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Neumann problem for singular degenerate parabolic equations

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In the memory of Yoshi's first daughter

Abstract. We prove a comparison theorem for viscosity solutions of singular degenerate parabolic equations with the Neumann boundary condition on a domain not necessarily convex. Our result applies to various level set equations including the Neumann problem for the mean curvature flow equations where every level set of solutions moves by its mean curvature and perpendicularly intersects the boundary of the domain.

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1. Introduction

This paper continues our investigation [S, GS] on the Neumann problem for singular degenerate parabolic equations. A typical example is

$$u_t - |\nabla u| \operatorname{div} (\nabla u/|\nabla u|) = 0 \quad \text{in} \quad (0,\infty) \times \Omega, \tag{1.1}$$
$$\partial u/\partial \nu = 0 \quad \text{on} \quad (0,\infty) \times \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain in $\mathbf{R}^n$ and $\partial/\partial\nu$ denotes the outer normal derivative on $\partial\Omega$. As explained in [CGG, ES] the first equation asserts each level set of u is evolving by its mean curvature in Ω at least formally. The boundary condition formally says that each level set of u intersects perpendicularly with the boundary $\partial\Omega$. We often call (1.1) the level set equation of the motion by mean curvature.

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The level set equation (1.1) with $\Omega = \mathbf{R}^n$ provides a new notion of generalized mean curvature motion of hypersurface in $\mathbf{R}^n$ as studied by [CGG], [ES] and others. The generalized motion is given as the zero level set of u as defined by [CGG] and [ES] (see also [CGG2] for corrections). One of key ingradients of their theory is to establish the comparison principle for viscosity solutions of (1.1) (see [CIL] for the theory of viscosity solutions).

In [S] the second author adapted the level set approach by [CGG] and [ES] to geometric evolutions of hypersurfaces (in Ω) intersecting with $\partial\Omega$ perpendicularly. He established the comparison principle for the Neumann problem a typical example of which is (1.1)-(1.2). However, his method needs the convexity of Ω.

Our goal in this paper is to extend the comparison principle even if Ω is non-convex. Our theory applies to a class of equations (cf. [CGG]) including (1.1). The basic strategy is the same as [S] where Crandall-Ishii's lemma[CIL, CI] is applied. However, we choose test functions in more clever way to handle nonconvex Ω.

There are extensive articles for generalized evolution without boundary conditions. We refer to [AAG], [ES2], [BSS], [OhS] and references cited there for a history of the level approach and its recent development. For the theory of viscosity solutions for the Neumann problem we refer to [Li], [S] and references therein.

Since existence of global solutions with continuous initial data is proved in [S] for geometric equations with (1.2) even for nonconvex Ω, our comparison result enables us to extend the level set approach for constructing generalized evolutions with the Neumann boundary condition (cf. [CGG]). In fact unique existence of global generalized evolutions for interface equations (cf. [GG]) with right angle boundary condition can be proved for general domain Ω. We do not state this result since it is already stated in [S, §4] for convex domain Ω.

If the initial hypersurface is the graph of a smooth function on a bounded domain D in $\mathbf{R}^n$, Huisken [Hu] constructed a global smooth evolution of hypersurfaces intersecting perpendicularly with $\partial\Omega$ and moving by mean curvature in $\Omega = D \times \mathbf{R}$.

Although we assumed that Ω is bounded in our comparison theorem, our theory is basically applicable to this cylindrical domain with technical modifications. This problem will be treated in the second author's forthcoming paper. We note that the motion by mean curvature with right contact angle at $\partial\Omega$ arises as a singular limit of the Allen-Cahn type reaction-diffusion equation with the Neumann condition [RSK].

The results in this paper has been announced in [GS]. However, it seems that the choice of test function appeared there does not achieve their comparison result. We choose test functions different from those in [GS].

2. Comparison principle

Let Ω be a bounded domain in $\mathbf{R}^n$ and T be a positive number. Let $\nu(x)$ be outer unit normal at $x \in \partial\Omega$. We consider the Neumann boundary value problem of form

$$u_t + F(t, \nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T) \times \Omega \tag{2.1a}$$

$$\partial u / \partial \nu = 0 \qquad \text{on} \quad S = (0,T) \times \partial\Omega, \tag{2.1b}$$

$u_t = \partial u/\partial t$, $\nabla u = \nabla_x u$; $\nabla^2 u (= \nabla_x^2 u)$ denotes the Hessian of u in the space variable x. We list assumptions on F.

(F1) $F : (0,T) \times (\mathbf{R}^n \setminus \{0\}) \times \mathbf{S}^n \to \mathbf{R}$ is continuous, where $\mathbf{S}^n$ denotes the space of $n \times n$ real matrices equipped with usual ordering.

(F2) F is degenerate elliptic, *i.e.* $F(t,p,X+Y) \leqslant F(t,p,X)$ for all $Y \geqslant 0, t \in (0,T)$.

(F3) $-\infty < F_*(t,0,O) = F^*(t,0,O) < \infty$, $t \in (0,T)$, where F_* and F^* are, respectively, the lower and upper semicontinuous envelope of F on $(0,T) \times \mathbf{R}^n \times \mathbf{S}^n$ (see [GGIS]).

THEOREM 2.1. *Let Ω be a bounded domain in $\mathbf{R}^n$ with C^2 boundary $\partial\Omega$. Suppose that F satisfies (F1)-(F3). Let u and v be, respectively, viscosity sub-and supersolutions of (2.1a)-(2.1b). If $u^*(0,x) \leqslant v_*(0,x)$ on $\overline{\Omega}$, then $u^* \leqslant v_*$ on $(0,T) \times \overline{\Omega}$.*

REMARK 2.2: As usual we may assume that u and v are bounded on $\overline{\Omega}$ (cf. [I], [S]) for the proof. We may assume that u and v are, respectively, upper and lower semicontinuous on $\overline{Q}$. We argue by contradiction. Suppose that $u > v$ at some point of $\overline{Q}$. Let (s,z) denote a maximizer of $u - v$ on $\overline{Q}$. By the initial condition we observe $s > 0$.

We shall find a good parabolic super 2-jet of $w(t,x,y) = u(t,x) - v(t,y)$ at some point where $w > 0$. We consider maximum of $\Phi(t,x,y) = \omega(t,x,y) - \Psi(t,x,y)$ with a suitable choice of Ψ. If z is an interior point of Ω, we may take $|x-y|^4/\varepsilon + \gamma/(T-t) + a|x-y|^2 + b|t-s|^2$ with some constant $\varepsilon, \gamma, a, b > 0$. In this case we easily lead a contradiction by applying Crandall-Ishii's lemma as in ([GGIS], [S], [cf. Section 3]). Even if $z \in \partial\Omega$, this choice of Ψ is good for our purpose provided that Ω is convex. The main reason is that

$$\langle \nu(x), \nabla_x |x-y|^4 \rangle \geqslant 0, \quad x \in \partial\Omega, \quad y \in \overline{\Omega}$$

holds, where $\langle , \rangle$ denotes the inner product in $\mathbf{R}^n$. Unfortunately, if Ω is not convex, the inequality is no longer valid, so we must replace Ψ by another function. The crucial step of our proof is its choice of test function as presented in the next section.

3. Proof of Comparison theorem

We will state several propositions and lemmas to be needed to prove Theorem 2.1. Assume that u and v are, respectively, upper semicontinuous and lower semicontinuous on $\overline{Q}$. Then the maximum of $u-v$ is attained on $\overline{Q}$, *i.e.*,

$$\sigma = \max_{\overline{Q}}(u-v) = (u-v)(s,z). \tag{3.1}$$

For $\varepsilon, \delta, \gamma, a, b > 0$ we set

$$\begin{aligned}
&\Phi(t,x,y) = w(t,x,y) - \Psi(t,x,y), w(t,x,y) = u(t,x) - v(t,y)\\
&\Psi(t,x,y) = \frac{\Xi(x-y)}{\varepsilon} + B(t,x,y) + S(t,x,y)\\
&B(t,x,y) = \delta(\varphi(x)+\varphi(y)+2\beta) + \frac{\gamma}{T-t}\\
&S(t,x,y) = a|x-z|^2 + b|s-t|^2.
\end{aligned} \tag{3.2}$$

The function B plays the role of a barrier for boundary and $t=T$. At present the text function Ξ is only assumed to be a C^2 function such that

$$c_1|x|^4 \leqslant \Xi(x) \leqslant c_2|x|^4. \tag{3.3}$$

The function $\varphi(x) \in C^2(\overline{\Omega})$ is taken so that $\varphi < 0$ in Ω, $\varphi = 0$ on $\partial\Omega$, $\nu(x) = \nabla\varphi(x)/|\nabla\varphi(x)|$ for all $x \in \partial\Omega$, $|\varphi(x)| \leqslant \beta$ for some $\beta > 0$ independent of $x \in \overline{\Omega}$, $|\nabla\varphi(x)| \geqslant 1$ for all $x \in \partial\Omega$.

PROPOSITION 3.1. *Suppose that u and v be, respectively, upper semicontinuous and lower semicontinuous. Assume that*

$$\sigma = \max_{\overline{Q}}(u-v) > 0. \tag{3.4}$$

Then there are positive constants γ_0, δ_0 such that

$$\sup_{\overline{U}}\Phi(t,x,y) > \sigma/2$$

holds for all $0<\delta<\delta_0$, $0<\gamma<\gamma_0, \varepsilon>0, a>0, b>0$, where $U = (0,T)\times\Omega\times\Omega$.

Proof: Since $\overline{\Omega}$ is compact, $u-v$ is bounded from above. This implies $\sigma<\infty$. We see

$$\sup_{\overline{Q}}\Phi(t,x,x) \geq \Phi(s,z,z) = \sigma - \gamma/(T-s) - \delta(2\varphi(z)+2\beta).$$

Since $\sup_{\overline{U}} \Phi(t,x,y) \geqslant \sup_{\overline{Q}} \Phi(t,x,x)$, we observe $\sup_{\overline{U}} \Phi(t,x,y) > \sigma/2$ if δ and γ is sufficiently small.

PROPOSITION 3.2. *Let u, v, δ_0, γ_0 be as in Proposition 3.1. Suppose that w is upper semicontinuous in $\overline{U}$.*
(i) Φ attains a maximum over $\overline{U}$ at $(\hat{t},\hat{x},\hat{y}) \in \overline{U}$ with $\hat{t} < T$.
(ii) $|\hat{x}-\hat{y}|$ is bounded as a function of $0<\varepsilon<1, 0<\delta<\delta_0, 0<\gamma<\gamma_0, a>0$ and $b>0$.
(iii) $|\hat{x}-\hat{y}|$ tends to zero as $\varepsilon \to 0$; the convergence is uniform in $0<\delta<\delta_0, 0<\gamma<\gamma_0$, $a>0$ and $b>0$.
(iv) $\Xi(\hat{x}-\hat{y})/\varepsilon$ tends to zero as $\varepsilon \to 0$; the convergence is uniform in $0<\delta<\delta_0, 0<\gamma<\gamma_0, a>0$ and $b>0$.
(v) $\hat{x} \to z, \hat{y} \to z$ and $\hat{t} \to s$ as $\delta, \gamma, \varepsilon \to 0$.

Proof: (i) This is obvious.
(ii) From (3.4) it follows $\Phi(\hat{t},\hat{x},\hat{y}) > 0$ for $0<\delta<\delta_0, 0<\gamma<\gamma_0, \varepsilon>0, a>0$ and $b>0$. By the property (3.3) of Ξ this yields

$$w(\hat{t},\hat{x},\hat{y}) \geqslant \frac{1}{\varepsilon}\Xi(\hat{x}-\hat{y}) \geqslant \frac{c_1}{\varepsilon}|\hat{x}-\hat{y}|^4. \tag{3.6}$$

The property (ii) now follows since $w(\hat{t},\hat{x},\hat{y}) \leqslant M$ for some $M>0$ implies

$$M \geqslant |\hat{x}-\hat{y}|^4/\varepsilon. \tag{3.7}$$

(iii) From (3.7) it follows (iii) as $\varepsilon \to 0$.
(iv) By (3.6) and (3.7) we observe that

$$\Xi(\hat{x}(\varepsilon)-\hat{y}(\varepsilon))/\varepsilon \to \xi \quad \text{as} \quad \varepsilon \to 0 \tag{3.8}$$

for some non-negative number ξ if we take some subsequence. From (ii) and boundedness of Ω it follows that

$$\hat{t}(\varepsilon) \to \bar{t}, \;\; \hat{x}(\varepsilon), \hat{y}(\varepsilon) \to \bar{z} \quad \text{as} \quad \varepsilon \to 0 \tag{3.9}$$

for some $\bar{t} \in [0,T], \bar{z} \in \overline{\Omega}$ if we take a subsequence $\varepsilon = \varepsilon_j \to 0$. By the definition of the point $(\hat{t},\hat{x},\hat{y})$ we have $\Phi(t,x,y) \leqslant \Phi(\hat{t_j},\hat{x_j},\hat{y_j})$, where $\hat{t_j} = \hat{t}(\varepsilon_j)$ and so on. Pluging $t=\bar{t}, x=y=\bar{z}$ in this inequality, we obtain

$$u(\bar{t},\bar{z}) - v(\bar{t},\bar{z}) - \frac{\gamma}{T-\bar{t}} \le u(\hat{t_j},\hat{x_j}) - v(\hat{t_j},\hat{y_j}) - \frac{1}{\varepsilon}\Xi(\hat{x_j}-\hat{y_j}) - \frac{\gamma}{T-\hat{t_j}}. \tag{3.10}$$

By (3.8) letting $\varepsilon_j \to 0$ in (3.10) yields

$$u(\bar{t},\bar{z}) - v(\bar{t},\bar{z}) - \frac{\gamma}{T-\bar{t}} \le \varlimsup_{\varepsilon_j\to 0}(u(\hat{t_j},\hat{x_j}) - v(\hat{t_j},\hat{y_j}) - \frac{\gamma}{T-\hat{t_j}}) - \xi.$$

From (3.9) it follows $\xi \leqslant 0$. Since the limit in (3.8) is independent of the choice of subsequence, the convergence (3.8) now yields (iv).
(v) From (iv) it is clear that $\hat{x},\hat{y} \to z$ and $\hat{t} \to s$ as $\delta,\gamma,\varepsilon \to 0$.

Construction of Ξ. Above two properties are standard (cf. [GGIS]). We have given a proof for completeness. We first state a key lemma to construct test function Ξ.

LEMMA 3.3. *Let Ω be a domain in $\mathbf{R}^n$ and let p be a point on $\partial\Omega$. Suppose that $\partial\Omega$ is C^2 near p. Then there is a continuous function $\phi = \phi_p$ on $\mathbf{R}^n$ such that*
(i) ϕ is positively homogeneous of degree one.
(ii) ϕ is smooth except the origin.
(iii) $\inf\{\phi(x); |x| = 1\} > 0$.
(iv) $\langle \nu(x), \nabla\phi(x-y)\rangle \geqslant 0$ for $x \in B_R(p)\cap\partial\Omega, y \in B_R(p)\cap\overline{\Omega}$ for small $R > 0$.

LEMMA 3.4. *Suppose that N is a cone such that $N = \{(x',x_n) \in \mathbf{R}^n; |x'| \leqslant cx_n\}$ with $c > 0$. Then there is a continuous function ϕ on $\mathbf{R}^n$ satisfying (i)-(iii) in Lemma 3.3 such that*

$$\partial\phi/\partial x_n \leqslant -c_0 \tag{3.11}$$

with some $c_0 > 0$ outside the interior of N.

Proof of Lemma 3.4: We set

$$L = \{(x',x_n); |x'|^2/c^2 + (x_n-2)^2/5 \le 1\}.$$

Let ϕ be the Minkowski function of L, *i.e.*, $\phi(x) = \inf\{\alpha; x \in \alpha L\}$. Since ∂L is smooth and the origin is an interior point of L, ϕ is smooth outside the origin and $\inf_{|x|=1}\phi > 0$.(Of course ϕ is continuous in $\mathbf{R}^n$.) It is not difficult to see that ϕ satisfies

$$|x'|^2/c^2 + (x_n - 2\phi)^2/5 = \phi^2.$$

Thus the function ϕ is

$$\phi = -2x_n + \sqrt{5c^2x_n^2 + 5|x'|^2}/c$$

which satisfies the properties of Lemma 3.3 (i)-(iii). A simple calculation shows (3.11) with $c_0 = (4-\sqrt{10})/2 > 0$ outside the interior of N. ∎

Proof of Lemma 3.3: We may assume $\nu(p) = (0, \cdots, 0, -1)$ by a rotation of coordinates. Since $\partial\Omega$ is Lipschitz near $p \in \partial\Omega$, there is $c > 0$ such that

$$(y + N) \cap B_R(p) \setminus \{y\} \subset \Omega \quad \text{for} \quad y \in \overline{\Omega}$$

with $N = \{(x', x_n) | x'| \leqslant c x_n\}$ provided that R is taken sufficiently small. We shall later take R smaller. We take ϕ as in Lemma 3.4 so that it satisfies (i)-(iii). The property (3.11) is rewritten as

$$\langle \nu(p), \nabla\phi(x - y)\rangle \geqslant c_0, \quad x \in \partial\Omega \cap B_R(p), \quad y \in \overline{\Omega} \cap B_R(p)$$

since x is outside the interior of $y + N$. Taking R smaller we may assume

$$|\nu(x) - \nu(p)| \leqslant c_0 / \sup_{x \neq 0} |\nabla\phi| \quad \text{for} \quad x \in \partial\Omega \cap B_R(p).$$

We thus obtain (iv) since

$$\begin{aligned}\langle \nu(x), \nabla\phi(x - y)\rangle &= \langle \nu(p), \nabla\phi(x - y)\rangle + \langle \nu(x) - \nu(p), \nabla\phi(x - y)\rangle,\\ &\geqslant c_0 - c_0 \geqslant 0. \blacksquare\end{aligned}$$

PROPOSITION 3.5. *Let Ω be a domain in $\mathbf{R}^n$ and let p be a point on $\partial\Omega$. Suppose that $\partial\Omega$ is C^2 near p. For $\phi = \phi_p$ in Lemma 3.3 set $\psi = \phi^4$. Then*
(i) ψ is a C^2 function.
(ii) $c_1'|x|^4 \leqslant \psi(x) \leqslant c_2'|x|^4$ with some positive constants c_1', c_2'.
(iii) $\langle \nu(x), \nabla\psi(x - y)\rangle \geqslant 0$ for $x \in B_R(p) \cap \partial\Omega, y \in B_R(p) \cap \overline{\Omega}$ for small $R > 0$.

Proof: (i) Since ϕ is positively homogeneous of degree one, so ψ is C^2 at the origin.
(ii) This follows from Lemma 3.3 (i)(iii).
(iii) This follows from Lemma 3.3 (iv). $\blacksquare$

We next define a C^2 function ρ_R for each $R > 0$ in Proposition 3.5 such that $0 \leqslant \rho_R \leqslant 1, \rho_R' \geqslant 0$ and

$$\rho_R(\zeta) = 1 \quad \text{for} \quad \zeta \geqslant 2R, \ \rho_R(\zeta) = 0 \quad \text{for} \quad 0 \leqslant \zeta \leqslant R. \tag{3.12}$$

For $p \in \partial\Omega$ we set $\psi = \phi_p^4$ and define $\Xi \in C^2(\overline{\Omega} \times \overline{\Omega})$ by

$$\Xi(x, y) = \rho_R(|x - y|)|x - y|^4 + (1 - \rho_R(|x - y|))\psi(x - y). \tag{3.13}$$

PROPOSITION 3.6. *(i) There are positive constants $c_1'', c_2'' > 0$ such that*

$$c_1''|x|^4 \leqslant \Xi(x) \leqslant c_2''|x|^4.$$

(ii) $$\langle \nu(x), \nabla\Xi(x-y)\rangle \geqslant 0, x \in B_R(p)\cap\partial\Omega, y \in B_R(p)\cap\overline{\Omega}$$

for sufficiently small $R > 0$.

This follows from Proposition 3.5. We state a version of localized parabolic Crandall-Ishii's lemma [CI,] which will be used in the proof of Theorem 2.1. The version presented below is interpreted as a localized version of [GGIS, Lemma 2.10] which can be easily reduced to usual parabolic Crandall-Ishii's lemma ([GGIS, Lemma][CI,]).

LEMMA 3.7. *Suppose that* u_i *is an upper semicontinuous viscosity solution of*

$$\begin{array}{lll} u_t + F_i(t, \nabla u, \nabla^2 u) = 0 & in & (0,T)\times\Omega_i \\ \partial u/\partial\nu = 0 & on & (0,T)\times\partial\Omega_i \end{array}$$

in a neighborhood of $(r, q_i) \in (0,T)\times\overline{\Omega}_i$, *where* Ω_i *is an open set in* $\mathbf{R}^{Ni}$ *and* $F_i : (0,T)\times\mathbf{R}^{Ni}\times\mathbf{S}^{Ni} \to \mathbf{R}\cup\{\pm\infty\}$ *is lower semicontinuous for* $i = 1,\cdots,k$. *Let* $\Psi(t, x_1, \cdots, x_k)$ *be a function on a neighborhood of* $(r,q) = (r, q_1, \cdots, q_k) \in (0,T)\times\mathbf{R}^{Ni}\times\cdots\times\mathbf{R}^{N_k}$ *such that* Ψ *is once continuously differentiable in* t *and twice continuously differentiable in* $x = (x_1, \cdots, x_k)$. *Suppose that*

$$\Phi(t, x_1, \cdots, x_k) \equiv \sum_{i=1}^{k} u:(t, x_i) - \Psi(t, x_1, \cdots, x_k) \leqslant \Phi(r, q, \cdots, q_k).$$

Suppose that

$$\langle \Psi_{xi}(r,q), \nu(q_i)\rangle > 0 \quad if \quad q_i \in \partial\Omega_i. \tag{3.14}$$

Then for each $\lambda > 0$ *there exists* $X_i \in \mathbf{S}^{Ni}$ *such that*

$$\Psi_t(r,q) + \sum_{i=1}^{k} F_i(\Psi_{x_i}(r,q), X) \leqslant 0$$

and

$$-(\frac{1}{\lambda} + |A|)I \leqslant \begin{pmatrix} X_1 & & O \\ & \ddots & \\ O & & X_k \end{pmatrix} \leqslant A + \lambda A^2,$$

where $A = \nabla_x^2 \Psi(r,q)$ and $|A|$ denotes the operator norm of A.

Proof of Theorem 2.1: We may assume that (F1)(F3) holds including $t = T$ by taking T smaller. We argue by contradiction. By Remark 2.2 we may assume

$$\sigma = \max_{\overline{Q}}(u - v) > 0$$

and the maximum is attained at some point $(s,z) \in (0,T] \times \overline{\Omega}$. Comparison result [CGG] implies that

$$\sigma = (u-v)(s,z) \quad \text{for some} \quad (s,z) \in (0,T] \times \partial\Omega.$$

We take Ξ with $p = z$ in (3.13). By Remark 2.2 and Proposition 3.6(i) we may use Proposition 3.1-3.2. By Proposition 3.2 (i) Φ attains a maximum over $\overline{U}$ at $(\widehat{t},\widehat{x},\widehat{y})(\in (0,T) \times \overline{\Omega} \times \overline{\Omega})$ close to (s,z,z) for small $\varepsilon, \delta, \gamma$. In other words

$$\Phi(t,x,y) = \omega(t,x,y) - \Psi(t,x,y) \leqslant \Phi(\widehat{t},\widehat{s},\widehat{y}) \quad \text{in} \quad (0,T] \times \overline{\Omega} \times \overline{\Omega}.$$

Let R be taken as in Proposition 3.6 (ii). We may assume that $\widehat{x}, \widehat{y} \in B_R(z)$ for sufficiently small $\varepsilon, \delta, \gamma$, say $\varepsilon < \varepsilon_0', \delta < \delta_0', \gamma < \gamma_0'$, by Proposition 3.2 (v). If $\widehat{x} \in \partial\Omega$ we have, by Proposition 3.6(ii),

$$\langle \nu(\widehat{x}), \nabla\Xi(\widehat{x} - \widehat{y})\rangle \geqslant 0 \tag{3.15}$$

for $\widehat{x} \in B_R(z) \cap \partial\Omega, \widehat{y} \in B_R(z) \cap \overline{\Omega}$. If $a < \delta/2R$, from (3.15) and the property of φ it follows

$$\begin{aligned} &\langle \nu(\widehat{x}), \nabla\Psi(\widehat{x}-\widehat{y})\rangle = \langle \nu(\widehat{x}), \frac{1}{\varepsilon}\nabla\psi(\widehat{x}-\widehat{y})\rangle + \delta + 2a\langle \nu(\widehat{x}), \widehat{x} - z\rangle \\ &\geqslant \delta - 2a|x - z| \geqslant \delta - 2aR > 0. \end{aligned} \tag{3.16}$$

By (3.16) the function Ψ satisfies the assumption (3.14) in Lemma 3.7, so we can apply Lemma 3.7 with $k = 2, u_1 = u, u_2 = -v, s = \widehat{t}, z = (\widehat{x},\widehat{y}), \Omega_i = \overline{\Omega}$. We thus conclude that for each $\lambda > 0$ there are $X, Y \in \mathrm{S}^n$ such that

$$\widehat{\Psi}_t + F_*(\widehat{t}, \widehat{\Psi}_x, X) - F^*(\widehat{t}, -\widehat{\Psi}_y, -Y) \leqslant 0 \tag{3.17}$$

and

$$-(\frac{1}{\lambda} + |A|)I \leqslant \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leqslant A + \lambda A^2, A \geqslant \nabla^2\Psi(\widehat{t},\widehat{x},\widehat{y}) \tag{3.18}$$

where $\widehat{\Psi}_t = \partial_t \Psi(\widehat{t}, \widehat{x}, \widehat{y})$, $\widehat{\Psi}_x = \nabla_x \Psi(\widehat{t}, \widehat{x}, \widehat{y})$, etc. By calculating $\widehat{\Psi}_t$ we observe that

$$\widehat{\Psi}_t = \frac{\gamma}{(T-\widehat{t})^2} + 2b(\widehat{t} - s) \geqslant \frac{\gamma}{T^2} - 2bT. \tag{3.19}$$

By $\widehat{\Psi}_x = -\widehat{\Psi}_y$ we see that (3.17) yields

$$0 \geqslant \gamma T^{-2} - 2bT + F_*(\widehat{t}, \widehat{\Psi}_x, X) - F^*(\widehat{t}, \widehat{\Psi}_x, -Y). \tag{3.20}$$

We next take a special A. Calculating $\nabla^2 \Psi$ yields

$$\begin{pmatrix} \widehat{\Psi}_{xx} & \widehat{\Psi}_{xy} \\ \widehat{\Psi}_{yx} & \widehat{\Psi}_{yy} \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} \nabla^2\psi(\hat{x}-\hat{y}) & -\nabla^2\psi(\hat{x}-\hat{y}) \\ -\nabla^2\psi(\hat{x}-\hat{y}) & \nabla^2\psi(\hat{x}-\hat{y}) \end{pmatrix} + \delta \begin{pmatrix} \nabla_x^2\varphi(\hat{x}) & O \\ O & \nabla_y^2\varphi(\hat{y}) \end{pmatrix}$$
$$+ \begin{pmatrix} 2aI & O \\ O & O \end{pmatrix} \leqslant \frac{1}{\varepsilon} \begin{pmatrix} J & -J \\ -J & J \end{pmatrix} + \ell \begin{pmatrix} I & O \\ O & I \end{pmatrix} \equiv A,$$

where $J = \nabla^2\psi(\widehat{x} - \widehat{y})$, $\ell = 2a + k\delta$ ($k = \max_{\overline{\Omega}} |\nabla^2\varphi|$). Clearly

$$A^2 = \frac{2}{\varepsilon^2} \begin{pmatrix} J^2 & -J^2 \\ -J^2 & J^2 \end{pmatrix} + \frac{2}{\varepsilon} \begin{pmatrix} J & -J \\ -J & J \end{pmatrix} + \ell^2 \begin{pmatrix} I & O \\ O & I \end{pmatrix}. \tag{3.21}$$

We take $\lambda = 1$ in (3.18) to get

$$A + \lambda A^2 = \frac{2}{\varepsilon^2} \begin{pmatrix} J^2 & -J^2 \\ -J^2 & J^2 \end{pmatrix} + \frac{3}{\varepsilon} \begin{pmatrix} J & -J \\ -J & J \end{pmatrix} + (\ell + \ell^2) \begin{pmatrix} I & O \\ O & I \end{pmatrix} \tag{3.22}$$

and

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leqslant A + \lambda A^2. \tag{3.23}$$

Since F has singular at $\widehat{\Psi}_x = 0$, we divide the situation in two cases.

Case 1. *$\widehat{\Psi}_x \to 0$ as $\delta \to 0, a \to 0$ with $a < \delta/2R$.* From (3.22) and (3.23) it follows

$$X, Y \leqslant \frac{2}{\varepsilon^2} J^2 + \frac{3}{\varepsilon} J + (\ell + \ell^2) I \equiv B.$$

By the degenerate ellipticity (F2) we have

$$F^*(\hat{t}, \hat{\Psi}_x\ X) \geqslant F_*(\hat{t}, \hat{\Psi}_x, B), F^*(\hat{t}, \hat{\Psi}_x, -Y) \leqslant F_*(\hat{t}, \hat{\Psi}_x, -B).$$

We may assume that $\hat{t} \geqslant s/2$ for $\varepsilon < \varepsilon_0', \delta < \delta_0', \gamma < \gamma_0'$ by taking $\varepsilon_0', \delta_0', \gamma_0'$ smaller. Since $B \to 0$ as $\delta \to 0, a \to 0$, we observe by taking a subsequence if necessary, that $\hat{t} \to \bar{t} \in (0, T]$ and that

$$\begin{aligned} &\liminf\ F_*(\hat{t}, \hat{\Psi}_x, X) \geqslant F_*(\bar{t}, 0, O) \\ &\limsup\ F^*(\hat{t}, \hat{\Psi}_x, -Y) \leqslant F^*(\bar{t}, 0, O) \end{aligned}$$

as $\delta \to 0, a \to 0$ with $a < \delta/2R$. From (3.20) we obtain

$$0 \geqslant \gamma T^{-2} - 2bT + F_*(\bar{t}, 0, O) - F^*(\bar{t}, 0, O).$$

By (F3) this yields $0 \geqslant \gamma/2T^2$ if $b \leq \gamma/4T^3$. This contradicts $\gamma > 0$.

Case 2: $\hat{\Psi} \to \alpha \neq 0$ *as* $\delta \to 0, a \to 0$, *with* $a < \delta/2R$ *at least for a subsequence.* This can be treated by a standard argument. We give a proof for completeness. From (3.22) and (3.23) we see $X + Y \leqslant (\ell + \ell^2)I$. We shall study (3.20). We choose $b < \gamma/4T^3$ and obtain, by (F2),

$$0 \geqslant \gamma/2T^2 + F_*(\hat{t}, \hat{\Psi}_x, -Y + (\ell + \ell^2)I) - F^*(\hat{t}, \hat{\Psi}_x, -Y). \tag{3.24}$$

Since Y is bounded as $\delta \to 0, a \to 0$ with $a < \delta/2R$ by (3.18) and (3.22), there is a subsequence $Y_j = Y(\delta)$ and $\overline{Y} \in \mathbf{S}^n$ such that $Y_j \to \overline{Y}, \hat{t} \to \bar{t}$ as $\delta_j \to 0, a_j \to 0$ with $a < \delta/2R$. Letting $\delta_j \to 0, a_j \to 0$ in (3.24) yields

$$0 \geqslant \gamma/2T^2 + F_*(\bar{t}, \alpha, \overline{Y}) - F^*(\bar{t}, \alpha, \overline{Y}).$$

Since F is continuous at $(\bar{t}, \alpha, \overline{Y})$ for $\alpha \neq 0$, this contradicts $\gamma > 0$. ∎

REMARK 3.8: The method presented above leads a contradiction even if z is not a boundary point.

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