THE EULER NUMBER OF A TOPOLOGICALLY STABLE SINGULAR SURFACE IN A 3-MANIFOLD

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THE EULER NUMBER OF A TOPOLOGICALLY STABLE SINGULAR SURFACE IN A 3-MANIFOLD

S. Izumiya and W. L. Marar

Abstract. A formula for the Euler number of a generic singular surface in a 3-manifold is given. This formula not only unifies the previous results but also allows some new applications.

1. Introduction

One of the themes in the global theory of singularities of mappings $f : N \to P$ is to study the relationship between the topology of N, P and $f(N)$ in the case when $\dim N < \dim P$ [8]. Recently, we have arrived at formulae for the Euler numbers of the wave front of a generic closed Legendrian surface [5] and the Euler number of the image of a C^∞-stable mapping from a closed surface to a 3-manifold [6]. The proofs in both cases are similar. Here we unify these results under the notion of topologically stable singular surfaces in a 3-manifold that is introduced below. We also give some new examples.

Let $f : N \to P$ be a C^∞-mapping from a closed surface to a 3-manifold. We say that *f gives a topologically stable singular surface* if the image of f is locally homeomorphic to the image of an immersion with normal crossing or cross-caps (Fig. 1). It follows that the number of the points which are homeomorphic to cross-caps is finite and we denote by $C(f)$. There also exist finitely many three-to-one points in $f(N)$ where three sheets of regular image are in general position. Such a point (Fig. 2) is called triple point of f and the number of triple points is denoted by $T(f)$.

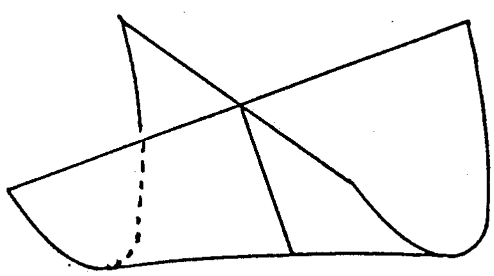

Fig.1

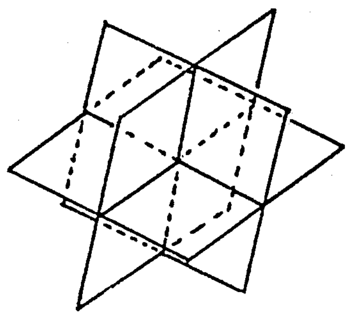

Fig.2

We denote the Euler number of a topological space X by $\chi(X)$. Our main result is the following :

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Theorem 1.1. *Suppose that a C^∞-mapping $f : N \to P$ gives a topologically stable singular surface. Then we have*

$$\chi(f(N)) = \chi(N) + T(f) + \frac{C(f)}{2}.$$

The proof of this theorem is given in the next section and follows the same steps as those of the theorems in [5,6]. As an application of this theorem, we can estimate the number of connected components of the complement of a topologically stable singular surface $f(N)$ in a 3-manifold P under some assumptions. We denote by $\sharp c.c(X)$ the number of connected components of X.

Theorem 1.2. *Let N be a connected closed surface and P be a connected closed 3-manifold with $H_1(P, \mathbb{Z}_2) = 0$. Suppose that $f : N \to P$ gives a topologically stable singular surface. Then*
(1) *If there are topological cross-caps in each connected component of the double point curve $D^2(f)$ of f in N, then we have*

$$\sharp c.c(P - f(N)) \leq 2 + \frac{C(f)}{2} + T(f).$$

Furthermor, if $D^2(f)$ is homologous to zero then $\sharp c.c(P - f(N)) = 2 + \frac{C(f)}{2} + T(f)$.
(2) *If $C(f) = 0$ and $T(f) = 0$, then we have*

$$\sharp c.c(P - f(N)) \leq 2 + \sharp c.c(f(D^2(f))).$$

Furthermore, if $D^2(f)$ is homologous to zero , then the equality holds.
Here, the precise definition of $D^2(f)$ is given in §2.

We can give some examples which suggest the above result.

Example 1.3. (1) Let N be the Klein bottle and P be the Euclidean space. Then we can construct a smooth mapping f with $C(f) = 2$ and $T(f) = 0$, whose image looks like the picture below (Fig. 3).

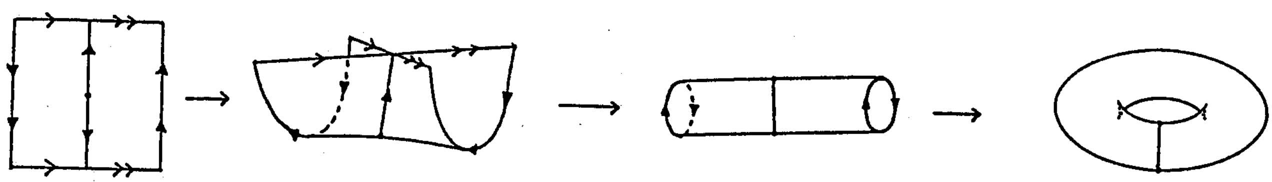

Fig.3

In this case we have $\sharp c.c(P - f(N)) = 2 < 2 + \frac{C(f)}{2} + T(f)$.
(2) Let N be the Torus and P be the Euclidean space. The following mapping f whose

image is shown in Fig. 4 has $C(f) = 2$ and $T(f) = 1$.

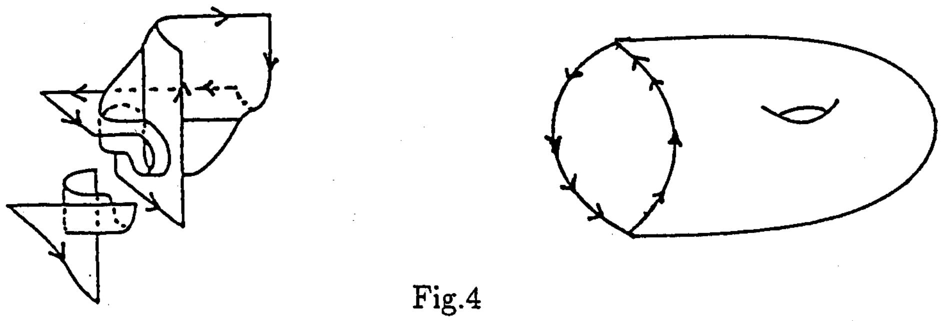

Fig.4

In this case we have $\sharp c.c(P - f(N)) = 4 = 2 + \frac{C(f)}{2} + T(f)$.
(3) Let N be the Torus and P be the Euclidean space. We can construct an immersion f with $T(f) = 0$ and $\sharp c.c(f(D^2(f)) = 2$ as follows (Fig. 5) :

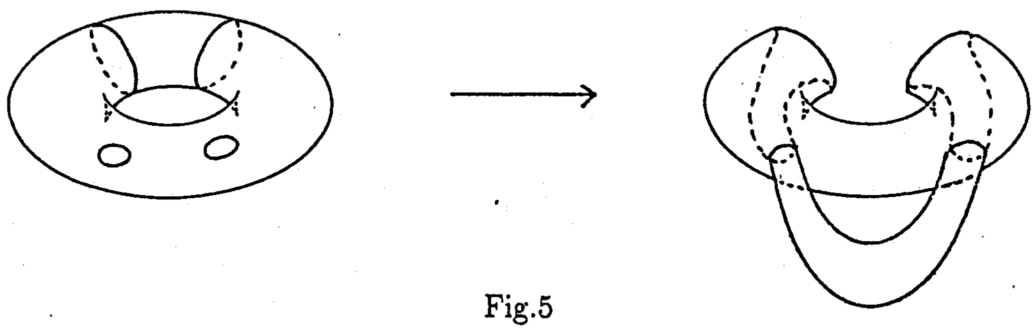

Fig.5

In this case we have $\sharp c.c(P - f(N)) = 3 < 2 + \sharp c.c(f(D^2(f))$.

We remark that there is a lower bound estimate for $\sharp c.c(P - f(N))$. In [17] Nuno Barestellos and Romero Fuster showed that $\sharp c.c(P - f(N)) \geq 2$ under a more general assumption than in here.

We shall give some new examples in §3 : A formula of the Euler number of the image of a stable mapping from a surface with boundary to a 3-manifold will be given. The restriction to the singular set of a C^∞-stable mapping from a closed 3-manifold to a 3-manifold gives a topologically stable singular surface. This example contains the classical dual of a generic closed space curve. The compactification of the tangent developable of a generic closed space curve is also another example.

All maps considered here are class C^∞ unless stated otherwise.

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2. Proof of theorems

Firstly we shall give a proof of Theorem 1.1. We now define the following sets:

$$D^2(f) = cl\{x \in N \mid \sharp f^{-1}f(x) \geq 2\},$$
$$D^3(f) = \{x \in D^2(f) \mid \sharp f^{-1}f(x) = 3\},$$
$$D^2(f,(2)) = \{x \in D^2(f) \mid \sharp f^{-1}f(x) = 1\},$$

where clX is the topological closure of X. Then we have the following diagram:

$$\begin{tikzcd} & D^3(f) \arrow[d, "h"] & \\ D^2(f,(2)) \arrow[r, "j"] & D^2(f) \arrow[d, "k"] & \\ & N \arrow[r, "f"] & f(N) \subset P, \end{tikzcd}$$

where h, j, k are inclusions.

By the definition, $D^2(f)$ is a union of curves on N with self-intersection and circles, $D^3(f)$ is the inverse image of triple points and $D^2(f,(2))$ is the set of topological cross-caps of $f(N)$, if not empty. It follows that these are homeomorphic to immersed submanifolds of N with dim $D^2(f) = 1$ and dim $D^3(f) =$ dim $D^2(f,(2)) = 0$ if not empty.

In order to prove the theorem, we need the following formula.

Lemma 2.1. $\chi(f(N)) = \chi(N) - \frac{1}{2}\chi(D^2(f)) + \frac{1}{2}\chi(D^2(f,(2))) - \frac{1}{6}\chi(D^3(f))$.

Proof. Consider the equation

$$(*) \qquad \chi(f(N)) = \alpha\,\chi(N) + \beta\,\chi(D^2(f)) + \gamma\,\chi(D^2(f,(2))) + \delta\,\chi(D^3(f)),$$

where α, β, γ and δ are unknown variables. We solve this by a purely combinatorial method.

We now construct a triangulation K_i of the stratified set $f(N)$ as follows: We start to triangulate $f(N)$ by including the image of $D^2(f,(2))$ and the image of $D^3(f)$ among the vertices of K_i. After this, we build up the one-skeleton $K_i^{(1)}$ of K_i so that the image of $D^2(f)$ is a subcomplex of $K_i^{(1)}$. We complete our procedure by constructing the two-skeleton $K_i^{(2)}$.

Since f and its restrictions to $D^2(f)$, $D^2(f,(2))$ and $D^3(f)$ are proper and finite-to-one mappings, then we can pull back K_i to obtain a triangulation for N, $D^2(f)$, $D^2(f,(2))$ and $D^3(f)$. Let C_j^X be the number of j-cells in X, where $X = f(N)$, N, $D^2(f)$, $D^2(f,(2))$ or $D^3(f)$. Then the equation (*) can be written by

$$\sum_j (-1)^j C_j^{f(N)} = \alpha \sum_j (-1)^j C_j^N + \beta \sum_j (-1)^j C_j^{D^2(f)}$$
$$+\gamma \sum_j (-1)^j C_j^{D^2(f,(2))} + \delta \sum_j (-1)^j C_j^{D^3(f)},$$

where $C_j^X = 0$ if $i > dim X$. So, if we can find real numbers α, β, γ and δ such that

$$(**) \qquad C_j^{f(N)} = \alpha\, C_j^N + \beta\, C_j^{D^2(f)} + \gamma\, C_j^{D^2(f,(2))} + \delta\, C_j^{D^3(f)},$$

for any j, then we have solutions of the equation (*). By the construction of the triangulation, we may concentrate on solving (**) in the case when $j = 0$. We remark that f is 3 to 1 over the points in the image of $D^3(f)$, 1 to 1 over the points in the image of $D^2(f,(2))$, 2 to 1 over the points in the image of $D^2(f) - (D^2(f,(2)) \cup D^3(f))$, and 1 to 1 over the points in the image of $N - D^2(f)$. It follows that the equation

$$C_0^{f(N)} = \alpha\, C_0^N + \beta\, C_0^{D^2(f)} + \gamma\, C_0^{D^2(f,(2))} + \delta\, C_0^{D^3(f)}$$

is equivalent to the system of linear equations :

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}.$$

We can easily solve this equation, so that $\alpha = 1$, $\beta = -1/2$, $\gamma = 1/2$ and $\delta = -1/6$. This completes the proof.

Then we can prove Theorem 1.1.

Proof of Theorem 1.1. By definition we have $\chi(D^2(f,(2))) = C(f)$ and $\chi(D^3(f)) = 3T(f)$. Since $D^2(f)$ is a union of closed curves on the surface N with $3T(f)$ crossings, then we can triangulate it with $3T(f) + n$ 0-cells and $6T(f) + n$ 1-cells, where n is the number of circles in $D^2(f)$. It follows that $\chi(D^2(f)) = -3T(f)$. If we substitute these on the formula in Lemma 2.1, then we have

$$\chi(f(N)) = \chi(N) + T(f) + \frac{1}{2}C(f).$$

This completes the proof.

Next we prove Theorem 1.2.

Proof of Theorem 1.2. Throughout we assume that the homology groups considered here are with $\mathbb{Z}_2$-coefficients. It is well-known that the number of connected components of $P - f(N)$ is given by $1 + \beta_2(f(N))$ under the assumption of the theorem (cf. [7] p205). Then we have

$$\sharp c.c(P - f(N)) = \chi(f(N)) + \beta_1(f(N)),$$

where β_i denote the ith Betti number.
(1) We remark that $\beta_1(f(N)) \leq \beta_1(N)$. Indeed, the mapping $\pi_1(N, x) \to \pi_1(f(N), f(x))$ is an epimorphism ([13]). Thus, the mapping $f_* : H_1(N) \to H_1(f(N))$ is also an epimorphism.

By Theorem 1.1, we have $\#c.c.(P - f(N)) = \chi(f(N)) + \beta_1(f(N)) = \chi(N) + C(f)/2 + T(f) + \beta_1(f(N)) \leq \chi(N) + C(f)/2 + T(f) + \beta_1(N)$. Since N is a closed, connected 2-manifold then $\chi(N) + \beta_1(N) = 2$. Therefore, $\#c.c.(P - f(N)) \leq 2 + C(f)/2 + T(f)$.

Furthermore, by the above argument, we have the following equality:

$$\#c.c.(P - f(N)) = 2 + C(f)/2 + T(f) - \mathrm{rank}\,\mathrm{Ker}\, f_*$$

Now, we consider the following commutative diagram:

$$\begin{tikzcd}
H_1(D^2(f)) \arrow[r, "i_*"] \arrow[d, "(f|D^2)_*"'] & H_1(N) \arrow[r, "\pi_*"] \arrow[d, "f_*"'] & H_1(N, D^2(f)) \arrow[d, "\tilde{f}_*"] \\
H_1(f(D^2(f))) \arrow[r, "i_{f*}"] & H_1(f(N)) \arrow[r, "\pi_{f*}"] & H_1(f(N), f(D^2(f))),
\end{tikzcd}$$

where the horizontal sequences are exact and $\tilde{f}_*$ is an isomorphism. Then we have the inclusion

$$\mathrm{Ker}\, f_* \subset i_*(H_1(D^2(f))).$$

So, if $D^2(f)$ is homologous to zero, then $\mathrm{Ker}\, f_* = 0$. This completes the proof of (i).
(2) We now consider the following diagram :

$$\begin{tikzcd}
D^2(f) \arrow[d, "i"] & \\
N \arrow[r, "f"] & f(N) \subset P
\end{tikzcd}$$

It follows that we have the following diagram whose rows are exact sequences :

$$\begin{tikzcd}
0 \arrow[r] & H_2(N) \arrow[r, "\pi"] \arrow[d, "f_1"'] & H_2(N, D^2(f)) \arrow[r, "\partial_1"] \arrow[d, "f_2"'] & H_1(D^2(f)) \arrow[r] \arrow[d, "f_3"'] & H_1(N) \arrow[d, "f_4"'] \\
0 \arrow[r] & H_2(f(N)) \arrow[r, "\pi"] & H_2(f(N), f(D^2(f))) \arrow[r, "\partial_2"] & H_1(f(D^2(f))) \arrow[r] & H_1(f(N)).
\end{tikzcd}$$

By the exactness of the sequences, we have

$$\beta_2(N) = \dim_{\mathbb{Z}_2} \mathrm{Ker}\, \partial_1 = \beta_2(N, D^2(f)) - \mathrm{rank}\, \partial_1$$

and

$$\beta_2(f(N)) = \dim_{\mathbb{Z}_2} \mathrm{Ker}\, \partial_2 = \beta_2(f(N), f(D^2(f))) - \mathrm{rank}\, \partial_2.$$

Since $f|N - D^2(f) : N - D^2(f) \to f(N) - f(D^2(f))$ is a homeomorphism, then we have $\beta_2(N, D^2(f)) = \beta_2(f(N), f(D^2(f)))$, so that $\beta_2(f(N)) - \beta_2(N) = \mathrm{rank}\, \partial_1 - \mathrm{rank}\, \partial_2$. We remark that $\dim \mathrm{Ker}\, f_3 = \frac{1}{2}\sharp c.c(D^2(f))$, then we have

$$\dim f_3^{-1}(\mathrm{Image}\, \partial_2) = \mathrm{rank}\, \partial_2 + \frac{1}{2}\sharp c.c(D^2(f)).$$

Since $f_3(\mathrm{Image}\,\partial_1) = \mathrm{Image}\,\partial_2$, then

$$\mathrm{rank}\,\partial_1 \leq \mathrm{rank}\,\partial_2 + \frac{1}{2}\sharp c.c(D^2(f)).$$

It follows that $\beta_2(f(N)) - \beta_2(N) \leq \frac{1}{2}\sharp c.c(D^2(f))$. Thus we have

$$\begin{aligned}\beta_1(f(N)) =&\beta_0(f(N)) + \beta_2(f(N)) - \chi(f(N)) \\ \leq&\beta_0(N) + \beta_2(N) + \frac{1}{2}\sharp c.c(D^2(f)) - \chi(N) \\ =&\beta_1(N) + \frac{1}{2}\sharp c.c(D^2(f)).\end{aligned}$$

On the other hand, by the first remark in this proof, we have $\sharp c.c(P - f(N)) = \chi(N) + \beta_1(f(N))$. So,

$$\begin{aligned}\sharp c.c(P - f(N)) \leq&\chi(N) + \beta_1(N) + \frac{1}{2}\sharp c.c(D^2(f)) \\ =&2 + \frac{1}{2}\sharp c.c(D^2(f)) \\ =&2 + \sharp c.c(f(D^2(f)).\end{aligned}$$

If $D^2(f)$ is homologous to zero in N, then ∂_1 is surjective, so that the equality holds

3 Examples

In this section we shall give some examples of our main results.

(3-1) The image of a C^∞-stable mapping [5]. It is well known that a mapping $f : N \to P$ from a closed surface to a 3-manifold is C^∞-stable if and only if it is an immersion with normal crossings except at the isolated singularities of cross-caps ([19]). Then we can directly apply Theorem 1.1 to this situation.

Theorem 3.1 [5]. *Let $f : N \to P$ be a C^∞-stable mapping from a closed surface to a 3-manifold. Then we have*

$$\chi(f(N)) = \chi(N) + T(f) + \frac{C(f)}{2}.$$

In [5] we have determined the set of Euler numbers of images of C^∞-stable mapping from a closed surface to a 3-manifold as an application of this theorem.

As a corollary of the above theorem, we can also get a formula in the case when the boundary of N is non-empty. In this case the stable mappings $f : N \to P$ we are going to deal with are a slight variation of the so called *completely semi-regular maps* ([19]) (see Fig. 6). Here we shall require that if $f(p) = f(q), p \neq q$ is a double point of f then $p \in \partial N$

if and only if $q \in \partial N$ (see Fig. 7).

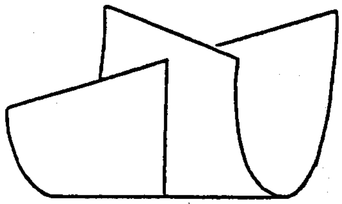

Fig.6

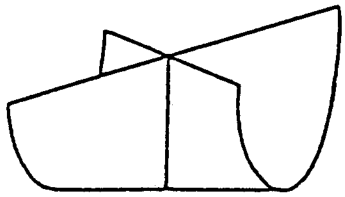

Fig.7

We denote the number of double points in $\partial f(N)$ by $\partial d(f)$. Then we can state the following as a corollary of the above theorem.

Corollary. *Let $f : N \to \mathbb{R}^3$ be a stable mapping such that if $f(p) = f(q)$, $p \neq q$ is a double point of f then $p \in \partial N$ if and only if $q \in \partial N$. Then we have*

$$\chi(f(N)) = \chi(N) + T(f) + \frac{C(f)}{2} - \frac{\partial d(f)}{2}.$$

This assertion can be used to classify map germs $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ with good perturbations [9].

Proof. Let us suppose that $\partial f(N)$ has k connected components $\partial_i f(N)$, for $i = 1, \dots, k$. Each component is a closed curve with a certain number of normal crossings. Suppose that $\partial_i f(N)$ has n_i normal crossings. Then $\sum_{i=1}^{k} n_i = \partial d(f)$. Now, triangulating each component of $\partial f(N)$ taking the crossings as 0-cells, we get $\chi(\partial f(N)) = \sum_{i=1}^{k} \chi(\partial f(N)) = \sum_{i=1}^{k}(n_i - 2n_i) = -\partial d(f)$.

Let $\mathcal{D}N$ denote the double of the manifold N and consider the stable mapping $h : \mathcal{D}N \to P$ given by $h|N = f$. So, by Theorem 3.1, we have $\chi(f(N)) = \chi(\mathcal{D}N) + C(h)/2 + T(h)$. However, $C(h) = 2C(f)$, $T(h) = 2T(f)$, $\chi(\mathcal{D}N) = 2\chi(f(N)) - \chi(\partial f(N))$. This completes the proof.

(3-2) The wave front of a closed Legendrian surface [6]. Let $\pi : E \to M$ be a Legendrian fibration over an $(n+1)$-manifold (i.e. the total space E is furnished with a contact structure and its fibers are Legendrian submanifold). For a Legendrian immersion $i : L \to E$, $\pi \circ i : L \to M$ is called *a Legendrian mapping* and the image of $\pi \circ i$ is called *a wave front* of i. We denote by $W(i)$ the wave front of i. We only consider the case of $n = 2$. In this case it is well known that a generic wave front has (semi cubic) cuspidal edges (A_2), swallowtails (A_3) and points of transversal self intersection (A_1A_1, A_1A_2, $A_1A_1A_1$) as singularities ([1], see Fig. 8). We shall refer to the $A_1A_1A_1$-type point as *a triple point* of

i.

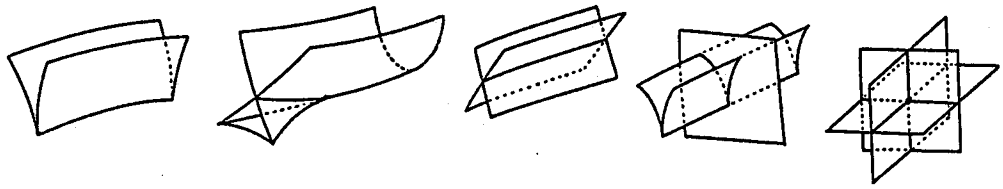

Fig. 8

Since the A_2-type point is homeomorphic to a regular point and the A_3-type point is homeomorphic to a cross-cap, then a generic Legendrian mapping gives a topological stable singular surface, so that we have the following :

Theorem 3.2. *Let $i : L \to E$ be a generic Legendrian immersion of a closed surface. Then we have*

$$\chi(W(i)) = \chi(L) + T(i) + \frac{S(i)}{2},$$

where $S(i)$ is the number of swallowtails.

(3-3) The discriminant set of a C^∞-stable mapping from a closed 3-manifold to a 3-manifold. It is well known that a mapping $f : N \to P$ from a closed 3-manifold to a 3-manifold is C^∞-stable if the singular set $\Sigma(f)$ is a closed surface and the discriminant set $D(f) = f(\Sigma(f))$ has (semi cubic) cuspidal edges (A_2), swallowtails (A_3) and points of transversal self intersection (A_1A_1, A_1A_2, $A_1A_1A_1$) as singularities ([10,11], see Fig. 8). We shall refer to the $A_1A_1A_1$-type point as *a triple point* of f. Since N is compact, then the singular set is a closed surface and the number of swallowtails and triple points are finite. By the same reason as that of the case (3-2), $f|\Sigma(f)$ gives a topologically stable singular surface, so that we have the following :

Theorem 3.3. *Let $f : N \to P$ be a C^∞-stable mapping from a closed 3-manifold to a 3-manifold. Then we have*

$$\chi(D(f)) = \chi(\Sigma(f)) + T(f) + \frac{S(f)}{2}.$$

We have an application of Theorem 3.3 to the dual of a space curve. Let $\gamma : S^1 \to \mathbb{R}^3$ be a simple closed C^∞-regular curve. *A tritangent plane* of γ is a plane in $\mathbb{R}^3$ which is tangent to γ at exactly three points. *A stall* of γ is a point of γ at which the torsion of γ is zero. It is known that, generically, γ is a curve with a finite number $\mathcal{T}(\gamma)$ of tritangent planes and a finite number $\mathcal{S}(\gamma)$ of stalls with some other properties [2]. We can also consider the classical dual surface γ^* in $(\mathbb{P}^3)^*$ consisting of all planes in $\mathbb{R}^3$ tangent to γ. Let $\mathbb{P}^2$ be the space of lines through the origin in $\mathbb{R}^3$, and let $L(\mathbb{P}^2)$ be the tautological line bundle over $\mathbb{P}^2$. For $\ell \in \mathbb{P}^2$ let $\Pi_\ell : C \to \ell$ be the restriction to C the orthogonal projection of $\mathbb{R}^3$ to ℓ, where $C = \gamma(S^1)$. Define $F_C : C \times \mathbb{P}^2 \to L(\mathbb{P}^2)$ by $F_C(x, \ell) = (\Pi_\ell(x), \ell)$. In [2], it has been proved that for a generic curve γ, F_C is C^∞-stable, $\Sigma(f) = C \times \mathbb{P}^1$ and $D(F_C) = \gamma^*$. It has also been proved that a triple point of γ^* corresponds to a tritangent plane of γ and a swallowtail point of γ^* corresponds to a stall of γ. Since $\chi(C \times \mathbb{P}^1) = 0$, then we have the following :

Corollary. *For a generic curve γ, we have*

$$\chi(\gamma^*) = \mathcal{T}(\gamma) + \frac{\mathcal{S}(\gamma)}{2}.$$

(3-4) The tangent developable of a space curve. For a simple closed C^∞-curve $\gamma : S^1 \to \mathbb{R}^3$ *the tangent developable of* γ, $D\gamma$, is defined to be the image of the mapping $D : S^1 \times \mathbb{R} \to \mathbb{R}^3$ defined by $D(s,t) = \gamma(s) + t\gamma'(s)$. In [3,12,16,18], it has been shown that the tangent developable of a generic space curve has (semi cubic) cuspidal edges (A_1), cuspidal cross-caps and points of transversal self intersection (A_1A_1, A_1A_2, $A_1A_1A_1$) as singularities (see Fig. 9). We shall refer to the $A_1A_1A_1$-type point as *a triple point* of $D\gamma$.

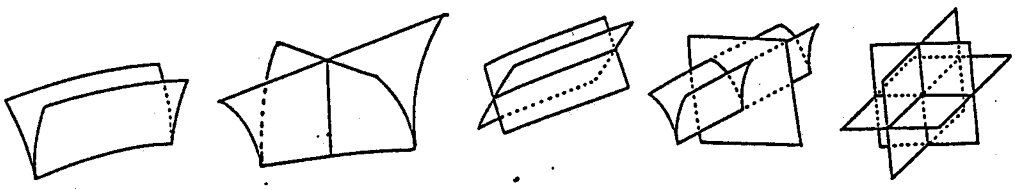

Fig. 9

In [16], J. J. Nuno Ballesteros has defined the compactification of the tangentdevelopable $\tilde{D} : S^1 \times \mathbb{R}^* \to \mathbb{P}^3$ as usual way, where $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ the extended real line. He has shown that $\tilde{D}$ is an immersion without triple points at the points (s, ∞) for generic curve γ. It is known that the triple point of the tangent developable corresponds to the "pyramid" of the curve (Fig. 10) and the cuspidal cross-cap corresponds to the stall of the curve.

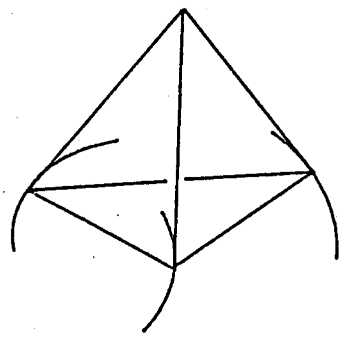

Fig. 10

We denote the number of the pyramid of the curve γ by $\mathcal{P}(\gamma)$. Since the cuspidal cross-cap is homeomorphic to the ordinary cross-cap and $\chi(S^1 \times \mathbb{R}^*) = 0$, then we have the following theorem.

Theorem 3.4. *For a generic curve γ, we have*

$$\chi(\tilde{D}\gamma) = \mathcal{P}(\gamma) + \frac{\mathcal{S}(\gamma)}{2}.$$

In relation to the tangent developable of a space curve, we give the following conjecture which is "the dual" of Freedman's conjecture on triple tangencies [4].

Conjecture. If there are no triple points on the tangent developable of a closed space curve, then this curve is unknotted.

Counter examples to Freedman's conjecture can be found in [14,15].

References

1. V. I. Arnol'd, *Singularities of Caustics and Wave fronts, Mathematics and Its Applications (Soviet Series)*, vol. 62, Kluwer Academic Publisher, 1990.
2. T. Banchoff, T. Gaffeney and C.McCrory, *Counting tritangent planes of space curves*, Topology 24 (1985), 15–23.
3. J. Cleave, *The form of the tangent developable at points of zero torsion on space curves*, Proc. Camb. Phil. Soc. 88 (1980), 403–407.
4. M. H. Freedman, *Planes triply tangent to curves with nonvanishing torsion*, Topology 19 (1980), 1–8.
5. S. Izumiya and W. L. Marar, *The Euler characteristic of a generic wave front in a 3-manifold*, to appear in Proceedings of AMS.
6. S. Izumiya and W. L. Marar, *The Euler characteristic of the image of a stable mapping from a closed n-manifold to a $(2n-1)$-manifold*, preprint (1991).
7. S. Lefschetz, *Introduction to Topology*, Princeton Univ. press, 1949.
8. H. Levine, *Stable maps: an introduction with low dimensional examples*, Bol.Soc.Bras.Mat. 7 (1976), 145–184.
9. W. L. Marar and D. Mond, *Mappings from $\mathbb{R}^2$ to $\mathbb{R}^3$ with good perturbations*, in preparations.
10. J. Mather, *Stability of C^∞-mappings V, Transversality*, Adv. in Math. 4 (1970), 301–336.
11. J. Mather, *Stability of C^∞-mappings VI, The nice dimensions*, Lecture Notes in mathematics 192, Springer-Verlag, 1972.
12. D. Mond, *Singularities of the tangent developable surface of a space curve*, Quart. J. Math. Oxford 40 (1989), 79–91.
13. D. Mond, *Vanishing cycles for analytic maps*, Springer Lecture Notes in Math. 1462 (1991), 221–234.
14. A. Montesinos Amilibia and J. J. Nuño Ballesteros, *A knot without tritangent planes*, preprint (1989).
15. H. R. Morton, *Trefoil knots without tritangent planes*, Bull. London Math. Soc. (1991), 78–80.
16. J. J. Nuño Ballesteros, *On the number of triple points of the tangent developable*, preprint (1991).
17. J. J. Nuño Ballesteros and M. C. Romero Fuster, *Separation properties of continuous maps in codimension 1*, to apear in Topology and its applications.
18. O. P. Scherbak, *Projectively dual space curve and Legendre singularities*, Sel. Math. Sov. 5 (1986), 391–421.
19. H. Whitney, *The singularities of a smooth n-manifold in $(2n-1)$-space*, Annals of Math. 45 (1944), 247–293.

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