MOMENTS OF RANDOM FIELDS OVER A FAMILY OF ELLIPTIC CURVES, AND MODULAR FORMS

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0. Introduction.

It is a rather general (and well known, see [A], [AIK 3], [W]) phenomenon that physical models with natural geometric requirements (e.g. in statistical mechanics or gauge field theory) give rise to invariants, e.g. in terms of the moments (correlations). In this article the invariants are, in certain special cases, modular forms.

The starting point is to solve, for a quite general class of random fields Y, the $\overline{\partial}$ equation $\overline{\partial}X = Y$ on the torus $\mathbb{C}/(\tau\mathbb{Z}+\mathbb{Z})$. (The same idea was used to construct conformally invariant fields in [AIK 1-2].) Then the realisations are elliptic functions of homogeneity degree -1. The reason for this is that the Green's function for $\overline{\partial}$ on a torus is, up to a correction term, the Weierstrass ζ-function.

In more detail, the solution $X = X(\tau, z, \omega)$ transforms under matrices γ in the modular group SL(2,$\mathbb{Z}$) as

$$X(\gamma\cdot\tau, \psi_{\gamma\cdot\tau}(a), \omega) = j(\gamma,\tau)^{-1}X(\tau, \psi_\tau(a\cdot\gamma), \omega\cdot\gamma),$$

where τ is in the complex upper half-plane, $\psi_\tau(a)$ denotes the natural coordinates of the torus $\mathbb{C}/(\tau\mathbb{Z}+\mathbb{Z})$ parametrised by $a\in\mathbb{R}^2/\mathbb{Z}^2$, and $j(\gamma,\tau)$ is the usual cocycle $c_{21}\tau + c_{22}$, if $\gamma = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix}$. It is easily seen from this that the moments of X evaluated at $z = \psi_\tau(a)$, a rational points in $\mathbb{R}^2/\mathbb{Z}^2$, are modular forms with respect to τ. The essential thing is of course a set of points which are invariant under the modular group. Now, there are arithmetic obstructions to finding invariant sets of given cardinality and to construct general modular forms (of integer weight) the model must be modified. This is done by

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renormalising the moments, and it provides, at least in principle, a possibility to construct modular forms of arbitrary weight.

At present, the method to identify moments as modular forms is by old-fashioned and long calculations. Some examples are given at the end of the article. A more general treatment will appear in [AIK 4].

1. We shall denote by H the open upper half-plane

$$H = \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$$

on which the group $\mathrm{SL}(2,\mathbb{R})$ (2×2 matrices with unit determinant and real entries) acts to the left by

$$\tau \mapsto \gamma \cdot \tau \equiv \frac{c_{11}\tau + c_{12}}{c_{21}\tau + c_{22}}, \quad \gamma = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

The function

$$(\gamma, \tau) \mapsto j(\gamma, \tau) \equiv c_{21}\tau + c_{22}$$

is a cocycle of the above action:

$$j(\gamma_1 \cdot \gamma_2, \tau) = j(\gamma_1, \gamma_2 \cdot \tau) j(\gamma_2, \tau).$$

For each $\tau \in H$ there is an $\mathbb{R}$-linear map

$$\psi_\tau \colon a = (a', a'') \mapsto \tau a' + a''$$

of $\mathbb{R}^2$ into $\mathbb{C}$. For $\tau \in H$ we denote by L_τ the lattice $\tau\mathbb{Z} + \mathbb{Z}$, and by M_τ the complex torus $\mathbb{C}/L_\tau$. The underlying real structure $\mathbb{R}^2/\mathbb{Z}^2$ will be denoted by M. ψ_τ induces a map (also denoted ψ_τ) from M to M_τ.

There is a right action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^2$:

$$a = (a', a'') \mapsto a \cdot \gamma \equiv (a'c_{11} + a''c_{21}, a'c_{12} + a''c_{22}). \tag{1.1}$$

Denote by Γ the modular group $\mathrm{SL}(2,\mathbb{Z})$ (matrices in $\mathrm{SL}(2,\mathbb{R})$ with integer entries). Γ acts on the lattices L_τ through $\tau \mapsto \gamma \cdot \tau$ and it yields the multiplication:

$$L_{\gamma\cdot\tau} = j(\gamma, \tau)^{-1} L_\tau, \quad \gamma \in \Gamma. \tag{1.2}$$

Similarly, we have

$$\psi_{\gamma\cdot\tau}(a) = j(\gamma, \tau)^{-1} \psi_\tau(a \cdot \gamma), \quad \gamma \in \Gamma. \tag{1.3}$$

2. Let L be a lattice in $\mathbb{C}$. The Weierstrass ζ-function associated with L is defined as

$$\zeta(L, z) = \frac{1}{z} + \sum_{l \in L}{}' \left\{ \frac{1}{z - l} + \frac{1}{l} + \frac{z}{l^2} \right\},$$

for $z \in \mathbb{C}\backslash L$, where $\sum'$ indicates summation over non-zero lattice points. When $L = L_\tau$, we shall also use the notation $\zeta(\tau, z)$, and similarly for other functions depending on lattices. The Weierstrass $\wp$-function is defined as

$$\wp(L, z) = \frac{1}{z^2} + \sum_{l \in L}' \left\{ \frac{1}{(z-l)^2} - \frac{1}{l^2} \right\}.$$

ζ and $\wp$ are homogeneous of degree -1 and -2, respectively: For any non-zero complex number λ we have

$$\zeta(\lambda L, \lambda z) = \frac{1}{\lambda}\zeta(L, z), \quad \wp(\lambda L, \lambda z) = \frac{1}{\lambda^2}\wp(L, z). \tag{2.1}$$

In contrast to ζ, $\wp$ is doubly periodic, hence can be identified with a (meromorphic) function on the torus $\mathbb{C}/L$. The derivative of ζ is $-\wp$, and the deviation from periodicity is expressed by

$$\eta(L, l) \equiv \zeta(L, z + l) - \zeta(L, z), \quad l \in L,$$

independently of $z \in \mathbb{C} \backslash L$. Since the vectors in L spans $\mathbb{C}$ over $\mathbb{R}$, we can naturally extend $\eta(L, \cdot)$ to an $\mathbb{R}$-linear map $\mathbb{C} \to \mathbb{C}$.

We now introduce the function

$$\phi(L, z) = \zeta(L, z) - \eta(L, z). \tag{2.2}$$

$\phi(L, \cdot)$ is doubly periodic w.r.t. L (but of course not meromorphic), hence can be looked upon as a function on $\mathbb{C}/L$. Put

$$\tilde{\phi}(\tau, a) \equiv \phi(\tau, \psi_\tau(a)).$$

Then

$$\tilde{\phi}(\gamma \cdot \tau, a) = j(\gamma, \tau)\tilde{\phi}(\tau, a \cdot \gamma), \quad \gamma \in \Gamma. \tag{2.3}$$

which follows from the transformation properties (1.1-3) and the homogeneity of ϕ inherited from ζ. (2.3) shows that for certain rational a, $\tilde{\phi}(\cdot, a)$ is automorphic of weight 1 for certain subgroups of Γ. Furthermore, it is holomorphic in H for fixed a. (See §4 below.)

3. We are now introduced to probability theory. The underlying probability space will be the disjoint union:

$$\Omega = \bigsqcup_{n \geq 2} \Omega_n, \quad \Omega_n = \mathbb{C}^n \times M^n,$$

with the natural σ-algebra ($M = \mathbb{R}^2/\mathbb{Z}^2$ is the standard torus). Points in Ω_n, the n-particle space, will be written

$$\omega = (\alpha_1, \ldots, \alpha_n; x_1, \ldots, x_n).$$

(1.1) induces a right action of Γ on Ω:

$$\omega \cdot \gamma = (\alpha_1, \ldots, \alpha_n; x_1 \cdot \gamma, \ldots, x_n \cdot \gamma). \tag{3.1}$$

The total probability measure P is the sum of

$$P_n = \mu_n \otimes \nu_n, \text{ on } \Omega_n,\ n \geq 2,$$

where μ_n is a measure on $\mathbb{C}^n$ which is symmetric under permutation of coordinates and satisfies

$$\mu_n\{\sum_{i=1}^{n} \alpha_i \neq 0\} = 0, \tag{3.2}$$

and

$$\nu_n(dx_1 \cdots dx_n) = dx_1 \cdots dx_n$$

is the Haar (Lebesgue) measure on M^n. Hence the x_i are independent and translation invariant.

With (P, Ω) as above we define, for fixed $\tau \in H$, a random field $\{X(\tau, z)\}$ on M_τ by

$$X(\tau, z, \omega) = \sum_{i=1}^{n} \alpha_i \phi(\tau, z - \psi_\tau(x_i)), \quad \omega \in \Omega_n.$$

This is done so that (in the sense of Schwartz' distribution)

$$\frac{\partial X}{\partial \bar{z}} = \pi \sum_{i=1}^{n} \alpha_i \delta_{z_i}, \quad z_i = \psi_\tau(x_i).$$

For a *finite* set $F \subset M$, define,

$$\xi(\tau, F, \omega) = \prod_{a \in F} X(\tau, \psi_\tau(a), \omega).$$

Using (2.3) we get

$$\xi(\gamma \cdot \tau, F, \omega) = j(\gamma, \tau)^{|F|} \xi(\tau, F \cdot \gamma, \omega \cdot \gamma). \tag{3.3}$$

The singularity of $\phi(\tau, z)$ is like z^{-1} and hence the singularity of $\xi(\tau, F, \cdot)$ is dominated by

$$\prod_{a \in F} \sum_{i=1}^{n} (\text{dist}\,(a, x_i))^{-1},$$

which is integrable w.r.t. the Lebesgue measure on M^n. It then follows that

$$\xi(\tau, F) \in L^1(\Omega, P). \tag{3.4}$$

The action (3.1) of Γ clearly preserves P (the Jacobian is always 1), so

$$E[\xi(\gamma \cdot \tau, F)] = j(\gamma, \tau)^{|F|} E[\xi(\tau, F \cdot \gamma)], \quad \gamma \in \Gamma. \tag{3.5}$$

The expectation $E[\xi(\tau, F)]$ can be looked upon as a symmetric function of the points in F. The symmetric group of $|F|$ letters decomposes this function, in the same way as a tensor product representation is decomposed into irreducible components summing over partitions (see e.g. [Ma], Ch. 13, or [Sa]). In our case, due to the fact that the points α_i and x_i are independent, we can express the moment on the form

$$E[\xi(\tau, F)] = \sum_{\Delta \in \mathcal{P}(F)} E[Y(F, \overline{\Delta})] g(\tau, F, \Delta). \tag{3.6}$$

Here $\mathcal{P}(F)$ denotes the partitions of the set F, the random variable $Y(F, \overline{\Delta})$ comes from the αs, is independent of τ, and of the equivalence class of the partition (hence the notation $\overline{\Delta}$), whereas $g(\tau, F, \Delta)$ corresponds to the points x_i.

Instead of writing down a general more detailed expression, we look at two examples. The second order moment can be written

$$E[\xi(\tau, \{a_1, a_2\})] = E[\sum_i \alpha_i^2] \int_M \tilde{\phi}(\tau, a_1 - x)\tilde{\phi}(\tau, a_2 - x) dx \tag{3.7}$$

When F consists of four points, say $a_1, \ldots a_4$, we have

$$\begin{aligned}
E[\xi(\tau, \{a_1, \ldots a_4\})] &= E[\sum_i \alpha_i^4] \int_M \tilde{\phi}(\tau, a_1 - x) \cdots \tilde{\phi}(\tau, a_4 - x) dx \\
+E[\sum_{i \neq j} \alpha_i^2 \alpha_j^2] \Big\{ &\int_M \tilde{\phi}(\tau, a_1 - x)\tilde{\phi}(\tau, a_2 - x)\, dx \int_M \tilde{\phi}(\tau, a_3 - x)\tilde{\phi}(\tau, a_4 - x)\, dx \\
+ &\int_M \tilde{\phi}(\tau, a_1 - x)\tilde{\phi}(\tau, a_3 - x)\, dx \int_M \tilde{\phi}(\tau, a_2 - x)\tilde{\phi}(\tau, a_4 - x)\, dx \\
+ &\int_M \tilde{\phi}(\tau, a_1 - x)\tilde{\phi}(\tau, a_4 - x)\, dx \int_M \tilde{\phi}(\tau, a_2 - x)\tilde{\phi}(\tau, a_3 - x)\, dx \Big\} \\
\equiv E[\sum_i \alpha_i^4]\, I + &E[\sum_{i \neq j} \alpha_i^2 \alpha_j^2]\, II.
\end{aligned} \tag{3.8}$$

In principle, there should be other terms, corresponding to partitioning 4 into 1+3, 1+1+2 and 1+1+1+1, but $\int_M \tilde{\phi}(\tau, x)\, dx = 0$, so these terms vanish (using independence again). In general, if $\Delta = \{I_1, \ldots, I_r\}$, we may write $g(\tau, F, \Delta) = \prod_{j=1}^r g(\tau, I_j)$, and $\sum_{|\Delta|=r} g(\tau, F, \Delta)$ is a symmetric function of r variables. We can now state the following result.

Theorem. *Suppose that F is invariant under a subgroup Γ' of the modular group, and let $n = |F|$. Then $f(\tau) = E[\xi(\tau, F)]$ is automorphic of weight n under Γ':*

$$f(\gamma \cdot \tau) = j(\gamma, \tau)^n f(\tau), \quad \gamma \in \Gamma'. \tag{3.9}$$

(3.10) Remarks. 1. The theorem can be strengthened: each term

$$\sum_{|\Delta|=r} g(\tau, F, \Delta)$$

transforms as in (3.9). Furthermore, the moments are holomorphic w.r.t. $\tau \in H$. (See the explicit expression for ϕ in the next section.)
2. The basic invariant sets are subsets of the *rational* points (the points of finite order) : for some integer N, $F \subset \{(j/N, k/N); 0 \leq j, k \leq N-1\} \subset M$.

Summing up, we have a way to produce modular forms from point processes, provided that we can verify the appropriate conditions at the cusps. We will comment more on this issue in the next section.

4. In this section we shall calculate several moments explicitly. It is then rather straightforward to verify the cusp condition directly.

Example A. The second order moment is given by (3.8). We make the choice $a_1 = 0$, $a_2 = (1/2, 0)$. The group $\Gamma_0(2) = \Gamma_1(2)$ (notation as in [K]), where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\}, \quad \Gamma_1(N) = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\},$$

leaves these points invariant. In general $\Gamma_1(N)$ fixes the point $(0, 1/N)$ of order N and $\Gamma_0(N)$ fixes the points $\{(0, k/N);\ 0 \leq k \leq N-1\}$, which is the cyclic group generated by $(0, 1/N)$. Therefore $E[\xi(\cdot, \{(0, k/2);\ k = 0, 1\})] \in M_2(\Gamma_0(2))$ (modular forms of weight 2 w.r.t. $\Gamma_0(2)$), provided the appropriate conditions at the cusps hold. Direct calculations show (see e.g. [L]) that

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}}\phi(\tau, z) &= \frac{\operatorname{Im} z}{\operatorname{Im} \tau} - \frac{1}{2} - \sum_{n=1}^{\infty} \left(1 + \sum_{m=1}^{\infty} e^{2\pi \mathrm{i} n m \tau}\right) e^{2\pi \mathrm{i} n z} \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} e^{2\pi \mathrm{i} n m \tau} e^{-2\pi \mathrm{i} n z}, \quad 0 < \operatorname{Im} z < \operatorname{Im} \tau, \end{aligned} \tag{4.1}$$

with a similar formula for $0 > \operatorname{Im} z > -\operatorname{Im} \tau$. Writing $q = e^{2\pi \mathrm{i} \tau}$ we get the following expression for the integral appearing in the second moment (3.8):

$$\begin{aligned} &\int_M \tilde{\phi}(\tau, -x)\tilde{\phi}(\tau, (0, 1/2) - x)\, dx \\ =&(2\pi \mathrm{i})^2 \left\{ \int_0^1 (t-1)^2\, dt - 2\sum_{n=1}^{\infty} (-1)^n \frac{q^n}{(1-q^n)^2} \right\} \\ =& -\frac{\pi^2}{3}\left\{1 - 24\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (-1)^n m q^{nm}\right\}. \end{aligned} \tag{4.2}$$

The last term within braces may be written as $1 + 24\sum_{k=1}^{\infty}(\sum_{d|k, d \text{ odd}} d)q^k$. By differentiating the expression for ζ from which (4.1) originates, we obtain the corresponding formula for $\wp$. Then we find that (4.2) is proportional to $\wp(\tau, 1/2)$. We therefore obtain

$$E[\xi(\tau, \{(0, k/2);\ k = 0, 1\})] = -\frac{1}{2}E[\sum_i \alpha_i^2]\wp(\tau, 1/2) = -\frac{4}{3}E[\sum_i \alpha_i^2]E_2^{\vartheta}(\tau), \tag{4.3}$$

where the notation E_2^ϑ is the one used in Mumford's lectures [Mu,pp. 78-82].

Example B. We shall now study modular forms for the whole modular group, and we start by considering automorphy of weight 4. We let F be the finite points of order 2: $a_1 = 0$, $a_2 = (0, 1/2)$, $a_3 = (1/2, 0)$, and $a_4 = (1/2, 1/2)$. As in (4.3) above, one finds

$$
(4.4)\qquad
\begin{aligned}
E[\xi(\tau, \{(k/2, 0);\ k = 0, 1\})] &= -\frac{1}{2}E[\sum_i \alpha_i^2]\wp(\tau, \tau/2)\\
E[\xi(\tau, \{(k/2, k/2);\ k = 0, 1\})] &= -\frac{1}{2}E[\sum_i \alpha_i^2]\wp(\tau, (1+\tau)/2).
\end{aligned}
$$

Using the notation e_1, e_2, e_3 for the value of $\wp(\tau, \cdot)$ at the points $1/2$, $\tau/2$, $(1+\tau)/2$, we see that the term II in (3.8) equals $\frac{1}{4}((e_1)^2 + (e_2)^2 + (e_3)^2)$ (using translation invariance). This symmetric polynomial of $\wp(\tau, \cdot)$ evaluated at the non-zero points of order 2 is proportional to E_4, the Eisenstein series of weight 4. (See e.g. [K].)

Using (4.1), and passing to the limit $\tau \to i\infty$ under the integral sign (this can be justified) we get

$$
(4.5)\qquad (2\pi i)^{-4} I(\tau) \to \int_0^{1/2} (t - 1/2)^2 t^2\, dt + \int_{1/2}^1 (t - 1/2)^2 (t - 1)^2\, dt = 2 \cdot 2^{-5} B(3, 3),
$$

where B is the Beta function. Hence the limit of I at $i\infty$ exists and is equal to $\pi^4/30$. We now know that I is proportional to E_4. The final result is

$$
E[\xi(\tau, \{(k/2, l/2);\ k, l = 0, 1\})] = \frac{\pi^4}{30}\Big(E[\sum_i \alpha_i^4] + 5E[\sum_{i \neq j} \alpha_i^2 \alpha_j^2]\Big) E_4(\tau).
$$

To represent E_4 as a moment we used an invariant set of four points. There is no direct generalisation of this to get E_6 (of weight 6), and similarly E_{10} (of weight 10). To obtain an invariant set of eight points, we remove the origin from the points of order three: $8 = 3^2 - 1$. Similarly we can get modular forms of weight twelve writing $12 = 4^2 - 2^2$ or $= 3^2 + 2^2 - 1$. Then the value at infinity of the components of the moments are given by Beta functions of several variables, generalising (4.5).

It is clear that sofar, the method depends on which numbers one may write as a combination of squares.

Example C. This example provides a method to construct modular forms of other weights, e.g. three or six. The idea is to allow coincident points in the moments. Since $\tilde{\phi}(\tau, \cdot)$ is not square integrable, we need to modify our model somewhat. This is done by renormalisation. Here we shall only consider the renormalised second power.

On Ω_n, with $z = \psi_\tau(a)$, we have

$$
X(\tau, z)^2 = \sum_{i=1}^n \alpha_i^2 \tilde{\phi}(\tau, a - x_i)^2 + \sum_{1 \leq i \neq j \leq n} \alpha_i \alpha_j \tilde{\phi}(\tau, a - x_i)\tilde{\phi}(\tau, a - x_j),
$$

where the second, but not the first, term is integrable. The dominating singularity of $\tilde{\phi}(\tau,\cdot)^2$ is the same as that of $\wp$ at the same point. We can use $\wp$ and simply subtract the quadratic singularities. We define a random field $:X^2:$ by

$$\begin{aligned}
:X^2:(\tau,z) &\equiv \sum_{i=1}^{n} \alpha_i^2(\tilde{\phi}(\tau,a-x_i)^2 - \wp(\tau,z-\psi_\tau(x_i))) \\
&+ \sum_{1\le i\ne j\le n} \alpha_i\alpha_j\tilde{\phi}(\tau,a-x_i)\tilde{\phi}(\tau,a-x_j) \\
&= X(\tau,z)^2 - \sum_{i=1}^{n}\alpha_i^2\wp(\tau,z-\psi_\tau(x_i)),
\end{aligned}$$

and note that $\tau \mapsto :X^2:(\tau,\psi_\tau(a))$ is holomorphic in H. It is obviously translation invariant. Finally, the homogeneity of $\wp$, Eq. (2.1), shows that its expectation transforms as a modular form of weight 2. In this case the expectation is actually zero for all z. One can however use $:X^2:(\tau,z)$ to construct non-trivial modular forms. For instance, to get E_6 we form

$$E[\prod_{a\in F} :X^2:(\tau,\psi_\tau(a))],$$

where F is the set of the non-zero points of order two in M. To obtain modular forms of weight three we may consider $E[:X^2:(\tau,\psi_\tau(a))\ X(\tau,\psi_\tau(b))]$ for suitable a,b. It is clear that the combinatorics behind formula (3.6) have counterparts involving renormalised powers.

Example D. Finally we consider another way to adjust our original random field. For $N=2,3,\ldots$, we put

$$B_N := \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2.$$

Then B_N induces a measure preserving map of the standard torus M onto itself. Define a random field $Y = Y^{(N)}$ as follows:

$$Y(\tau,z) := \sum_{i=1}^{n} \alpha_i\phi(\tau,z-\psi_{N\tau}(B_N x_i)).$$

Then $\{Y(\tau,z)\}$ is a random field of weight 1 and level N.

References

[Ah] L.V. Ahlfors, *Complex analysis, 2nd edition*, McGraw-Hill, New York, 1966.

[AIK 1] S. Albeverio, K. Iwata and T. Kolsrud, *Conformally invariant random fields and Processes- Old and new*, Stochastic analysis and applications (Proc. of Lisbon Conference, September 1989) (A.B. Cruzeiro and J. C. Zambrini, eds.), Birkhäuser, New York, 1991.

[AIK 2] S. Albeverio, K. Iwata and T. Kolsrud, *Conformally invariant and reflection positive random fields in two dimensions*, Stochastic analysis (E. Mayer-Wolf, E. Merzbach and A. Shwartz, eds.), Academic Press, New York, 1991.

[AIK 3] S. Albeverio, K. Iwata and T. Kolsrud, *Random parallel transport on surfaces of finite type, and relations to homotopy*, Proc of Siena Conference.

[AIK 4] S. Albeverio, K. Iwata and T. Kolsrud, *Modular forms, renormalisation and random fields (Preliminary title).*

[At] M.F. Atiyah, *The geometry and physics of knots, Lincei lectures*, Cambridge Univ. Press, Cambridge, 1991.

[FK] H.M. Farkas and I. Kra, *Riemann surfaces*, Springer-Verlag, New York, 1980.

[K] N. Koblitz, *Introduction to elliptic curves and modular forms*, New York, 1984.

[L] S. Lang, *Elliptic functionss*, Addison-Wesley, New York, 1973.

[Ma] G.W. Mackey, *Unitary group representations in physics, probability, and number theory*, Benjamin-Cummings, 1978.

[Mu] D. Mumford, *Tata lectures on theta.* I, Birkhäuser, 1983.

[S] B.E. Sagan, *The symmetric group*, Wadsworth, 1991.

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