a II_1 factora and $\alpha : G \to \mathrm{Aut}\, N$ be an outer action of a finite group G. Consider a crossed product $M = N \rtimes_\alpha G$. Let H be a subgroup of G and $A = N \rtimes H$ an intermediate subfactor for $N \subset M$. Then $A = N \rtimes H$ is double commuting if and only if H is quasi-normal. In particular if H is a normal subgroup of G, then $A = N \rtimes H$ is double commuting.

Proof. Any intermediate subfactor B has a form $B = N \rtimes_\alpha K$ for some subgroup K of G by Nakamura-Takeda [NT] . Therefore the diagram $\begin{matrix} A & \subset & A\vee B \\ \cup & & \cup \\ A\wedge B & \subset & B \end{matrix}$ is always commuting square. Hence $\begin{matrix} A & \subset & A\vee B \\ \cup & & \cup \\ A\wedge B & \subset & B \end{matrix}$ is co-commuting square if and only if $[A:A\wedge B] = [A\vee B:B]$ by Proposition 3.2. Since this means that $[H:H\cap K] = [H\vee K:K]$, it is equivalent to that H and K are permutable. Therefore A is double commuting if and only if H is a quasinormal subgroup. q.e.d.

Remark. It is also known that H is a normal subgroup of G if and only if M is a crossed product of A by a certain group . See for example Teruya [Te].

As a bonus we can characterize the simplicity of groups interms of subfactors using commuting and co-commuting square conditions:

Proposition 3.5. Let N be a II_1 factor and $\alpha : G \to \mathrm{Aut}\, N$ be an outer action of a finite group G. Then G is simple if and only if any double commuting intermediate subfactor for $N \subset M$ is N or M.

Proof. Suppose that G is not simple, then there exists a non-trivial normal subgoup H of G. Then $A = N \rtimes_\alpha H$ is a non-trivial double commuting intermediate subfactor by Proposition 3.4. Conversely suppose that G is simple . By [Su; Proposition 1.3] , if

H is a maximal quasi-normal subgroup of G, then H is a normal subgroup of G. Therefore there exist no non-trivial quasi-normal subgroups. Thus there exist no non-trivial double commuting subfactor by Proposition 3.4. Q.E.D.

Definition 3.6 Let M be a factor and N a subfactor of M with $N \neq M$. Then N is a *maximal* subfactor of M if for any intermediate subfactor A for $N \subset M$ we have $A = N$ or $A = M$.

Remark. If $[M:N] < 4$, then N is a maximal subfactor of M. But even if G is a simple group, N is *not* a maximal subfactor of $N \rtimes_\alpha G$ in general, because G has many subgroups in general, where α is an outer action of a finite group G on a II_1 factor N. Note that N is a maximal subfactor of $M = N \rtimes_\alpha G$ if and only if G is a cyclic group of prime order. Therefore maximality of subfactor is much more stronger than simplicity of the corresponding group. If $M = N \otimes M_n(\mathbb{C})$ then N is a maximal subfactor of M if and only if n is a prime number. Therefore you may think that maximal subfactor behaves like prime numbers. But I do not know any in general. For example consider the tectrahedral group $G = \langle\langle x,y \mid x^3 = 1,\ y^3 = 1,\ (xy)^2 = 1 \rangle\rangle$. Let $K = \langle\langle x \rangle\rangle$ be the subgroup of G generated by x. Then K is a maximal subgroup of G and $[G:K] = 4$, [Wei; Result 4.6.10]. Therefore $N = R \rtimes K$ is a maximal subfactor of $M = R \rtimes G$ but $[M:N] = 4$. Since maximal subgroups have a rich geometrical structure as in [KL], we may expect a similar structure for maximal subfactor.

Remark. We also have fixed point algebra version of Proposition

3.4 and 3.5, because the commutant of a double commuting intermediate subfactor is also double commuting.

Proposition 3.7. Let N_i be a subfactor of a type II_1-factor M_i with $[M_i:N_i] < \infty$ for $i = 1,2$. Put $M = M_1 \otimes M_2$ and $N = N_1 \otimes N_2$. If $N_i' \cap M_i = \mathbb{C}$, then $M_1 \otimes N_2$ and $N_1 \otimes M_2$ are double commuting intermediate subfactors for $N \subset M$.

Proof. Let e_i be the Jones projections for $N_i \subset M_i$ ($i = 1,2$). Then e_i is a central projection in $N_i' \cap \langle M_i, e_i\rangle$ by [PP1;1.9.Proposition]. For any intermediate subfactor B for $N \subset M$, let e_B^M be the Jones projection for $B \subset M$. Then

$$e_B^M \in N' \cap M = (N_1' \cap \langle M_1, e_1\rangle) \otimes (N_2' \cap \langle M_2, e_2\rangle) \quad .$$

Since the Jones projection for $M_1 \otimes N_2 \subset M$ is $I \otimes e_2$ and $I \otimes e_2$ is central in $N' \cap M$, it commutes with e_B^M. Thus the diagram

$$\begin{array}{ccc} M_1 \otimes N_2 & \subset & M \\ \cup & & \cup \\ (M_1 \otimes N_2) \cap B & \subset & B \end{array}$$

is a commuting square. Applying the same argument for the commutant $(M_1 \otimes N_2)' = M_1' \otimes N_2'$ in the inclusion $M' = M_1' \otimes M_2' \subset N' = N_1' \otimes N_2'$, we conclude that $M_1 \otimes N_2$ is a double commuting intermediate subfactor. Q.E.D.

Now we come back to a relation between modular identity and commuting square condition. The following is a key lemma.

Lemma 3.8. Let M be a type II_1 factor and N a subfactor of M such that $[M:N] < \infty$ and $N' \cap M = \mathbb{C}$. Let P, Q and X be intermediate subfactors for $N \subset M$ such that $P \subset X$. Suppose that the diagram

$$\begin{array}{ccc} P & \subset & P \vee Q \\ \cup & & \cup \\ P \wedge Q & \subset & Q \end{array}$$ is a commuting and co-commuting square. If the diagram $$\begin{array}{ccc} X & \subset & M \\ \cup & & \cup \\ X \wedge Q & \subset & Q \end{array}$$ is a commuting square, then we have

$$(P \vee Q) \wedge X = P \vee (Q \wedge X) \quad .$$

Proof. The inclusion $(P \vee Q) \wedge X \supset P \vee (Q \wedge X)$ is trivial. Take w in $(P \vee Q) \wedge X$. By Proposition 3.2, there exist $p_1, \ldots, p_n \in P$ and $q_1, \ldots, q_n \in Q$ such that $w = \Sigma_i \; p_i q_i$. Since w is also in X,

$$w = E^M_X(w) = \Sigma_i \; E^M_X(p_i q_i) = \Sigma_i \; p_i E^M_X(q_i)$$
$$= \Sigma_i \; p_i E^M_X E^M_Q(q_i) = \Sigma_i \; p_i E^M_{X \wedge Q}(q_i) \in P \vee (Q \wedge X)$$ Q.E.D.

Recall the second homomorphism theorem in group theory. Let G be a group, H is a normal subgoup of G and K is a subgoup of G, then HK = KH is a subgroup of G and we have a canonical ismorphism $K/(H \cap K) \simeq HK/H$. The following theorem is an analog of that in the level of intermediate subfactor lattice.

Theorem 3.9 Let M be a type II_1 factor and N a subfactor of M such that $[M:N] < \infty$ and $N' \cap M = \mathbb{C}$. Let P and Q be double commutaing intermediate subfactors for $N \subset M$. Then $PQ = QP = P \vee Q$ and (P,Q) is a modular pair and also dual modular pair in the lattice $\mathcal{L}at(N \subset M)$, that is, the follwing modular identities hold:
For any X and $Y \in \mathcal{L}at(N \subset M)$,

if $Y \subset Q$, then $(Q \wedge P) \vee Y = Q \wedge (P \vee Y)$

and if $P \subset X$, then $(P \vee Q) \wedge X = P \vee (Q \wedge X)$.

Furthermore we have a canonical lattice isomorphism

$\varphi : \mathcal{L}at(P\wedge Q \subset Q) \longrightarrow \mathcal{L}at(P \subset P\vee Q)$ such that $\varphi(Y) = P \vee Y$ and and $\varphi^{-1}(X) = Q \wedge X$ for $Y \in \mathcal{L}at(P\wedge Q \subset Q)$ and $X \in \mathcal{L}at(P \subset P\vee Q)$. The ismorphism φ preserves Jones index, that is, for any $A, B \in \mathcal{L}at(P\wedge Q \subset Q)$ with $A \supset B$, we have that $[\varphi(A):\varphi(B)] = [A:B]$.

Proof. Applying Lemma 3.8. appropriately, we have that (P,Q) is a modular and dual modular pair. Put $\phi(X) = Q \wedge X$. then by the modular identities, we have $\phi\varphi(Y) = Q \wedge (P \vee Y) = (Q \wedge P) \vee Y = Y$ and similarly we have $\varphi\phi(X) = X$. Hence $\phi = \varphi^{-1}$. We show that φ preserves Jones index. It is enough to show that $[\varphi(Q):\varphi(B)] = [Q:B]$ for all $B \in \mathcal{L}at(P\wedge Q \subset Q)$. Since $\varphi(Q) = P \vee Q$, $\varphi(B) = P \vee B$ and Q is double commuting, the diagram $\begin{matrix} P\vee B & \subset & P\vee Q \\ \cup & & \cup \\ B & \subset & Q \end{matrix}$ is a commuting and co-commuting square. Thus by Proposition 3.2, $[\varphi(Q):\varphi(B)] = [Q:B]$.

Q.E.D.

Remark. The assumption of Theorem 3.9. that P and Q are double commuting intermediate subfactors can be weakened as follows: Suppose that $\begin{matrix} P & \subset & P\vee B \\ \cup & & \cup \\ P\wedge B & \subset & B \end{matrix}$ is co-commuting and $\begin{matrix} B & \subset & B\vee Q \\ \cup & & \cup \\ B\wedge Q & \subset & Q \end{matrix}$ is commuting for any $B \in \mathcal{L}at(N\subset M)$. Then we have the same conclusion except that the lattice ismorphism φ preserves Jones index. First note that $\begin{matrix} P' & \subset & P'\vee B' \\ \cup & & \cup \\ P'\wedge B' & \subset & B' \end{matrix}$ is commuting and $\begin{matrix} B' & \subset & B'\vee Q' \\ \cup & & \cup \\ B'\wedge Q' & \subset & Q' \end{matrix}$ is co-commuting for any $B' \in \mathcal{L}at(M' \subset N')$ by taking commutants on $L^2(M, tr)$. Therefore by Lemma 3.8, we have that for any $X \in \mathcal{L}at(N\subset M)$

$$\text{if } P \subset X, \text{ then } (P \vee Q) \wedge X = P \vee (Q \wedge X) .$$

And for any $Y \in \mathcal{L}at(N \subset M)$, we have $Y' \in \mathcal{L}at(M' \subset N')$ and

if $Q' \subset Y'$, then $(Q' \vee P') \wedge Y' = Q' \vee (P' \wedge Y')$.

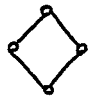
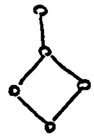
Thus if $Q \supset Y$, then $(Q \wedge P) \vee Y = Q \wedge (P \vee Y)$. Therefore we have a lattice isomorphism $\varphi : \mathcal{L}at(P \wedge Q \subset Q) \to \mathcal{L}at(P \subset P \vee Q)$ as well as Theorem 3.9. Moreover suppose that P or Q is double commuting, then the lattice isomorphism φ preserves Jones index as above.

Example 3.10 Let N be a type II_1 factor, $\alpha : G \longrightarrow \mathrm{Aut}\, N$ is an outer action of a finite group G. Let H be a normal subgroup of G and K a subgroup of G. Then intermediate subfactors $P = N \rtimes_\alpha H$ and $Q = N \rtimes_\alpha K$ satisfy the weakened assumption of Theorem 3.9. in the Remark. The lattice isomorphism $\varphi : \mathcal{L}at(P \wedge Q \subset Q) \longrightarrow \mathcal{L}at(P \subset P \vee Q)$ corresponds to group isomorphism $f : K/(H \cap K) \longrightarrow HK/H$ exactly because $P \wedge Q = N \rtimes_\alpha (H \cap K)$ and $P \vee Q = N \rtimes_\alpha (HK)$.

Example 3.11 Let M_i be a type II_1 factor and N_i is a subfactor of M_i with $[M_i : N_i] < \infty$ and $N_i' \cap M = \mathbb{C}$. for $i = 1,2$. Put $M = M_1 \otimes M_2$ and $N = N_1 \otimes N_2$. Then intermediate subfactors $P = M_1 \otimes N_2$ and $Q = N_1 \otimes M_2$ satisfy the assumption of Theorem 3.9. Then the lattice isomorphism $\mathcal{L}at(P \wedge Q \subset Q) \simeq \mathcal{L}at(P \subset P \vee Q)$ is nothing but the lattice isomorphism $\mathcal{L}at(N_1 \otimes N_2 \subset N_1 \otimes M_2) \simeq \mathcal{L}at(M_1 \otimes N_2 \subset M_1 \otimes M_2) \simeq \mathcal{L}at(N_2 \subset M_2)$.

Corollary 3.12. Let M be a type II_1 factor and N a subfactor of M such that $[M:N] < \infty$ and $N' \cap M = \mathbb{C}$. Let $M_1 = \langle M, e_N \rangle$ and $M_2 = \langle M_1, e_M \rangle$ Assume that $N' \cap M_1$ and $M' \cap M_2$ are abelian, Then $\mathcal{L}at(N \subset M)$ is a modular lattice.

4. Adding chains

We shall show that we can add chains on the top or the bottom in the class $\mathcal{L}$(Subfactors). For example, let $X =$  and $X^{\sim} =$  the lattice added a chain  on the top of X. Since X is in $\mathcal{L}$(Subfactors), $X^{\sim}$ will be shown in $\mathcal{L}$(Subfactors). We starts from elementary lemmas.

Lemma 4.1 Let M be a finite factor and N a subfactor of M with $[M:N] < \infty$. Let K be a finite factor. Then for any subfactor D with $K \otimes N \subset D \subset K \otimes M$, there exists a subfactor B such that $N \subset B \subset M$ and $D = K \otimes B$. Moreover $B \longmapsto K \otimes B$ is an order isomorphism between the sets of the intermediate subfactors for $N \subset M$ and that for $K \otimes N \subset K \otimes M$.

Proof. First consider the case that M is a type II_1 factor. Then There exists a type II_1 factor P such that $M = \langle N, e_P \rangle$. For any subfactor D with $K \otimes N \subset D \subset K \otimes M$, put $A = (J_K \otimes J_N) D' (J_K \otimes J_N)$ on $L^2(K \otimes N) = L^2(K) \otimes L^2(N)$. Then $D = \langle K \otimes N, e_A^{K \otimes N} \rangle$ and we have that $e_A^{K \otimes N} \in A' \cap D \subset (K \otimes P)' \cap K \otimes M = (K' \cap K) \otimes (P' \cap M) = \mathbb{C} \otimes (P' \cap M)$. Therefore there exists a projection $r \in P' \cap M$ such that $e_A^{K \otimes N} = I \otimes r$. Let B be the von Neumann algebra of M generared by N and r. Then $D = K \otimes B$ and B is also a factor because D is a factor. For two intermediate subfactors B and C with $B \neq C$, we have

$$e_{K \otimes B}^{K \otimes M} = I \otimes e_B^M \neq I \otimes e_C^M = e_{K \otimes C}^{K \otimes M} .$$

Thus the map $B \longmapsto K \otimes B$ is an order isomorphism.

Next consider the case that M is a type I_n factor. We may not have a tunnel P for $N \subset M$. So we consider the inclusions

$$K' \otimes M' = (K \otimes M)' \subset D' \subset K' \otimes N' \quad \text{on } L^2(K) \otimes L^2(M).$$

Applying the proceeding argument, there exists a subfactor B' such that $M' \subset B'$ N' and $D' = K' \otimes B'$. Therefore $D = K \otimes B$ and $N \subset B \subset M$. Q.E.D.

Corollary 4.2 Let N be a type II_1 factor and p be a prime number. Then N is a maximal subfactor of $M = N \otimes M_p(\mathbb{C})$.

See M. Choda [Cho] for the related fact.

Lemma 4.3 Let M be a type II_1 factor and N a subfactor of M with $N' \cap M = \mathbb{C}$. Take $b \in M \otimes M$. Suppose that

$$(n\otimes m)b = b(m\otimes n) \quad \text{for all } m \in M \text{ and } n \in N .$$

Then we have $b = 0$.

Proof. Assume that $(n\otimes m)b = b(m\otimes n)$ for all $m \in M$ and $n \in N$. We shall show that $b = 0$. On the contrary suppose that $b \neq 0$. Then

$$b^*b(m\otimes n) = b^*(n\otimes m)b = (m\otimes n)b^*b .$$

Thus $b^*b \in (M \otimes N)' \cap (M \otimes M) = (M' \cap M) \otimes (N' \cap M) = \mathbb{C} \otimes \mathbb{C}$.

Since $b \neq 0$, there exists a non-zero $\lambda \in \mathbb{C}$ and a unitary $v \in M \otimes M$ such that $b = \lambda v$. Then $(x\otimes y)v = v(y\otimes x)$ for all $x,y \in N$, so that $(x\otimes y)v^2 = v(y\otimes x)v = v^2(x\otimes y)$. Hence $v^2 \in (N \otimes N)' \cap (M \otimes M) = \mathbb{C} \otimes \mathbb{C}$. Thus there exist $z \in \mathbb{C}$ such that $|w| = 1$ and $v^2 = z^2$. Put $w = z\, v = z\lambda^{-1}b$. Then w is a untary in $M \otimes M$ such that $w^2 = 1$ and

$(n\otimes m)w = w(m\otimes n)$ for all $n \in N$ and $m \in M$. Since we have $w^*(n^*\otimes m^*) = (m^*\otimes n^*)w^*$ and $w = w^*$, $w(n\otimes m) = (m\otimes n)w$. Therefore for all $x,y \in M$, $w(x\otimes y)w^* = w(x\otimes 1)w^*w(1\otimes y)w^* = (1\otimes x)(y\otimes 1) = y\otimes x$. Since Sakai's flip flop is outer by [Sa], this is a contradiction. Therefore $b = 0$. Q.E.D.

Lemma 4.4 Let M be a type II_1 factor and N a subfactor of M with $N' \cap M = \mathbb{C}$. Let $\alpha \in \mathrm{Aut}(M \otimes M)$ be the Sakai's flip flop. Let $L = (M \otimes M) \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}$ and $K = M \otimes N$. Then $K' \cap L = \mathbb{C}I$.

Proof. Let $z = a + bu \in L$ for $a,b \in M \otimes M$ and the unitary u with $u^2 = 1$ which impliments the automorphism α. Suppose that $z \in K' \cap L$. Then for all $m \in M$ and $n \in N$, $(m\otimes n)(a+bu) = (a+bu)(m\otimes n)$. Since $\alpha(m\otimes n) = u(m\otimes n)u^* = n\otimes m$, we have that $(m\otimes n)a = a(m\otimes n)$ and $(m\otimes n)b = b(n\otimes m)$. Then $b = 0$ by Lemma 4.3. Since $a \in (M\otimes N)' \cap M\otimes M = \mathbb{C} \otimes \mathbb{C}$, we have $z = a + bu \in \mathbb{C}$. Thus $K' \cap L = \mathbb{C}$. Q.E.D.

Lemma 4.5 Let M be a type II_1 factor and N a subfactor of M with $N' \cap M = \mathbb{C}$. Let $H = L^2(M\otimes M, tr)$. For $\xi \in H$, let K_ξ be the closed subspace generated by $(M \otimes N)((J_MNJ_M) \otimes (J_MMJ_M))\xi$. If $\xi \neq 0$, then we have $K_\xi = H$.

Proof. Let P_ξ be the projection of H onto K_ξ. Since K_ξ is invariant under $M \otimes N$ and $(J_MNJ_M) \otimes (J_MMJ_M)$,

$$\begin{aligned} P_\xi &\in (M \otimes N)' \cap ((J_MNJ_M) \otimes (J_MMJ_M))' \\ &= (M' \cap \langle M,e_N\rangle) \cap (N' \cap M) = \mathbb{C} \otimes \mathbb{C} \end{aligned}$$

Since $P_\xi \neq 0$, we have $P_\xi = I$, that is, $K_\xi = H$. Q.E.D.

Theorem 4.6 Let X be a finite lattice in $\mathcal{L}$(Subfactors). Let $X^{\sim}$ (resp. $X_{\sim}$) be the finite lattice adding a chain  on the top (resp. bottom) of X. Then $X^{\sim}$ and $X_{\sim}$ are also in $\mathcal{L}$(Subfactors).

Proof. By considering a duality of Proposition 2.3, it suffices to show that $X^{\sim}$ is in $\mathcal{L}$(Subfactors). Let M be a type II_1 factor and N a subfactor of M such that $N' \cap M = \mathbb{C}$, $[M:N] < \infty$ and $X \simeq \mathcal{L}at(N \subset M)$. Let $\alpha \in \mathrm{Aut}(M \otimes M)$ be the Sakai's flip flop. Put $L = (M \otimes M) \rtimes_{\alpha} \mathbb{Z}/2\mathbb{Z}$ and $Q = M \otimes N$. It is sufficient to show that $Q' \cap L = \mathbb{C}$ and

$$\mathcal{L}at(Q \subset L) = \{ M \otimes B \mid B \in \mathcal{L}at(N \subset M)\} \cup \{ L \} .$$

The fact that $Q' \cap L = \mathbb{C}$ is proved in Lemma 4.4. Let $D \in \mathcal{L}at(Q \subset L)$. If $D \subset M \otimes M$, then there exists $B \in \mathcal{L}at(N \subset M)$ such that $D = M \otimes B$ by Lemma 4.1. Now suppose that D is not contained in $M \otimes M$. We have to show that $D = L$. There exists $z \in D$ such that $z \notin M \otimes M$. Let $H = L^2(M \otimes M, tr)$ and we shall identify $L^2(L, tr)$ with $H \oplus H$ by the formula $\eta(x + yu) = (\eta(x), \eta(y))$ for $x + yu \in L = (M \otimes M) \rtimes_{\alpha} \mathbb{Z}/2\mathbb{Z}$ and $x, y \in M \otimes M$. Let K be the $\| \ \|_2$-closure of $\eta(D)$ in $L^2(L, tr)$ and K_z the $\| \ \|_2$-closure of $QJ_L QJ_L \eta(z)$. Since $z \notin M \otimes M$, there exist $a, b \in M \otimes M$ such that $z = a + bu$ and $b \neq 0$. Therefore K_z is not contained in $H \oplus 0$. Since $z \in D$ and $Q \subset D$, we have $K_z \subset K$. Let P_z (resp. P) be the projection of $L^2(L, tr)$ onto K_z (resp. K). Then P_z is not domminated by $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \in B(H \oplus H)$. Note that $P_z \leq P$ and $\begin{pmatrix} e_{M \otimes N}^{M \otimes M} & 0 \\ 0 & 0 \end{pmatrix} \leq P$. Let $\pi : L \longrightarrow B(L^2(L, tr)) = B(H \oplus H)$ be the

GNS representation. Then for $a \in M \otimes M$, we have

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad \text{and} \quad J_L\pi(a)J_L = \begin{pmatrix} J_0aJ_0 & 0 \\ 0 & J_0\alpha(a)J_0 \end{pmatrix},$$

where $J_0 = J_{M\otimes M}$ on $H = L^2(M\otimes M, tr)$, since $\pi(a)\eta(x + yu) = \eta(ax + ayu)$ and $J_L\pi(a)J_L\eta(x + yu) = \eta((x + yu)a^*) = \eta(xa^* + y\alpha(a^*)u)$ for $x + yu \in L = (M \otimes M) \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}$.

Put $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ for some $p,q,r,s \in B(H)$ with $q = r^*$. Since $Q = M \otimes N \subset K$, we have $P \in \pi(M\otimes M)' \cap (J_L\pi(M\otimes M)J_L)'$. Thus we have

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} \text{ and } \begin{pmatrix} p & q \\ q & s \end{pmatrix}\begin{pmatrix} J_0aJ_0 & 0 \\ 0 & J_0\alpha(a)J_0 \end{pmatrix} = \begin{pmatrix} J_0aJ_0 & 0 \\ 0 & J_0\alpha(a)J_0 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

for all $a \in M \otimes N$. Therefore we have that

p,q,r and s is in $(M \otimes N)' = M' \otimes N'$ on $H = L^2(M\otimes M) = L^2(M)\otimes L^2(M)$,

$pJ_0aJ_0 = J_0aJ_0p$, $qJ_0\alpha(a)J_0 = J_0aJ_0q$, $rJ_0aJ_0 = J_0\alpha(a)J_0r$ and $sJ_0\alpha(a)J_0 = J_0\alpha(a)J_0s$ for all $a \in M \otimes N$.

Then $s \in (M' \otimes N') \cap ((J_MNJ_M)' \otimes (J_MMJ_M)')$

$= (M' \cap \langle M,e_N\rangle) \otimes (N' \cap M) = \mathbb{C} \otimes \mathbb{C}$.

Similarly we have $p \in (M' \cap M) \otimes (N' \cap \langle M,e_N\rangle)$.

Since P is a projection, s is a scalar such that $0 \leq s \leq 1$.

We consider the three cases such that $s = 0$, $0 < s < 1$, or $s = 1$.

First assume that $s = 0$. Since P is a projection,

$$P^2 = \begin{pmatrix} p & r^* \\ r & 0 \end{pmatrix}^2 = \begin{pmatrix} p^2 + r^*r & pr^* \\ rp & rr^* \end{pmatrix} = P = \begin{pmatrix} p & r^* \\ r & 0 \end{pmatrix}.$$

Hence $r = 0$ and p is a projection. Then we have

$$P_z \leq P = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

This is a contradiciton. Thus $s \neq 0$.

Next we assume that s is a scalar with $0 < s < 1$. Since

$$P^2 = \begin{pmatrix} p^2 + r^*r & pr^* + r^*s \\ rp + sr & rr* + s^2 \end{pmatrix} = P = \begin{pmatrix} p & r^* \\ r & s \end{pmatrix} ,$$

$rr^* + s^2 = s$. Thus $rr^* = s - s^2$ is a non-zero scalar. Since r is in a finite factor $M' \otimes N'$ on H, there exist a unitary $w \in M' \otimes N'$ and a non-zero scalar c such that $r = cw$. Since $r(p + s) = r$ and r is invertible, we have $p + s = I$. Thus $p = I - s$ is a scalar with $0 < p < 1$. Recall that $\begin{pmatrix} e_{M\otimes N}^{M\otimes M} & 0 \\ 0 & 0 \end{pmatrix} \leq P = \begin{pmatrix} p & r^* \\ r & s \end{pmatrix}$. Thus we have $e_{M\otimes N}^{M\otimes M} \leq p$ and $p < 1$. This is a contradiction. Therefore only the case that $s = 1$ occurs. We suppose that $s = 1$. Then

$$P^2 = \begin{pmatrix} p^2 + r^*r & pr^* + r^* \\ rp + r & rr^* + 1 \end{pmatrix} = \begin{pmatrix} p & r^* \\ r & 1 \end{pmatrix} = P .$$

Since $rr^* + 1 = 1$, we have $r = 0$. Then we have

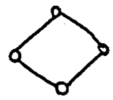
$$\begin{pmatrix} e_{M\otimes N}^{M\otimes M} & 0 \\ 0 & 0 \end{pmatrix} \leq P = \begin{pmatrix} p & 0 \\ 0 & I \end{pmatrix} .$$

Since $tr_{M'\otimes N'}(p) \neq 0$, we have $tr_{\pi(D)'}(P) = \dfrac{tr(p) + 1}{2} > \dfrac{1}{2}$.
Therefore $[L:D] = \dfrac{1}{tr_{\pi(D)'}(P)} < 2$. This implies that $[L:D] = 1$ and thus we have shown that $L = D$. Q.E.D

5. Tensor products

As we know that the subgroup lattice $\mathcal{L}(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$ is not isomorphic to $\mathcal{L}(\mathbb{Z}/2\mathbb{Z}) \times \mathcal{L}(\mathbb{Z}/2\mathbb{Z})$, we do *not* have the lattice isomorphism that $\mathcal{L}at(N_1\otimes N_2 \subset M_1\otimes M_2) \simeq \mathcal{L}at(N_1\subset M_1) \times \mathcal{L}at(N_2\subset M_2)$ in general. But the formula holds in some interesting case.

Proposition 5.1 Let M_i be a type II_1 factor such that $N_i' \cap M_i = \mathbb{C}$ and $2 < [M_i:N_i] < \infty$ for $i = 1,2$. Denote the Jones projection by $e_i = e_{N_i}^{M_i}$ for $i = 1,2$. Assume that $N_i' \cap \langle M_i, e_i\rangle$ is generated by $\{1, e_i\}$, i,e, isomorphic to $\mathbb{C}^2$. Put $M = M_1 \otimes M_2$ and $N = N_1 \otimes N_2$. Then $\mathcal{L}at(N_i \subset M_i) = \{N, M\} \simeq$  $i = 1,2$ and

$$\mathcal{L}at(N \subset M) = \{ N,\ M_1 \otimes N_2,\ N_1 \otimes M_2,\ M\} \simeq$$  .

In particular $\mathcal{L}at(N \subset M) \simeq \mathcal{L}at(N_1 \subset M_1) \times \mathcal{L}at(N_2 \subset M_2)$.

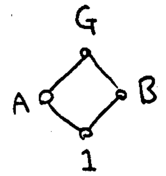
Proof. Note that $e_N^M = e_1 \otimes e_2$ and $\langle M, e_N^M\rangle \simeq \langle M_1, e_1\rangle \otimes \langle M_2, e_2\rangle$. Therefore we have $N' \cap \langle M, e_N^M\rangle = (N_1' \cap \langle M_1, e_1\rangle) \otimes (N_2' \cap \langle M_2, e_2\rangle)$ $= (\mathbb{C}I + \mathbb{C}e_1) \otimes (\mathbb{C}I + \mathbb{C}e_2) \simeq \mathbb{C}^4$ has a linear basis $\{I\otimes I, e_1\otimes I, I\otimes e_2, e_1\otimes e_2\}$ Let $K \in \mathcal{L}at(N \subset M)$, then $e_K^M \geq e_N^M$. Put $r = I - (e_1\otimes I \vee I \otimes e_2)$.

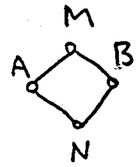
First consider the case that the equality $e_K^M \geq e_N^M + r$ holds. Then

$$\begin{aligned} tr(e_K^M) &\geq tr(e_N^M) + tr(r) \\ &= tr(e_N^M) + 1 - (tr(e_1) + tr(e_2) - tr(e_1 \otimes e_2)) \\ &= 1 + 2[M_1:N_1]^{-1}[M_2:N_2]^{-1} - [M_1:N_1]^{-1} - [M_2:N_2]^{-1} \\ &= 2(\tfrac{1}{2} - [M_1:N_1]^{-1})(\tfrac{1}{2} - [M_2:N_2]^{-1}) + \tfrac{1}{2} > \tfrac{1}{2} . \end{aligned}$$

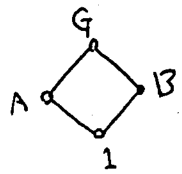
Thus $[M:K] = tr(e_K^M)^{-1} < 2$. This implies that $[M:K] = 1$ and $K = M$.

Next we consider the case that the equality $e_K^M \geq e_N^M + r$ does not hold. Then we have $e_N^M = e_1 \otimes e_2 \leq e_K^M \leq e_1 \otimes I \vee I \otimes e_2$. Therefore $e_K^M = e_1\otimes I \vee I\otimes e_2$, $e_K^M = e_1 \otimes I$, $e_K^M = I \otimes e_2$ or $e_K^M = e_1 \otimes e_2$. Suppose that $e_K^M = e_1 \otimes I \vee I \otimes e_2$. Then we have $K \supset N_1 \otimes M_2$ and $K \supset M_1 \otimes N_2$. Hence $K = M_1 \otimes M_2$. Then $e_K^M = I \geq e_N^M + r$. This is a contradiction. Thus $e_K^M \neq e_1 \otimes I \vee I \otimes e_2$. The other cases actually occur and $K = N_1 \otimes M_2$, $K = M_2 \otimes N_2$ or $K = N_1 \otimes N_2 = N$. Q.E.D.

Corollary 5.2 There exist no finite group G such that $\mathcal{L}(G) \simeq$ 

with $[G:A] = [G:B] = [A:1] = [B:1]$. There exists an inclusion $N \subset M$ of type II_1 factors such that $[M:N] < \infty$, $N' \cap M = \mathbb{C}$ and $\mathcal{L}at(N \subset M) \simeq$ 

with $[M:A] = [M:B] = [A:N] = [B:N]$.

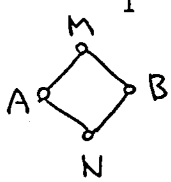
Proof. Let G be a finite group such that $\mathcal{L}(G) \simeq$  .

Take $x \in G \setminus A$ and $y \in G \setminus B$. Put $c = xy$ and consider the subgroup $C = \langle\langle c \rangle\rangle$ of G generated by c. Since C is not contained in A nor B, we have $C = G$, so that G is a cyclic group. Therefore $G = A \oplus B = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$ for some prime numbers p and q with $p \neq q$. Therefore $[A:1] \neq [B:1]$. Thus there exist no such a group that $[G:A] = [G:B] = [A:1] = [B:1]$.

Nextly let $\{e_1, e_2, e_3, \ldots\}$ be a sequence of Jones projections such that $e_i e_j = e_j e_i$ ($|i-j| \geq 2$) and $e_i e_{i\pm 1} e_i = \tau e_i$ for some $\tau^{-1} = 4\cos^2 \frac{\pi}{n}$ $(n = 5,6,7,\ldots)$. Consider type II_1 factors

$$M_i = \{e_1, e_2, e_3, \ldots\}'' \supset N_i = \{e_2, e_3, \ldots\}'' \quad \text{for } i = 1,2 .$$

Put $N = N_1 \otimes N_2$, $M = M_1 \otimes M_2$, $A = M_1 \otimes N_2$ and $B = N_1 \otimes M_2$.

Then $\mathcal{L}at(N \subset M) \simeq$  with $[M:A] = [M:B] = [A:N] = [B:N]$ by Proposition 5.1. Q.E.D.

6. Some examples.

As we have shown in Theorem 2.2, the intermediate subfactor lattice $\mathcal{L}at(N\subset M)$ is a finite lattice if $[M:N] < \infty$ and $N' \cap M = \mathbb{C}$. It is an interesting problem to determine which finite lattices are realized as intermediate subfactor lattices. Here we shall examine the lattices with at most six elements. Let us describe the Hasse diagrams of them in figure 1, c.f., [St;Chapter 3] .

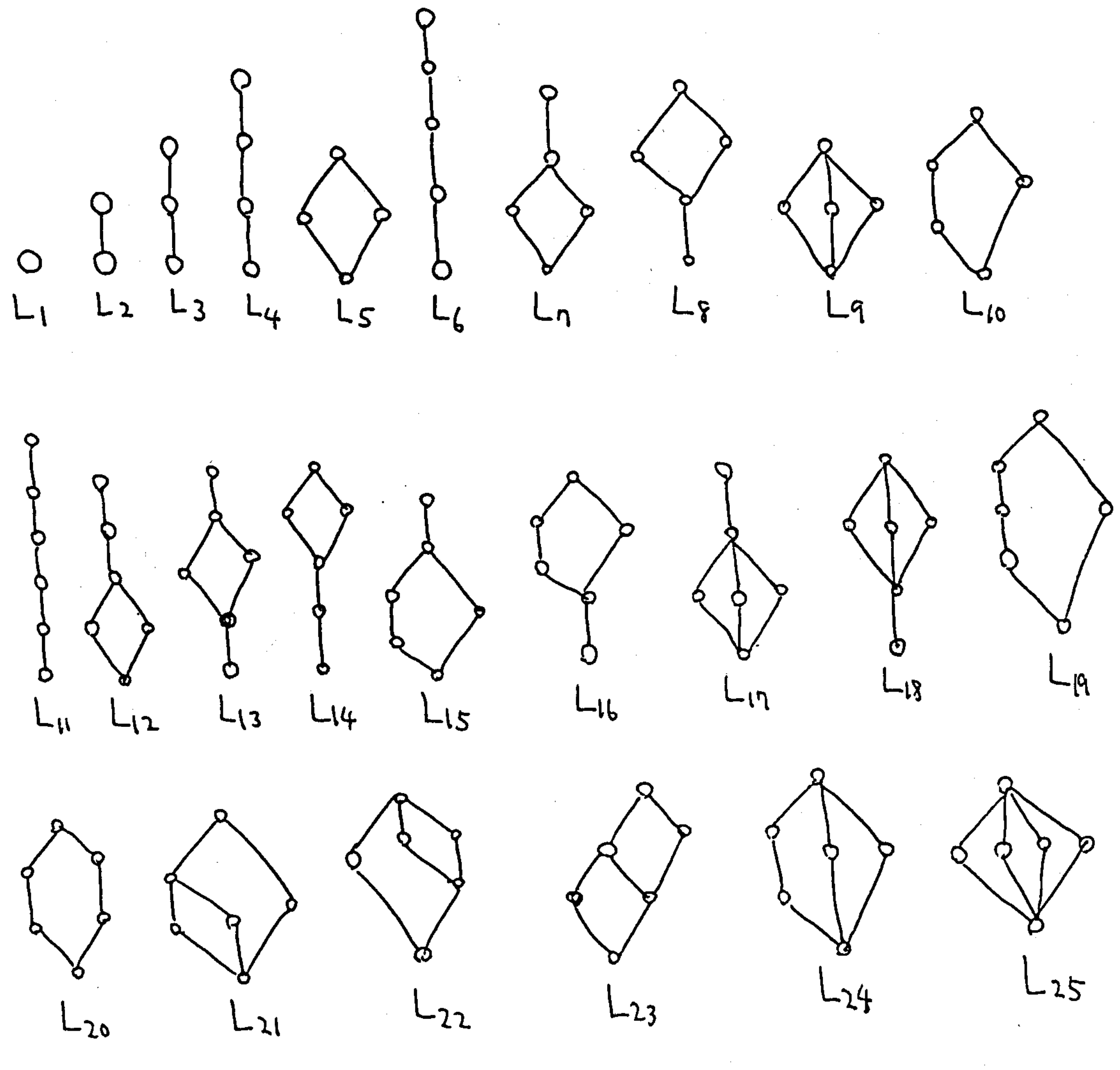


figure 1

We shall give a certain realization of some of the above lattices by intermediate subfactor lattices as far as we know.

Example 6.1 The lattice $L_1 \simeq \mathcal{L}at(N \subset N)$, $L_2 \simeq \mathcal{L}at(N \subset N \rtimes \mathbb{Z}/2\mathbb{Z})$, $L_3 \simeq \mathcal{L}at(N \subset N \rtimes \mathbb{Z}/4\mathbb{Z})$, $L_4 \simeq \mathcal{L}at(N \subset N \rtimes \mathbb{Z}/8\mathbb{Z})$, $L_5 \simeq \mathcal{L}at(N \subset N \rtimes (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}))$, and $L_6 \simeq \mathcal{L}at(N \subset N \rtimes \mathbb{Z}/16\mathbb{Z})$.
The lattices L_7 and L_8 are obtained by adding a chain to L_5. Hence L_7 and $L_8 \in \mathcal{L}(\text{Subfactors})$ by Theorem 5.1. $L_9 \simeq \mathcal{L}at(N \subset N \rtimes (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$
The lattice $L_{10} \notin \mathcal{L}(\text{Groups})$ and $L_{10} \notin \mathcal{L}(\text{Group duals})$. We do not know wheather $L_{10} \in \mathcal{L}(\text{Subfactors})$ or not. And $L_{11} \simeq \mathcal{L}at(N \subset N \rtimes \mathbb{Z}/32\mathbb{Z})$.
The lattices L_{12}, L_{13} and L_{14} are obtained by adding chains to L_5. Hence L_{12}, L_{13} and $L_{14} \in \mathcal{L}(\text{Subfactors})$ by Theorem 5.1. We do not know wheather L_{15}, $L_{16} \in \mathcal{L}(\text{Subfactors})$ or not.

Let $G = \langle\langle x,y \mid x^4 = 1,\ x^2y^{-2} = 1,\ y^{-1}xyx = 1\rangle\rangle$ be the quaternion group, see [Wei;Example 4.4]. Then The lattice $L_{17} \simeq \mathcal{L}at(M^G \subset M)$ and $L_{18} \simeq \mathcal{L}at(N \subset N \rtimes G)$.

The Lattices L_{19} , L_{20} $\notin \mathcal{L}(\text{Groups})$ and L_{19}, $L_{20} \notin \mathcal{L}(\text{Group duals})$. We do not know wheather L_{19}, L_{20} $\in \mathcal{L}(\text{Subfactors})$ or not.

Let S_3 (resp. S_2) be the symmetric group of order 3 (resp. 2). Let $\alpha \in \mathrm{Aut}\,(S_3 \times S_3)$ be the flip flop. Put $G = (S_3 \times S_3) \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}$ and $H = S_2 \times S_2$. Then H is a subgroup of G and the lattices $L_{21} \simeq \mathcal{L}at(N \rtimes H \subset N \rtimes G)$ and $L_{22} \simeq \mathcal{L}at(M^G \subset M^H)$.

The lattice $L_{23} \simeq \mathcal{L}at(N \subset N \rtimes \mathbb{Z}/12\mathbb{Z})$. We consider S_2 is a subgroup of S_4 and the lattice $L_{24} \simeq \mathcal{L}at(N \rtimes S_2 \subset N \rtimes S_4)$.

The lattice $L_{25} \simeq \mathcal{L}at(N \subset N \rtimes (\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}))$. Or we consider

the dihedral group $D_6 = \langle\langle x, y \mid x^3 = 1,\ y^2 = 1,\ (xy)^2 = 1 \rangle\rangle$.
Then $L_{25} \simeq \mathcal{L}at(N \subset N \rtimes D_6)$.

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