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**PERESTROIKAS OF OPTICAL WAVE  
FRONTS AND GRAPHLIKE  
LEGENDRIAN UNFOLDINGS**

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# PERESTROIKAS\* OF OPTICAL WAVE FRONTS AND GRAPHLIKE LEGENDRIAN UNFOLDINGS

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## 0. Introduction

It is well known that parallels of given smooth hypersurface in  $\mathbb{R}^n$  have Legendrian singularities ([1,3]). If we are concerned the way in which these parallels change as we alter the distance, it will be appeared perestroikas of wave front sets. This problem has been considered by several people ([2,3,4,5]). Roughly speaking, it has been shown that generic perestroikas of the singularities of parallels of smooth hypersurfaces are generic perestroikas of Legendrian singularities. In [15] Zakalyukin proved that generic perestroikas of Legendrian singularities are stable perestroikas of Legendrian singularities in the case  $n \leq 5$  and classified these perestroikas. In this paper we shall consider the realization problem of the perestroikas of Legendrian singularities. In relation to this problem, Arnol'd ([2], p40) mentioned "One may find in the literature the statement that the local perestroikas of the wave fronts generated by the general Legendre mapping over space-time and of the equidistants (i.e. parallel) of the smooth hypersurface are same. It seems this has never been correctly proved." However, he has corrected his mistakes in his another book ([3], p60) "Indeed, the non-trivial perestroikas  $A_1$  and  $A_2$  change the number of connected components of the Legendrian manifold. Hence they cannot occur as perestroikas of equidistant hypersurfaces." This fact has already known by Bill Bruce in 1983 ([4]). Then we have the following natural question.

**Question.** What sort of a class of one-parameter families of Legendrian immersion germs is the correct class to describe perestroikas of parallels of hypersurfaces?

Here we shall give a candidate of this class which we call *graphlike Legendrian unfoldings*. Roughly speaking, a graphlike Legendrian unfolding is a Legendrian submanifold germ with submersive generating function. The general properties of graphlike Legendrian unfoldings will be studied in §2.

On the other hand, it should be natural to consider this problem in the framework of Hamiltonian systems which also contains the situation of parallels of hypersurfaces in a Riemannian manifold (cf. [5]).

All maps considered here are class  $C^\infty$  unless stated otherwise.

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\* The author uses this word in honor of Professor V. I. Arnol'd.

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## 1. Optical wave fronts

In this section we shall introduce the notion of optical wave fronts which is a generalization of parallels of hypersurfaces. We need some notions and results on Legendrian singularity theory. For this we refer [1,2,3,12,14].

Let  $J^1(\mathbb{R}^n, \mathbb{R})$  be the 1-jet bundle of functions of  $n$ -variables. Since we only consider the local situation, the 1-jet bundle  $J^1(\mathbb{R}^n, \mathbb{R})$  may be consider as  $\mathbb{R}^{2n+1}$  and the canonical coordinate is given by  $(x_1, \dots, x_n, t, p_1, \dots, p_n)$ . Then the canonical 1-form is given by  $\theta = dt - \sum_{i=1}^n p_i dx_i$ . Let  $T^*\mathbb{R}^n$  be the cotangent bundle over  $\mathbb{R}^n$  whose canonical coordinate is given by  $(x_1, \dots, x_n, p_1, \dots, p_n)$ . We also have the canonical 1-form  $\alpha = \sum_{i=1}^n p_i dx_i$  on  $T^*\mathbb{R}^n$ . There are natural projections. Namely,

$$\begin{aligned} \pi : J^1(\mathbb{R}^n, \mathbb{R}) &\rightarrow \mathbb{R}^n \times \mathbb{R} ; \pi(x, t, p) = (x, t), \\ \tilde{\pi} : T^*\mathbb{R}^n &\rightarrow \mathbb{R}^n ; \tilde{\pi}(x, p) = x, \\ \Pi : J^1(\mathbb{R}^n, \mathbb{R}) &\rightarrow T^*\mathbb{R}^n ; \Pi(x, t, p) = (x, p), \\ \pi_2 : J^1(\mathbb{R}^n, \mathbb{R}) &\rightarrow \mathbb{R} ; \pi_2(x, t, p) = t. \end{aligned}$$

Let  $H : (T^*\mathbb{R}^n, (x_0, p_0)) \rightarrow (\mathbb{R}, c)$  be a function germ. We call it a *Hamiltonian function germ*. We say that  $H$  is *starshaped* if it satisfies  $\sum_{i=1}^n p_i \frac{\partial H}{\partial p_i}(x, p) \neq 0$  at any  $(x, p) \in H^{-1}(c)$ . We remark that if  $H$  is starshaped then  $H^{-1}(c)$  is transverse to fibres of  $\tilde{\pi}$ . It is easy to prove the following proposition.

**Proposition 1.1.** *If  $H$  is starshaped, then  $\alpha|_{H^{-1}(c)}$  gives a contact structure on  $H^{-1}(c)$  and  $\tilde{\pi}|_{H^{-1}(c)} : H^{-1}(c) \rightarrow \mathbb{R}^n$  is a Legendrian fibration.*

Let  $(\ell, (x_0, p_0)) \subset (T^*\mathbb{R}^n, (x_0, p_0))$  be a submanifold germ. We say that  $\ell$  is an *optical Legendrian submanifold relative to  $H$*  if  $\dim \ell = n - 1$ ,  $\alpha|_{\ell} = 0$  and  $\ell \subset H^{-1}(c)$ . We call the image  $\tilde{\pi}(\ell)$  an *optical wave front relative to  $H$* .

By this proposition, if  $\ell$  is an optical Legendrian submanifold relative to  $H$ , then  $\ell$  is a Legendrian submanifold of  $H^{-1}(c)$  and  $\tilde{\pi}(\ell)$  is a wave front set in the ordinary sense (see [1,2,3,14]). We now give examples of optical wave front sets.

**Example 1.2.** Let  $g = \sum_{ij} g_{ij}(x) dx_i dx_j$  be a Riemannian metric on  $\mathbb{R}^n$ . We define a Hamiltonian function  $H$  on  $T^*\mathbb{R}^n$  by

$$H(x_1, \dots, x_n, p_1, \dots, p_n) = \sum_{ij} g_{ij}(x) p_i p_j.$$

Then we can show that  $\sum_{k=1}^n p_k \frac{\partial H}{\partial p_k} = 2 \sum_{ik} g_{ik}(x) p_i p_k = 2$  on  $H^{-1}(1)$ .

For a Hamiltonian function germ  $H : (T^*\mathbb{R}^n, (x_0, p_0)) \rightarrow (\mathbb{R}, c)$ , we define a function germ  $\tilde{H} : (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, t_0, p_0)) \rightarrow (\mathbb{R}, c)$  by  $\tilde{H}(x, t, p) = H(x, p)$ . We call  $\tilde{H}$  an induced contact Hamiltonian function germ from  $H$ . By the definition, we have the Hamiltonian vector field :

$$X_H = - \sum_{i=1}^n \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x_i} + \sum_{j=1}^n \frac{\partial H}{\partial x_j} \frac{\partial}{\partial p_j} \text{ on } T^*\mathbb{R}^n.$$

For an isotropic  $(n-1)$ -dimensional submanifold germ  $(\ell, (x_0, p_0)) \subset (T^*\mathbb{R}^n, (x_0, p_0))$  (i.e.  $\alpha|_\ell = 0$ ), we say that  $\ell$  is non-characteristic relative to  $H$  at  $(x_0, p_0)$  if  $X_{H, (x_0, p_0)} \notin T_{(x_0, p_0)}\ell$ . Then we have the following lemma.

**Lemma 1.3.** *If  $\sum_{i=1}^n p_i \frac{\partial H}{\partial p_i} \neq 0$  at  $(x_0, p_0) \in \ell$ , then  $\ell$  is non-characteristic relative to  $H$  at  $(x_0, p_0)$ .*

*Proof.* If  $X_{H, (x_0, p_0)} \in T_{(x_0, p_0)}\ell$ , then  $0 = \alpha(T_{(x_0, p_0)}\ell) = \alpha(X_{H, (x_0, p_0)}) = - \sum_{i=1}^n p_i \frac{\partial H}{\partial p_i}$  at  $(x_0, p_0)$ . This contradicts to the assumption.

By this lemma, if  $\ell$  is an optical Legendrian submanifold relative to a starshaped Hamiltonian function  $H$ , then  $\ell$  is automatically non-characteristic at any points relative to  $H$ . Then we can apply characteristic method to solve Cauchy problem of the first order partial differential equation  $H = c$  with the given initial condition  $\ell$  (see [12]). We consider the embedding  $i : \mathcal{O} \subset H^{-1}(c)$ , where  $\mathcal{O} \subset \ell \times (-\varepsilon, \varepsilon)$  is some open neighbourhood of  $\ell \times 0$ ,  $\varepsilon > 0$ ; here  $i(u, t) = T_t(u)$ ,  $u \in \ell$ ,  $(u, t) \in \mathcal{O}$ , and  $T_t$  is an one-parameter group of translations along  $X_H$ . Then we have a family of optical wave fronts  $\tilde{\pi}(\ell_t)$ .

**Example 1.4.** We now consider the Hamiltonian function  $H = p_1^2 + \dots + p_n^2$ . Let  $F : (\mathbb{R}^n, x_0) \rightarrow (\mathbb{R}, 0)$  be a submersion germ. We define an  $(n-1)$ -dimensional submanifold of  $T^*\mathbb{R}^n$  by

$$\ell_F = \{(x, p) | p = \frac{1}{(\sum_{i=1}^n (F_{x_i})^2)^{\frac{1}{2}}} (F_{x_1}, \dots, F_{x_n}) \text{ and } F(x) = 0\},$$

where  $F_{x_i} = \frac{\partial F}{\partial x_i}$ .

By the definition,  $\ell \subset H^{-1}(1)$  and  $\alpha|_\ell = 0$ . Parallels of hypersurface  $F^{-1}(0)$  in  $\mathbb{R}^n$  are  $\tilde{\pi}(\ell_t)$  which are constructed by the above argument with the initial submanifold  $\ell_F$ .

Our purpose is to construct a framework to describe the perestroika of singularities of  $\tilde{\pi}(\ell_t)$ . Then we consider what the time  $t$  is. We can construct a Legendrian submanifold  $\mathcal{L} \subset J^1(\mathbb{R}^n, \mathbb{R})$  associated with the Lagrangian submanifold  $L \subset T^*\mathbb{R}^n$  by the following way : We fix a point  $(x_0, p_0) \in L$ . Then for every point  $(x, p) \in L$  and every path  $\gamma = \gamma(t)$ ,  $\gamma(0) = (x_0, p_0)$ ,  $\gamma(1) = (x, p)$ , we define a function

$$S(\gamma, (x, p)) = \int_\gamma \alpha|_L.$$

Then

$$\mathcal{L} = \bigcup_{([\gamma], (x, p))} \{(x, S(x, p), p)\},$$

where  $[\gamma]$  is the homotopy class represented by the path  $\gamma$  joining  $(x_0, p_0)$  and  $(x, p)$ . Since we consider the local situation, then  $\mathcal{L}$  does not depend on the choice of  $\gamma$ . For any  $(x, p) \in \ell_t \subset L$ , we put  $(x', p') = T_{-1}(x, p) \in \ell$ . Then there exists a path  $\gamma$  in  $\ell$  such that  $\gamma(0) = (x_0, p_0)$  and  $\gamma(1) = (x', p')$ . We denote  $\phi(t) = (x(t), p(t))$  which is defined by  $\phi(\tau) = T_\tau(x', p')$ . Since  $\alpha|_\ell = 0$ , then we have

$$\begin{aligned} S(x(t), p(t)) &= \int_{\phi \cdot \gamma} \alpha|_L = \int_0^t \sum_{i=1}^n p_i(\tau) \frac{dx_i}{dt}(\tau) d\tau \\ &= \int_0^t \sum_{i=1}^n p_i(\tau) \frac{\partial H}{\partial p_i}(x(\tau), p(\tau)) d\tau. \end{aligned}$$

Since  $H$  is starshaped, then we have  $\frac{dS(x(t), p(t))}{dt}|_{t=0} \neq 0$ . This means that  $\pi_2|_{\mathcal{L}} : \mathcal{L} \rightarrow \mathbb{R}$  is a submersion germ. According to this fact, we will introduce the notion of graphlike Legendrian unfoldings in the next section.

## 2. Graphlike Legendrian unfoldings

In which we shall introduce the notion of graphlike Legendrian unfoldings and study some fundamental properties. In ([10,11]) we defined the notion of Legendrian unfoldings which is very useful in the theory of first order partial differential equations. However it is still too general for our purpose.

Let  $(\mathcal{L}, (x_0, t_0, p_0)) \subset J^1(\mathbb{R}^n, \mathbb{R})$  be a Legendrian submanifold germ (i.e.  $\dim \mathcal{L} = n$  and  $\theta|_{\mathcal{L}} = 0$ ), we say that  $\mathcal{L}$  is a *graphlike Legendrian unfolding* if  $\pi_2|_{\mathcal{L}}$  is a submersion germ. We remark that  $\mathcal{L}_{t_0} = \mathcal{L} \cap \{t = t_0\}$  is an  $(n-1)$ -dimensional submanifold and  $\Pi(\mathcal{L}_{t_0}) = \ell_{t_0}$  is an isotropic submanifold in  $(T^*\mathbb{R}^n, (x_0, p_0))$ . Here we call  $\tilde{\pi}(\ell_{t_0})$  a *wave front* of  $\ell_{t_0}$ . In order to study perestroikas of wave fronts of graphlike Legendrian unfoldings, we now introduce the following equivalence relation among graphlike Legendrian unfoldings. Let  $(\mathcal{L}_i, (x_i, t_i, p_i)) \subset J^1(\mathbb{R}^n, \mathbb{R})$  ( $i = 0, 1$ ) be graphlike Legendrian unfoldings. We say that  $(\mathcal{L}_0, (x_0, t_0, p_0))$  and  $(\mathcal{L}_1, (x_1, t_1, p_1))$  are *P-Legendrian equivalent* if there exists a contact diffeomorphism

$$\Phi : (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, t_0, p_0)) \rightarrow (J^1(\mathbb{R}^n, \mathbb{R}), (x_1, t_1, p_1))$$

of the form

$$\Phi(x, t, p) = (\phi_1(x, t), \phi_2(t), \phi_3(x, t, p))$$

such that  $\Phi(\mathcal{L}_0) = \mathcal{L}_1$ .

This equivalence relation preserves the diffeomorphic type of perestroikas of wave fronts of graphlike Legendrian unfoldings. We can define the notion of the stability with respect to P-Legendrian equivalence exactly the same way as that of the ordinary Legendrian stability (see [1,2,3,14]).

By the Arnol'd-Zakalyukin's theory of Legendrian singularities, we can construct generating families of graphlike Legendrian unfoldings. Let  $F : (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k, 0) \rightarrow (\mathbb{R}, 0)$  be a function germ such that  $\frac{\partial F}{\partial t} \neq 0$  and  $(d_2 F, F)|_{\{0\} \times \mathbb{R}^n \times \mathbb{R}^k}$  is non-singular,

where  $d_2F(x, q) = (\frac{\partial F}{\partial q_1}(x, q), \dots, \frac{\partial F}{\partial q_k}(x, q))$ . Then  $(d_2F, F)$  is a submersion germ, so that  $C_*(F) = (d_2F, F)^{-1}(0)$  is an  $n$ -dimensional smooth submanifold. We now define a map germ

$$\Phi_F : (C_*(F), 0) \rightarrow J^1(\mathbb{R}^n, \mathbb{R})$$

by

$$\Phi_F(t, x, q) = (x, t, \frac{\partial F}{\partial t}, \dots, \frac{\partial F}{\partial t}).$$

It is easy to show that  $\Phi_F^*\theta = 0$  and  $\pi_2 \circ \Phi_F$  is a submersion germ, so that  $\Phi_F$  is a graphlike Legendrian unfolding. By the theory of Arnol'd-Zakalyukin, we can show the following :

**Proposition 2.1.** *All graphlike Legendrian unfolding germs are constructed by the above method.*

We call  $F$  a *generating family* of  $\Phi_F$ . We also consider an equivalence relation among generating families of graphlike Legendrian unfoldings. Let

$$F_i : (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k, 0) \rightarrow (\mathbb{R}, 0) \quad (i = 0, 1)$$

be generating families of  $\Phi_{F_i}$ . We say that  $F_0$  and  $F_1$  are *t-P-K-equivalent* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k, 0) \rightarrow (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k, 0)$$

of the form

$$\Phi(t, x, q) = (\phi_1(t), \phi_2(t, x), \phi_3(t, x, q))$$

such that

$$\langle F_1 \circ \Phi \rangle_{\mathcal{E}(t, x, q)} = \langle F_0 \rangle_{\mathcal{E}(t, x, q)},$$

where  $\langle F_0 \rangle_{\mathcal{E}(t, x, q)}$  denotes the ideal generated by  $F_0$  in the ring of function germs of  $(t, x, q)$ -variables at the origin. The definition of *the stable t-P-K-equivalence* is given by the usual way (see [1,14]).

For a generating family  $F$  of  $\Phi_F$ , we put

$$T_e(\text{P-K})(f) = \langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k}, f \rangle_{\mathcal{E}(x, q)} + \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle_{\mathcal{E}_x},$$

where  $f = F|_{0 \times \mathbb{R}^n \times \mathbb{R}^k}$ . We also say that  $F$  is a *P-K-versal deformation* of  $f$  if

$$\mathcal{E}_{(x, q)} = \langle \frac{\partial F}{\partial t} |_{0 \times \mathbb{R}^n \times \mathbb{R}^k} \rangle_{\mathbb{R}} + T_e(\text{P-K})(f).$$

Then we have the following proposition whose proof is just like as that of the ordinary theory of Legendrian singularities ([1,14]).



**Proposition 2.2.** (1) Let  $F_i$  ( $i = 0, 1$ ) be generating families of  $\Phi_{F_i}$ . Then  $\Phi_{F_0}$  and  $\Phi_{F_1}$  are  $P$ -Legendrian equivalent if and only if  $F_0$  and  $F_1$  are stably  $t$ - $P$ - $\mathcal{K}$ -equivalent.  
(2) Let  $F$  be a generating family of  $\Phi_F$ , then  $\Phi_F$  is stable with respect to the  $P$ -Legendrian equivalence if and only if  $F$  is a  $P$ - $\mathcal{K}$ -versal deformation of  $f$ .

We can classify generic graphlike Legendrian unfoldings by the  $P$ -Legendrian equivalence for  $n \leq 5$  by the aid of classifications of one parameter perestroikas of wave fronts. In [15] Zakalyukin has given a generic classification of function germs  $F : (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k, 0) \rightarrow (\mathbb{R}, 0)$  by the stable  $t$ - $P$ - $\mathcal{K}$ -equivalence. Since the set of function germs  $F(t, x, q)$  which satisfy  $\frac{\partial F}{\partial t} \neq 0$  is an open subset, then such a function germ is stably  $t$ - $P$ - $\mathcal{K}$ -equivalent to one of germs in the following list :

$$({}^0A_r) \quad q_1^{r+1} \pm q_2^2 + \sum_{i=1}^r x_i q_1^{i-1} + t \quad (1 \leq r \leq n)$$

$$({}^0D_r) \quad q_1^2 q_2 \pm q_2^{r-1} + \sum_{i=2}^r x_i q_2^{i-2} + x_1 q_1 + t \quad (4 \leq r \leq n)$$

$$({}^1A_r) \quad q_1^{r+1} \pm q_2^2 + q_1^{r-1} (t \pm x_r^2 \pm \dots \pm x_n^2) + \sum_{i=1}^{r-1} x_i q_1^{i-1} + t \quad (2 \leq r \leq n+1)$$

$$({}^1D_r) \quad q_1^2 q_2 \pm q_2^{r-1} + q_2^{r-2} (t \pm x_r^2 \pm \dots \pm x_n^2) + \sum_{i=2}^{r-1} x_i q_2^{i-2} + x_1 q_1 + t \quad (4 \leq r \leq n+1)$$

$$({}^1E_6) \quad q_1^3 + q_2^4 + (q_1 q_2^2 + 1)t + x_5 q_1 q_2 + x_4 q_2^2 + x_3 q_1 + x_2 q_2 + x_1.$$

Since  $(d_2 F, F)|_0 \times \mathbb{R}^n \times \mathbb{R}^k$  for germs  ${}^1A_1$  and  ${}^1A_2$  are not submersive, then these germs can be removed from the list of generating families of graphlike Legendrian unfoldings. Thus we have the following :

**Theorem 2.3.** For  $n \leq 5$ , the generic graphlike Legendrian unfolding is  $P$ -Legendrian equivalent to one of germs of the following type :

$${}^0A_r \quad (1 \leq r \leq n), \quad {}^0D_r \quad (4 \leq r \leq n), \quad {}^1A_r \quad (3 \leq r \leq n), \quad {}^1D_r \quad (4 \leq r \leq n), \quad {}^1E_6.$$

We remark that we can choose a generating family of the graphlike Legendrian unfolding  $\Phi_F$  of the form  $F(t, x, q) = f(x, q) - t$  by the implicit function theorem. In this case the function  $f(x, q)$  should satisfy that  $(d_2 f, f)$  is non-singular, so that it gives a generating family of the initial isotropic submanifold  $\Phi_F|_{t=0}$ .

### 3. Optical graphlike Legendrian unfoldings

In this section we return to study perestroikas of optical wave fronts. For our purpose, we now introduce the following : Let  $H : (T^*\mathbb{R}^n, (x_0, p_0)) \rightarrow (\mathbb{R}, c)$  be a starshaped Hamiltonian function germ and  $(\mathcal{L}, (x_0, t_0, p_0)) \subset J^1(\mathbb{R}^n, \mathbb{R})$  be a graphlike Legendrian unfolding. We say that  $\mathcal{L}$  is *optical relative to  $H$*  if  $\mathcal{L} \subset \tilde{H}^{-1}(c)$ , where  $\tilde{H}$  is the induced contact Hamiltonian function germ from  $H$ . Then we have the following theorem

**Theorem 3.1.** (1) Let  $H : (T^*\mathbb{R}^n, (x_0, p_0)) \rightarrow (\mathbb{R}, c)$  be a starshaped Hamiltonian function germ and  $(\ell, (x_0, p_0)) \subset T^*\mathbb{R}^n$  be an optical Legendrian submanifold relative to  $H$ . Then there exists a graphlike Legendrian unfolding  $(\mathcal{L}, (x_0, 0, p_0))$  which is optical relative to  $H$  such that  $\tilde{\pi}(\mathcal{L} \cap \{t = 0\}) = \ell$ .

(2) Let  $(\mathcal{L}, (x_0, t_0, p_0)) \subset J^1(\mathbb{R}^n, \mathbb{R})$  be a graphlike Legendrian unfolding. Then there exists a starshaped Hamiltonian function germ  $H : (T^*\mathbb{R}^n, (x_0, p_0)) \rightarrow (\mathbb{R}, c)$  such that  $\mathcal{L}$  is optical relative to  $H$ .

In order to prove this theorem, we need the following quite useful theorem in symplectic geometry.

**Theorem 3.2.** (Kostant-Sternberg [8]) Let  $(P, \omega)$  be a symplectic manifold,  $L$  a Lagrangian submanifold and  $\beta$  a smooth 1-form on  $P$  with  $\beta|_L = 0$  and  $d\beta = \omega$ . Then there exists a tubular neighbourhood  $V$  of  $L$  in  $P$ , and a unique vector bundle isomorphism  $K : V \rightarrow (T^*L, \alpha_L)$  such that  $K$  is the identity on  $L$  and  $K^*\alpha_L = \beta$ . Here,  $\alpha_L$  is the canonical 1-form on  $T^*L$ .

*Proof.* (1) This assertion has been already proved in §1.

(2) Since  $\mathcal{L}$  is a graphlike Legendrian unfolding, then we can choose a local parametrization  $\phi : (\mathbb{R}^n, (u_0, t_0)) \rightarrow (\mathcal{L}, (x_0, t_0, p_0))$  of  $\mathcal{L}$  of the form  $\phi(u, t) = (x(u, t), t, p(u, t))$ , where  $u = (u_1, \dots, u_{n-1})$ . Furthermore, we have  $\phi^*\theta = 0$ , so that it is equivalent to

$$(a) \quad \sum_{i=1}^n p_i(u, t) \frac{\partial x_i}{\partial u_j}(u, t) = 0 \quad j = 1, \dots, n-1,$$

$$(b) \quad \sum_{i=1}^n p_i(u, t) \frac{\partial x_i}{\partial t}(u, t) = 1.$$

We now consider a vector field

$$X = \left( \frac{\partial x}{\partial t}(u, t), 1, \frac{\partial p}{\partial t}(u, t) \right)$$

on  $\mathcal{L}$ , so that we have

$$\tilde{\pi}_*X = \left( \frac{\partial x}{\partial t}(u, t), \frac{\partial p}{\partial t}(u, t) \right) \in TL,$$

where  $L = \tilde{\pi}(\mathcal{L})$ . Since  $L$  is a Lagrangian submanifold germ in  $(T^*\mathbb{R}^n, (x_0, p_0))$ , then there exist a decomposition  $I \cup J = \{1, \dots, n\}$  and a function germ  $S(x_I, p_J)$  such that

$$L = \left\{ (x_I, x_J, p_I, p_J) \mid x_J = -\frac{\partial S}{\partial p_J}, p_I = \frac{\partial S}{\partial x_I} \right\},$$

(for detail see [1,14]). If we define an 1-form  $\beta$  on  $T^*\mathbb{R}^n$  by  $\beta = dS - p_I dx_I + x_J dp_J$ , then we have  $\beta|_L = 0$ . Thus we can use  $\beta$  as the 1-form in Theorem 3.2. It follows that there exists a tubular neighbourhood  $V$  of  $L$  in  $T^*\mathbb{R}^n$  and a unique vector bundle isomorphism  $K : V \rightarrow (T^*L, \alpha_L)$  such that  $k|_L = id_L$  and  $K^*\alpha_L = \beta$ .

Since  $\tilde{\pi}_*X$  is a vector field on  $L$ , then we can lift this vector field  $X_{H'}$  on  $T^*L$ , where  $H' : (T^*L, (x_0, p_0)) \rightarrow \mathbb{R}$  is a Hamiltonian function germ. If we adopt the canonical local coordinate  $(\bar{x}, \bar{p})$  of  $T^*L$  around  $(x_0, p_0)$ , where  $\bar{x}$  is a local coordinate of  $L$ , then  $L$  is the submanifold defined by  $\bar{p}_i = 0$  ( $i = 1, \dots, n$ ). We now define a Hamiltonian function germ  $H'' : (T^*L, (x_0, p_0)) \rightarrow (\mathbb{R}, c)$  by  $H''(\bar{x}, \bar{p}) = H'(\bar{x}, \bar{p}) - H'(\bar{x}, 0) + c$ . By the definition, we have  $H''(L) = c$ . The Hamiltonian vector field  $X_{H''}$  on  $T^*L$  has the following component :

$$\begin{cases} \frac{\partial \bar{p}_i}{\partial t} = \frac{\partial H''}{\partial \bar{x}_i} \\ \frac{\partial \bar{x}_i}{\partial t} = -\frac{\partial H''}{\partial \bar{p}_i}, \end{cases}$$

$i = 1, \dots, n$ .

On the other hand, we have  $\frac{\partial H''}{\partial \bar{p}_i} = \frac{\partial H'}{\partial \bar{p}_i}$  and

$$\frac{\partial H''}{\partial \bar{x}_i}(\bar{x}, 0) = \frac{\partial H'}{\partial \bar{x}_i}(\bar{x}, 0) - \frac{\partial H'}{\partial \bar{x}_i}(\bar{x}, 0) = 0 \text{ on } L,$$

so that  $X_{H''}|_L = X_{H'}|_L = \tilde{\pi}_*X$ .

Finally, we define a Hamiltonian function germ  $H : (T^*\mathbb{R}^n, (x_0, p_0)) \rightarrow (\mathbb{R}, c)$  by  $H = H'' \circ K$ . Since  $K|_L = id_L$ , then the Hamiltonian vector field  $X_H$  is an extension of  $\tilde{\pi}_*X$  on  $T^*\mathbb{R}^n$  and  $H(L) = c$ . It follows that we have relations

$$\begin{cases} \frac{\partial x_i}{\partial t}(u, t) = \frac{\partial H}{\partial p_i}(x(u, t), p(u, t)), \\ \frac{\partial p_i}{\partial t}(u, t) = -\frac{\partial H}{\partial x_i}(x(u, t), p(u, t)) \end{cases}$$

on  $L$ . By the equality (b), we have

$$1 = \sum_{i=1}^n p_i(u, t) \frac{\partial x_i}{\partial t}(u, t) = \sum_{i=1}^n p_i(u, t) \frac{\partial H}{\partial p_i}(x(u, t), p(u, t)).$$

Thus,  $H$  is starshaped near  $(x_0, p_0)$ . This completes the proof

This theorem guarantees that the class of graphlike Legendrian unfoldings supplies the correct class which describes perestroikas of optical wave fronts. However, for our first question, we must deal with in the case when the Hamiltonian function germ  $H$  is fixed.

#### 4. Optical graphlike Legendrian unfoldings relative to a fixed Hamiltonian

In this section we consider the perestroikas of optical wave fronts relative to a fixed Hamiltonian function  $H$ . Let  $H : (T^*\mathbb{R}^n, (x_0, p_0)) \rightarrow (\mathbb{R}^n, 0)$  be a starshaped Hamiltonian function germ. Since  $\tilde{\pi} : H^{-1}(c) \rightarrow \mathbb{R}^n$  is a Legendrian fibration, then there exists a

local contact diffeomorphism  $K : PT^*\mathbb{R}^n \rightarrow H^{-1}(c)$  and a diffeomorphism  $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $\tilde{\pi} \circ K = h \circ p$ , where  $p : PT^*\mathbb{R}^n \rightarrow \mathbb{R}^n$ , is the projective cotangent bundle (see [1,2,3,14]). If  $\ell$  is a Legendrian submanifold germ in  $PT^*\mathbb{R}^n$ , then  $K(\ell)$  is also a Legendrian submanifold germ in  $H^{-1}(c)$ . Of course,  $\ell$  and  $K(\ell)$  are Legendrian equivalent, so that any wave front can be realized by an optical one. However, for the perestroikas of wave fronts, the situation is different from the above. Consider the following example :

**Example 4.1.** Let  $H : (T^*\mathbb{R}^n, (x_0, p_0)) \rightarrow (\mathbb{R}, c)$  be a Hamiltonian function germ defined by  $H(x, p) = p_n$ , where  $(x_0, p_0) = (0, \dots, 0, 1)$ . Let  $F(t, x, q) = f(x, q) - t$  be a generating family of the graphlike Legendrian unfolding  $\Phi_F$ . Suppose that  $\Phi_F|_{t=0}$  is an optical Legendrian submanifold relative to  $H$ . Then we have

$$H(x, \frac{\partial f}{\partial x}(x, q)) \equiv 1 \pmod{\langle f, \frac{\partial f}{\partial q} \rangle_{\mathcal{E}(x, q)}}.$$

Because  $H(x, \frac{\partial f}{\partial x}(x, q)) = \frac{\partial f}{\partial x_n}(x, q)$ , we have

$$-1 \in \langle \frac{\partial f}{\partial x_n} \rangle_{\mathcal{E}_x} + \langle f, \frac{\partial f}{\partial x} \rangle_{\mathcal{E}(x, q)} \subset T_e(p\mathcal{K})(f).$$

It follows that  $F(t, x, q)$  is not a  $P\mathcal{K}$ -versal deformation of  $f(x, q)$  if  $P\mathcal{K}\text{-cod}(f) \geq 1$ . Thus, for example, types of  ${}^1A_r$  and  ${}^1D_r$  in Theorem 2.3 cannot be realized by an optical one relative to  $H$ .

Hence, we should assume a kind of non-degeneracy condition on a Hamiltonian function germ. We say that a Hamiltonian function germ  $H : (T^*\mathbb{R}^n, (x_0, p_0)) \rightarrow (\mathbb{R}, c)$  is *non-degenerated* if

(1)  $H$  is starshaped.

(2) The quadratic form  $\sum_{1 \leq i, j \leq n} \frac{\partial^2 H}{\partial p_i \partial p_j}(x_0, p_0) \xi_i \xi_j$  does not vanish on the null space of  $(\frac{\partial H}{\partial p_1}(x_0, p_0), \dots, \frac{\partial H}{\partial p_n}(x_0, p_0))$ .

The second condition of the above definition is given by Duistermaat (Proposition 5.2.1 in [7]) in order to study optical Lagrangian submanifolds. It is easy to check that the Hamiltonian function in Example 1.2 satisfies the above conditions. Then we have the following theorem which gives an almost complete answer to our first question.

**Theorem 4.2.** Let  $H : (T^*\mathbb{R}^n, (x_0, p_0)) \rightarrow (\mathbb{R}, c)$  be a non-degenerated Hamiltonian function and  $(\mathcal{L}, (x_1, t_1, p_1)) \subset J^1(\mathbb{R}^n, \mathbb{R})$  be a  $P$ -Legendrian stable graphlike Legendrian unfolding. Then there exists an optical graphlike Legendrian unfolding  $(\mathcal{L}', (x_0, t_0, p_0))$  relative to  $H$  such that  $\mathcal{L}$  and  $\mathcal{L}'$  are  $P$ -Legendrian equivalent.

*Proof.* Without loss of generality, we may assume that  $(x_0, t_0) = (0, 0)$ . Let

$$G : (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k, 0) \rightarrow (\mathbb{R}, 0)$$

be a generating family of the graphlike Legendrian unfolding  $\mathcal{L}$ . Since  $(d_2G, G)|_{0 \times \mathbb{R}^n \times \mathbb{R}^k}$  is non-singular, then the set

$$\phi_g = \{((x_1, \dots, x_n), [\frac{\partial g}{\partial x_1}(x, q); \dots; \frac{\partial g}{\partial x_n}(x, q)]) | d_2g(x, q) = g(x, q) = 0\}$$

is a Legendrian submanifold in  $PT^*\mathbb{R}^n$ , where  $g = G|_{0 \times \mathbb{R}^n \times \mathbb{R}^k}$ . Then  $\ell_0 = K(\psi_g)$  is an optical Legendrian submanifold relative to  $H$  which is Legendrian equivalent to  $\psi_g$ , where  $K$  is the contact diffeomorphism like as the previous arguments. It follows from Theorem 3.1 that there exists an optical graphlike Legendrian unfolding  $\mathcal{L}'$  relative to  $H$  such that  $\Pi(\mathcal{L}'|_{t=0}) = \ell_0$ .

We now choose a generating family  $F(t, x, q) = f(x, q) - t$  of  $\mathcal{L}'$ . by the definition,  $\mathcal{L}'|_{t=0}$  is given by

$$\Phi_F|_{t=0} = \{(x, 0, \frac{\partial f}{\partial x_1}(x, q), \dots, \frac{\partial f}{\partial x_n}(x, q) | d_2 f(x, q) = f(x, q) = 0\},$$

so that  $f$  and  $g$  are stably P- $\mathcal{K}$ -equivalent by the Arnol'd-Zakalyukin's theory ([1,2,3,14]). Thus we may assume that  $f$  and  $g$  are P- $\mathcal{K}$ -equivalent.

If P- $\mathcal{K}$ -cod( $f$ ) = 0, then P- $\mathcal{K}$ -cod( $g$ ) = 0, so that  $f(x, q)$  is already a P- $\mathcal{K}$ -versal deformation of itself. Hence,  $F(t, x, q)$  is also a P- $\mathcal{K}$ -versal deformation of  $f(x, q)$ . It follows from the uniqueness theorem of P- $\mathcal{K}$ -versal deformations (see [6,8]) that  $F(t, x, q)$  and  $G(t, x, q)$  are t-P- $\mathcal{K}$ -equivalent. Then  $\mathcal{L}' = \Phi_F$  and  $\mathcal{L} = \Phi_G$  are P-Legendrian equivalent.

Thus, we now assume that P- $\mathcal{K}$ -cod( $g$ ) = 1, so that P- $\mathcal{K}$ -cod( $f$ ) = 1. If  $-1 = \frac{\partial F}{\partial t}|_{t=0} \notin T_e(\text{P-}\mathcal{K})(f)$ , then we can get the required assertion by the uniqueness of the P- $\mathcal{K}$ -versal deformation like as the previous case.

Suppose that  $1 \in T_e(\text{P-}\mathcal{K})(f)$  for any generating family of  $\mathcal{L}'$  of the form  $F(t, x, q) = f(x, q) - t$ . Since  $H(\ell_0) = c$ , then we have a relation

$$H(x, \frac{\partial f}{\partial x}(x, q)) \equiv c \pmod{\langle f, \frac{\partial f}{\partial q} \rangle_{\mathcal{E}_{(x,q)}}},$$

so that

$$H(0, \frac{\partial f}{\partial x}(0, q)) \equiv c \pmod{\langle f_0, \frac{\partial f_0}{\partial q} \rangle_{\mathcal{E}_q}},$$

where  $f_0 = f|_{x=0}$ . We may assume that  $f_0 \in \mathfrak{M}_k^3$ . We now consider the Taylor polynomial of  $H(x, p)$  of degree 2 at  $(x, p_0)$  with respect to  $p = (p_1, \dots, p_n)$ -variables as follows :

$$\begin{aligned} H(x, p) &= H(x, p_0) + \sum_{i=1}^n \frac{\partial H}{\partial p_i}(x, p_0)(p_i - p_{0,i}) \\ &\quad + \frac{1}{2} \sum_{i,j} \frac{\partial^2 H}{\partial p_i \partial p_j}(x, p_0)(p_i - p_{0,i})(p_j - p_{0,j}) + \text{higher term.} \end{aligned}$$

Since  $H(0, p_0) = c$ , then we have

$$\begin{aligned} H(0, \frac{\partial f}{\partial x}(0, q)) &= c + \sum_{i=1}^n \frac{\partial H}{\partial p_i}(0, \frac{\partial f}{\partial x}(0, q))(\frac{\partial f}{\partial x_i}(0, q) - p_{0,i}) \\ &\quad + \frac{1}{2} \sum_{i,j} \frac{\partial^2 H}{\partial p_i \partial p_j}(0, \frac{\partial f}{\partial x}(0, q))(\frac{\partial f}{\partial x_i}(0, q) - p_{0,i})(\frac{\partial f}{\partial x_j}(0, q) - p_{0,j}) + \text{higher term.} \end{aligned}$$

Because  $H(0, \frac{\partial f}{\partial x}(0, q)) - c \in \langle \frac{\partial f_0}{\partial q}, f_0 \rangle_{\mathcal{E}_q} \subset \mathfrak{M}_k^2$ , the right hand side of the above equality is a function of  $q = (q_1, \dots, q_k)$ -variables whose order is at least two. Since  $F(t, x, q) = f(x, q) - t$  is a generating family of the graphlike Legendrian unfolding  $\mathcal{L}'$ , then

$$\text{rank} \left( \frac{\partial^2 f}{\partial q_i \partial x_j}(0) \right) = k.$$

Then we may assume that  $(\frac{\partial^2 f}{\partial q_i \partial x_j}(0) = \delta_{ij}$  for  $i, j = 1, \dots, k$  and  $\frac{\partial f}{\partial x_\ell}(0, q) \in \mathfrak{M}_k^2$  for  $\ell = k+1, \dots, n$ , then  $\frac{\partial f}{\partial x_i}(0, q) = q_i + \psi(q)$ , where  $\psi(q) \in \mathfrak{M}_k^2$ , so that

$$\sum_{i=1}^k \frac{\partial H}{\partial p_i}(0, p_0) q_i = 0.$$

Since  $H$  is non-degenerated, then

$$\sum_{i=1}^k \frac{\partial H}{\partial p_i \partial p_j}(0, p_0) q_i q_j \neq 0.$$

On the other hand, we have

$$1 \in \langle \frac{\partial f}{\partial x_1}(0, q), \dots, \frac{\partial f}{\partial x_n}(0, q) \rangle_{\mathbb{R}} + \langle f_0, \frac{\partial f_0}{\partial q} \rangle_{\mathcal{E}_q}$$

by the assumption. It follows that

$$\begin{aligned} & \frac{1}{2} \sum_{1 \leq i, j \leq n} \frac{\partial^2 H}{\partial p_i \partial p_j}(0, p_0) \left( \frac{\partial f}{\partial x_i}(0, q) - p_{0,i} \right) \left( \frac{\partial f}{\partial x_j}(0, q) - p_{0,j} \right) \\ & \equiv H(0, \frac{\partial f}{\partial x}(0, q)) - c - \sum_{i=1}^n \frac{\partial H}{\partial p_i}(0, p_0) \left( \frac{\partial f}{\partial x_i}(0, q) - p_{0,i} \right) \text{ mod } \mathfrak{M}_k^3 \\ & = H(0, \frac{\partial f}{\partial x}(0, q)) - c - \sum_{i=1}^n \frac{\partial H}{\partial p_i}(0, p_0) \frac{\partial f}{\partial x_i}(0, q) + \sum_{i=1}^n p_{0,i} \frac{\partial H}{\partial p_i}(0, p_0) \\ & \in \langle \frac{\partial f}{\partial x_1}(0, q), \dots, \frac{\partial f}{\partial x_n}(0, q) \rangle_{\mathbb{R}} + \langle f_0, \frac{\partial f_0}{\partial q} \rangle_{\mathcal{E}_q} \text{ mod } \mathfrak{M}_k^3. \end{aligned}$$

Let  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear isomorphism, then we have

$$\frac{\partial f(Ax, q)}{\partial x_i} = \sum_{j=1}^n A_{ij} \frac{\partial f}{\partial x_j}(Ax, q).$$

Since

$$\begin{aligned} & \sum_{1 \leq i, j \leq n} \frac{\partial^2 H}{\partial p_i \partial p_j}(0, p_0) \left( \frac{\partial f}{\partial x_i}(0, q) - p_{0,i} \right) \left( \frac{\partial f}{\partial x_j}(0, q) - p_{0,j} \right) \\ & \equiv \sum_{1 \leq i, j \leq k} \frac{\partial^2 H}{\partial p_i \partial p_j}(0, p_0) q_i q_j \text{ mod } \mathfrak{M}_k^3 \end{aligned}$$

and the vector space

$$\left\langle \frac{\partial f}{\partial x_1}(0, q), \dots, \frac{\partial f}{\partial x_n}(0, q) \right\rangle_{\mathbb{R}} + \left\langle f_0, \frac{\partial f_0}{\partial q} \right\rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_k^3$$

is invariant under the action of linear isomorphisms  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , then every quadratic forms of  $q = (q_1, \dots, q_k)$ -variables are contained in the above vector space. If there exists a quadratic form of  $q$ -variables such that it is contained in  $\langle f_0, \frac{\partial f_0}{\partial q} \rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_k^3$ , then every quadratic forms are contained in it by the same reason as that of the above. In this case, since the vector space  $\langle f_0, \frac{\partial f_0}{\partial q} \rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_k^3$  has dimension  $k$ , then  $k$  should be 1 and  $f_0$  is an  $A_2$ -type function germ. It follows from Theorem 2.3 that  $F(t, x, q)$  is  ${}^0A_2$ -type, so that this case is contained in the case of  $P\text{-}\mathcal{K}\text{-cod}(f) = 0$ .

We may assume that every quadratic forms of  $q$ -variables are contained in

$$\left\langle \frac{\partial f}{\partial x_1}(0, q), \dots, \frac{\partial f}{\partial x_n}(0, q) \right\rangle_{\mathbb{R}} \bmod \mathfrak{M}_k^3.$$

Since  $\mathcal{K}\text{-cod}(f_0)$  is finite (for the definition of  $\mathcal{K}$ -finiteness, see [13]), then there exists  $r \in \mathbb{N}$  such that  $\mathfrak{M}_k^r \subset \langle f_0, \frac{\partial f_0}{\partial q} \rangle_{\mathcal{E}_q}$ . By the same arguments as those of the previous paragraph, we can assert that every monomials of  $q$ -variables of degree 3 are contained in the vector space

$$\left\langle \frac{\partial f}{\partial x_1}(0, q), \dots, \frac{\partial f}{\partial x_n}(0, q) \right\rangle_{\mathbb{R}} + \left\langle f_0, \frac{\partial f_0}{\partial q} \right\rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_k^4.$$

If there exists a monomial of degree 3 which is contained in  $\langle f_0, \frac{\partial f_0}{\partial q} \rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_k^4$ , then  $k$  should be 1 and  $f_0$  is an  $A_3$ -type function germ. It follows that

$$\dim_{\mathbb{R}} \left\langle \frac{\partial f}{\partial x_1}(0, q), \dots, \frac{\partial f}{\partial x_n}(0, q) \right\rangle_{\mathbb{R}} \geq 3 \bmod \mathfrak{M}_k^4.$$

By Theorem 2.3,  $F(t, x, q)$  should be  ${}^0A_3$ -type, then this case is contained in the case of  $P\text{-}\mathcal{K}\text{-cod}(f) = 0$ .

Since we can remove the assumption  $n \leq 5$  for  ${}^0A_\ell$  or  ${}^1A_\ell$ -type germs in Theorem 2.3 (cf. Theorem 2.2 in [15]), we can continue this procedure up to degree  $r - 1$ . Eventually, it remains the case that every polynomial of degree  $r - 1$  are contained in the vector space

$$\left\langle \frac{\partial f}{\partial x_1}(0, q), \dots, \frac{\partial f}{\partial x_n}(0, q) \right\rangle_{\mathbb{R}} + \left\langle f_0, \frac{\partial f_0}{\partial q} \right\rangle_{\mathcal{E}_q} \bmod \mathfrak{M}_k^r.$$

Since  $\mathfrak{M}_k^r \subset \langle f_0, \frac{\partial f_0}{\partial q} \rangle_{\mathcal{E}_q}$ , then we have

$$\mathcal{E}_q = \left\langle \frac{\partial f}{\partial x_1}(0, q), \dots, \frac{\partial f}{\partial x_n}(0, q) \right\rangle_{\mathbb{R}} + \left\langle f_0, \frac{\partial f_0}{\partial q} \right\rangle_{\mathcal{E}_q}.$$

It follows that

$$\mathcal{E}_{(x, q)} = T_e(P\text{-}\mathcal{K})(f) + \mathfrak{M}_k \mathcal{E}_{(x, q)},$$

so that we have

$$\mathcal{E}_{(x, q)} = T_e(P\text{-}\mathcal{K})(f)$$

by the Malgrange preparation theorem. This contradicts to the fact that  $P\text{-}\mathcal{K}\text{-cod}(f) = 1$ . This completes the proof.

**Remark.** The above proof is motivated by the proof of Proposition 5.2.1 in [7]. In fact, Theorem 4.2 is invented by the attempt to understand the proof of the proposition.

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