A new algorithm driven from the view-point of the fluctuation-dissipation theorem in the theory of KM_2O-Langevin equations

Yasunori Okabe

Series ♯166. October 1992

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

♯ 138: S. Izumiya, W.L. Marar, The Euler characteristic of the image of a stable mapping from a closed n-manifold to a $(2n-1)$-manifold, 5 pages. 1992.

♯ 139: Y. Giga, Z. Yoshida, A bound for the pressure integral in a plasma equilibrium, 20 pages. 1992.

♯ 140: S. Izumiya, What is the Clairaut equation ?, 13 pages. 1992.

♯ 141: H. Takamura, Weighted deformation theorem for normal currents, 27 pages. 1992.

♯ 142: T. Morimoto, Geometric structures on filtered manifolds, 104 pages. 1992.

♯ 143: G. Ishikawa, T. Ohmoto, Local invariants of singular surfaces in an almost complex four-manifold, 9 pages. 1992.

♯ 144: K. Kubota, K. Mochizuki, On small data scattering for 2-dimensional semilinear wave equations, 22 pages. 1992.

♯ 145: T. Nakazi, K. Takahashi, Hyponormal Toeplitz operators and extremal problems of Hardy spaces, 30 pages. 1992.

♯ 146: N. Hayashi, T. Ozawa, Remarks on nonlinear Schrödinger equations in one space dimension, 10 pages. 1992.

♯ 147: M. Sato, Interface evolution with Neumann boundary condition, 16 pages. 1992.

♯ 148: Y. Okabe, Langevin equations and causal analysis, 49 pages. 1992.

♯ 149: Y. Giga, S. Takahashi, On global weak solutions of the nonstationary two-phase Stokes flow, 25 pages. 1992.

♯ 150: G. Ishikawa, Determinacy of envelope of the osculating hyperplanes to a curve, 9 pages. 1992.

♯ 151: G. Ishikawa, Developable of a curve and determinacy relative to osculation-type, 15 pages. 1992.

♯ 152: H. Kubo, Global existence of solutions of semilinear wave equations with data of non compact support in odd space dimensions, 25 pages. 1992.

♯ 153: Y. Watatani, Lattices of intermediate subfactors, 33 pages. 1992.

♯ 154: T. Ozawa, On critical cases of Sobolev inequalities, 11 pages. 1992.

♯ 155: M. Ohnuma, M. Sato, Singular degenerate parabolic equations with applications to geometric evolutions, 20 pages. 1992.

♯ 156: S. Izumiya, Perestroikas of optical wave fronts and graphlike Legendrian unfoldings, 13 pages. 1992.

♯ 157: A. Arai, Momentum operators with gauge potentials, local quantization of magnetic flux, and representation of canonical commutation relations, 11 pages. 1992.

♯ 158: S. Izumiya, W.L. Marar, The Euler number of a topologically stable singular surface in a 3-manifold, 11 pages. 1992.

♯ 159: T. Hibi, Cohen-Macaulay types of Cohen-Macaulay complexes, 26 pages. 1992.

♯ 160: A. Arai, Properties of the Dirac-Weyl operator with a strongly singular gauge potential, 26 pages. 1992.

♯ 161: A. Arai, Dirac operators in Boson-Fermion Fock spaces and supersymmetric quantum field theory, 30 pages. 1992.

♯ 162: S. Albeverio, K. Iwata, T. Kolsrud, Random parallel transport on surfaces of finite type, and relations to homotopy, 8 pages. 1992.

♯ 163: S. Albeverio, K. Iwata, T. Kolsrud, Moments of random fields over a family of elliptic curves, and modular forms, 9 pages. 1992.

♯ 164: Y. Giga, M. Sato, Neumann problem for singular degenerate parabolic equations, 12 pages. 1992.

♯ 165: J. Wierzbicki, Y. Watatani, Commuting squares and relative entropy for two subfactors, 18 pages. 1992.

A new algorithm driven from the view-point of the fluctuation-dissipation theorem in the theory of KM_2O-Langevin equations

YASUNORI OKABE

§1. Introduction

We have constructed in [4] a theory of KM_2O-Langevin equations for multi-dimensional weakly stationary processes with discrete time and found, from the point of view of the so-called fluctuation-dissipation theorem in irreversible statistical physics ([2]), *a fluctuation-dissipation theorem* which gives a relation between the fluctuant terms and the deterministic terms in the KM_2O-Langevin equations. *The fluctuation-dissipation theorem* had been already found as *Levinson-Whittle-Wiggins-Robinson algorithm* for *the fitting of AR-models* in the field of system, control and information ([3],[1],[10],[11]). By sublimating certain philosophical structure of the fluctuation-dissipation theorem into *the fluctuation-dissipation principle*, we have applied the theory of KM_2O-Langevin equations to data analysis to develop *a stationary analysis* and *a causal analysis* ([7],[6]). Furthermore, we have also used it to solve the non-linear prediction problem for one-dimensional strictly stationary processes with discrete time and developped *a prediction analysis* as the third project in data analysis ([5],[9],[8]).

Let $\mathbf{X} = (X(n); n \in \mathbf{Z})$ be an $\mathbf{R}^d$-valued weakly stationary process on a probability space $(\Omega, \mathcal{B}, P)$ with expectation vector zero and covariance matrix function R:

$$R(m-n) \equiv E(X(m)\,{}^tX(n)) \qquad (m, n \in \mathbf{Z}), \tag{1.1}$$

where d is any fixed natural number.

This research was partially supported by Grant-in-Aid for Science Research No. 03452011 and No. 04352003, the Ministry of Education, Science and Culture, Japan.

We define, for each $n \in \mathbf{N}$, a block Toeplitz matrix $S_n \in M(nd; \mathbf{R})$ by

$$(1.2) \qquad S_n \equiv \begin{pmatrix} R(0) & R(1) & \dots & R(n-1) \\ R(-1) & R(0) & \dots & R(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ R(-(n-1)) & R(-(n-2)) & \dots & R(0) \end{pmatrix}.$$

In this paper we shall treat the case where the following Toeplitz condition holds:

$$(1.3) \qquad S_n \in GL(nd; \mathbf{R}) \qquad \text{for any } n \in \mathbf{N}.$$

It then follows from the theory of $\mathrm{KM_2O}$-Langevin equations that the time evolution *in the future* (resp. *in the past*) of the process $\mathbf{X}$ is governed by *the forward* (resp. *the backward*) *KM_2O-Langevin equation* (1.5_+) (resp. (1.5_-)) with (1.4):

$$(1.4) \qquad X(0) = \nu_+(0) = \nu_-(0)$$

$$(1.5_+) \qquad X(n) = -\sum_{k=1}^{n-1} \gamma_+(n,k)X(k) - \delta_+(n)X(0) + \nu_+(n) \qquad (n \in \mathbf{N})$$

$$(1.5_-) \qquad X(-n) = -\sum_{k=1}^{n-1} \gamma_-(n,k)X(-k) - \delta_-(n)X(0) + \nu_-(-n) \quad (n \in \mathbf{N}).$$

Here the random force $\nu_+ = (\nu_+(\ell); \ell \in \mathbf{N}^*)$ (resp. $\nu_- = (\nu_-(\ell); \ell \in -\mathbf{N}^*)$ is said to be *the forward* (resp. *the backward*) *KM_2O-Langevin force* associated with $\mathbf{X}$. We call the system $\{\gamma_+(n,k), \gamma_-(n,k), \delta_+(m), \delta_-(m), V_+(\ell), V_-(\ell); \ell \in \mathbf{N}^*, k, m, n \in \mathbf{N}, n > k\}$ whose elements belong to $M(d; \mathbf{R})$ *the KM_2O-Langevin data* associated with the covariance matrix function R of $\mathbf{X}$, where $V_\pm(\ell)$ are the covariance matrices of $\mathrm{KM_2O}$-Langevin forces $\nu_\pm(\pm\ell)$ $(\ell \in \mathbf{N}^*)$:

$$(1.6) \qquad V_+(\ell) \equiv E(\nu_+(\ell)^t\nu_+(\ell)) \quad \text{and} \quad V_-(\ell) \equiv E(\nu_-(-\ell)^t\nu_-(-\ell)).$$

In particular, the subsystem $\{\delta_+(n), \delta_-(n); n \in \mathbf{N}\}$ is called *the partial autocorrelation coefficient* in the field of system, control and information.

The fluctuation-dissipation theorem stated in the first paragraph consists of the following relations $(1.7_\pm)$–(1.11):

Dissipation-Dissipation Theorem ([3],[1],[10],[11],[4]). *For any $n, k \in \mathrm{N}$, $n > k$,*

$$(1.7_{\pm}) \qquad \gamma_{\pm}(n,k) = \gamma_{\pm}(n-1,k-1) + \delta_{\pm}(n)\gamma_{\mp}(n-1,n-k-1),$$

where

$$(1.8) \qquad \gamma_{+}(n,0) \equiv \delta_{+}(n) \quad \textit{and} \quad \gamma_{-}(n,0) \equiv \delta_{-}(n).$$

Fluctuation-Dissipation Theorem ([3],[1],[10],[11],[4]). *For any $n \in \mathrm{N}$,*

$$(1.9_{\pm}) \qquad V_{\pm}(n) = (I - \delta_{\pm}(n)\delta_{\mp}(n))V_{\pm}(n-1)$$

$$(1.10) \qquad \delta_{-}(n)V_{+}(n-1) = V_{-}(n-1)^{t}\delta_{+}(n)$$

$$(1.11) \qquad \delta_{-}(n)V_{+}(n) = V_{-}(n)^{t}\delta_{+}(n).$$

The review of the theory of KM_2O-Langevin equations reminds us that the relations $(1.9_{\pm})$–(1.11) result from the following *Burg's relation*:

Burg's relation ([3],[1],[10],[11],[4]). *For any $n \in \mathrm{N}$,*

$$(1.12) \qquad \sum_{k=0}^{n-1} \gamma_{+}(n,k)R(k+1) = \sum_{k=0}^{n-1} R(k+1)^{t}\gamma_{-}(n,k).$$

As will be shown in §2, Burg's relation is really equivalent to the following relation:

$$(1.13) \qquad E(\nu_{+}(n)^{t}\nu_{-}(-1)) = E(\nu_{+}(1)^{t}\nu_{-}(-n)) \quad (n \in \mathrm{N}^{*}).$$

From certain philosophical understanding of the fluctuation-dissipation theorem , relation (1.13) can be regarded as a special case of the so-called *fluctuation-fluctuation theorem.* The purpose of this paper is to obtain a general algorithm which holds among the mutual covariance matrix functions $I(m,n)$ of the forward KM_2O-Langevin force ν_{+} and the backward KM_2O-Langevin force ν_{-}:

$$(1.14) \qquad I(m,n) \equiv E(\nu_{+}(m)^{t}\nu_{-}(-n)) \qquad (m,n \in \mathrm{N}^{*}).$$

The main theorem is the following:

Fluctuation-Fluctuation Theorem.

$$
\begin{aligned}
&\text{(i)} && I(0,0) = V_+(0) \\
&\text{(ii)} && I(m,0) = I(0,m) = 0 \qquad (m \in \mathrm{N}) \\
&\text{(iii)} && I(m,1) = I(1,m) = -\delta_+(m+1)V_-(m) \quad (m \in \mathrm{N}) \\
&\text{(iv)} && I(m,n) = I(m+1,n-1) + \{\sum_{k=1}^{n-2} I(m+1,k)^t\delta_+(k+1)\}^t\delta_-(n) - \\
& && \qquad - \delta_+(m+1)\{\sum_{k=1}^{m-1}\delta_-(k+1)I(k,n)\} \quad (m,n \geq 2).
\end{aligned}
$$

This algorithm has been already announced in [5] and [9]. We shall show in the future that *the fluctuation-fluctuation theorem*, with *the dissipation-dissipation theorem* and *the fluctuation-dissipation theorem*, will give a characterlization of the weak stationarity of the stochastic process $\mathbf{X}$ in terms of the $\mathrm{KM_2O}$-Langevin forces ν_+ and ν_-.

§2. Proof of Fluctuation-Fluctuation Theorem

For any fixed natural number d, let $\mathbf{X} = (X(n); n \in \mathrm{Z})$ be the same $\mathbf{R}^d$-valued weakly stationary process as stated in §1.

We shall briefly remind of the definition of $\mathrm{KM_2O}$-Langevin forces.

For any d-dimensional stochastic process $\mathbf{Y} = ({}^t(Y_1(n), \cdots, Y_d(n)); \ell \leq n \leq r)$ on the probability space $(\Omega, \mathcal{B}, P)$ $(-\infty \leq \ell < r \leq \infty)$, we define, for each $n_1, n_2, \ell \leq n_1 \leq n_2 \leq r$, the closed subspace $\mathrm{M}_{n_1}^{n_2}(\mathbf{Y})$ of $L^2(\Omega, \mathcal{B}, P)$ by

$$
\text{(2.1)} \qquad \mathrm{M}_{n_1}^{n_2}(\mathbf{Y}) \equiv \text{the closed linear hull of } \{Y_j(n); 1 \leq j \leq d, n_1 \leq n \leq n_2\}.
$$

The forward (resp. the backward) $\mathrm{KM_2O}$-Langevin force $\nu_+ = (\nu_+(n); n \in \mathrm{N}^*)$ (resp. $\nu_- = (\nu_-(\ell); \ell \in -\mathrm{N}^*)$) is an R^d-valued stochastic process defined by

$$
\text{(2.2)} \qquad \begin{cases} \nu_+(n) & \equiv X(n) - P_{\mathrm{M}_0^{n-1}(\mathrm{X})}X(n) \qquad (n \in \mathrm{N}^*) \\ \nu_-(-n) & \equiv X(-n) - P_{\mathrm{M}_{-n+1}^0(\mathrm{X})}X(-n) \qquad (n \in \mathrm{N}^*), \end{cases}
$$

where $\mathrm{M}_0^{-1}(\mathrm{X}) = \mathrm{M}_1^0(\mathrm{X}) = \{0\}$ and $P_{\mathrm{M}_0^{n-1}(\mathrm{X})}$ (resp. $P_{\mathrm{M}_{-n+1}^0(\mathrm{X})}$) stands for the orthogonal projection on the space $\mathrm{M}_0^{n-1}(\mathrm{X})$ (resp. $\mathrm{M}_{-n+1}^0(\mathrm{X})$). In particular, it holds that

(2.3) $\nu_+(0) = \nu_-(0) = X(0)$

(2.4) The stochastic processes $\nu_\pm$ are orthogonal with mean vector zero

(2.5) $\mathrm{M}_0^n(\mathrm{X}) = \mathrm{M}_0^n(\nu_+)$ and $\mathrm{M}_{-n}^0(\mathrm{X}) = \mathrm{M}_{-n}^0(\nu_-)$ $(n \in \mathrm{N}^*)$.

As stated as in §1, the stochastic process X satisfies the forward (resp. the backward) $\mathrm{KM_2O}$-Langevin equation (1.5_+) (resp. (1.5_-)). The dissipation-dissipation theorem $(1.7_\pm)$ and the fluctuation-dissipation theorem $(1.9_\pm)$–(1.11) hold among the $\mathrm{KM_2O}$-Langevin data.

On the other hand, the fundamental quantities $\delta_\pm(\cdot)$ can be calculated from the covariance matrix function R by the following algorithm:

Partial Autocorrelation Coefficient ([3],[1],[10],[11],[4]). *For any* $n \in \mathrm{N}$,

$$(2.6_\pm) \qquad \delta_\pm(n) = -\{R(\pm n) + \sum_{k=0}^{n-2} \gamma_\pm(n-1,k)R(\pm(k+1))\}V_\mp(n-1)^{-1}.$$

Now we shall take eleven steps to prove Fluctuation-Fluctuation Theorem .

(Step 1) We claim that Burg's relation (1.12) is equivalent to a special case of the fluctuation-fluctuation theorem: for any $m \in \mathrm{N}$,

$$I(m,1) = I(1,m).$$

Multiplying both-hand sides of equation (1.5_+) with $n = m$ by ${}^tX(-1)$ from the right and taking an expectation with respect to the probability measure P, we have

$$(2.7) \qquad R(m+1) = -\sum_{k=0}^{m-1} \gamma_+(m,k)R(k+1) + E(\nu_+(m)\,{}^tX(-1)).$$

Since it follows from the weak stationarity of X that $R(m+1) = E(X(m)\,{}^tX(-1)) = E(X(1)\,{}^tX(-m))$, similarly, we multiply both-hand sides of equation (1.5_-) with $n = m$ by $X(1)$ from the left to obtain

$$(2.8) \qquad R(m+1) = -\sum_{k=0}^{m-1} R(k+1)\,{}^t\gamma_-(m,k) + E(X(1)\,{}^t\nu_-(-m)).$$

Therefore, we apply Burg's relation (1.12) to (2.7) and (2.8) to obtain

$$(2.9) \qquad E(\nu_+(m)^t X(-1)) = E(X(1)^t \nu_-(-m)).$$

On the other hand, it follows immediately from ($1.5_\pm$) and (2.3)–(2.5) that

$$(2.10) \qquad E(\nu_+(m)^t X(-1)) = I(m,1) \text{ and } E(X(1)^t \nu_-(-m)) = I(1,m).$$

Hence Step 1 follows from (2.9) and (2.10).

(Step 2) We claim that for any $m \in \mathrm{N}$,

$$I(m,1) = I(1,m) = -\delta_+(m+1)V_-(m).$$

Immediately from (2.6_+), we have

$$R(m+1) = -\sum_{k=0}^{m-1} \gamma_+(m,k)R(k+1) - \delta_+(m+1)V_-(m),$$

which with (2.7) and (2.10) gives Step 2.

(Step 3) We claim that for any $n \in \mathrm{N}$,

$$\text{(i)} \qquad V_+(n) = R(0) + \sum_{k=0}^{n-1} R(n-k)^t \gamma_+(n,k)$$

$$\text{(ii)} \qquad V_-(n) = R(0) + \sum_{k=0}^{n-1} \gamma_-(n,k)^t R(n-k).$$

These follow immediately from (4.5) and (4.6) in the proof of Lemma 4.2 in [4]. Actually Step 3 can be proved from multiplyng both-hand sides of equations ($1.5_\pm$) by ${}^t X(\pm n)$ from the right, taking an expectation with respect to P and using (2.3)–(2.5).

(Step 4) For any $m, n \in \mathrm{N}^*$, put

$$(2.11) \qquad F_n(m) \equiv R(n) + \sum_{k=1}^{m} \gamma_-(m, m-k)R(n+k).$$

Immediately from (ii) in Step 3, we have

$$F_0(m) = V_-(m).$$

(Step 5) We claim that for any $m, n \in \mathrm{N}$,

$$R(m+n) = -\sum_{k=1}^{n} \delta_+(m+k)F_{n-k}(m+k-1) - \sum_{k=0}^{m-1} \gamma_+(m,k)R(k+n).$$

We shall prove Step 5 by mathematical induction. By (2.6_+), we have

$$R(m+n) = -\delta_+(m+n)V_-(m+n-1) - \sum_{k=0}^{m+n-2} \gamma_+(m+n-1,k)R(k+1).$$

In particular, it follows from Step 4 that Step 5 holds for any $m \in \mathrm{N}$ and $n = 1$. Let n_0 be any natural number greater than or equal to 2 and assume that Step 5 holds for any $m \in \mathrm{N}$ and $n = n_0 - 1$. It then follows that

(2.12)
$$\begin{aligned} R(m+n_0) &= R((m+1)+(n_0-1)) = \\ &= -\sum_{k=1}^{n_0-1} \delta_+(m+k)F_{n_0-1-k}(m+k) - \sum_{k=0}^{m} \gamma_+(m+1,k)R(k+n_0-1). \end{aligned}$$

By relation (1.7_+) in the dissipation-dissipation theorem, we have

$$\begin{aligned} &\sum_{k=0}^{m} \gamma_+(m+1,k)R(k+n_0-1) \\ &= \delta_+(m+1)F_{n_0}(m) + \sum_{k=0}^{m-1} \gamma_+(m,k)R(k+n_0). \end{aligned} \tag{2.13}$$

Therefore, we see from (2.12) and (2.13) that Step 5 holds for any $m \in \mathrm{N}$ and $n = n_0$. Hence, we can prove Step 5.

(Step 6) We claim that for any $m, n \in \mathrm{N}$,

$$E(\nu_+(m)^t X(-n)) = -\sum_{k=1}^{n} \delta_+(m+k)F_{n-k}(m+k-1).$$

Multiplying both-hand sides of equation (1.5_+) with $n = m$ by ${}^tX(-n)$ from the right and taking an expectation with respect to P, we have

$$R(m+n) = -\sum_{k=0}^{m-1} \gamma_+(m,k)R(k+n) + E(\nu_+(m)^t X(-n)),$$

which with Step 5 yields Step 6.

(Step 7) We claim that for any $m, n \in \mathrm{N}, m \geq 2$,

$$F_{n-1}(m) = F_{n-1}(m-1) + \delta_-(m)E(\nu_+(m-1)^t X(-n)).$$

Applying (1.7_-) to each term $\gamma_-(m, m-k)$ $(1 \leq k \leq m-1)$ in the definition of $F_{n-1}(m)$, we have

$$\begin{aligned}
F_{n-1}(m) &= R(n-1) + \delta_-(m)R(n-1+m) + \\
&\quad + \sum_{k=1}^{m-1}(\gamma_-(m-1, m-1-k) + \delta_-(m)\gamma_+(m-1, k-1))R(n-1+k) \\
&= F_{n-1}(m-1) + \delta_-(m)\{R(n-1+m) + \sum_{\ell=0}^{m-2}\gamma_+(m-1, \ell)R(n+\ell)\} \\
&= F_{n-1}(m-1) - \delta_-(m)\{\sum_{k=1}^{n}\delta_+(m-1+k)F_{n-k}(m+k)\}.
\end{aligned}$$

Hence, Step 7 follows from Step 6.

(Step 8) We claim that for any $m, n \in \mathrm{N}, m \geq 2$,

$$\begin{aligned}
E(\nu_+(m)^t X(-n)) &= E(\nu_+(m+1)^t X(-n+1)) - \\
&\quad - \delta_+(m+1)\{\sum_{k=2}^{m}\delta_-(k)E(\nu_+(k-1)^t X(-n))\} - \delta_+(m+1)F_{n-1}(1).
\end{aligned}$$

By Step 6, we have

$$E(\nu_+(m)^t X(-n)) = E(\nu_+(m+1)^t X(-n+1)) - \delta_+(m+1)F_{n-1}(m).$$

Hence, a repeat substitution of this into Step 7 yields Step 8.

(Step 9) We claim that for any $m, n \in \mathrm{N}, m, n \geq 2$,

$$\begin{aligned}
E(\nu_+(m)^t \nu_-(-n)) &= E(\nu_+(m+1)^t X(-n+1) - \\
&\quad - \delta_+(m+1)\{\sum_{k=2}^{m}\delta_-(k)E(\nu_+(k-1)^t \nu_-(-n))\} - \delta_+(m+1)F_{n-1}(1) + \\
&\quad + \sum_{\ell=1}^{n-1}\{E(\nu_+(m+1)^t X(-\ell+1)) - \delta_+(m+1)F_{\ell-1}(1)\}^t \gamma_-(n, \ell).
\end{aligned}$$

Substituting the right-hand side of equation (1.5_-) into the term $X(-n)$ of both-hand sides in Step 8, we have

$$\begin{aligned}
&E(\nu_+(m)^t\nu_-(-n)) = E(\nu_+(m+1)^t X(-n+1)) - \\
&\quad -\delta_+(m+1)\{\sum_{k=2}^{m}\delta_-(k)E(\nu_+(k-1)^t\nu_-(-n))\} - \delta_+(m+1)F_{n-1}(1) + \\
&\quad +\sum_{\ell=1}^{n-1}\{E(\nu_+(m)^t X(-\ell)) + \delta_+(m+1)(\sum_{k=2}^{m}\delta_-(k)E(\nu_+(k-1)^t X(-\ell)))\}^t\gamma_-(n,\ell).
\end{aligned}$$

On the other hand, it follows from Step 8 that

$$\begin{aligned}
&\text{the coefficient of } {}^t\gamma_-(n,\ell) \text{ in the relation above} \\
&= E(\nu_+(m+1)^t X(-\ell+1)) - \delta_+(m+1)F_{\ell-1}(1).
\end{aligned}$$

Hence, we have Step 9.

(Step 10) We claim that for any $m, n \in \mathrm{N}, n \geq 2$,

$$\begin{aligned}
&E(\nu_+(m+1)^t X(-n+1)) + \sum_{\ell=1}^{n-1} E(\nu_+(m+1)^t X(-\ell+1))^t\gamma_-(n,\ell) \\
&= E(\nu_+(m+1)^t\nu_-(-n+1)) + \sum_{k=1}^{n-2} E(\nu_+(m+1)^t\nu_-(-k))^t(\delta_-(n)\delta_+(k+1)).
\end{aligned}$$

It is easy to see from (2.3)–(2.5) and (2.20) that Step 10 hods for $n = 2$. Let $n \geq 3$. By equation (1.5_-) with $n = n-1$ and (1.7_-), we have

$$\begin{aligned}
&\text{the upper-hand side in Step 10} \\
=&E(\nu_+(m+1)^t\nu_-(-(n-1))) + \sum_{k=0}^{n-2} E(\nu_+(m+1)^t X(-k))^t(\gamma_-(n,k+1) - \gamma_-(n-1,k)) \\
=&E(\nu_+(m+1)^t\nu_-(-(n-1))) + E(\nu_+(m+1)^t X(-(n-2)))^t(\delta_-(n)\delta_+(n-1)) + \\
&\quad +\sum_{k=0}^{n-3} E(\nu_+(m+1)^t X(-k))^t(\delta_+(n)\gamma_+(n-1,n-k-2)).
\end{aligned}$$

By using equation (1.5_-) with $n = n-2$ and (1.7_-) again, we have

the upper-hand side in Step 10

$$
\begin{aligned}
=&E(\nu_+(m+1)^t\nu_-(-(n-1))) + E(\nu_+(m+1)^t\nu_-(-(n-2)))^t(\delta_-(n)\delta_+(n-1))+ \\
&+\sum_{k=0}^{n-3} E(\nu_+(m+1)^t X(-k))^t\{\delta_+(n)(\gamma_+(n-1,n-k-2)-) - \delta_+(n-1)\gamma_-(n-2,k))\} \\
=&E(\nu_+(m+1)^t\nu_-(-(n-1))) + E(\nu_+(m+1)^t\nu_-(-(n-2)))^t(\delta_-(n)\delta_+(n-1))+ \\
&\qquad + E(\nu_+(m+1)^t X(-(n-3)))^t(\delta_-(n)\delta_+(n-2))+ \\
&\qquad + \sum_{k=0}^{n-4} E(\nu_+(m+1)^t X(-k))^t(\delta_-(n)\gamma_+(n-2,n-k-3)).
\end{aligned}
$$

Hence, repeating the same prodedure, we can prove Step 10.

(Step 11) We claim that for any $n \in \mathrm{N}, n \geq 2$,

$$F_{n-1}(1) + \sum_{\ell=1}^{n-1} F_{\ell-1}(1)^t\gamma_-(n,\ell) = 0.$$

Since it follows from the definition of $F_m(n)$ that for any $m \in \mathrm{N}$,

$$F_m(1) = R(m) + \delta_-(1)R(m+1),$$

we have

$$F_{n-1}(1) + \sum_{\ell=1}^{n-1} F_{\ell-1}(1)^t\gamma_-(n,\ell) = I + \delta_-(1)II, \tag{2.14}$$

where

$$I = R(n-1) + \sum_{\ell=1}^{n-1} R(\ell-1)^t\gamma_-(n,\ell)$$

and

$$II = R(n) + \sum_{\ell=1}^{n-1} R(\ell)^t\gamma_-(n,\ell).$$

We shall show

$$II = -R(0)^t\delta_-(n). \tag{2.15}$$

By (1.7_-) and (2.6_-),

$$\begin{aligned} II &= -V_+(n-1)^t\delta_-(n) + \sum_{\ell=1}^{n-1} R(\ell)^t(\gamma_-(n,\ell) - \gamma_-(n-1,\ell-1)) \\ &= -\{V_+(n-1) - \sum_{\ell=1}^{n-1} R(\ell)^t\gamma_+(n-1,n-\ell-1)\}^t\delta_-(n). \end{aligned}$$

Hence, (2.15) follows from (i) in Step 3.

Further we shall show

$$I = R(0)^t\delta_+(1)^t\delta_-(n). \tag{2.16}$$

When $n = 2$, it follows from (1.7_-) and (2.6_+) that

$$\begin{aligned} I &= R(1) + R(0)^t\gamma_-(2,1) \\ &= -\delta_+(1)R(0) + R(0)^t(\delta_-(1) + \delta_-(2)\delta_+(1)). \end{aligned}$$

Hence, by (1.10), we see that (2.16) holds for $n = 2$.

Let $n \geq 3$. By (1.7_-) and (2.6_-),

$$I = -V_+(n-2)^t\delta_-(n-1) - \sum_{k=0}^{n-3} R(k+1)^t\gamma_-(n-2,k) - \sum_{\ell=1}^{n-1} R(\ell-1)^t\gamma_-(n,\ell).$$

Applying (1.7_-) to the third term in the right-hand side of the relation above, we have

$$\begin{aligned} I = &-V_+(n-2)^t\delta_-(n-1) + R(0)^t\delta_-(n-1) + \\ &+ \sum_{k=0}^{n-3} R(k+1)^t(\gamma_-(n-1,k+1) - \gamma_-(n-2,k)) + \\ &+ \sum_{\ell=1}^{n-1} R(\ell-1)^t\gamma_-(n-1,n-\ell-1)^t\delta_-(n). \end{aligned}$$

By using (1.7_-) again, we have

(2.17)

$$\begin{aligned} I = &-\{V_+(n-2) - R(0) - \sum_{k=0}^{n-3} R(k+1)^t\gamma_+(n-2,n-k-3)\}^t\delta_-(n-1) + \\ &+ \sum_{\ell=1}^{n-1} R(\ell-1)^t\gamma_-(n-1,n-\ell-1)^t\delta_-(n). \end{aligned}$$

It follows from (i) in Step 3 that

$$\text{(2.18)} \qquad \text{the coefficient of } {}^t\delta_-(n-1) \text{ in (2.17)} = 0.$$

Furthemore, we claim

$$\text{(2.19)} \qquad \text{the coefficient of } {}^t\delta_-(n) \text{ in (2.17)} = R(0)^t\delta_+(1).$$

By (1.7_+),

$$\begin{aligned}&\text{the coefficient of } {}^t\delta_-(n) \text{ in (2.17)}\\ &=\sum_{k=0}^{n-3} R(k)^t\gamma_+(n-2,n-3-k)++\{R(n-2)+\sum_{k=0}^{n-3} R(k)^t\gamma_-(n-2,k)\}^t\delta_+(n-1).\end{aligned}$$

On the other hand, by $(3.5)_0$ in [4], we get

$$R(n-2) = -\sum_{k=0}^{n-3} R(k)^t\gamma_-(n-2,k),$$

which results from multiplying both-hand sides of equation (1.5_-) with $n = n-2$ by ${}^tX(0)$ from the right and taking an expectation with respect P. Therefore, we get

$$\text{the coefficient of } {}^t\delta_-(n) \text{ in (2.17)} = \sum_{k=0}^{n-3} R(k)^t\gamma_+(n-2,n-3-k).$$

By repeating the same procedure and using (1.7_+),(1.10) and (2.6_+), we can show that

$$\begin{aligned}\text{the coefficient of } {}^t\delta_-(n) \text{ in (2.17)} &= \sum_{k=0}^{1} R(k)^t\gamma_+(2,1-k)\\ &= R(0)^t(\delta_+(1)+\delta_+(2)\delta_-(1)) - \delta_+(1)R(0)\delta_+(2)\\ &= R(0)^t\delta_+(1)\end{aligned}$$

and so (2.19) holds.

Hence, (2.16) follows from (2.17)-(2.19). Consequently, we see from (2.14)-(2.16) that Step 11 holds.

(Step 12) We are now in the final position to prove Fluctuation-Fluctuation Theorem. (i) and (ii) follow from (2.3)–(2.5). (iii) has been proved in Step 2. By Step 9 and Step 10, we have

$$
\begin{aligned}
&I(m,n)\\
=&I(m+1,n-1)+\{\sum_{k=1}^{n-2} I(m+1,k)^t\delta_+(k+1)\}^t\delta_-(n)-\\
&-\delta_+(m+1)\{\sum_{k=2}^{m}\delta_-(k)I(k-1,n)\}-\delta_+(m+1)\{F_{n-1}(1)+\sum_{\ell=1}^{n-1}F_{\ell-1}(1)^t\gamma_-(n,\ell)\}.
\end{aligned}
$$

Hence, by virtue of Step 11, (iv) holds

Thus we complete the proof of Fluctuation-Fluctuation Theorem. (Q.E.D.)

References

1. Durbin, J., *The fitting of time series model*, Rev. Int. Stat. 28 (1960), 233-244.
2. Kubo, R., *Statistical mechanical theory of irreversible processes I, general theory and simple applications to magnetic and conduction problem*, J. Phys. Soc. Japan 12 (1957), 570-586.
3. Levinson, N., *The Wiener RMS error criterion in filter design and prediction*, J. Math. Phys. 25 (1947), 261-278.
4. Okabe, Y., *On stochastic difference equations for the multi-dimensional weakly stationary time series*, Prospect of Algebraic Analysis (ed. by M. Kashiwara and T. Kawai), Academic Press, 1988, vol.2,601-645.
5. ———, *Application of the theory of KM_2O-Langevin equations to the linear prediction problem for the multi-dimensional weakly stationary time series*, to appear in J. Math. Soc. of Japan.
6. Okabe, Y. and A. Inoue, *The theory of KM_2O-Langevin equations and its applications to data analysis (II): Causal analysis*, to be submitted in Nagoya Math. J.
7. Okabe, Y. and Y. Nakano, *The theory of KM_2O-Langevin equations and its applications to data analysis (I): Stationary analysis*, Hokkaido Math. J. 20 (1991), 45-90.
8. Okabe, Y. and T. Ootsuka, *The theory of KM_2O-Langevin equations and its applications to data analysis (III): Prediction analysis*, in preparation.
9. ———, *Application of the theory of KM_2O-Langevin equations to the non-linear prediction problem for the one-dimensional strictly stationary time series*, to be submitted in Acta Math..
10. Whittle, P., *On the fitting of multivariate autoregressions, and the approximate canonical factorization of a spectral density matrix*, Biometrika 50 (1963), 129-134.
11. Wiggins, R.A. and E.A. Robinson, *Recursive solution to the multichannel fitting problem*, J. Geophys. Res. 70 (1965), 1885-1891.

Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060
Japan