Cohen-Macaulay types of Cohen-Macaulay complexes

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Cohen-Macaulay types of Cohen-Macaulay complexes *)

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Abstract. We say that a Cohen-Macaulay poset (partially ordered set) is "superior" if every open interval (x, y) of $P^\wedge$ with $\mu_{P^\wedge}(x,y) \neq 0$ is doubly Cohen-Macaulay. For example, if $L = P^\wedge$ is a modular lattice, then the Cohen-Macaulay poset P is superior. We present a formula for the computation of the Cohen-Macaulay type of the Stanley-Reisner ring of the order complex of a Cohen-Macaulay poset which is superior.

Introduction

The Cohen-Macaulay type $\mathrm{type}(R)$ of a Cohen-Macaulay local ring $(R,\underline{m},k)$ is defined to be the dimension of the socle $\mathrm{Soc}(R/\underline{q}R) := \{ y \in R/\underline{q}R \, ; \, \underline{m}y = 0 \}$ of $R/\underline{q}R$ as a vector space over the residue field k. Here $\underline{q}$ is a parameter ideal of R. This invariant $\mathrm{type}(R)$ has rich background. For example, the minimal number of the generators of the canonical module K_R (see, e.g., [H-K]) of R is equal to $\mathrm{type}(R)$ and, moreover, $\mathrm{type}(R)$ coincides with the last non-zero Betti number which appears in the minimal free resolution of R over a regular local ring A with $R = A/I$ for an ideal I of A.

We study Cohen-Macaulay types of Stanley-Reisner rings of Cohen-Macaulay complexes. It would, of course, be of great interest to find a "combinatorial" formula (e.g., [H4] and [H5]) to compute Cohen-Macaulay types of Cohen-Macaulay complexes. The purpose of the paper is to present such a formula for an excellent class of Cohen-Macaulay complexes. We believe that the information about Cohen-Macaulay types of Cohen-Macaulay complexes is one of the most important foundations for finding explicit expressions of canonical modules (e.g., [Bac3]) and for constructing the minimal free resolutions of Stanley-Reisner rings of Cohen-Macaulay complexes.

Here is a brief sketch of this paper. First, in Section 1, we recall fundamental material for algebra and topology as well as combinatorics on simplicial complexes. The topological formulae Eq. (1) and Eq. (2), both of them are due to Hochster, for Betti numbers and for Hilbert series of local cohomology modules of Stanley-Reisner rings play essential roles in the paper. Also, in Section 2, we define the concept "superior" for Cohen-Macaulay complexes, see (2.4). Our Theorem (2.7) supplies a combinatorial formula for the explicit computation of the Cohen-Macaulay type of the Stanley-Reisner ring of a Cohen-Macaulay complex which is superior. The proof of Theorem (2.7) is given in Section 3. Our proof relies on Eq. (1) and is based on the long exact sequence of local cohomology modules in the theory of commutative algebra.

On the other hand, in Section 4, we are interested in Cohen-Macaulay posets (partially ordered sets) whose order complexes are superior. As a corollary to Theorem (2.7), the Cohen-Macaulay type of the Stanley-Reisner ring of a Cohen-Macaulay poset which is superior can be computed by means of the Möbius function of the poset, see Theorem (4.8). Moreover, many interesting Cohen-Macaulay posets, e.g., modular lattices, turn out to be superior.

The formula in Theorem (4.8) was pointed out, with the form of a conjecture, to the author by Stanley while we were discussing Cohen-Macaulay types of distributive lattices (cf. [H4]) at M. I. T. (June, 1991). The author is grateful to Professor Richard P. Stanley for valuable suggestions and exciting discussions on the topic of Cohen-Macaulay types of Cohen-Macaulay posets.

§1. Algebraic background

We here summarize fundamental material and notation for algebra, topology and combinatorics on simplicial complexes.

(1.1) Fix a finite set $V = \{ x_1, x_2, \ldots, x_v \}$, called the *vertex set*, and let Δ be a *simplicial complex* on V. Thus Δ is a family of subsets of V such that (i) $\{ x_i \} \in \Delta$ for each $1 \le i \le v$ and (ii) $\sigma \in \Delta$, $\tau \subset \sigma$ imply $\tau \in \Delta$. Each element σ of Δ is called a *face* of Δ. More precisely, if $\#(\sigma) = i + 1$, then σ is called an i-face of Δ. Here $\#(\sigma)$ is the cardinality of σ as a finite set. Let $d := \max\{\#(\sigma) ; \sigma \in \Delta\}$. Then the *dimension* of Δ is defined by $\dim \Delta := d - 1$. We say that a face σ of Δ is a *facet* of Δ if $\#(\sigma) = d$. A simplicial complex Δ is called *pure* if every maximal face, with respect to inclusion, is a facet of Δ.

Given a face σ of Δ we define the subcomplex $\mathrm{link}_\Delta(\sigma)$ of Δ by

$$\mathrm{link}_\Delta(\sigma) := \{ \tau \in \Delta ; \sigma \cap \tau = \emptyset \text{ and } \sigma \cup \tau \in \Delta \}.$$

Thus, in particular, $\mathrm{link}_\Delta(\emptyset) = \Delta$. Moreover, if $\tau \in \Delta' = \mathrm{link}_\Delta(\sigma)$, then $\mathrm{link}_{\Delta'}(\tau) = \mathrm{link}_\Delta(\sigma \cup \tau)$.

On the other hand, if W is a subset of V , then the subcomplex Δ_W of Δ is defined to be the simplicial complex

$$\Delta_W := \{ \sigma \in \Delta ; \sigma \subset W \}$$

on the vertex set W .

(1.2) We fix a base field k . Let $\widetilde{H}_i(\Delta;k)$ be the i-th *reduced homology group* of Δ with coefficients k . Recall that the *reduced Euler characteristic* $\widetilde{\chi}(\Delta)$ of Δ is defined by

$$\widetilde{\chi}(\Delta) := \sum_{i \geq -1} (-1)^i \dim_k(\widetilde{H}_i(\Delta;k)) .$$

In particular, $\widetilde{\chi}(\emptyset) = -1$. Note that $\widetilde{\chi}(\Delta)$ is independent of the field characteristic $\mathrm{char}(k)$ of the base field k .

(1.3) Let $A = k[x_1, x_2, \ldots, x_v]$ be the polynomial ring in v ($= \#(V)$) variables over k whose indeterminates are the elements of V . We define I_Δ to be the ideal of A which is generated by those square-free monomials $x_{i1} x_{i2} \ldots x_{ir}$ such that $\{ x_{i1}, x_{i2}, \ldots, x_{ir} \} \notin \Delta$, and set $k[\Delta] := A/I_\Delta$. The algebra $k[\Delta]$ over k is called the *Stanley-Reisner ring* of Δ over k in commemoration of Stanley [Sta1] and Reisner [Rei].

From now on, we fix the "fine grading" on A given by setting each $\deg x_i = (0, \ldots, 0, 1, 0, \ldots 0) \in \mathbb{Z}^v$, the "1" in the i-th component , and regard $k[\Delta]$ as a graded module over A with the "quotient grading."

When $\mathcal{M} = \oplus_{\rho \in \mathbb{Z}^v} \mathcal{M}_\rho$ is a graded module over A with the fine grading such that each $\dim_k \mathcal{M}_\rho < \infty$, where $\dim_k \mathcal{M}_\rho$ is the dimension of $\mathcal{M}_\rho$ as a vector space over k , the *Hilbert series* $F(\mathcal{M}, \lambda)$ of $\mathcal{M}$ is defined to be the formal power series

$$F(\mathcal{M}, \lambda) := \sum_{\rho \in \mathbb{Z}^v} (\dim_k \mathcal{M}_\rho) \lambda^\rho$$

in the variables $\lambda_1, \lambda_2, \ldots, \lambda_v$. Here $\lambda^\rho := \lambda_1{}^{\rho_1}\lambda_2{}^{\rho_2}\ldots\lambda_v{}^{\rho_v}$ if $\rho = (\rho_1,\rho_2,\ldots,\rho_v)$.

(1.4) A ***minimal free resolution*** of $k[\Delta]$ over A is an exact sequence

$$0 \to A^{\beta_t} \xrightarrow[\delta_t]{} \ldots \to A^{\beta_i} \xrightarrow[\delta_i]{} A^{\beta_{i-1}} \to \ldots \xrightarrow[\delta_1]{} A^{\beta_0} \xrightarrow[\delta_0]{} k[\Delta] \to 0$$

of modules over A, where each A^{β_i} is the free module $\{(a_1,a_2,\ldots,a_{\beta_i})\,;\,a_q \in A, 1 \le q \le \beta_i\}$ over A of rank $\beta_i > 0$, and where each δ_i is a $\beta_i \times \beta_{i-1}$ matrix any of whose entries is either 0 or a monomial of A of the form $\xi x_1{}^{n_1}x_2{}^{n_2}\ldots x_v{}^{n_v}$ with $0 \neq \xi \in k$ and $n_j > 0$ for some $1 \le j \le v$.

It is known that there exists a minimal free resolution of $k[\Delta]$ over A, and that t and each β_i are independent of the chosen minimal free resolution. Moreover, $v - d \le t \le v$. The ***homological dimension*** $\mathrm{hd}_A(k[\Delta])$ of $k[\Delta]$ over A is the non-negative integer t, and each $\beta_i = \beta_i{}^A(k[\Delta])$ is called the i-th ***Betti number*** of $k[\Delta]$ over A. Thus, in the language of homological algebra, we have

$$\beta_i{}^A(k[\Delta]) = \dim_k \underline{\mathrm{Tor}}_i{}^A(k[\Delta],k)\,.$$

The following topological formula on the i-th Betti number $\beta_i{}^A(k[\Delta])$ is given by Hochster [Hoc, Theorem (5.1)] :

$$\beta_i{}^A(k[\Delta]) = \sum_{W \subset V} \dim_k(\widetilde{H}_{v-\#(W)-i-1}(\Delta_{V-W};k))\,. \qquad (1)$$

(1.5) Let m be the irrelevant maximal ideal $(x_1,x_2,\ldots,x_v)$ of A and $\underline{H}_m{}^i(k[\Delta])$ the i-th ***local cohomology module*** of $k[\Delta]$ over A, i.e.,

$$\underline{H}_m{}^i(k[\Delta]) := \varinjlim_n \underline{\mathrm{Ext}}_A{}^i(A/m^n,k[\Delta])\,.$$

Then (i) $H_m^i(k[\Delta]) = 0$ unless $v - hd_A(k[\Delta]) \le i \le d$ and (ii) $H_m^i(k[\Delta]) \ne 0$ for $i = d$, $v - hd_A(k[\Delta])$. Consult, e.g., [S-V] for basic facts on local cohomology modules $H_m^i(k[\Delta])$.

The Hilbert series $F(H_m^i(k[\Delta]),\lambda)$ of $H_m^i(k[\Delta])$ as a module over A with the fine grading is

$$F(H_m^i(k[\Delta]),\lambda)$$
$$= \sum_{\sigma \in \Delta} \dim_k(\widetilde{H}_{i-\#(\sigma)-1}(\mathrm{link}_\Delta(\sigma);k)) \prod_{x_i \in \sigma} \frac{\lambda_i^{-1}}{1-\lambda_i^{-1}}. \quad (2)$$

See [Sta3, pp.70-73].

(1.6) We say that a simplicial complex Δ is *Cohen-Macaulay* over k if $hd_A(k[\Delta]) = v - d$, i.e., $H_m^i(k[\Delta]) = 0$ for every $i \ne d$. A topological criterion [Rei] says that a simplicial complex Δ is Cohen-Macaulay over k if and only if $\widetilde{H}_i(\mathrm{link}_\Delta(\sigma);k) = 0$ for every face $\sigma \in \Delta$ (possibly, $\sigma = \emptyset$) and for each $i \ne \dim(\mathrm{link}_\Delta(\sigma))$. For example, a simplicial complex whose geometric realization is homeomorphic to either a ball or a sphere is Cohen-Macaulay. Every Cohen-Macaulay complex is pure. If Δ is Cohen-Macaulay, then $\mathrm{link}_\Delta(\sigma)$ is Cohen-Macaulay for every face σ of Δ. Refer to, e.g., [Hoc], [Sta3] for the detailed information about Cohen-Macaulay complexes.

§2. Cohen-Macaulay types

We inherit the notation in the preceding section.

(2.1) Suppose that a simplicial complex Δ is Cohen-Macaulay over k, i.e., $hd_A(k[\Delta]) = v - d$. Then we write $\mathrm{type}(k[\Delta])$ for the $(v-d)$-th Betti number $\beta_{v-d}^A(k[\Delta])$ of $k[\Delta]$ over A, and call $\mathrm{type}(k[\Delta])$ the *Cohen-Macaulay type* of $k[\Delta]$. By virtue of Eq. (1) we have the topological formula for $\mathrm{type}(k[\Delta])$ as follows :

$$\mathrm{type}(k[\Delta]) = \sum_{W \subset V} \dim_k(\widetilde{H}_{d-\#(W)-1}(\Delta_{V-W};k)) \qquad (3)$$

A Cohen-Macaulay complex with $\mathrm{type}(k[\Delta]) = 1$ is called *Gorenstein.* For example, a simplicial complex whose geometric realization is homeomorphic to a sphere is Gorenstein. If Δ is Gorenstein, then $\mathrm{link}_\Delta(\sigma)$ is Gorenstein for every face σ of Δ. Some characterization of Gorenstein complexes is obtained in, e.g., [Hoc, pp.210-211] and [Sta3, p.75].

(2.2) A Cohen-Macaulay complex Δ is called *doubly Cohen-Macaulay* [Bac2] if the subcomplex $\Delta_{V-\{x\}}$ is Cohen-Macaulay of the same dimension as Δ for each $x \in V$. A Gorenstein complex Δ with $\widetilde{\chi}(\Delta) \neq 0$ is doubly Cohen-Macaulay. Every subcomplex $\mathrm{link}_\Delta(\sigma)$ of a doubly Cohen-Macaulay complex Δ is again doubly Cohen-Macaulay. Moreover, a Cohen-Macaulay complex Δ is doubly Cohen-Macaulay if and only if $(-1)^{d-1}\widetilde{\chi}(\Delta) = \mathrm{type}(k[\Delta])$.

REMARK. (a) The non-negative integer $v - \mathrm{hd}_A(k[\Delta])$ depends on the field characteristic $\mathrm{char}(k)$ of k. On the other hand, if we fix a base field k, $v - \mathrm{hd}_A(k[\Delta])$ is topological [Mun], i.e., depends only on the geometric realization of Δ. Thus, in particular, the Cohen-Macaulay condition for Δ is topological. Moreover, the doubly Cohen-Macaulay condition for Δ is topological [Wal].

(b) Whether Δ is Gorenstein or not depends on $\mathrm{char}(k)$. Moreover, even though we fix a base field, the Gorenstein condition for Δ is *not* topological. However, if $\widetilde{\chi}(\Delta) \neq 0$, $\widetilde{\chi}(\Delta') \neq 0$ and if we suppose that the geometric realization of Δ is homeomorphic to that of Δ', then Δ is Gorenstein if and only if Δ' is Gorenstein. On the other hand, the problem of whether a Cohen-Macaulay complex Δ is Gorenstein or not is combinatorial (cf. [H3]).

(2.3) EXAMPLE. Let $v = 7$, $d = 3$ and Δ the simplicial complex of Figure 1. Then Δ is Cohen-Macaulay over an arbitrary field [Bjö3, p.277]. However, $\mathrm{type}(k[\Delta]) = 7$ if $\mathrm{char}(k) \neq 2$ and $\mathrm{type}(k[\Delta]) = 8$ if $\mathrm{char}(k) = 2$.

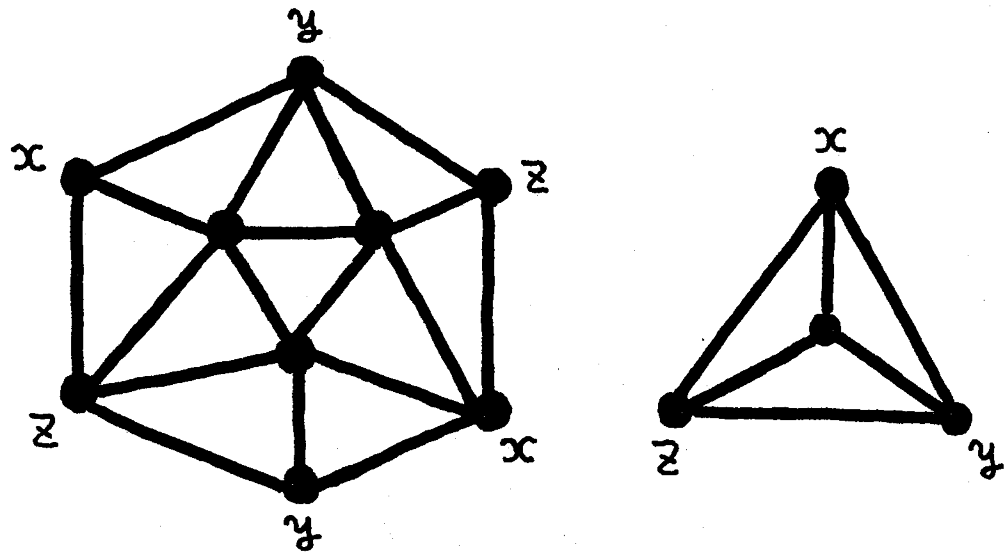

Figure 1

(2.4) We now come to the crucial definitions in the paper.

DEFINITION. A face σ of a Cohen-Macaulay complex Δ is called *fundamental* if (i) $\tilde{\chi}(\mathrm{link}_\Delta(\sigma)) \neq 0$, and (ii) $\tilde{\chi}(\mathrm{link}_\Delta(\tau)) = 0$ for every face $\tau \subsetneqq \sigma$ of Δ.

We write $\mathcal{F}(\Delta)$ for the set of fundamental faces of Δ.

REMARK. (a) The empty face $\emptyset$ of a Cohen-Macaulay complex Δ is fundamental if and only if $\tilde{\chi}(\Delta) \neq 0$.

(b) The existence of a fundamental face of a Cohen-Macaulay complex Δ is guaranteed by the fact that, for each facet $\sigma \in \Delta$, we have $\tilde{\chi}(\mathrm{link}_\Delta(\sigma))$ ($= \tilde{\chi}(\emptyset) = -1$) $\neq 0$.

DEFINITION. We say that a Cohen-Macaulay complex Δ is *superior* if $\mathrm{link}_\Delta(\sigma)$ is doubly Cohen-Macaulay for every fundamental face σ of Δ.

REMARK. (a) Every doubly Cohen-Macaulay complex is superior.

(b) When a Cohen-Macaulay complex Δ is superior, the Cohen-Macaulay subcomplex $\mathrm{link}_\Delta(\tau)$ of Δ is doubly Cohen-Macaulay for every face τ of Δ with $\widetilde{\chi}(\mathrm{link}_\Delta(\tau)) \neq 0$.

(c) The Cohen-Macaulay subcomplex $\mathrm{link}_\Delta(\sigma)$ of a Cohen-Macaulay complex Δ which is superior is again superior for every face σ of Δ.

(2.5) **EXAMPLE.** Suppose that the geometric realization of a simplicial complex Δ of dimension $d-1$ is homeomorphic to the $(d-1)$-ball. The *boundary* $\partial\Delta$ of Δ is defined to be the subcomplex of Δ which consists of those faces τ of Δ such that there exists a $(d-2)$-face ρ which is contained in exactly one $(d-1)$-face of Δ with $\rho \supset \tau$. Then, a face σ of Δ satisfies $\widetilde{\chi}(\mathrm{link}_\Delta(\sigma)) \neq 0$ if and only if $\sigma \in \Delta - \partial\Delta$. Hence, the set $\mathcal{F}(\Delta)$ of fundamental faces of Δ is equal to the set of minimal elements of $\Delta - \partial\Delta$. Since the geometric realization of $\mathrm{link}_\Delta(\sigma)$ is homeomorphic to the $(d-1-\#(\sigma))$-sphere if $\sigma \in \Delta - \partial\Delta$, the Cohen-Macaulay complex Δ is superior.

(2.6) Let Δ' (resp. Δ'') be a simplicial complex on the vertex set V' (resp. V''). Then the *simplicial join* $\Delta' * \Delta''$ is the simplicial complex

$$\Delta' * \Delta'' := \{ \sigma' \cup \sigma'' ; \sigma' \in \Delta' \text{ and } \sigma'' \in \Delta'' \}$$

whose vertex set is the disjoint union $V' \cup V''$ of V' and V''. Thus $k[\Delta' * \Delta''] = k[\Delta'] \otimes_k k[\Delta'']$. Also, we have $\dim \Delta' * \Delta'' = \dim \Delta' + \dim \Delta'' + 1$ and $\widetilde{\chi}(\Delta' * \Delta'') = - \widetilde{\chi}(\Delta')\widetilde{\chi}(\Delta'')$. On the other hand, the simplicial join $\Delta' * \Delta''$ of Δ' and Δ'' is Cohen-Macaulay (resp. Gorenstein, doubly Cohen-Macaulay) if and only if both Δ' and Δ'' are Cohen-Macaulay (resp. Gorenstein, doubly Cohen-Macaulay). Moreover, if Δ and Δ' are Cohen-Macaulay, then $\Delta' * \Delta''$ is superior if and only if both Δ' and Δ'' are superior (since $\mathrm{link}_{\Delta' * \Delta''}(\sigma' \cup \sigma'') = \mathrm{link}_{\Delta'}(\sigma') * \mathrm{link}_{\Delta''}(\sigma'')$).

We are now in the position to state the main result of this paper.

(2.7) THEOREM. Suppose that a simplicial complex Δ of dimension $d-1$ is Cohen-Macaulay over a field k, and let $\mathcal{F}(\Delta)$ be the set of fundamental faces of Δ.

(a) Then we have the lower bound inequality

$$\mathrm{type}(k[\Delta]) \geq \sum_{\sigma \in \mathcal{F}(\Delta)} (-1)^{d-1-\#(\sigma)} \tilde{\chi}(\mathrm{link}_\Delta(\sigma)).$$

for the Cohen-Macaulay type of $k[\Delta]$.

(b) Moreover, in order that the equality holds in the above inequality, it is sufficient (however, not necessary) that the Cohen-Macaulay complex Δ is superior.

(2.8) EXAMPLE. Let $v = 9$, $d = 3$ and Δ the Cohen-Macaulay complex of Figure 2. The fundamental faces of Δ are $\sigma_1 = \{x\}$, $\sigma_2 = \{y\}$ and $\sigma_3 = \{z\}$ with each $\tilde{\chi}(\mathrm{link}_\Delta(\sigma_i)) = -1$. Since $\mathrm{type}(k[\Delta]) = 3$, we have the equality in the lower bound inequality for $\mathrm{type}(k[\Delta])$. However, the Cohen-Macaulay complex Δ is *not* superior. In fact, $\mathrm{link}_\Delta(\sigma_1)$ and $\mathrm{link}_\Delta(\sigma_2)$ are not doubly Cohen-Macaulay.

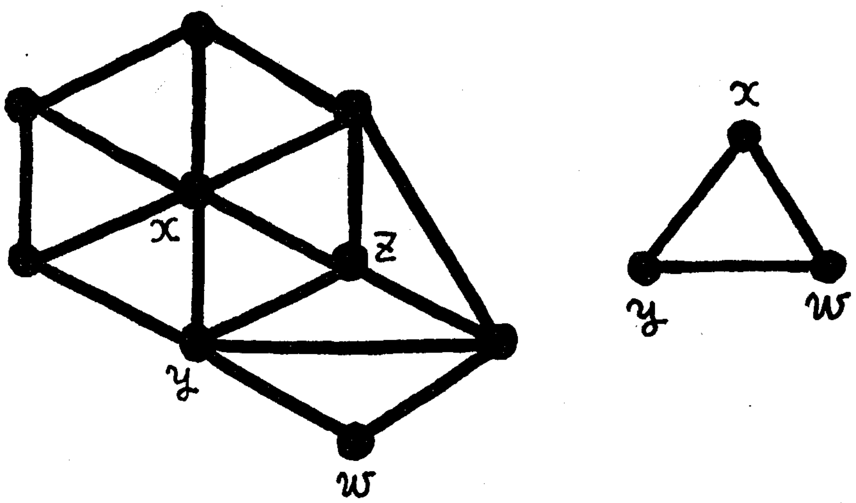

Figure 2

REMARK. The Cohen-Macaulay complex Δ of Figure 2 is a "level complex" in the sense of Stanley [Sta2]. However, $\mathrm{link}_\Delta(\sigma_1)$ and $\mathrm{link}_\Delta(\sigma_2)$ are not level. Thus, a subcomplex $\mathrm{link}_\Delta(\sigma)$ of a level complex Δ is not necessarily level. Note that a simplicial complex Δ is doubly Cohen-Macaulay if and only if Δ is level with $\tilde{\chi}(\Delta) \neq 0$. See [Bac2].

§3. Proof of Theorem (2.7)

In this section, based on Hochster's topological formula (3) for $\mathrm{type}(k[\Delta])$, we give a proof of Theorem (2.7). Our basic tool is the long exact sequence of local cohomology modules in the theory of commutative algebra.

We inherit the notation in (1.1) and (1.3), and, moreover, for each face σ of Δ, we define the subcomplex $\mathrm{star}_\Delta(\sigma)$ by

$$\mathrm{star}_\Delta(\sigma) := \{ \tau \in \Delta ; \sigma \cup \tau \in \Delta \} .$$

Thus, in particular, if $\Delta' = \mathrm{star}_\Delta(\sigma)$, then $\mathrm{link}_{\Delta'}(\sigma) = \mathrm{link}_\Delta(\sigma)$.

Let $x \in V$ and $\rho = \deg x \in \mathbb{Z}^V$. Then, we have the exact sequence

$$0 \longrightarrow k[\mathrm{star}_\Delta(\{x\})](\rho) \longrightarrow k[\Delta] \longrightarrow k[\Delta_{V-\{x\}}] \longrightarrow 0 \tag{4}$$

as graded modules over A with the fine grading. Here $k[\mathrm{star}_\Delta(\{x\})](\rho)$ is the shift in grading of $k[\mathrm{star}_\Delta(\{x\})]$ of degree $\rho \in \mathbb{Z}^V$, i.e., the ν-th homogeneous part of $k[\mathrm{star}_\Delta(\{x\})](\rho)$ is the $(\nu - \rho)$-th homogeneous part of $k[\mathrm{star}_\Delta(\{x\})]$ for each $\nu \in \mathbb{Z}^V$. Hence, there exists the long exact sequence

$$
\begin{aligned}
0 \longrightarrow \underline{H}_m{}^0(k[\mathrm{star}_\Delta(\{x\})])(\rho) &\longrightarrow \underline{H}_m{}^0(k[\Delta]) \longrightarrow \underline{H}_m{}^0(k[\Delta_{V-\{x\}}]) \\
\longrightarrow \underline{H}_m{}^1(k[\mathrm{star}_\Delta(\{x\})])(\rho) &\longrightarrow \underline{H}_m{}^1(k[\Delta]) \longrightarrow \underline{H}_m{}^1(k[\Delta_{V-\{x\}}]) \\
\longrightarrow \cdots & \\
\longrightarrow \underline{H}_m{}^q(k[\mathrm{star}_\Delta(\{x\})])(\rho) &\longrightarrow \underline{H}_m{}^q(k[\Delta]) \longrightarrow \underline{H}_m{}^q(k[\Delta_{V-\{x\}}]) \\
\longrightarrow \cdots &
\end{aligned}
$$

of local cohomology modules as graded modules over A **with the fine grading.**

In what follows, suppose that a simplicial complex Δ **of dimension** $d - 1$ **on the vertex set** V **with** $\#(V) = v$ **is Cohen-Macaulay over a field** k **. Let** $\mathcal{F}(\Delta)$ **be the set of fundamental faces of** Δ **.**

(3.1) LEMMA. The local cohomology module $\underline{H}_m{}^i(k[\Delta_{V-W}])$ **vanishes for every subset** W **of** V **and for each** $i < d - \#(W)$ **. Moreover,** $\underline{H}_m{}^{d-\#(W)}(k[\Delta_{V-W}]) = 0$ **if** Δ **is doubly Cohen-Macaulay and** $W \neq \emptyset$ **.**

Proof. The local cohomology module $\underline{H}_m{}^i(k[\Delta])$ **vanishes for each** $i < d$ **since** Δ **is Cohen-Macaulay. Suppose that** $\underline{H}_m{}^i(k[\Delta_{V-W}]) = 0$ **for a subset** W **of** V **and for each** $i < d - \#(W)$ **. Then, thanks to Eq. (2), it is routine work to see that** $\underline{H}_m{}^i(k[\mathrm{star}_{\Delta_{V-W}}(\{x\})]) = 0$ **for every** $x \in V - W$ **and for each** $i < d - \#(W)$ **. Thus, replace** Δ **with** Δ_{V-W} **in (4), and it follows immediately from the long exact sequence of local cohomology modules that** $\underline{H}_m{}^i(k[\Delta_{V-(W\cup\{x\})}]) = 0$ **for each** $i < d - \#(W) - 1$ **.**

Moreover, if Δ **is doubly Cohen-Macaulay and** $W \neq \emptyset$ **, then** $\underline{H}_m{}^{d-\#(W-\{x\})-1}(k[\Delta_{(V-\{x\})-(W-\{x\})}]) = 0$ **since** $\Delta_{V-\{x\}}$ **,** $x \in W$ **, is Cohen-Macaulay of dimension** $d - 1$ **. Q.E.D.**

(3.2) LEMMA. Let τ be a non-empty face of Δ, $x \in \tau$ with $\rho = \deg x \in \mathbb{Z}^V$ and set $\sigma = \tau - \{x\} \in \Delta$. Then there exists the exact sequence

$$0 \longrightarrow \underline{H}_m{}^{d-\#(\tau)}(k[\Delta_{V-\tau}]) \longrightarrow \underline{H}_m{}^{d-\#(\sigma)}(k[\mathrm{star}_{\Delta_{V-\sigma}}(\{x\})])(\rho)$$

$$\longrightarrow \underline{H}_m{}^{d-\#(\sigma)}(k[\Delta_{V-\sigma}]) \longrightarrow \underline{H}_m{}^{d-\#(\sigma)}(k[\Delta_{V-\tau}])$$

of local cohomology modules as graded modules over A with the fine grading.

Proof. If we replace Δ with $\Delta_{V-\sigma}$ in (4), then the long exact sequence of local cohomology modules guarantees the desired exact sequence since $\underline{H}_m{}^{d-\#(\tau)}(k[\Delta_{V-\sigma}]) = 0$ by Lemma (3.1).

Q.E.D.

(3.3) COROLLARY. Let τ be a non-empty face of Δ, $x \in \tau$ and set $\sigma = \tau - \{x\} \in \Delta$. Also, let $\Delta' = \mathrm{link}_\Delta(\{x\})$ and V' the vertex set of Δ'.

(a) If $\widetilde{H}_{d-\#(\sigma)-1}(\Delta_{V-\sigma};k) = 0$, then

$$\dim_k(\widetilde{H}_{d-\#(\tau)-1}(\Delta_{V-\tau};k)) = \dim_k(\widetilde{H}_{d-\#(\tau)-1}(\Delta'_{V'-\sigma};k)).$$

(b) When $\widetilde{H}_{d-\#(\tau)-1}(\Delta'_{V'-\sigma};k) = 0$, the reduced homology group $\widetilde{H}_{d-\#(\tau)-1}(\Delta_{V-\tau};k)$ vanishes.

Proof. By virtue of Eq. (2), the dimension of the homogeneous part of degree $(0,0,\ldots,0) \in \mathbb{Z}^V$ of the local cohomology module $\underline{H}_m{}^{d-\#(\tau)}(k[\Delta_{V-\tau}])$ as a vector space over k coincides with $\dim_k(\widetilde{H}_{d-\#(\tau)-1}(\Delta_{V-\tau};k))$. Moreover, the dimension of the homogeneous part of degree $(0,0,\ldots,0) \in \mathbb{Z}^V$ of the local cohomology module $\underline{H}_m{}^{d-\#(\sigma)}(k[\mathrm{star}_{\Delta_{V-\sigma}}(\{x\})])(\rho)$ with $\rho = \deg x \in \mathbb{Z}^V$ is equal to $\dim_k(\widetilde{H}_{d-\#(\tau)-1}(\Delta'_{V'-\sigma};k))$ since $\mathrm{link}_{\Delta_{V-\sigma}}(\{x\}) = \Delta'_{V'-\sigma}$. Hence, Lemma (3.2) enables us to see (a) and (b) immediately.

Q.E.D.

(3.4) LEMMA. Let τ be a face of Δ and suppose that $\widetilde{\chi}(\mathrm{link}_\Delta(\sigma)) = 0$ for every face $\sigma \subsetneq \tau$. Then

$$\dim_k(\widetilde{H}_{d-\#(\tau)-1}(\Delta_{V-\tau};k)) = (-1)^{d-1-\#(\tau)}\,\widetilde{\chi}(\mathrm{link}_\Delta(\tau)).$$

Proof. First, $\widetilde{H}_i(\Delta;k) = 0$ for each $i \neq d-1$ since Δ is Cohen-Macaulay. Thus $\widetilde{\chi}(\Delta) = (-1)^{d-1}\dim_k(\widetilde{H}_{d-1}(\Delta;k))$. Hence the required equality holds when $\tau = \varnothing$.

We now employ induction on $\#(\tau)$ in the proof. Let τ be a non-empty face of Δ and suppose that $\widetilde{\chi}(\mathrm{link}_\Delta(\sigma)) = 0$ for every face $\sigma \subsetneq \tau$. Let $x \in \tau$ and set $\sigma = \tau - \{x\} \in \Delta$. Also, let $\Delta' = \mathrm{link}_\Delta(\{x\})$ and V' the vertex set of Δ'. It follows from assumption of induction that we have the equality $\dim_k(\widetilde{H}_{d-\#(\sigma)-1}(\Delta_{V-\sigma};k)) = (-1)^{d-1-\#(\sigma)}\,\widetilde{\chi}(\mathrm{link}_\Delta(\sigma))$. Hence $\widetilde{H}_{d-\#(\sigma)-1}(\Delta_{V-\sigma};k) = 0$. Thus, Corollary (3.3) guarantees that $\dim_k(\widetilde{H}_{d-\#(\tau)-1}(\Delta_{V-\tau};k)$ is equal to $\dim_k(\widetilde{H}_{d-\#(\tau)-1}(\Delta'_{V'-\sigma};k))$. On the other hand, since $\widetilde{\chi}(\mathrm{link}_{\Delta'}(\sigma')) = \widetilde{\chi}(\mathrm{link}_\Delta(\sigma' \cup \{x\})) = 0$ for every face $\sigma' \subsetneq \sigma$, and since $\#(\sigma) < \#(\tau)$, we can apply assumption of induction to Δ' and $\sigma \in \Delta'$. Thus

$$\begin{aligned}
\dim_k(\widetilde{H}_{d-\#(\tau)-1}(\Delta_{V-\tau};k) &= \dim_k(\widetilde{H}_{d-\#(\tau)-1}(\Delta'_{V'-\sigma};k)) \\
&= \dim_k(\widetilde{H}_{(d-1)-\#(\sigma)-1}(\Delta'_{V'-\sigma};k)) \\
&= (-1)^{d-1-\#(\sigma)-1}\,\widetilde{\chi}(\mathrm{link}_{\Delta'}(\sigma)) \\
&= (-1)^{d-1-\#(\sigma)-1}\,\widetilde{\chi}(\mathrm{link}_\Delta(\tau)) \\
&= (-1)^{d-1-\#(\tau)}\,\widetilde{\chi}(\mathrm{link}_\Delta(\tau))
\end{aligned}$$

as desired. **Q.E.D.**

(3.5) COROLLARY. Let τ be a face of Δ and suppose that $\sigma \subset \tau$ for no fundamental face σ of Δ. Then the reduced homology group $\widetilde{H}_{d-\#(\tau)-1}(\Delta_{V-\tau};k))$ vanishes.

(3.6) LEMMA. Suppose that Δ is superior and let τ be a face of Δ with $\sigma \subsetneq \tau$ for some $\sigma \in \mathcal{F}(\Delta)$. Then the reduced homology group $\widetilde{H}_{d-\#(\tau)-1}(\Delta_{V-\tau};k)$ vanishes.

Proof. First, suppose that $\emptyset \in \mathcal{F}(\Delta)$, i.e., $\tilde{\chi}(\Delta) \neq 0$. Then $\Delta = \mathrm{link}_\Delta(\emptyset)$ is doubly Cohen-Macaulay since Δ is superior. Recall from Eq. (2) that the constant term of the Hilbert series $F(\underline{H}_m{}^i(k[\Delta_{V-\tau}]),\lambda)$ of the local cohomology module $\underline{H}_m{}^i(k[\Delta_{V-\tau}])$ is equal to the dimension of $\tilde{H}_{i-1}(\Delta_{V-\tau};k))$ as a vector space over k. Thus, by Lemma (3.1), we have $\tilde{H}_i(\Delta_{V-\tau};k) = 0$ for every $i < d - \#(\tau)$.

We now proceed with the proof by induction on $\#(\tau)$. Let τ be a face of Δ with $\#(\tau) > 1$ and suppose that $\sigma \subsetneqq \tau$ for some $\emptyset \neq \sigma \in \mathcal{F}(\Delta)$. Let $x \in \sigma$ and set $\Delta' = \mathrm{link}_\Delta(\{x\})$. Then the face $\sigma - \{x\}$ of Δ' is a fundamental face of Δ' since $\mathrm{link}_{\Delta'}(\sigma'-\{x\}) = \mathrm{link}_\Delta(\sigma')$ if $\sigma' \subset \sigma$. On the other hand, the Cohen-Macaulay complex Δ' is superior. Thus, we can apply assumption of induction to Δ' and $\sigma - \{x\} \subsetneqq \tau - \{x\}$. Hence $\tilde{H}_{(d-1)-\{\#(\tau)-1\}-1}(\Delta'_{V'-(\tau-\{x\})};k)) = 0$, where V' is the vertex set of Δ', i.e., $\tilde{H}_{d-\#(\tau)-1}(\Delta'_{V'-(\tau-\{x\})};k)) = 0$. Then, thanks to Corollary (3.3), we have $\tilde{H}_{d-\#(\tau)-1}(\Delta_{V-\tau};k)) = 0$ as required. Q.E.D.

(3.7) LEMMA. If a non-empty subset W of V is not a face of Δ, then the reduced homology group $\tilde{H}_{d-\#(W)-1}(\Delta_{V-W};k)$ vanishes.

Proof. Since W is not a face of Δ, there exist $x, y \in W$ such that $\{x, y\} \notin \Delta$. Let $\rho = \deg x \in \mathbb{Z}^V$ and $\nu = \deg y \in \mathbb{Z}^V$. Then, there exists the exact sequence (cf. [H2, p.338])

$$0 \longrightarrow k[\mathrm{star}_\Delta(\{x\})](\rho) \oplus k[\mathrm{star}_\Delta(\{y\})](\nu)$$

$$\longrightarrow k[\Delta] \longrightarrow k[\Delta_{V-\{x,y\}}] \longrightarrow 0$$

as graded modules over A with the fine grading. Since both $\mathrm{star}_\Delta(\{x\})$ and $\mathrm{star}_\Delta(\{y\})$ are Cohen-Macaulay, the local cohomology group $\underline{H}_m{}^i(k[\Delta_{V-\{x,y\}}])$ vanishes for each $i < d - 1$. Hence, the similar technique as in the proof of Lemma (3.1) enables us to see $\underline{H}_m{}^{d-\#(W)}(k[\Delta_{V-W}]) = 0$. Thus, by Eq. (2), the reduced homology group $\tilde{H}_{d-\#(W)-1}(\Delta_{V-W};k)$ vanishes. Q.E.D.

We have established all the necessary preparations for the proof of Theorem (2.7), and we can finish our work as follows :

$$\mathrm{type}(k[\Delta]) = \sum_{W \subset V} \dim_k(\widetilde{H}_{d-\#(W)-1}(\Delta_{V-W};k))$$

(by Eq. (3))

$$= \sum_{\sigma \in \Delta} \dim_k(\widetilde{H}_{d-\#(\sigma)-1}(\Delta_{V-\sigma};k))$$

(by Lemma (3.7))

$$= \sum_{\sigma \in \Delta_f} \dim_k(\widetilde{H}_{d-\#(\sigma)-1}(\Delta_{V-\sigma};k))$$

(by Corollary (3.5))

$$\geq \sum_{\sigma \in \mathcal{F}(\Delta)} \dim_k(\widetilde{H}_{d-\#(\sigma)-1}(\Delta_{V-\sigma};k)) \qquad (5)$$

(since $\mathcal{F}(\Delta) \subset \Delta_f$)

$$= \sum_{\sigma \in \mathcal{F}(\Delta)} (-1)^{d-1-\#(\sigma)}\, \widetilde{\chi}(\mathrm{link}_\Delta(\sigma))$$

(by Lemma (3.4)).

Here Δ_f is the set of those faces τ of Δ such that $\tau \supset \sigma$ for some $\sigma \in \mathcal{F}(\Delta)$. On the other hand, Lemma (3.6) guarantees the equality in (5) when the Cohen-Macaulay complex Δ is superior.

§4. Möbius functions

Theorem (2.7) is a powerful tool for the explicit computation of Cohen-Macaulay types of certain Cohen-Macaulay partially ordered sets.

(4.1) Every partially ordered set (*poset* for short) to be studied is finite. A *chain* is a totally ordered set. The *length* of a chain C is $\ell(C) := \#(C) - 1$. A totally ordered subset in a poset P is also called a chain of P. The *rank* of a poset P is defined to be $\mathrm{rank}(P) := \max\{ \ell(C) ; C \text{ is a chain of } P \}$. When $x, y \in P$, we say that y *covers* x if $x < y$ and $x < z < y$ for no element $z \in P$. A chain $x_1 < x_2 < \ldots < x_s$ of P is called *saturated* if x_{i+1} covers x_i for each $1 \leq i < s$.

Given a poset P, we write $\Delta(P)$ for the set of chains of P. Then $\Delta(P)$ is a simplicial complex on the vertex set P. We say that $\Delta(P)$ is the *order complex* of P. Note that $\dim \Delta(P) = \mathrm{rank}(P)$. A poset P is called *pure* if the order complex $\Delta(P)$ is pure.

On the other hand, for a poset P, we define the poset $\hat{P}$ by $\hat{P} := P \cup \{ \hat{0}, \hat{1} \}$ such that $\hat{0} < x < \hat{1}$ for every $x \in P$.

When $x \leq y$ in a poset P, the *open interval* (x, y) (resp. *closed interval* $[x, y]$) of P is the induced subposet $\{ z \in P ; x < z < y \}$ (resp. $\{ z \in P ; x \leq z \leq y \}$) of P. In particular, $(x, x) = \emptyset$ and $[x, x] = \{ x \}$ for every $x \in P$.

(4.2) The *Möbius function* $\mu_P(x,y)$ of a poset P, where $x \leq y$ in P, is defined as follows:

(i) $\mu_P(x,x) = 1$ for each $x \in P$, and

(ii) $\mu_P(x,y) = - \sum_{x \leq z < y} \mu_P(x,z)$ for every $x \leq y$ in P.

One of the most important formula for us on Möbius functions is

$$\mu_{\hat{P}}(\hat{0},\hat{1}) = \tilde{\chi}(\Delta(P)). \qquad (6)$$

Here $\tilde{\chi}(\Delta(P))$ is the reduced Euler characteristic of the order complex $\Delta(P)$ of P. See, e.g., [Sta4, p.120] on the above Eq. (6). Moreover, let $x < y$ in P, and suppose that both $\hat{0} = x_0 < x_1 < \ldots < x_s = x$ and $y = y_0 < y_1 < \ldots < y_t = \hat{1}$ are saturated chains of P. Also, set $\sigma = \{ x_0 , \dots , x_s , y_0 , \dots , y_t \} \in \Delta(P)$. Then $\mathrm{link}_\Delta(\sigma)$ is just the order complex of the open interval (x , y) of P. Hence

$$\mu_{\hat{P}}(x,y) = \tilde{\chi}(\mathrm{link}_{\Delta(P)}(\sigma)) . \tag{7}$$

(4.3) A poset P is called *Cohen-Macaulay* (resp. *Gorenstein, doubly Cohen-Macaulay*) over a field k if the order complex $\Delta(P)$ of P is Cohen-Macaulay (resp. Gorenstein, doubly Cohen-Macaulay) over k. When $x < y$ in a Cohen-Macaulay (resp. Gorenstein, doubly Cohen-Macaulay) poset P, the open interval (x , y) is also Cohen-Macaulay (resp. Gorenstein, doubly Cohen-Macaulay).

Moreover, we say that a Cohen-Macaulay poset P is *superior* if the Cohen-Macaulay complex $\Delta(P)$ is superior.

(4.4) EXAMPLE. The *face poset* $P(\Delta)$ of a simplicial complex Δ is the poset which consists of all the non-empty faces of Δ, ordered by inclusion. Moreover, the *barycentric subdivision* $\mathrm{sd}(\Delta)$ of a simplicial complex Δ is defined to be the order complex of the face poset $P(\Delta)$ of Δ, i.e., $\mathrm{sd}(\Delta) = \Delta(P(\Delta))$. Note that the geometric realization of $\mathrm{sd}(\Delta)$ is homeomorphic to that of Δ. Thus, in particular, a simplicial complex Δ is Cohen-Macaulay (resp. doubly Cohen-Macaulay) if and only if $\mathrm{sd}(\Delta)$ is Cohen-Macaulay (resp. doubly Cohen-Macaulay). It is not difficult to show that a Cohen-Macaulay complex Δ is superior if and only if the barycentric subdivision $\mathrm{sd}(\Delta)$ of Δ is superior. We do not know whether the superior condition for Δ is topological or not.

(4.5) LEMMA. A cohen-Macaulay poset P is superior if and only if the following condition (#) is satisfied :

(#) If $(\hat{0} \leq)\ x < y\ (\leq \hat{1})$ in $\hat{P}$ with $\mu_{\hat{P}}(x,y) \neq 0$, then the open interval (x , y) of $\hat{P}$ is doubly Cohen-Macaulay.

Proof. The "only if" part follows from Eq. (7) and the fact (b) of the second Remark in (2.4). On the other hand, since each subcomplex $\mathrm{link}_{\Delta(P)}(\sigma)$ of $\Delta(P)$ is equal to the simplicial join of order complexes of open intervals of $P^\wedge$, the "if" part is guaranteed by (2.6). Q.E.D.

(4.6) LEMMA. Suppose that a Cohen-Macaulay poset P is superior and let (x, y) be an open interval of $P^\wedge$ with $\mu_{P^\wedge}(x,y) \neq 0$. Then $\mu_{P^\wedge}(x,z) \neq 0$ and $\mu_{P^\wedge}(z,y) \neq 0$ for every $z \in P$ with $x \leq z \leq y$.

(4.7) EXAMPLE. The Cohen-Macaulay poset P of Figure 3 is not doubly Cohen-Macaulay since the subposet $P - \{x\}$ is not Cohen-Macaulay. We have $\mu_{P^\wedge}(0^\wedge,y) = 0$, even though $\mu_{P^\wedge}(0^\wedge, 1^\wedge) \neq 0$.

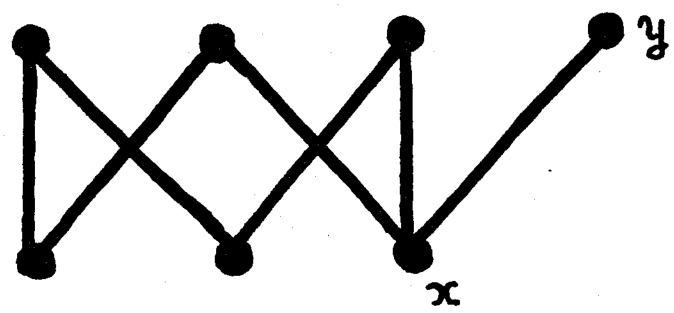

Figure 3

The poset-version of the second result (b) of Theorem (2.7) can be stated as follows :

(4.8) THEOREM. Suppose that a Cohen-Macaulay poset P of rank $r-1$ is superior and let $\mathrm{type}(k[\Delta(P)])$ be the Cohen-Macaulay type of the Stanley-Reisner ring $k[\Delta(P)]$ of the order complex $\Delta(P)$ of P over a field k. Then

$$\operatorname{type}(k[\Delta(P)]) = (-1)^{r-1} \sum_{\substack{\hat{0} = x_0 < x_1 < \ldots < x_s < x_{s+1} = \hat{1} \\ \mu_{\hat{P}}(x_{i-1},x_i) \neq 0 \text{ for each } 1 \le i \le s+1 \\ \mu_{\hat{P}}(x_{i-1},x_{i+1}) = 0 \text{ for each } 1 \le i \le s}} \prod_{1 \le i \le s} \mu_{\hat{P}}(x_{i-1},x_i) .$$

Here, the summation runs over every chain $\hat{0} = x_0 < x_1 < \ldots < x_s < x_{s+1} = \hat{1}$ of $\hat{P}$ with (i) $\mu_{\hat{P}}(x_{i-1},x_i) \neq 0$ for each $1 \le i \le s+1$ and (ii) $\mu_{\hat{P}}(x_{i-1},x_{i+1}) = 0$ for each $1 \le i \le s$.

Proof. Let $\hat{0} = x_0 < x_1 < \ldots < x_s < x_{s+1} = \hat{1}$ be a chain of $\hat{P}$ and set $\sigma = \{ x_1, x_2, \ldots, x_s \} \in \Delta(P)$. Then, by Lemma (4.6), the face σ of $\Delta(P)$ is fundamental if and only if $\mu_{\hat{P}}(x_{i-1},x_i) \neq 0$ for each $1 \le i \le s+1$ and $\mu_{\hat{P}}(x_{i-1},x_{i+1}) = 0$ for each $1 \le i \le s$. Moreover,

$$\begin{aligned}
&(-1)^{r-1-\#(\sigma)} \tilde{\chi}(\operatorname{link}_{\Delta(P)}(\sigma)) \\
&= (-1)^{r-1-\#(\sigma)} \tilde{\chi}(\Delta((x_0,x_1)) * \Delta((x_1,x_2)) * \ldots * \Delta((x_s,x_{s+1}))) \\
&= (-1)^{r-1} \tilde{\chi}(\Delta((x_0,x_1)))\tilde{\chi}(\Delta((x_1,x_2))) \ldots \tilde{\chi}(\Delta((x_s,x_{s+1}))) \\
&= (-1)^{r-1} \mu_{\hat{P}}(x_0,x_1)\mu_{\hat{P}}(x_1,x_2) \ldots \mu_{\hat{P}}(x_s,x_{s+1}) .
\end{aligned}$$

Hence, thanks to Theorem (2.7), we have the above formula for $\operatorname{type}(k[\Delta(P)])$ as desired. **Q.E.D.**

We now turn to the problem of finding interesting classes of Cohen-Macaulay posets which are superior.

(4.9) A *lattice* is a poset L for which every pair of elements α and β has a least upper bound (or "join") denoted by $\alpha \vee \beta$, and a greatest lower bound (or "meet") denoted by $\alpha \wedge \beta$. Thus, in particular, every (finite) lattice has a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. Every closed interval of a lattice is again a lattice.

An *atom* of a lattice L is an element which covers $\hat{0}$ in L. We say that a lattice L is *atomic* if every element is the join of atoms of L. A lattice L is called *complemented* if, for every $x \in L$, there exists $y \in L$ such that $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. Moreover, a lattice L is called *relatively complemented* if every closed interval of L is complemented.

We say that a lattice L is *semimodular* if the following condition is satisfied : If $x, y \in L$ both cover $x \wedge y$, then $x \vee y$ covers both x and y. Moreover, we say that a lattice L is *modular* if, for all elements x, y and z in L with $x \leq z$, we have $x \vee (y \wedge z) = (x \vee y) \wedge z$. Every modular lattice is semimodular. A semimodular lattice L is atomic if and only if L is relatively complemented. A *geometric lattice* is a lattice which is both relatively complemented and semimodular.

(4.10) LEMMA. (a) (e.g., [Bac1], [B-G-S]) Every semimodular lattice is Cohen-Macaulay.

(b) ([Bac2]) If $L = \hat{P}$ is a geometric lattice, then P is doubly Cohen-Macaulay.

(4.11) PROPOSITION. If $L = \hat{P}$ is a modular lattice, then the Cohen-Macaulay poset P is superior.

Proof. Suppose that $L = \hat{P}$ is a modular lattice with $\mu_{\hat{P}}(\hat{0},\hat{1}) \neq 0$. Then, by [Sta4, Corollary (3.9.5)], in L, the element $\hat{1}$ is the join of atoms. Hence, thanks to [Bir, Thoerem 6, p.88], the lattice L is complemented. Then, [Bir, Theorem 14, p.16] guarantees that L is relatively complemented. Hence, the lattice L is geometric. Thus, the poset P is doubly Cohen-Macaulay. Since every closed interval of a modular lattice is again modular, every Cohen-Macaulay poset P such that $L = \hat{P}$ is a modular lattice satisfies the condition (#) in Lemma (4.5) as desired. Q.E.D.

(4.12) EXAMPLE. Let $L = \hat{P}$ be the modular lattice of Figure 4. Then the fundamental faces of the order complex $\Delta(P)$ of P are $\{x\}$, $\{y, z\}$, $\{p, q\}$ and $\{p', q'\}$. Thus $\mathrm{type}(k[\Delta(P)]) = 10$.

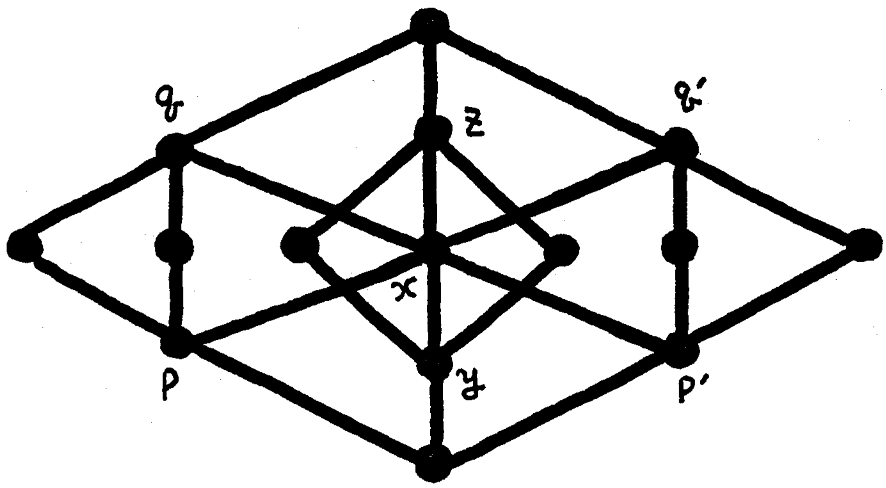

Figure 4

(4.13) EXAMPLE. Even though $L = \hat{P}$ is a semimodular lattice, the Cohen-Macaulay poset P is not necessarily superior. For instance, if P is the Cohen-Macaulay poset of Figure 3, then $L = \hat{P}$ is a semimodular lattice with $\mu_{\hat{P}}(\hat{0},\hat{1}) \neq 0$, however, the poset P is not doubly Cohen-Macaulay. The problem of finding a combinatorial formula to compute Cohen-Macaulay types of semimoular lattices would be of great interest. On the other hand, it follows from [Sta4, Corollary (3.9.5)] and [Bir, Theorem 6, p.88] that, when $L = \hat{P}$ is a semimodular lattice, the Cohen-Macaulay poset P is superior if and only if the following condition is satisfied : If $\hat{0} \leq x < y \leq \hat{1}$ in $\hat{P}$ and $\mu_{\hat{P}}(x,y) \neq 0$, then $\mu_{\hat{P}}(z,w) \neq 0$ for every $z, w \in \hat{P}$ with $x \leq z < w \leq y$ in $\hat{P}$.

(4.14) EXAMPLE. The *boolean algebra* of rank N is the poset (in fact, lattice) which consists of the subsets of the N-element set $\{1, 2, \ldots, N\}$, ordered by inclusion. Then $\mathcal{B}_N = \hat{P}$ is Gorenstein with $\mu_{\hat{P}}(\hat{0},\hat{1}) = (-1)^N$. A lattice L is called *meet-distributive* (e.g., [Sta4, p.156]) if every closed interval $[x, y]$ of L such that x is the meet of the elements of $[x, y]$ covered by y is a boolean algebra. Note that the lattice of Figure 3-41 in [Sta4, p.156] is meet-distributive, but not modular.

If we have $x < y$ in a meet-distributive lattice $L = P^\wedge$ with $\mu_{P^\wedge}(x,y) \neq 0$, then x is the meet of elements which are covered by y ([Sta4, Corollary (3.9.5)]), hence the closed interval $[x, y]$ of $P^\wedge$ is a boolean algebra. Thus, if $L = P^\wedge$ is a Cohen-Macaulay meet-distributive lattice, then the Cohen-Macaulay poset P is superior. Moreover, the Cohen-Macaulay type $\mathrm{type}(\Delta(P))$ of the Stanley-Reisner ring of the order complex $\Delta(P)$ of P is equal to the number of chains $0^\wedge = x_0 < x_1 < \ldots < x_s < x_{s+1} = 1^\wedge$ of $P^\wedge$ such that (i) each closed interval $[x_{i-1}, x_i]$ of $P^\wedge$ is a boolean algebra and (ii) the closed interval $[x_{i-1}, x_{i+1}]$ of $P^\wedge$ is not a boolean algebra for each $1 \leq i \leq s$. This enumerative result is a generalization of [H4, Therorem (2.10)].

(4.15) EXAMPLE. Some familiarity with Coxeter groups is assumed. See, e.g., [Bjö2]. Let (W,S) be a Coxeter group with the Bruhat ordering. If $J \subset S$, then define $\mathcal{D}_J := \{ w \in W ; ws < w$ if and only if $s \in J$, for every $s \in S \}$. Moreover, for $I \subset K \subset S$, set $\mathcal{D}_I^K := \cup_{I \subset J \subset K} \mathcal{D}_J$. It is known that the subposet $\mathcal{D}_I^K$ of W is Cohen-Macaulay if $\mathcal{D}_I^K$ is finite. Furthermore, the Cohen-Macaulay poset $\mathcal{D}_I^K$ is superior. It would be of interest to find the Cohen-Macaulay type of $\mathcal{D}_I^K$ when, e.g., W is a symmetric group S_n.

(4.16) EXAMPLE. Let G be a finite group and $\mathcal{L}(G)$ the lattice of subgroups of G ordered by inclusion. Then $\mathcal{L}(G)$ is Cohen-Macaulay if and only if G is supersolvable [Bjö1]. Moreover, $\mathcal{L}(G)$ is Gorenstein if and only if G is a cyclic group whose order is either square-free of a prime power [H1]. We do not know for which supersolvable groups G the lattice $\mathcal{L}(G)$ is superior. Also, it might be of interest to classify all the supersolvable groups G with, e.g., $\mathrm{type}(k[\Delta(\mathcal{L}(G))]) = 2$. For example, if G is the non-cyclic group with $\#(G) = 4$, then $\mathrm{type}(k[\Delta(\mathcal{L}(G))]) = 2$.

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