Commuting Squares and Relative Entropy for Two Subfactors

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Commuting Squares and Relative Entropy for Two Subfactors

Jerzy Wierzbicki and Yasuo Watatani

Abstract. We compute Connes-Störmer relative entropy $H(M|N)$ for two subfactors M and N of the type II_1 factor without assuming $N \subset M$. If they form a commuting square, then we have $H(M|N) = H(M|M \cap N)$. If their commutants form a commuting square, then we have $H(M|N) = H(M \vee N|N)$.

1.Introduction. The entropy in the non-commutative frame was recently a subject of numerous papers. A.Connes and E.Störmer [CS] introduced the concept of entropy for automorphisms of a finite von Neumann algebra L. They also defined the relative entropy $H(M|N)$ for finite dimensional von Neumann subalgebras M and N of L. When L is commutative, M and N are generated by some finite partitions P and Q then, $H(M|N)$ coincides with the classical conditional entropy $h(P, Q)$ ([B]). The relative entropy was helpful in proof of their Kolmogoroff-Sinai type theorem. However later, M.Pimsner and S.Popa [PP1] related the relative entropy $H(M|N)$ with the Jones index $[M : N]$, where N is a subfactor of a type II_1 factor M. M.Pimsner and S.Popa's investigation was followed by [PP2], [KY], [Bis] and also by [Hi], [T], [K], where a more general von Neumann algebra set up was considered. The dynamical aspect of the notion, in a C^*-algebra context, can be found, for example in [V], [CNT], [ST] or [Hu]. It is also possible to start with a different (from [Ar1],[Ar2]) formula of relative entropy for a pair of operators ([FK]).

The purpose of this paper is to relate the relative entropy $H(M|N)$ with com-

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muting square conditions ([GHJ]). In particular we will extend the classical formula

$$h(P, Q) = h(P \vee Q, Q)$$

in ergodic theory to the non-commutative case. Even, if M and N are commuting, finite-dimensional von Neumann subalgebras of L,

$$H(M|N) = H(M \vee N|N)$$

is not always true. Here $M \vee N$ means the von Neumann algebra generated by M and N. We show that commuting square condition for commutants ensures validity of the formula. We give also a dual formula $H(M|N) = H(M|N \cap M)$, which hold in commuting square case, which extends the classical independence of two σ-fields.

Our recognition of duality between $H(M \vee N|N)$ and $H(M|M \cap N)$ is new, even in commutative case, as it seems to be difficult to capture the commuting square condition for commutants without recent development of classification of subfactors. Our results suggest that the relative entropy $H(M|N)$ may have a deep connection with the relative position of two subfactors M and N.

2.Relative entropy

Let L be a finite von Neumann algebra with a fixed (faithful, normal, normalized) trace τ. Let N and M be von Neumann subalgebras of L. We denote by E_N and E_M the unique (faithful, normal) τ - preserving conditional expectations onto N and M.

$$S(L) = \{(x_1, \ldots, x_n)| \quad n \text{ - a positive integer }, x_i \in L_+, \sum_{i=1}^{n} x_i = 1\}$$

is the set of finite partitions of the unity in L. A.Connes and E.Störmer [CS] defined the relative entropy by

$$H(M|N) = \sup_{S(L)} \sum_i \tau\eta E_N(x_i) - \tau\eta E_M(x_i).$$

Here η is the function : $\mathbb{R}_+ \to \mathbb{R}$ defined by $\eta(t) = -t \ln t$.

Remark. In the above definition we may replace $S(L)$ by $S(D)$ for any von Neumann algebra D with $M \vee N \subset D \subset L$ without affecting the value $H(M|N)$. Here $M \vee N$ is the von Neumann algebra generated by M and N in L.

If M is a type II_1 - factor and N is a subfactor of M, then as shown in [PP1], the relative entropy $H(M|N)$ depend on both, the relative commutant $N' \cap M$

and the Jones index $[M : N]$. In particular, if $N' \cap M = \mathbb{C}$, then they prove that $H(M|N) = \log[M : N]$. In this paper we do not assume $N \subset M$, but still, we use some modified technical ideas of their paper [PP1].

3. Commuting squares and co-commuting squares.

In [P1] S.Popa introduced notion of orthogonal pairs of finite von Neumann algebras. It corresponds to the independence of two σ-fields in the probability theory. It was later generalized to the notion of commuting square, which plays an important role in the classification of subfactors [GDJ]. Recall that a diagram

$$\begin{array}{ccc} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{array}$$

of finite von Neumann algebras with a fixed (finite, faithful, normal) trace τ on L is a commuting square, if $E_M E_N = E_N E_M$ and $K = M \cap N$. (See equivalent conditions in [GHJ].) We shall introduce a dual notion of co-commuting square.

Definition 1. A diagram $\begin{array}{ccc} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{array}$ of finite von Neumann algebras with a fixed (finite, faithful, normal) trace τ' on K' is a co-commuting square, if their commutants $\begin{array}{ccc} M' & \subset & K' \\ \cup & & \cup \\ L' & \subset & N' \end{array}$ form a commuting square.

Since $L' = M' \cap N'$, it is necessary for $L = M \vee N$ to hold. Throughout the paper we consider only the case of K' being a finite factor, therefore we do not worry about dependence on the choice of trace τ' on K'. The following example shows that the commutative case always satisfies the co-commuting square condition by suitable representation.

Example 2. If $(X, \mathbf{F}, p)$ is a probability space,

$$\mathbf{A} = \{A_1, \dots, A_m\}, \qquad \mathbf{B} = \{B_1, \dots, B_n\} \quad \mathbf{D} = \{A_i \cap B_k\}_{i \le m, k \le n}$$

are finite partitions of X, then we may consider the following von Neumann algebras:

$$M = L^\infty(X, \sigma(\mathbf{A})), \quad N = L^\infty(X, \sigma(\mathbf{B})), \quad L = L^\infty(X, \sigma(\mathbf{D})).$$

The trace on L corresponds to the expected value of a random variable, $\tau(g) = \int_X g dp$ and E_N, E_M are conditional expectations in the sense of the probability theory. In this case

$$H(M|N) = h(\mathbf{A}, \mathbf{B}), \text{ where}$$

$$h(\mathbf{A},\mathbf{B})=\sum_{k=1}^{n}p(B_k)\sum_{i=1}^{m}\eta(p(A_i|B_k))$$

is the ergodic theory conditional entropy of the partition $\mathbf{A}$ given $\mathbf{B}$([B]). When we represent N, M and L on $L^2(X,\sigma(\mathbf{D})$, we see that

$$M'=\bigoplus_{i=1}^{m}\mathcal{B}(L^2(A_i,\sigma(\{A_i\cap B_1,\ldots,A_i\cap B_n\})$$

$$N'=\bigoplus_{k=1}^{n}\mathcal{B}(L^2(B_k,\sigma(\{A_1\cap B_k,\ldots,A_m\cap B_k\})\text{ and }L'=L,$$

where $\mathcal{B}(H)$ means all linear bounded operators on a Hilbert space H.

It occurs that the diagram $\begin{matrix} M & \subset & L \\ \cup & & \cup \\ \mathbb{C} & \subset & N \end{matrix}$ is a co-commuting square.

Indeed, if $I=\{(i,j)|A_i\cap B_j$ is not empty $\}$ then operators from K' correspond to matrices $(X_a^b)_{a,b\in I}$ and the trace preserving conditional expectations are as follows: $E_{M'}((X_{(i,j)}^{(k,l)}))=(\delta_{i,k}X_{(i,j)}^{(k,l)})$ and $E_{L'}((X_{(i,j)}^{(k,l)}))=(\delta_{i,k}\delta_{j,l}X_{(i,j)}^{(k,l)})$. Hence obviously $E_{M'}E_{N'}=E_{N'}E_{M'}=E_{L'}$.

So we see that, the commutative situation implies, in a sense, the co-commuting square property. It can be shown that the co-commuting square property is independent from choice of the representation on a Hilbert space, provided K' is a finite factor.

Lemma 3. Consider a diagram $\begin{matrix} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{matrix}$ of finite von Neumann algebras. assume that K is a finite factor. Let $\pi_i:L\longrightarrow\mathcal{B}(H_i),(i=1,2)$, be faithful normal representations of L, such that $\pi_i(K)'$ are finite factors. Then, the diagram

$$\begin{matrix} \pi_1(M)' & \subset & \pi_1(K)' \\ \cup & & \cup \\ \pi_1(L)' & \subset & \pi_1(N)' \end{matrix}\qquad\text{is a commuting square,}$$

if and only if

$$\text{the diagram}\quad\begin{matrix} \pi_2(M)' & \subset & \pi_2(K)' \\ \cup & & \cup \\ \pi_2(L)' & \subset & \pi_2(N)' \end{matrix}\quad\text{is.}$$

Proof. According to [T]5.5 there is a Hilbert space F such that $\pi_2(L)$ is spatially isomorphic to $(\pi_1(L)\otimes 1\!\!1_{\mathcal{B}(F)})_e$, where $e\in\pi_1(L)'\otimes\mathcal{B}(F)$ is a projection. Since $\pi_2(K)'$ is assumed to be finite, F must have finite dimension, so that $\mathcal{B}(F)\approx M_n(\mathbb{C})$.

It is clear that spatial isomorphism can not spoil the commuting square property. Let us show now that the diagram

$$\begin{array}{ccc} \pi_1(M)' \otimes M_n(\mathbb{C}) & \subset & \pi_1(K)' \otimes M_n(\mathbb{C}) \\ \cup & & \cup \\ \pi_1(L)' \otimes M_n(\mathbb{C}) & \subset & \pi_1(N)' \otimes M_n(\mathbb{C}) \end{array}$$

is a commuting square, if the diagram $\begin{array}{ccc} \pi_1(M)' & \subset & \pi_1(K)' \\ \cup & & \cup \\ \pi_1(L)' & \subset & \pi_1(N)' \end{array}$ is. Indeed, for $x \otimes a \in \pi(K_1)' \otimes M_n(\mathbb{C})$ we have:

$$E_{\pi_1(M)'\otimes M_n(\mathbb{C})}(x \otimes a) = E_{\pi_1(M)'}(x) \otimes a,$$

with similar formulas for projections on $\pi_1(N)' \otimes M_n(\mathbb{C})$ and on $\pi_1(L)' \otimes M_n(\mathbb{C})$, which makes the above statement obvious.

Also, if $e \in \pi_1(L)'$ is a projection, then the commuting square property of the diagram $\begin{array}{ccc} \pi_1(M)' & \subset & \pi_1(K)' \\ \cup & & \cup \\ \pi_1(L)' & \subset & \pi_1(N)' \end{array}$ implies that the diagram: $\begin{array}{ccc} \pi_1(M)'_e & \subset & \pi_1(K)'_e \\ \cup & & \cup \\ \pi_1(L)'_e & \subset & \pi_1(N)'_e \end{array}$ is a commuting square too. In fact, for $x \in \pi_1(K)'_e$, $E_{\pi_1(M)'_e}(x) = E_{\pi_1(M)'}(x)$, with similar formulas for projections on other subalgebras in question. Hence, we immediately get:

$$E_{\pi_1(M)'_e} E_{\pi_1(N)'_e} = E_{\pi_1(L)'_e},$$

which completes the proof.

Remark. T.Sano and Y.Watatani ([SW]) introduced angles between two subfactors. By definition, $\mathrm{Ang}_L(M,N) = \frac{\pi}{2}$, iff the diagram $\begin{array}{ccc} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{array}$ is a commuting square, and the opposite angles $\text{Op-ang}_L(M,N) = \frac{\pi}{2}$ iff the diagram $\begin{array}{ccc} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{array}$ is a co-commuting square.

Example 4. Let P be a type II_1 factor. Let G be a finite group and A, B subgroups of G. Let $\alpha : G \longrightarrow \mathrm{Aut}(P)$ be an outer action. Then the crossed product algebras

$$\begin{array}{ccc} P \rtimes_\alpha A & \subset & P \rtimes_\alpha G \\ \cup & & \cup \\ P \rtimes_\alpha (A \cap B) & \subset & P \rtimes_\alpha B \end{array}$$ form a commuting square.

The fixed point algebras

$$\begin{array}{ccc} P^A & \subset & P^{A\cap B} \\ \cup & & \cup \\ P^G & \subset & P^B \end{array}$$ form a co-commuting square.

We need the following lemma to compute relative entropy. It follows from [SW]Lemmas 3.9, 7.1 and 7.5.

Lemma 5. Suppose, that the diagram

$$\begin{array}{ccc} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{array} \quad \text{is a co-commuting square with } [L:K] < \infty.$$

Let Q be the downward basic construction for $K \subset M$ and $e_Q = e_Q^K$ its Jones projection. Put $a = [L:N]E_N(e_Q)$. Then a is a projection in N and $\tau(a) = \frac{[L:N]}{[M:K]}$.

Proof. Since $[L:K] < \infty$, we can consider downward basic constructions. From [SW] there are type II_1 factors Q, R and S such that the diagram

$$\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array} \quad \text{is a commuting square}$$

and $L = \langle K, e_S \rangle$, $M = \langle K, e_Q \rangle$ and $N = \langle K, e_R \rangle$.

Let E_M, E_N, E_K be τ-preserving conditional expectations onto M, N, K and by $\{u_i\}_{i=1}^q$ we denote the Pimsner-Popa basis of Q over S.

[SW]3.9 says that

$$(*) \qquad e_Q = \sum_{i=1}^{q} u_i e_S u_i^*.$$

By [SW]7.1 we have also:

$$(**) \qquad E_N(e_S) = [L:N]^{-1} e_R.$$

Now by $(*), (**)$ we see that the operator $a = [L:N]E_N(e_Q)$, we are interested in, may be written as follows:

$$a = [L:N]E_N(\sum_i u_i e_S u_i^*) = [L:N]\sum u_i E_N(e_S) u_i^* = \sum_i u_i e_R u_i^*.$$

Similarly as in [SW]7.5 we can show that a is a projection. This ends the proof.

4.A duality for the entropy.

We show here our two basic theorems, which are in a dual relation. The case of commuting square is easy to prove.

Theorem 6. Let diagram $\begin{matrix} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{matrix}$ be a commuting square of finite von Neumann algebras. Then we have $H(M|N) = H(M|K)$.

Proof.

$$H(M|K) = \sup_{(x_i)\in S(L)} \sum_i \tau\eta E_K(x_i) - \tau\eta E_M(x_i) =$$

$$= \sup_{(x_i)\in S(L)} \sum_i \tau\eta E_N E_M(x_i) - \tau\eta E_M E_M(x_i) =$$

$$= \sup_{(y_i)\in S(M)} \sum_i \tau\eta E_N(y_i) - \tau\eta E_M(y_i) \leq$$

$$\leq \sup_{(y_i)\in S(L)} \sum_i \tau\eta E_N(y_i) - \tau\eta E_M(y_i) = H(M|N).$$

As $H(M|N)$ is decreasing in N, this gives the theorem.

The case of co-commuting square is more involved.

Theorem 7. Let diagram $\begin{matrix} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{matrix}$ be a co-commuting square of type II_1 - factors with $[L:K] < \infty$. Then $H(M|N) = H(L|N)$.

Proof. Suppose that $f_1, \dots, f_n$ are minimal projections in $N' \cap M$ with $\sum_{i=1}^n f_i = 1$. Let us use the following notations: the reduced algebras $K_{f_i} = K_i$ and $M_{f_i} = M_i$, the traces on M_i's are denoted by τ_i. So $\tau_i(x) = \frac{\tau(x)}{\tau(f_i)}$. E_{K_j} will be the τ_j preserving conditional expectations of M_j onto K_j, $E_{N'\cap M}$ will be the τ preserving conditional expectation of M onto $N' \cap M$, E_N - τ preserving conditional expectation of L onto N and E_K - of M onto K.

Since $[L:K] < \infty$, by [J1] all the factors L, M, N and K may be represented on $L^2(K,\tau)$. If J is the modular conjugation on $L^2(K,\tau)$, then let us also use notation $\hat{x} = JxJ$.

Similarly like in [PP1] (proof of Theorem 4.4), the following hold.

(1) There are $e_j \in P(M_j)$ such that $E_{K_j}(e_j) = [M_j : K_j]^{-1} f_j$ and (the commutator) $[\hat{f}_j, e_j] = 0$.

(2) If $q_j = \hat{f}_j e_j$, then $\tau(q_j) = [M:K]^{-1}$.

(3) If $M \ni u_j : q_1 \sim q_j$, $\quad v_{ij} = u_i u_j^*$ and $q = \sum_{ij} (t_i t_j)^{\frac{1}{2}} v_{ij}$, where $t_i = \tau(f_i)$, then $q \in P(M)$ and $f_i q f_i = f_i \hat{f}_i q f_i \hat{f}_i = t_i q_i$.

(4) $E_{N' \cap M}(q) = \tau(q) = [M : K]^{-1}$.

If the trace on K' is denoted by τ' and $t'_k = \tau'(f_k) = \tau(\hat{f}_k)$ then the projection $e = \sum_{k,l}(t'_k t'_l)^{\frac{1}{2}} v_{kl}$ is an element of M and $E_K(e) = [M : K]^{-1}$. Indeed

$$E_K(e) = E_K(\sum_j E_{K_j}(f_j e f_j)) = E_K(\sum_j E_{K_j}(t'_j q_j))$$

$$= \sum_j t'_j E_K(E_{K_j}(\hat{f}_j e_j)) = \sum_j t'_j E_K(\hat{f}_j f_j)[M_j : K_j]^{-1}$$

$$= \sum_j t'_j [M_j : K_j]^{-1} \hat{f}_j E_K(f_j) = \sum_j t'_j t_j [M_j : K_j]^{-1} \hat{f}_j = [M : K]^{-1}.$$

The last inequality comes from [J1]Lemma 2.2.1 and the fact that $\sum_j \hat{f}_j = 1$. So, by [PP1]Corollary 1.8 $e = e_Q$ for some downward basic construction $Q = K \cap \{e\}'$. It is easy to see that if $b = \sum_i (t_i {t'}_i^{-1})^{\frac{1}{2}} f_i \in N' \cap M$, then $q = beb$.

We obtain now similar estimate for the entropy $H(M|N)$ like in [PP1](Lemma 4.2 and Theorem 4.4). Let R be the downward basic construction for the inclusion $K \subset N$ with corresponding Jones projection e_R. We define set

$$\Omega = \overline{\mathrm{conv}\{vxv^* | v \in U(K) \cup \{u_R\}\}}^{\mathrm{weak}},$$

where $U(K)$ is set of all unitary operators in K, $u_R = 2e_R - 1$ and $x = q - \tau(q)$. Similarly like in [PP1] we see that Ω is a convex subset of L, which is closed in the topology induced by the Hilbert space structure. Moreover, as e_R and e_Q commute ([SW]), then by $u_R q u_R^* = u_R beb u_R = b u_R u_R^* eb = q \in M$ we obtain $\Omega \subset M$.

Ω is a convex closed subset of a Hilbert space, therefore there exists a unique $y_0 \in \Omega$ such that

$$\|y_0\|_2 = \inf\{\|y\|_2 : y \in \Omega\}$$

Hence, $y_0 \in (U(K) \cup \{u_R\})' = N'$, where the last equality comes from [J1]. Note also that $E_{N' \cap M}(vxv^*) = 0$ for $v \in U(K) \cup \{u_R\}$ which implies $\Omega \subset \ker E_{N' \cap M}$. Therefore, $y_0 = E_{N' \cap M}(y_0) = 0 \in \Omega$. Using the same argument as in [PP1](Lemma 4.2) we obtain the following inequality:

$$(5) \quad H(M|N) \geq \frac{1}{\tau(q)} \tau\eta E_N(q).$$

Now the job to do is the calculation of its right-hand side. First note that by (4) $\tau(q) = [M : K]^{-1}$.

We define isomorphisms $\omega_j : N \longrightarrow N_j$ by $\omega_j(x) = x f_j$. Note that for $x \in N_j$ we have the formula:

$$E_N(x) = t_j \omega_j^{-1}(x), \text{ where } t_j = \tau(f_j).$$

So that, if we denote $a_j = [L_j : N_j]E_{N_j}(e_j)$ then

$$\tau\eta E_N(q) = \tau\eta(\sum_j \frac{t_j^2}{[L_j : N_j]}\omega_j^{-1}(\hat{f}_j a_j)) =$$

$$\sum_j \eta(\frac{t_j^2}{[L_j : N_j]})\tau\omega_j^{-1}(\hat{f}_j a_j) + \frac{t_j^2}{[L_j : N_j]}\tau\eta\omega_j^{-1}(\hat{f}_j a_j).$$

The last equality is a consequence of the derivative property, the additivity and the fact that $\omega_j^{-1}(\hat{f}_j a_j)$ are pairwise orthogonal. Indeed $[\hat{f}_j, a_j] = 0$ and

$$\omega_j^{-1}(\hat{f}_j a_j) = \omega_j^{-1}(\hat{f}_j f_j)\omega_j^{-1}(a_j) = \hat{f}_j\omega_j^{-1}(a_j)$$

and $\hat{f}_j$'s are obviously pairwise orthogonal. Let us calculate the trace

$$\tau\omega_j^{-1}(\hat{f}_j a_j) = \tau_j(\hat{f}_j a_j) = \tau_j(\hat{f}_j f_j E_{K_j}(a_j)) =$$

$$= \frac{[L_j : N_j]}{[M_j : K_j]}\tau_j(\hat{f}_j f_j) = \frac{[L_j : N_j]}{t_j[M : K]}, \text{ by [J1]2.2.1.}$$

Hence

$$(6) \quad \tau\eta E_N(q) = [M : K]^{-1}\sum_j t_j \ln\frac{[L_j : N_j]}{t_j^2} + \sum_j \frac{t_j^2}{[L_j : N_j]}\tau_j\eta(\hat{f}_j a_j).$$

We also need to show that if $f \in P(N' \cap M)$, then $\begin{matrix} M_f & \subset & L_f \\ \cup & & \cup \\ K_f & \subset & N_f \end{matrix}$ is also a co-commuting square. Take $x \in K_f'$ and note such formulas:

$$E_{N_f'}(x) = E_{N'}(x), \quad E_{M_f'}(x) = \tau(f)^{-1}fE_{M'}(x),$$

$$E_{L_f'}(x) = \tau(f)^{-1}fE_{L'}(x).$$

It follows that

$$E_{N_f'}E_{M_f'}(x) = E_{N'}(\tau(f)^{-1}fE_M'(x)) = \tau(f)^{-1}fE_{N'}E_{M'}(x) =$$

$$= \tau(f)^{-1}E_{L'}(x) = E_{L_f'}(x)$$

and similarly $E_{M_f'}E_{N_f'}(x) = E_{L_f'}(x)$.

Since a_j commute with $\hat{f}_j$ then by Lemma 5. $\hat{f}_j a_j$ is a projection in N_j. This and (6) give:

$$\tau\eta E_N(q) = [M : K]^{-1}\sum_j t_j \ln\frac{[L_j : N_j]}{t_j^2}.$$

So, the inequality (5) gives: $H(M|N) \geq \sum_j t_j \ln \frac{[L_j:N_j]}{t_j^2}$. Let us pick up now such minimal projections g_j^i in $N' \cap L$ that $f_j = \sum_i g_j^i$ and denote $L_j^i = L_{g_j^i}, N_j^i = N_{g_j^i}$. By [J1]2.2.2 $[L_j : N_j] = \sum_i [L_j^i : N_j^i] \tau_j(g_j^i)^{-1}$ and the concavity of "ln" results in:

$$\ln \frac{[L_j : N_j]}{t_j^2} = \ln(\sum_i \tau_j(g_j^i) \frac{[L_j^i : N_j^i]}{t_j^2 \tau_j(g_j^i)^2}) \geq \sum_i \tau_j(g_j^i) \ln \frac{[L_j^i : N_j^i]}{\tau(g_j^i)^2},$$

hence, by [PP1]4.4

$$\sum_j t_j \ln \frac{[L_j : N_j]}{t_j^2} \geq \sum_{j,i} \tau(g_j^i) \ln \frac{[L_j^i : N_j^i]}{\tau(g_j^i)^2} = H(L|N),$$

which completes the proof, because always $H(L|N) \geq H(M|N)$ for $M \subset L$.

5. Several examples.

Using the theorems 6 and 7 we can compute the relative entropy $H(M|N)$ of many interesting cases, without assuming $N \subset M$.

Example 8. Let P be a type II_1. Let G be a finite group and A, B subgroups of G. Let $\alpha : G \longrightarrow \mathrm{Aut}(P)$ be an outer action. Consider crossed product $P \rtimes_\alpha G$. Then we have

$$H(P \rtimes_\alpha A | P \rtimes_\alpha B) = H(P \rtimes_\alpha A | P \rtimes_\alpha (A \cap B)) =$$

$$= \log[P \rtimes_\alpha A : P \rtimes_\alpha (A \cap B)] = \log[A : A \cap B]$$

by theorem 6, [PP1] and [J1]. Similarly, by theorem 7, we have

$$H(P^A | P^B) = H(P^{A \cap B} | P^B) = \log[P^{A \cap B} : P^B] = \log[B : A \cap B].$$

In particular, we give an example, which shows that the formula $H(M|N) = H(M \vee N|N)$ does not hold in general.

Example 9. Let $D_n = \langle x, y | x^n = y^2 = (xy)^2 = 1 \rangle$ be the dihedral group of order $2n$ for $n > 2$.Let A and B be the subgroup of D_n singly generated by y and by xy. Let P be a type II_1 factor and $\alpha : D_n \longrightarrow \mathrm{Aut}(P)$ an outer action. Consider crossed products $L = P \rtimes_\alpha D_n$, $M = P \rtimes_\alpha A$ and $N = P \rtimes_\alpha B$. Then $M \cap N = P$ and $M \vee N = P \rtimes_\alpha D_n = L$. Then,

$$H(M|N) = H(M|M \cap N) = \log[M : P] = \log 2$$

$$H(M \vee N|N) = \log[L : M] = \log n.$$

Therefore $H(M \vee N|N) \neq H(M|N)$.

In fact, under some conditions we may say much more.

Proposition 10.

Let us denote the diagram $\begin{array}{ccc} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{array}$ by $\mathcal{S}$.

(1) If $\mathcal{S}$ is a co-commuting square and the condition $(*)$ $K' \cap M = \mathbb{C}$ is satisfied, then

$$\mathcal{S} \text{ is commuting square } \Leftrightarrow \quad H(M|N) = H(M|K).$$

(2) If $\mathcal{S}$ is a commuting square and the condition $(**)$ $N' \cap L = \mathbb{C}$ is satisfied, then

$$\mathcal{S} \text{ is co-commuting square } \Leftrightarrow \quad H(M|N) = H(L|N)$$

Proof of (1).
"$\Rightarrow$" follows from Theorem 6.
"$\Leftarrow$". From Theorem 7 and [PP1] we see that

$$H(M : K) = \ln[M : K] = H(L|N) \leq \ln[L : N].$$

So that $[M : K] \leq [L : N]$. Now, by [SW]7.7 and 7.8, the proof is completed.
Proof of the second part is analogous.

Remark. The condition $(*)$ in the above proposition may be and replaced by $(*)'$ $H(M|K) = \ln[M : K]$ or even by $(*)''$ $H(M_p|K_p) = \ln[M_p : K_p]$ for any minimal projection in $N' \cap M$.
Also, $(**)$ can be relaxed in the same way.

Example 11. If P, G and α are as in Example 8. and Q, R are intermediate subfactors between P and $P \rtimes_\alpha G$, then the diagram

$\begin{array}{ccc} Q & \subset & Q \vee R \\ \cup & & \cup \\ Q \cap R & \subset & R \end{array}$ is a co-commuting square, iff $H(Q|R) = H(Q \vee R|R)$.

Remark. We note that in finite dimensional case the theorem 7 is not always true. This is connected to the fact that the downward basic construction may not exist in that case. For example, let $M = M_k(\mathbb{C}) \otimes \mathbb{C}$ and $N = \mathbb{C} \otimes M_n(\mathbb{C})$. Then the diagram

$\begin{array}{ccc} M & \subset & M \vee N \\ \cup & & \cup \\ M \cap N & \subset & N \end{array}$ is a commuting square and co-commuting square, so, by Theorem 6 and [PP1] we obtain:

$$H(M|N) = H(M|M \cap N) = H(M|\mathbb{C}) = \log k^2,$$

but

$$\text{but} \quad H(M \vee N|N) = \begin{cases} \log k^2, & \text{if } k \leq n \\ \log kn, & \text{if } k > n; \end{cases}$$

so that always $H(M|N) < H(M \vee N|N)$.

Combining theorems 6 and 7, we calculate the relative entropy in some cases, where there is no any commuting or co-commuting condition.

Example 12.
Let a finite group G be the direct product of its subgroups F and H, $G = FH$. Suppose that the subgroups A, D generate H and similarly the subgroups B, C generate F. Let $A \cap D = B \cap C = \{e\}$ – trivial subgroup. Let G act outerly on a II_1 factor P, $\alpha : G \ni g \mapsto \alpha_g \in \mathrm{Aut}(P)$. Under these assumptions

$$H(P^A \rtimes_\alpha B|P^D \rtimes_\alpha C) = \ln(|B| \cdot |D|)$$

.

Before we sketch the proof, let us note two lemmas.

Suppose that K is a II_1 factor and a finite group G is a direct product of its subgroups A and F. Let $B \subset F$ be any subgroup of F. By $\tilde{g}$ we denote a projection of $g \in G$ onto F. Let α be an outer action of G on K, implemented by λ_g's. We define an action μ on the von Neumann algebra $K \otimes L^\infty(F/B)$ this way

$$\mu_g(x \otimes \chi_{fB}) = \alpha_g(x) \otimes \chi_{\tilde{g}fB},$$

where χ_{fB} is the characteristic function:

$$\chi_{fB}(cB) = 1 \text{ if } fB = cB, \quad \text{if otherwise then } \chi_{fB}(cB) = 0.$$

We fix the trace on $L^\infty(F/B)$ by the formula:

$$\tau(\chi_{fB}) = [F : B]^{-1}, \text{ where } [F : B] \text{ - is the index of } B \text{ in } F.$$

Lemma 13. Under above assumptions, the von Neumann algebra $L = (K \otimes L^\infty(F/B)) \rtimes_\mu G$ is the basic construction for the pair $K^A \rtimes_\alpha B \subset K \rtimes_\alpha F$ of II_1 factors.
Proof.
One can check that the mapping Φ, which is defined below, is an isomorphism.

$$\Phi : \langle K \rtimes_\alpha F, e_{K^A \rtimes_\alpha B} \rangle \to (K \otimes L^\infty(F/B)) \rtimes_\mu G$$

$$\Phi((x\lambda_s)e_{K^A\rtimes_\alpha B}(y\lambda_t)) = |A|^{-1}\sum_{a\in A}(x\alpha_{sa}(y)\otimes\chi_{sB})\lambda_{sat},$$

where $x, y \in K$, $s, t \in F$. Also

$$\Phi(e_{K^A\rtimes_\alpha B}) = |A|^{-1}\sum_{a\in A}(1_K\otimes\chi_B)\lambda_a \text{ with } \Phi(x) = x, \text{ for } x\in K\rtimes_\alpha F.$$

Lemma 14. Suppose that L, M, N, K, Q, R, S are type II_1 factors, with $[L : K] < \infty$, such that

$$\begin{array}{ccccccc} M & \subset & L & & Q & \subset & K \\ \cup & & \cup & \text{and} & \cup & & \cup \\ K & \subset & N & & S & \subset & R \end{array}$$

and that M is extension of K by Q, N is extension of K by R and L is extension of K by S. Then $H(M|N) = H(Q'|R')$, where the commutants are taken over $L^2(K,\tau)$.

Proof. Let represent all the algebras on $L^2(K,\tau)$ and let J be the modular conjugation on $L^2(K,\tau)$. The algebras L, M, N are antiisomorphic to S', Q', R' via the mapping $x \mapsto JxJ$. The definition of the relative entropy and the observation, that

$$E_{Q'}(x) = JE_M(JxJ)J \text{ and } \tau'\eta E_{Q'}(x) = \tau\eta E_M(JxJ)$$

with similar formulas for other pairs of factors, gives the lemma.

Proof of the Example 12. Let denote $L = (P\otimes L^\infty(F))\rtimes_\mu G$, where μ is defined as in the Lemma 13 (with $B = \{e\}$ there). Let $M = (P\otimes L^\infty(F/B))\rtimes_\mu AF$ and $N = (P\otimes L^\infty(F/C))\rtimes_\mu DF$ be subalgebras of L. From the lemma 11 we know that L is extension of $P\rtimes_\alpha F$ by P^H, M is extension of $P\rtimes_\alpha F$ by $P^A\rtimes_\alpha B$ and N is extension of $P\rtimes_\alpha F$ by $P^D\rtimes_\alpha C$. In particular $q = |A|^{-1}\sum_{a\in A}(1_P\otimes\chi_B)\lambda_a$ is the Jones projection such that $\langle P\rtimes_\alpha F, q\rangle = M$ is the basic construction for $P^A\rtimes_\alpha B\subset P\rtimes_\alpha F$. Let $E_N = E^L_N$ be the trace preserving conditional expectation of L onto N. Let us calculate $\tau\eta E_N(q)$. The formula for E_N is easy to guess:

for $x\in P, g\in G$ and a subset $Z\subset F$

$$E_N((x\otimes\chi_Z)\lambda_g) = \delta_{g,DF}(x\otimes(\sum_{i=1}^{[F:C]}\frac{|Z\cap f_iC|}{|C|}\chi_{f_iC}))\lambda_g$$

where $\{f_iC\}_{i=1}^{[F:C]}$ are all cosets of the subgroup C, $\delta_{g,DF} = 1$, if $g\in DF$ and $\delta_{g,DF} = 0$, if otherwise. So we obtain in particular:

$$E_N(q) = |C|^{-1}|A|^{-1}(1\otimes\sum_{i=1}^{[F:C]}|B\cap f_iC|\chi_{f_iC}).$$

Note that $|B \cap f_i C| \leq 1$, for $1 \leq i \leq [F:C]$. Because, if $b_1, b_2 \in B \cap f_i C$, then

$$b_1 = f_i c_1 \text{ and } b_2 = f_i c_2 \Rightarrow b_2^{-1} b_1 = c_2^{-1} c_1 \Rightarrow b_1 = b_2$$

as $B \cap C$ was assumed to be trivial. In other words the operator $1_P \otimes \sum_i |B \cap f_i C| \chi_{f_i C}$ is a projection. Note also

$$B = B \cap F = \bigcup_i (B \cap f_i C), \text{ hence } \sum_{i=1}^{[F:C]} |B \cap f_i C| = |B|,$$

so there are exactly $|B|$ non-zero $|B \cap f_i C|$. This allows us to evaluate the trace:

$$\tau(1_P \otimes \sum_i |B \cap f_i C| \chi_{f_i C}) = \frac{|B|}{[F:C]}.$$

Hence

$$(*) \quad \tau\eta E_N(q) = \eta(\frac{1}{|A| \cdot |C|}) \cdot |B|[F:C]^{-1} = \frac{|B|}{|A| \cdot |F|} \ln(|A| \cdot |C|).$$

Using similar method as in [PP1]4.2 and the fact that $\tau_{|K' \cap M} = \tau'_{|K' \cap M}$ we obtain the following inequality:

$$H(M|N) \geq [M:K]\tau\eta E_N(q).$$

From $(*)$ its right-hand side gets calculated:

$$H(M|N) \geq [M:K]\tau\eta E_N(q) = [F:B]|A| \frac{|B|}{|A| \cdot |F|} \ln(|A| \cdot |C|) =$$
$$= \ln(|A| \cdot |C|).$$

From Lemma 14., we obtain $H((P')^B \rtimes_\alpha A | (P')^C \rtimes_\alpha D) \geq \ln(|A| \cdot |C|)$. The reverse inequality is obtained from the following:

$$H((P')^B \rtimes_\alpha A | (P')^C \rtimes_\alpha D) \leq H((P')^B \rtimes_\alpha A | (P')^B \rtimes_\alpha D) +$$
$$+ H((P')^B \rtimes_\alpha D | (P')^C \rtimes_\alpha D) = \ln|A| + \ln|C|,$$

the last equality being a consequence of theorems 6 and 7. Now by a symmetry we can exchange A with B and C with D and put P for P' in the above formula in order to get $H(P^A \rtimes_\alpha B | P^D \rtimes_\alpha C) = \ln(|B| \cdot |D|)$.

Remark. Let us note a property of relative entropy in the Example 12. Let denote $K = P \rtimes_\alpha F$, $Q = P^A \rtimes_\alpha B$, $R = P^D \rtimes_\alpha C$ and $S = P^H$. Then the diagram

$$\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array} \text{ is: } \begin{cases} (1) & \text{a commuting square, iff } H(Q|R) = H(Q|S); \\ (2) & \text{a co-commuting square, iff } H(Q|R) = H(K|R). \end{cases}$$

Indeed, the equalities for entropy follow from theorems 6 and 7. Suppose now that $H(Q|R) = H(Q|S)$. From Example 12. and [PP1] $H(Q|R) = \ln(|B| \cdot |D|)$ and $H(Q|S) = \ln \frac{|B|\cdot|H|}{|A|}$. So that $|H| = |A| \cdot |D|$ and consequently $H = AD$. This and a direct computation ends the argument.

If $H(Q|R) = H(K|R)$, then we similarly obtain $F = BC$. Hence the operator $E_N(q)$ in the above proof is a scalar, so by [SW]lemma 7.2 the diagram in question is a co-commuting square.

This property, as well as the Proposition 10., suggests, that under some conditions, the inverses of theorems 6 and 7 are true. Unfortunately, we have not identified them completely.

Another example, which indicates that the relative entropy $H(M|N)$ controls, in some sense, the commuting square property of a diagram $\begin{array}{ccc} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{array}$, arises from the commutative case. Indeed, with the notations as in Example 2. the equality $h(\mathbf{A}, \mathbf{B}) = h(\mathbf{D},\mathbf{B})$ is equivalent to the independence of the σ-fields $\sigma(\mathbf{A})$ and $\sigma(\mathbf{B})$. We saw in Example 2., that the commutative situation reflects in the co-commuting square property, therefore, the above mentioned equivalence can be considered as a commutative version of the Proposition 10(1).

Example 15. Let the diagram $\begin{array}{ccc} M_1 & \subset & L_1 \\ \cup & & \cup \\ K_1 & \subset & N_1 \end{array}$ be a commuting square of II_1 factors and let $\begin{array}{ccc} M_2 & \subset & L_2 \\ \cup & & \cup \\ K_2 & \subset & N_2 \end{array}$ be a co-commuting square of type II_1 factors, with $[L_2 : K_2] < \infty$.

Then $H(M_1 \otimes M_2|N_1 \otimes N_2) = H(M_1|M_2) + H(M_2|N_2)$.

Proof.

$$H(M_1 \otimes M_2|N_1 \otimes N_2) \geq H(M_1|N_1) + H(M_2|N_2)$$

follows easily from definition of the relative entropy. On the other hand,

$$H(M_1 \otimes M_2|N_1 \otimes N_2) \leq H(M_1 \otimes M_2|M_1 \otimes N_2) + H(M_1 \otimes N_2|N_1 \otimes N_2).$$

By Theorem 7. and [PP1]4.4

$$H(M_1 \otimes M_2|M_1 \otimes N_2) = H(M_1 \otimes L_2|M_1 \otimes N_2) = H(M_2|N_2).$$

Similarly, by theorem 6 and [PP1]4.4

$$H(M_1 \otimes N_2|N_1 \otimes N_2) = H(M_1 \otimes N_2|K_1 \otimes N_2) =$$

$$= H(M_1|K_1) = H(M_1|N_1).$$

Finally we show that the relative entropy depends on the angle between two subfactors.

Example 16. Let compute $H(A|B_\theta)$, where $A = C \oplus C$ is the algebra of 2×2 diagonal matrices and $B_\theta = u_\theta A u_\theta^*$, where $u_\theta \in M_2(\mathbb{C})$ is the rotation by the angle θ.

$$u_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

If the 2×2 matrices $(x_i)_i$ form a partition of the unit then

$$\sum_i \tau\eta E_{B_\theta}(x_i) - \tau\eta E_A(x_i) = \sum_i \tau\eta E_A(u_\theta^* x_i u_i) - \tau\eta E_A(x_i) =$$

$$\sum_i \tau(x_i)[\tau\eta(\tau(x_i)^{-1} E_A(u_\theta^* x_i u_\theta)) - \tau\eta(\tau(x)^{-1} E_A(x_i))] \leq$$

$$\leq \sup_{x \geq 0}[\tau\eta(\tau(x)^{-1} E_A(u_\theta^* x u_\theta)) - \tau\eta(\tau(x)^{-1} E_A(x))],$$

$$\text{where } \; E_A(u_\theta^* x u_\theta) =$$

$$\begin{pmatrix} x_{1,1}\cos^2\theta + \Re(x_{1,2})\sin 2\theta + x_{2,2}\sin^2\theta & 0 \\ 0 & x_{1,1}\sin^2\theta - \Re(x_{1,2})\sin 2\theta + x_{2,2}\cos^2\theta \end{pmatrix}$$

After suitable change of variables we get

$$H(A|B_\theta) \leq \sup_\alpha[h(\cos^2(\theta - \alpha)) - h(\cos^2\alpha)],$$

where $h(t) = \eta(t) + \eta(1-t)$, $t \geq 0$. The equality holds by the following observation. If the matrix $(x_{i,j})$ realizes (may be approximately) the supremum on the right, then the matrix

$$y = \begin{pmatrix} x_{2,2} & -x_{2,1} \\ -x_{1,2} & x_{1,1} \end{pmatrix}$$

has the same property. Now take the partition:

$$(2\tau(x))^{-1}x + (2\tau(y))^{-1}y = \mathbb{1}_2.$$

It gives (may be approximately) the same value for the defining sum of the entropy as the supremum on the right. So

$$H(A|B_\theta) = \sup_\alpha[h(\cos^2(\theta - \alpha)) - h(\cos^2\alpha)].$$

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