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On Critical Cases of Sobolev Inequalities

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On Critical Cases of Sobolev Inequalities

Dedicated to Professor R.Iino on his seventieth birthday

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Abstract. We present a new form of Trudinger type inequality, which shows an explicit dependence of functions in the Sobolev space of critical order. Moreover, we give a proof of the Brezis-Gallouet-Wainger inequality which is independent of the Fourier representation, thereby arriving at a solution to Brezis' problem.

1. Introduction.

Trudinger's inequality states that compactly supported functions f in the unit ball of the Sobolev space $W^{n/p,p}(\mathbb{R}^n)$ of fractional order n/p with $1 < p < \infty$ satisfy

$$\int_{\mathbb{R}^n} (\exp(\alpha|f(x)|^{p'}) - 1)dx \le C \tag{1.1}$$

for some positive constants α and C, where $1/p' + 1/p = 1$ (see [1,7,11,13]). This replaces Sobolev's inequalities in a limiting case of the imbedding theorem. In fact, $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ if $1/q = 1/p - m/n$ and $0 < m < n/p$; $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ if $m > n/p$; $W^{n/p,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ if $p \le q < \infty$, whereas $W^{n/p,p}(\mathbb{R}^n) \not\subset L^\infty(\mathbb{R}^n)$ except $p = 1$. Trudinger's inequality would, therefore, fill a gap in the Sobolev imbedding theorem. It is not completely clear how functions depend on the right hand side of (1.1), however.

Our first goal in this paper is to present an inequality of Trudinger type, which shows an explicit dependence of functions and holds for all functions in $W^{n/p,p}(\mathbb{R}^n)$. We have the following result:

Theorem 1. *Let p and p' satisfy $1/p + 1/p' = 1$ with $1 < p < \infty$. Then there exist positive constants α and C depending only on p and n such that for all $f \in W^{n/p,p}(\mathbb{R}^n)$*

$$\begin{aligned}\int_{\mathbb{R}^n} \Big(\exp\Big(\frac{\alpha|f(x)|^{p'}}{\|(-\Delta)^{n/2p}f\|_{L^p}^{p'}}\Big) - \sum_{\substack{0\le j<p-1\\ j\in\mathbb{Z}}} \frac{1}{j!}\Big(\frac{\alpha|f(x)|^{p'}}{\|(-\Delta)^{n/2p}f\|_{L^p}^{p'}}\Big)^j\Big)dx \\ \le C\frac{\|f\|_{L^p}^p}{\|(-\Delta)^{n/2p}f\|_{L^p}^p}.\end{aligned} \tag{1.2}$$

Remark 1. (1) The constant C in (1.2) is given by

$$C = C(\alpha) = \sum_{\substack{j\ge p-1\\ j\in\mathbb{Z}}} \frac{j^j}{j!}(p'M^{p'}\alpha)^j$$

where M depends only on p and n. Therefore (1.2) holds for any α and $C(\alpha)$ with $0 < \alpha < \alpha_0$, where α_0 is defined by $p'M^{p'}\alpha_0 = e^{-1}$.
(2) $\|(-\Delta)^{n/2p}f\|_{L^p}$ may be replaced by $\mu + \|(-\Delta)^{n/2p}f\|_{L^p}$ for any $\mu \ge 0$ with α and $C(\alpha)$ unchanged (see the proof below). Similarly, $\|(-\Delta)^{n/2p}f\|_{L^p}$ may be replaced by any equivalent norm on $W^{n/p,p}(\mathbb{R}^n)$ by making α small if it is needed.

(3) The same result holds for $W^{n/p,p}(\Omega)$ for any domain Ω in $\mathbb{R}^n$ with smooth and compact boundary if one replaces $||(-\triangle)^{n/2p}f||_{L^p}$ by $||f||_{W^{n/p,p}(\Omega)}$ in (1.2). The only assumption on Ω that we need here is in fact the existence of an extension operator E_Ω with the following properties: E_Ω is bounded from $W^{n/p,p}(\Omega)$ to $W^{n/p,p}(\mathbb{R}^n)$ and from $L^p(\Omega)$ to $L^p(\mathbb{R}^n)$; $R_\Omega E_\Omega = id$, where R_Ω is the restriction operator from $L^p(\mathbb{R}^n)$ to $L^p(\Omega)$.
(4) When $p = 2$, (1.2) is obtained in [8] for $n = 2$ and in [9] for all dimensions. When $p = n \geq 2$, (1.2) is obtained in [12].
(5) When $p = n \geq 2$, the optimal bound of α for which Trudinger's inequality holds is obtained in [7] for (1.1) (limit included) and in [12] for (1.2) (limit excluded).

The usual proof uses the power series expansion of the exponential function in the integrand in (1.1) and reduces the problem to majorizing each term of the expansion in terms of the Sobolev norms in order that the resulting power series should converge. The method in this paper proceeds in the same way, though the proof of (1.2) requires more precise estimates of the individual terms of the expansion concerning the growth order with respect to the index q of the $L^q(\mathbb{R}^n)$ norm. To be more specific, the proof relies on the following proposition:

Proposition. *Let $1 < p < \infty$. Then there exists a constant M depending only on p and n such that*

$$||f||_{L^q} \leq M q^{1-1/p} ||(-\triangle)^{n/2p} f||_{L^p}^{1-p/q} ||f||_{L^p}^{p/q} \tag{1.3}$$

for any q with $p \leq q < \infty$.

An essential difference between (1.3) and the previous estimates (see, e.g. [1,11,13]) consists in the explicit dependence of $||(-\triangle)^{n/2p}f||_{L^p}$ with sharp exponent on the dominant term. When $p = n$, (1.3) is obtained in [6] through Sobolev's best constant. When $p = 2$, (1.3) has a simple proof based on the Fourier transform [9], while a similar technique seems to be almost useless for proving (1.3). In order to prove (1.3), we trace back to the classical proof of the Hardy-Littlewood-Sobolev inequality and investigate the precise dependence of every estimate on the indices.

We next consider the Brezis-Gallouet-Wainger (BGW) inequality, which arises naturally in a limiting case of the Sobolev estimates. The BGW inequality states that all functions

$f \in W^{n/p,p}(\mathbb{R}^n) \cap W^{m,q}(\mathbb{R}^n)$ with $||f||_{W^{n/p,p}} \leq 1$, $m > n/q$, $1 < p < \infty$, $1 \leq q \leq \infty$, satisfy

$$(1.4) \qquad ||f||_{L^\infty} \leq C(1 + \mathrm{Log}(1 + ||f||_{W^{m,q}}))^{1/p'}$$

where C depends only on p, q, m, n (see [2,3,4]). The BGW inequality is an improvement of the inequality of Gagliardo-Nirenberg type as regards the growth rate with respect to the $W^{m,q}(\mathbb{R}^n)$ norm in the dominant term. In fact, the Gagliardo-Nirenberg inequality proves that the $L^\infty(\mathbb{R}^n)$ norm is bounded by an arbitrarily small power of the $W^{m,q}(\mathbb{R}^n)$ norm, while the BGW inequality proves that the order may be replaced by $O((\mathrm{Log}\, ||f||_{W^{m,q}})^{1/p'})$. Moreover, in [4] it is shown that the last order is optimal. The original proof of (1.4) relies heavily on the Fourier transform. In [2], Brezis conjectured that we could prove (1.4) without the Fourier representation of functions. In [5], Engler gave an answer to this question in the special case where n/p and m are integers.

Our second goal in this paper is to give a proof of the BGW inequality in the general case, which is independent of the Fourier representation.

We prove:

Theorem 2. *Let $p, q, m \in \mathbb{R}$ satisfy $1 < p < \infty$, $1 \leq q < \infty$, $m > n/q$. Then there exists a constant C depending only on p, q, m, n such that for all $f \in W^{n/p,p}(\mathbb{R}^n) \cap W^{m,q}(\mathbb{R}^n)$*

$$(1.5) \qquad ||f||_{L^\infty} \leq C||f||_{W^{n/p,p}}(1 + \mathrm{Log}(1 + \frac{||(-\triangle)^{m/2} f||_{L^q}}{||f||_{W^{n/p,p}}}))^{1/p'}.$$

Remark 2. (1) When $p = 1$ or $p = \infty$, (1.5) is clear.
(2) $||f||_{W^{n/p,p}}$ may be replaced by $\mu + ||f||_{W^{n/p,p}}$ for any $\mu \geq 0$ (see the proof below).
(3) As in Theorem 2, the same result holds for $W^{n/p,p}(\Omega) \cap W^{m,q}(\Omega)$ if one replaces $||(-\triangle)^{m/2} f||_{L^q}$ by $||f||_{W^{m,q}(\Omega)}$.

Our proof depends again on (1.3), though the idea based on Morrey's estimate is due to Engler [5].

The third inequality that we are interested in is the Brezis-Wainger (BW) inequality which concerns another limiting case where the modulus of continuity of Lipschitz (or

Hölder) type needs a modification. The BW inequality states that all functions $f \in W^{n/p+1,p}(\mathbb{R}^n)$ with $1 < p < \infty$ satisfy

$$(1.6) \qquad |f(x) - f(y)| \leq C\|f\|_{W^{n/p+1,p}}|x-y|(1+|\operatorname{Log}|x-y||)^{1/p'}$$

for almost every $x, y \in \mathbb{R}^n$, where C depends only on n and p (see [4]). The BW inequality is a substitute for the Zygmund type estimates of the form:

$$(1.7) \qquad |f(x) + f(y) - 2f(\frac{x+y}{2})| \leq C\|f\|_{W^{n/p+1,p}}|x-y|.$$

The Morrey type estimates assure that $W^{n/p+\sigma,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \cap C^\sigma(\mathbb{R}^n)$ for any σ and p with $0 < \sigma < 1$, $1 < p < \infty$, while in general, all functions $f \in W^{n/p+1,p}(\mathbb{R}^n)$ need not satisfy the Lipschitz condition. In this sense (1.7) is critial and (1.6) shows that all functions $f \in W^{n/p+1,p}(\mathbb{R}^n)$ are almost Lipschitz.

In [4] the proof of (1.6) uses the following inequality:

Theorem 3([4]). *Let $1 < p < \infty$. Then there exists a constant C depending only on p and n such that for all $f \in W^{n/p,p}(\mathbb{R}^n)$ and for all Lebesgue measurable sets E with finite measure*

$$(1.8) \qquad \int_E |f(x)|dx \leq C\|f\|_{W^{n/p,p}}|E|(1+|\operatorname{Log}|E||)^{1/p'},$$

where $|E|$ denotes the Lebesgue measure of E.

Given (1.8), the desired estimate (1.6) follows easily by Morrey's technique (see [4]). Our third purpose in this paper is to give a simple proof of (1.8) which uses (1.3) again.

As Brezis has observed in [2], it is hard to recognize a simple relation between Trudinger's inequality and the BGW inequality. Still, we could take the point of view that both inequalities are corollaries to the proposition in this paper.

2. Proof of the Proposition.

We use the following notations: For a Lebesgue measurable set $E \subset \mathbb{R}^n$, $|E|$ denotes its Lebesgue measure. $L^p(\mathbb{R}^n)$ will be denoted simply by L^p with norm denoted accordingly by $||\cdot||_p$. The integral sign without subscript denotes the integral over $\mathbb{R}^n$. Γ denotes the gamma function. Different positive constants might be denoted by the same letter C.

We first recall the Marcinkiewicz interpolation theorem:

Let $1 \le p_j \le q_j < \infty$, $j = 0, 1$, $q_0 < q_1$, and let T be a sublinear operator of weak type (p_j, q_j) with constant M_j for both $j = 0$ and $j = 1$, i.e.,

$$\operatorname*{Sup}_{t>0} t|\{x \in \mathbb{R}^n; |Tf(x)| > t\}|^{1/q_j} \le M_j ||f||_{p_j}, \quad j = 0, 1.$$

Then for any θ with $0 < \theta < 1$, T is bounded from L^{p_θ} to L^{q_θ} with $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$, $1/q_\theta = (1-\theta)/q_0 + \theta/q_1$. Moreover,

$$||Tf||_{q_\theta} \le 2\left(\frac{q_\theta}{q_1 - q_\theta} + \frac{q_\theta}{q_\theta - q_0}\right)^{1/q_\theta} M_0^{1-\theta} M_1^{\theta} ||f||_{p_\theta}$$

for all $f \in L^{p_\theta}$.

We next consider the Riesz potential I_λ of order $\lambda \in (0, n)$, defined by

$$(I_\lambda f)(x) = \int |x - y|^{\lambda - n} f(y) dy = (K_\lambda * f)(x),$$

where $*$ denotes the convolution and $K_\lambda(x) = |x|^{\lambda - n}$. For $s > 0$, we decompose $K_\lambda = K_{\lambda,s}^{(0)} + K_{\lambda,s}^{(1)}$, where $K_{\lambda,s}^{(0)}(x) = K_\lambda(x)$ if $|x| < s$ and $K_{\lambda,s}^{(0)}(x) = 0$ otherwise. We denote by $I_\lambda = I_\lambda^{(0)}(s) + I_\lambda^{(1)}(s)$ the corresponding decomposition of the Riesz potential. By Young's inequality, for any p with $1 \le p < \infty$

$$||I_\lambda^{(0)}(s) f||_p \le ||K_{\lambda,s}^{(0)}||_1 ||f||_p = \frac{\omega_{n-1}}{\lambda} s^\lambda ||f||_p,$$
$$||I_\lambda^{(1)}(s) f||_\infty \le ||K_{\lambda,s}^{(1)}||_{p'} ||f||_p = \left(\frac{\omega_{n-1} q}{n p'}\right)^{\frac{1}{p'}} s^{\lambda - \frac{n}{p}} ||f||_p,$$

where $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$, $1/q = 1/p - \lambda/n$, and we follow the convention $(\frac{\omega_{n-1} q}{n p'})^{\frac{1}{p'}} = 1$ when $p = 1$. For $t > 0$, we define

$$J_\lambda^{(k)}(t) = I_\lambda^{(k)}\left(\left(\frac{2}{t}\left(\frac{\omega_{n-1} q}{n p'}\right)^{\frac{1}{p'}}\right)^{\frac{q}{n}}\right), \quad k = 0, 1.$$

We have thus made the decomposition $I_\lambda = J_\lambda^{(0)}(t) + J_\lambda^{(1)}(t)$ such that for any $t > 0$

$$||J_\lambda^{(0)}(t)f||_p \le \frac{\omega_{n-1}}{\lambda}\Big(\frac{2}{t}\Big(\frac{\omega_{n-1}q}{np'}\Big)^{\frac{1}{p'}}\Big)^{\frac{q}{p}-1}||f||_p,$$
$$||J_\lambda^{(1)}(t)f||_\infty \le \frac{t}{2}||f||_p,$$

which implies

$$\begin{aligned}
&|\{x \in \mathbb{R}^n; |(I_\lambda f)(x)| > t||f||_p\}| \\
&\le |\{x \in \mathbb{R}^n; |(J_\lambda^{(0)}(t)f)(x)| > \frac{t}{2}||f||_p\}| \\
&\quad + |\{x \in \mathbb{R}^n; |(J_\lambda^{(1)}(t)f)(x)| > \frac{t}{2}||f||_p\}| \\
&= |\{x \in \mathbb{R}^n; |(J_\lambda^{(0)}(t)f)(x)| > \frac{t}{2}||f||_p\}| \\
&\le \Big(\frac{\omega_{n-1}}{\lambda}\Big(\frac{2}{t}\Big(\frac{\omega_{n-1}q}{np'}\Big)^{\frac{1}{p'}}\Big)^{\frac{q}{p}-1}||f||_p\Big)^p\Big(\frac{t}{2}||f||_p\Big)^{-p} \\
&= 2^q\Big(\frac{\omega_{n-1}}{\lambda}\Big)^p\Big(\frac{\omega_{n-1}q}{np'}\Big)^{\frac{q-p}{p'}}t^{-q},
\end{aligned}$$

where we have used Chebyshev's inequality. Therefore,

$$\begin{aligned}
(2.1)\qquad &\operatorname*{Sup}_{t>0} t|\{x \in \mathbb{R}^n; |(I_\lambda f)(x)| > t\}|^{1/q} \\
&= \operatorname*{Sup}_{t>0} t||f||_p|\{x \in \mathbb{R}^n; |(I_\lambda f)(x)| > t||f||_p\}|^{1/q} \\
&\le 2\Big(\frac{\omega_{n-1}}{\lambda}\Big)^{\frac{p}{q}}\Big(\frac{\omega_{n-1}q}{np'}\Big)^{\frac{q-p}{p'q}}||f||_p \\
&= 2\Big(\frac{\omega_{n-1}q}{n}\Big)^{1-\frac{1}{p}+\frac{1}{q}}\frac{(p-1)^{(p-1)(\frac{1}{p}-\frac{1}{q})}}{p^{1-\frac{1}{p}-\frac{2p-1}{q}}(q-p)^{\frac{p}{q}}}||f||_p
\end{aligned}$$

whenever $\frac{1}{q} = \frac{1}{p} - \frac{\lambda}{n}$, $1 \le p < q < \infty$, $0 < \lambda < n$. When $(\frac{1}{p_0}, \frac{1}{q_0}) = (1, 1 - \frac{\lambda}{n})$ with $0 < \lambda < n$, (2.1) leads to

$$(2.2)\qquad \operatorname*{Sup}_{t>0} t|\{x \in \mathbb{R}^n; |(I_\lambda f)(x)| > t\}|^{\frac{1}{q_0}} \le 2\Big(\frac{\omega_{n-1}q_0}{n(q_0-1)}\Big)^{\frac{1}{q_0}}||f||_1.$$

When $(\frac{1}{p_1}, \frac{1}{q_1}) = (\frac{1}{p} - \frac{(n-\lambda p)^2}{pn(pn+n-\lambda p)}, \frac{n-\lambda p}{pn+n-\lambda p})$ with $0 < \lambda < n$ and $1 < p < n/\lambda$, (2.1) leads to

$$\begin{aligned}
(2.3)\qquad &\operatorname*{Sup}_{t>0} t|\{x \in \mathbb{R}^n; |(I_\lambda f)(x)| > t\}|^{\frac{1}{q_1}} \\
&\le 2\Big(\frac{\omega_{n-1}q_1}{n}\Big)^{1-\frac{1}{p_1}+\frac{1}{q_1}}\frac{(p_1-1)^{(p_1-1)(\frac{1}{p_1}-\frac{1}{q_1})}}{p_1^{1-\frac{1}{p_1}-\frac{2p_1-1}{q_1}}(q_1-p_1)^{\frac{p_1}{q_1}}}||f||_{p_1}.
\end{aligned}$$

Now we fix $p \in (1,\infty)$. For any q with $p < q < \infty$ we define $\lambda = n(\frac{1}{p}-\frac{1}{q})$. Then $0 < \lambda < n$, $1 < p < \frac{n}{\lambda}$, $(\frac{1}{p_0}, \frac{1}{q_0}) = (1, 1-\frac{1}{p}+\frac{1}{q})$, $(\frac{1}{p_1}, \frac{1}{q_1}) = (\frac{1}{p}-\frac{1}{q}+\frac{1}{q+1}, \frac{1}{q+1})$, so that (2.2) and (2.3) prove that $I_{n(\frac{1}{p}-\frac{1}{q})}$ is of weak types (p_0,q_0) and (p_1,q_1) simultaneously. Moreover, if we set $\theta = \theta(q) = (1-\frac{1}{p})/(1-\frac{1}{p}+\frac{1}{q}-\frac{1}{q+1})$, then $0<\theta<1$, $\frac{1}{p} = \frac{1-\theta}{p_0}+\frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0}+\frac{\theta}{q_1}$. Consequently, the Marcinkiewicz interpolation theorem shows

$$||I_{n(\frac{1}{p}-\frac{1}{q})}f||_q \leq 4(q+\frac{pq}{q-p})^{\frac{1}{q}} M_0(q)^{1-\theta} M_1(q)^{\theta} ||f||_p, \tag{2.4}$$

where

$$M_0(q) = (\frac{\omega_{n-1}q_0}{n(q_0-1)})^{\frac{1}{q_0}}$$

$$M_1(q) = (\frac{\omega_{n-1}q_1}{n})^{1-\frac{1}{p_1}+\frac{1}{q_1}} \frac{(p_1-1)^{(p_1-1)(\frac{1}{p_1}-\frac{1}{q_1})}}{p_1^{1-\frac{1}{p_1}-\frac{2p_1-1}{q_1}}(q_1-p_1)^{\frac{p_1}{q_1}}}.$$

We consider the growth property of the right hand side of (2.4) with respect to q. We have

$$\lim_{q\to\infty} \theta(q) = 1,$$

$$\lim_{q\to\infty} M_0(q) = (\frac{\omega_{n-1}p}{n})^{1-\frac{1}{p}},$$

$$\lim_{q\to\infty} q^{\frac{1}{p}-1}M_1(q) = (\frac{\omega_{n-1}(p-1)}{np})^{1-\frac{1}{p}},$$

and therefore for any q with $p<q<\infty$

$$||I_{n(\frac{1}{p}-\frac{1}{q})}f||_q \leq Cq^{1-\frac{1}{p}}||f||_p \tag{2.5}$$

with some positive constant C depending only on p and n.

Since $(-\triangle)^{-\mu/2} = \frac{\Gamma((n-\mu)/2)}{2^{\mu}\pi^{n/2}\Gamma(\mu/2)}I_\mu$ for $0<\mu<n$, we conclude from (2.5) that for any q with $p<q<\infty$

$$||f||_q \leq Cq^{1-\frac{1}{p}}||(-\triangle)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}f||_p, \tag{2.6}$$

where C depends only on p and n.

We are now in a position to prove the proposition.

Proof of the Proposition. In view of (2.6), it suffices to prove the interpolation inequality

$$\|(-\triangle)^{\frac{\sigma}{2}(1-\theta)}f\|_p \leq C\|(-\triangle)^{\frac{\sigma}{2}}f\|_p^{1-\theta}\|f\|_p^{\theta}, \; f \in W^{\sigma,p}(\mathbb{R}^n), \tag{2.7}$$

where $0 \leq \sigma \leq n/p$, $0 < \theta < 1$, $1 < p < \infty$, and C depends only on p and n. To prove (2.7), we use a dyadic decomposition. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative function such that $\operatorname{Supp}\psi \subset \{\xi; 1/2 < |\xi| < 2\}$ and $\sum_{j=-\infty}^{\infty} \psi(2^{-j}\xi) = 1$ for any $\xi \neq 0$ with at most two nonvanishing terms in the summation. We set $\psi_j(\xi) = \psi(2^{-j}\xi)$ and denote by F the Fourier transform. Then by the Hörmander-Mihlin multiplier theorem and Littlewood-Paley theory (see, e.g. [10]), there exist two positive constants C and C' such that

$$\begin{aligned} C\|(-\triangle)^{s/2}f\|_p &\leq \|(\sum_{j=-\infty}^{\infty} |2^{js}F^{-1}\psi_j Ff|^2)^{1/2}\|_p \\ &\leq C'\|(-\triangle)^{s/2}f\|_p \end{aligned}$$

for all $f \in W^{s,p}(\mathbb{R}^n)$, where C and C' depend only on p, n, and s, yet it is possible to make those two constants independent of s if s ranges over a bounded subset of $\mathbb{R}$. Hence (2.7) follows by applying Hölder's inequality to the norm expressed in terms of the dyadic decomposition and by using the equivalence of norms as above.

3. Proof of the Theorems.

Proof of Theorem 1. The theorem follows by taking the power series expansion of the exponential function in the integrand in (1.1), by using (1.3) in the estimate of each integral, and finally by applying the monotone convergence theorem. We note here that $j \geq p-1$ implies $p'j \geq p$, which makes the subtraction in the integrand in (1.2) reasonable.

Proof of Theorem 2. It suffices to prove (1.5) when $0 < m - n/q < 1$ since the case $m - n/q \geq 1$ follows by means of the inequalities

$$\begin{aligned} \|(-\triangle)^{s/2}f\|_q &\leq C\|(-\triangle)^{m/2}f\|_q^{s/m}\|f\|_q^{1-s/m} \\ &\leq C\|(-\triangle)^{m/2}f\|_q^{s/m}\|f\|_{W^{n/p,p}}^{1-s/m} \\ &\leq C\|(-\triangle)^{m/2}f\|_q + C\|f\|_{W^{n/p,p}} \end{aligned}$$

for $0 < s < m$. By the imbedding theorem of the homogeneous Besov sapces, we have

$$|f(x) - f(y)| \leq C\|(-\triangle)^{m/2} f\|_q |x-y|^\sigma \tag{3.1}$$

with $\sigma = m - n/q \in (0,1)$. Let $0 < \varepsilon < e^{-p}$ and let $\xi \in \mathbb{R}^n$ satisfy $|\xi| \leq 1$. By (3.1),

$$|f(x) - f(x+\varepsilon\xi)| \leq C\varepsilon^\sigma \|(-\triangle)^{m/2} f\|_q. \tag{3.2}$$

On the other hand, by Hölder's inequality and (1.3)

$$\begin{aligned} \int_{|\xi|\leq 1} |f(x+\varepsilon\xi)| d\xi &\leq \left(\frac{\omega_{n-1}}{n}\right)^{1-1/r} \left(\int_{|\xi|\leq 1} |f(x+\varepsilon\xi)|^r d\xi\right)^{1/r} \\ &\leq \left(\frac{\omega_{n-1}}{n}\right)^{1-1/r} \varepsilon^{-n/r} \|f\|_r \\ &\leq C\varepsilon^{-n/r} r^{1-1/p} \|f\|_{W^{n/p,p}} \end{aligned} \tag{3.3}$$

for any r with $p \leq r < \infty$, where C depends only on p and n. Let $r = \mathrm{Log}(\frac{1}{\varepsilon})$. Then $p \leq r < \infty$ and (3.3) implies

$$\int_{|\xi|\leq 1} |f(x+\varepsilon\xi)| d\xi \leq C(\mathrm{Log}(\frac{1}{\varepsilon}))^{1-1/p} \|f\|_{W^{n/p,p}} \tag{3.4}$$

for any ε with $0 < \varepsilon < e^{-p}$, where C depends only on p and n since $\varepsilon^{-n/r} = n$. By (3.2) and (3.4),

$$\begin{aligned} |f(x)| &= \frac{n}{\omega_{n-1}} \int_{|\xi|\leq 1} |f(x)| d\xi \\ &\leq \frac{n}{\omega_{n-1}} \int_{|\xi|\leq 1} (|f(x) - f(x+\varepsilon\xi)| + |f(x+\varepsilon\xi)|) d\xi \\ &\leq C\varepsilon^\sigma \|(-\triangle)^{m/2} f\|_q + C(\mathrm{Log}(\frac{1}{\varepsilon}))^{1-1/p} \|f\|_{W^{n/p,p}}. \end{aligned} \tag{3.5}$$

The result now follows by setting

$$\varepsilon = 1/(e^p + (\|(-\triangle)^{m/2} f\|_q / \|f\|_{W^{n/p,p}})^{1/\sigma})$$

in (3.5).

Proof of Theorem 3. By Hölder's inequality and (1.3),

$$\begin{aligned} \int_E |f(x)| dx &\leq |E|^{1-1/q} \|f\|_q \\ &\leq C|E|^{1-1/q} q^{1-1/p} \|f\|_{W^{n/p,p}} \end{aligned} \tag{3.6}$$

for any q with $p \leq q < \infty$, where C depends only on p and n. When $0 < |E| \leq e^{-p}$, we set $q = \mathrm{Log}\,\frac{1}{|E|}$ in (3.6). Then

$$\int_E |f(x)|dx \leq C|E||\,\mathrm{Log}\,|E||^{1/p'}||f||_{W^{n/p,p}}. \tag{3.7}$$

When $|E| > e^{-p}$, we set $q = p$ in the first inequality of (3.6). Then

$$\begin{aligned} \int_E |f(x)|dx &\leq |E|^{1-1/p}||f||_p \\ &\leq e|E|||f||_p. \end{aligned} \tag{3.8}$$

The result now follows from (3.7) and (3.8).

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