RANDOM COLLISION MODEL FOR INTERACTING POPULATIONS OF TWO SPECIES AND ITS STRONG LAW OF LARGE NUMBERS

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RANDOM COLLISION MODEL FOR INTERACTING POPULATIONS OF TWO SPECIES AND ITS STRONG LAW OF LARGE NUMBERS

BY YASUNORI OKABE, HAJIME MANO, AND YOSHIAKI ITOH

1. Introduction

Problems of interspecific competitions have been studied by many authors since Lotka [6] and Volterra [8]. Ehrenfest's urn model was discussed by Kac [3] and Moran [7] studied an urn model for the random genetic drift. Itoh [1, 2] introduced a random collision model which is an urn model for competing species in finite numbers of individuals of several types interacting with each other and studied the problem of coexistence of species.

Kogan, Liptser, Shiryayev and Smorodinski [4, 5] investigated a queuing model which is formulated in the framework of semi-martingales and proved both the weak law of large numbers and the central limit theorem for the model by using stochastic calculus.

In this paper we shall discuss the random collision model of two species in [1, 2] which is described by the random time change of a Poisson process. At first we aim to show that our model has a similar stochastic structure to the one of the queuing model in [4, 5]. Therefore the weak law of large numbers for our model can be proved by the same method as in [4, 5]. The second purpose of this paper is to prove the strong law of large numbers for our random collision model.

2. Model and Solution

Let us consider a population of two types of individuals in which individuals randomly interact with each other. Changes occur by interactions only between particles of different types. If two individuals of different types interact, then two individuals of the dominant type result from the interaction. Hence the total number of the particles is invariant under interactions.

We set any positive integer M which denotes the total number of the particles. For each j, $j = 1, 2$, let $X_j^{(M)}(*)$ be a stochastic process which denotes the number of individual of type j. We assume that $X_1^{(M)}(*)$ is dominant and that each of the individuals is described by the time change of a standard Poisson process $N(*)$ in a differential form as

$$
(2.1) \qquad \begin{cases} dX_1^{(M)}(t) = dN(\frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds), \\ dX_2^{(M)}(t) = -\, dN(\frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds), \end{cases}
$$

where λ is a positive constant. This is also written in the integral form as

$$
(2.2) \qquad \begin{cases} X_1^{(M)}(t) = X_1^{(M)}(0) + N(\frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds), \\ X_2^{(M)}(t) = X_2^{(M)}(0) - N(\frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds), \\ X_1^{(M)}(0) + X_2^{(M)}(0) = M, \end{cases}
$$

where $X_j^{(M)}(0)$ are initial values of $X_j^{(M)}(*)$ $(j = 1, 2\ )$.

Now we shall prove the existence and uniqueness of the solution of equation (2.2). We denote by $\{\tau_i\}_{i \geq 0}$ the set of the jump time of the standard Poisson process $N(*)$ $(\tau_0 = 0)$.

Theorem 2.1. *There exists a unique solution of equation* (2.2) *and it is represented in the form*

$$
(2.3) \qquad X_1^{(M)}(t) = X_1^{(M)}(0) - 1 + \sum_{i=0}^{M - X_1^{(M)}(0)} \chi_{[\sigma_i^{(M)},\infty)}(t),
$$

$$
(2.4) \qquad X_2^{(M)}(t) = X_2^{(M)}(0) + 1 - \sum_{i=0}^{M - X_1^{(M)}(0)} \chi_{[\sigma_i^{(M)},\infty)}(t),
$$

where $\sigma_k^{(M)}$ $(0 \leq k \leq M)$ *are defined by*

$$
(2.5) \qquad \begin{cases} \sigma_0^{(M)} = 0, \\ \sigma_k^{(M)} = \infty \quad for \ \ 1 \leq k \leq M, \ \ X_1^{(M)}(0) = 0 \ \ or \ \ M, \\ \sigma_k^{(M)} = \sum_{i=1}^{k} \frac{\tau_i - \tau_{i-1}}{\lambda(X_1^{(M)}(0) + i - 1)(1 - (X_1^{(M)}(0) + i - 1)/M)} \\ \qquad\qquad for \ \ 1 \leq k \leq M - X_1^{(M)}(0), \ \ X_1^{(M)}(0) \neq 0, M, \\ \sigma_k^{(M)} = \infty \quad for \ \ k \geq M - X_1^{(M)}(0) + 1\ , \ X_1^{(M)}(0) \neq 0, M. \end{cases}
$$

Proof. Let $X_j^{(M)}(*)$ $(j = 1,2)$ be the solution of equation (2.2). For each fixed $t \in [0,\infty)$, we define

$$T^{(M)}(t) = \frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds. \tag{2.6}$$

Every time when the function $T^{(M)}(*)$ comes to the jump time τ_k of the standard Poisson process $N(*)$, the stochastic process $X_1^{(M)}(*)$ increases in the width of one. We define $\sigma_k^{(M)}$ by

$$\begin{aligned} \sigma_0^{(M)} &= 0, \\ \sigma_k^{(M)} &= \inf\{t \geq 0; T^{(M)}(t) = \tau_k\} \quad (1 \leq k \leq M). \end{aligned} \tag{2.7}$$

When $X_1^{(M)}(0) = 0$ or M, we see that $T^{(M)}(t) = 0$, and so, $X_1^{(M)}(t) = X_1^{(M)}(0)$. It is clear that (2.3), (2.4) and (2.5) hold.

When $X_1^{(M)}(0) \neq 0, M$, if $\sigma_{k-1}^{(M)} \leq t < \sigma_k^{(M)}$ for $1 \leq k \leq M - X_1^{(M)}(0)$, and so, $\tau_{k-1} \leq T^{(M)}(t) < \tau_k$, then

$$X_1^{(M)}(t) = X_1^{(M)}(0) + N(T^{(M)}(t)) = X_1^{(M)}(0) + k - 1.$$

Hence

$$\begin{aligned} X_1^{(M)}(t) &= X_1^{(M)}(0) - 1 + \sum_{i=0}^{k-1} 1 \\ &= X_1^{(M)}(0) - 1 + \sum_{i=0}^{M} \chi_{[\sigma_i^{(M)},\infty)}(t). \end{aligned}$$

If $t \geq \sigma^{(M)}_{M-X_1^{(M)}(0)}$, then $T^{(M)}(t) = \tau^{(M)}_{M-X_1^{(M)}(0)}$, and so,

$$\begin{aligned} X_1^{(M)}(t) &= X_1^{(M)}(0) + N(T^{(M)}(t)) \\ &= X_1^{(M)}(0) - 1 + \sum_{i=0}^{M-X_1^{(M)}(0)} 1 \\ &= X_1^{(M)}(0) - 1 + \sum_{i=0}^{M} \chi_{[\sigma_i^{(M)},\infty)}(t). \end{aligned}$$

Therefore we see that (2.3) and (2.4) hold. It can be seen from (2.7) that the random times $\sigma_k^{(M)}$ satisfy a recursive relation (2.5).

Conversely let $X_1^{(M)}(*)$ be a stochastic process defined by (2.3) and put $X_2^{(M)}(*) = M - X_1^{(M)}(*)$. It is easy to see that $X_j^{(M)}(*)$ are right continuous and have the limit from the left-hand side $(j = 1,2)$.

When the time t is involved in the interval $[\sigma_{k-1}^{(M)}, \sigma_k^{(M)})$ $(1 \le k \le M - X_1^{(M)}(0))$, we have the estimate:

$$\begin{aligned}\tau_{k-1} \le T^{(M)}(t) =& \frac{\lambda}{M}\sum_{i=1}^{k-1}\int_{\sigma_{i-1}^{(M)}}^{\sigma_i^{(M)}} X_1^{(M)}(s)X_2^{(M)}(s)ds + \frac{\lambda}{M}\int_{\sigma_{k-1}^{(M)}}^{t} X_1^{(M)}(s)X_2^{(M)}(s)ds \\ =& \sum_{i=1}^{k-1}\frac{\lambda}{M}(X_1^{(M)}(0)+(i-1))(M-X_1^{(M)}(0)-(i-1))(\sigma_i^{(M)}-\sigma_{i-1}^{(M)}) \\ &+ \frac{\lambda}{M}(X_1^{(M)}(0)+(k-1))(M-X_1^{(M)}(0)-(k-1))(t-\sigma_{k-1}^{(M)}) \\ =& \tau_{k-1} + \frac{\lambda}{M}(X_1^{(M)}(0)+(k-1))(M-X_1^{(M)}(0)-(k-1))(t-\sigma_{k-1}^{(M)}) < \tau_k.\end{aligned}$$

Hence $N(T^{(M)}(t)) = k-1$.

On the other hand, it follows from (2.3) that when the time t is involved in the interval $[\sigma_{k-1}^{(M)}, \sigma_k^{(M)})$, $X_1^{(M)}(t) = X_1^{(M)}(0) - 1 + k$, and so, $X_1^{(M)}(t) = X_1^{(M)}(0) + N(T^{(M)}(t))$.

When $t \ge \sigma_{M-X_1^{(M)}(0)}$, we find that

$$\begin{aligned}T^{(M)}(t) =& \frac{\lambda}{M}\sum_{i=1}^{M-X_1^{(M)}(0)}\int_{\sigma_{i-1}^{(M)}}^{\sigma_i^{(M)}} X_1^{(M)}(s)X_2^{(M)}(s)ds + \frac{\lambda}{M}\int_{\sigma_{M-X_1^{(M)}(0)}^{(M)}}^{t} X_1^{(M)}(s)X_2^{(M)}(s)ds \\ =& \sum_{i=1}^{M-X_1^{(M)}(0)}\frac{\lambda}{M}(X_1^{(M)}(0)+(i-1))(M-X_1^{(M)}(0)-(i-1))(\sigma_i^{(M)}-\sigma_{i-1}^{(M)}) \\ =& \tau_{M-X_1^{(M)}(0)}.\end{aligned}$$

Hence $X_1^{(M)}(t) = X_1^{(M)}(0) + (M - X_1^{(M)}(0)) = X_1^{(M)}(0) + N(T^{(M)}(t))$.

Consequently $X_j^{(M)}(*)$ $(j = 1, 2)$ satisfy (2.2). $\square$

3. A stochastic structure of the model and its weak law of large numbers

From now on, we assume that $X_1^{(M)}(0)$ is independent of $N(*)$ and define a reference family $(\mathcal{F}_t^{(M)})_{t\ge 0}$ by

$$\mathcal{F}_t^{(M)} = \sigma(X_1^{(M)}(0)) \vee \sigma(N(s); 0 \le s \le t). \tag{3.1}$$

Similarly as in (2.6), we define for each $t \in [0, \infty)$ a random time $T^{(M)}(t)$ by

$$T^{(M)}(t) = \frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds. \tag{3.2}$$

At first we shall prove the following lemma for investigating certain stochastic structure of our random collision model.

Lemma 3.1. *For each $t \in [0,\infty)$, $T^{(M)}(t)$ is a stopping time with respect to the reference family $(\mathcal{F}_t^{(M)})_{t\geq 0}$.*

Proof. To be proved is that for any $u \in [0,\infty)$,

$$(T^{(M)}(t) \leq u) \equiv \{\omega; T^{(M)}(t)(\omega) \leq u\} \in \mathcal{F}_u^{(M)}. \tag{3.3}$$

Since $T^{(M)}(*)$ is a strictly increasing and continuous function in $[0, \sigma_{M-X_1^{(M)}(0)})$, we see that for any u, $\tau_k \leq u < \tau_{k+1} \leq \tau_{M-X_1^{(M)}(0)}$,

$$\begin{aligned} t_u(\tau_0,\tau_1,\tau_2,\cdots,\tau_k) &\equiv T^{(M)-1}(u) \\ &= \sum_{i=1}^{k} \frac{\tau_i - \tau_{i-1}}{\lambda(X_1^{(M)}(0)+i-1)(1-(X_1^{(M)}(0)+i-1)/M)} \\ &\quad + \frac{u-\tau_k}{\lambda(X_1^{(M)}(0)+k-1)(1-(X_1^{(M)}(0)+k-1)/M)}. \end{aligned}$$

Now we decompose $(T^{(M)}(t) \leq u)$ into

$$\begin{aligned} (T^{(M)}(t) \leq u) =& [\cup_{l=1}^{M-1}\{(X_1^{(M)}(0)=l) \\ &\cap \{\cup_{k=0}^{M-l-1}(\tau_k \leq u < \tau_{k+1}, T^{(M)}(t) \leq u)\} \cup (\tau_{M-l} \leq u, T^{(M)}(t) \leq u)\}] \\ &\cup [\{(X_1^{(M)}(0)=0) \cap (T^{(M)}(t) \leq u)\} \cup \{(X_1^{(M)}(0)=M) \cap (T^{(M)}(t) \leq u)\}] \\ =& \{\cup_{l=1}^{M-1}(A_l \cup B_l)\} \\ &\cup [(X_1^{(M)}(0)=0) \cup (X_1^{(M)}(0)=M)], \end{aligned}$$

where

$$\begin{aligned} A_l \equiv& (X_1^{(M)}(0)=l) \cap (\cup_{k=0}^{M-l-1}(\tau_k \leq u < \tau_{k+1}, t \leq t_u(\tau_0,\tau_1,\tau_2,\cdots,\tau_k)), \\ B_l \equiv& (X_1^{(M)}(0)=l) \cap (\tau_{M-l} \leq u, T^{(M)}(t) \leq u). \end{aligned}$$

Here we fix any l, $1 \leq l \leq M-1$. Then

$$\begin{aligned} A_l = (X_1^{(M)}(0)=l) \cap \cup_{k=0}^{M-l-1}\{(\tau_k \leq u) \cap \\ (t \leq t_u(\tau_0,\tau_1,\tau_2,\cdots,\tau_k) = \sum_{i=1}^{k}\frac{\tau_i-\tau_{i-1}}{\lambda(l+i-1)(1-(l+i-1)/M)} \\ + \frac{u-\tau_k}{\lambda(l+i-1)(1-(l+i-1)/M)}) \cap (u < \tau_{k+1})\}. \end{aligned}$$

Since for any k, $0 \le k \le M - l$,

$$(t \le t_u(\tau_0, \tau_1, \tau_2, \cdots, \tau_k)) \in \mathcal{F}_{\tau_k} \equiv \{C \in \mathcal{F}_\infty; C \cap (\tau_k \le v) \in \mathcal{F}_v \quad for \quad any \quad v > 0\},$$

we see that $(\tau_k \le u) \cap (t \le t_u(\tau_0, \tau_1, \tau_2, \cdots, \tau_k)) \in \mathcal{F}_u$. Moreover, since $(u < \tau_{k+1}) \in \mathcal{F}_u$, it follows that $A_l \in \mathcal{F}_u$.

In addition to this, for any fixed l, $1 \le l \le M-1$, $(X_1^{(M)}(0) = l) \cap (\tau_{M-l} \le u, T^{(M)}(t) \le u) = (X_1^{(M)}(0) = l) \cap (\tau_{M-l} \le u)$, because $T^{(M)}(*)$ has the maximal value $\tau_{M - X_1^{(M)}(0)}$ when $X_1^{(M)}(0) \ne 0, M$. Since $(\tau_{M-l} \le u) \in \mathcal{F}_u$, we see that $B_l \in \mathcal{F}_u$.

Consequently, (3.3) is proved.$\diamondsuit$

Set

$$\mathcal{M}^{(M)}(t) = N(T^{(M)}(t)) - T^{(M)}(t), \tag{3.4}$$

and

$$\tilde{\mathcal{F}}_t^{(M)} = \mathcal{F}^{(M)}_{T^{(M)}(t)}. \tag{3.5}$$

We have

$$X_1^{(M)}(t) = X_1^{(M)}(0) + \mathcal{M}^{(M)}(t) + T^{(M)}(t). \tag{3.6}$$

Theorem 3.1. *The stochastic process $X_1^{(M)}(*)$ is an $(\tilde{\mathcal{F}}_t^{(M)})_{t\ge 0}$-semi-martingale such that*

(i) *$\mathcal{M}^{(M)}(t)$ is a square-integrable $(\tilde{\mathcal{F}}_t^{(M)})_{t\ge 0}$-martingale,*
(ii) *$T^{(M)}(t)$ is a continuous increasing and $(\tilde{\mathcal{F}}_t^{(M)})_{t\ge 0}$-adapted process,*
(iii) *$< \mathcal{M}^{(M)} >_t = T^{(M)}(t)$.*

Proof. The martingale part and the quadratic variational part of the standard Poisson process $N(*)$ is expressed as $N(t) - t$ and t, respectively.

By virtue of Lemma 3.1, we can apply the optional sampling theorem due to Doob to $X_1^{(M)}(*)$ to get the above facts. $\square$

Let $u_1 = u_1(t)$ and $u_2 = u_2(t)$ $(t \in [0, \infty))$ be the solution of the deterministic system

$$\begin{cases} \dfrac{du_1(t)}{dt} = \lambda u_1(t) u_2(t), \\ \dfrac{du_2(t)}{dt} = -\lambda u_1(t) u_2(t). \end{cases} \tag{3.7}$$

Now, we shall discuss the convergence of $\frac{X_1^{(M)}(t)}{M}$ and $\frac{X_2^{(M)}(t)}{M}$ to $u_1(t)$ and $u_2(t)$, as M tends to infinity. By the same method as in the queuing model by Liptser-Shiryayev [5], we can show the following lemma.

Lemma 3.2. *Let $w = w(t)$ $(t \in [0, \infty))$ be a solution of the differential equation*

$$\frac{dw(t)}{dt} = f(w(t)) \tag{3.8}$$

satisfying the property $\inf_{s \le t} w(s) > 0$ for any $t > 0$. Here $f = f(x)$ is a non-negative function on $[0, \infty)$ with local Lipschitz condition.

For each $M > 0$, let the stochastic process $Y^{(M)}(*)$ be an $(\mathcal{H}_t^{(M)})_{t \ge 0}$-semi-martingale such that

(i) $Y^{(M)}(t) = Y^{(M)}(0) + \mathcal{M}^{(M)}(t) + \mathcal{A}^{(M)}(t)$,

(ii) $\mathcal{M}^{(M)}(t)$ is a square-integrable $(\mathcal{H}_t^{(M)})_{t \ge 0}$-martingale,

(iii) $\mathcal{A}^{(M)}(t)$ is a continuous increasing $(\mathcal{H}_t^{(M)})_{t \ge 0}$-adapted process,

(iv) $\mathcal{A}^{(M)}(t) = \int_0^t M f(\frac{Y^{(M)}(s)}{M}) ds$,

(v) $< \mathcal{M}^{(M)} >_t = \mathcal{A}^{(M)}(t)$.

Moreover we assume

$$\lim_{M \to \infty} \frac{Y^{(M)}(0)}{M} = w(0) \quad \textit{in} \quad \textit{probability}.$$

Then for any $t \in (0, \infty)$

$$\lim_{M \to \infty} \frac{Y^{(M)}(t)}{M} = w(t) \quad \textit{in} \quad \textit{probability}.$$

By applying Lemma 3.2 to Theorem 3.1, we have

Theorem 3.2 (The weak law). *We assume*

$$\begin{cases} \lim_{M \to \infty} \dfrac{X_1^{(M)}(0)}{M} = u_1(0) \quad \textit{in} \quad \textit{probability}, \\ 0 < u_1(0) < 1 \quad \textit{and} \quad u_1(0) + u_2(0) = 1. \end{cases}$$

Then for any $t \in (0, \infty)$,

$$\begin{cases} \lim_{M \to \infty} \sup_{0 < s \le t} \left| \dfrac{X_1^{(M)}(s)}{M} - u_1(s) \right| = 0 \quad \textit{in} \quad \textit{probability}, \\ \lim_{M \to \infty} \sup_{0 < s \le t} \left| \dfrac{X_2^{(M)}(s)}{M} - u_2(s) \right| = 0 \quad \textit{in} \quad \textit{probability}. \end{cases}$$

4. The strong law of large numbers

As a strong assertion of Theorem 3.2, we shall show

Theorem 4.1 (The strong law). *We assume*

$$\begin{cases} \lim_{M\to\infty} \dfrac{X_1^{(M)}(0)}{M} = u_1(0) \quad a.s., \\ 0 < u_1(0) < 1 \quad and \quad u_1(0) + u_2(0) = 1. \end{cases}$$

Then for any $t \in (0, \infty)$,

$$\begin{cases} \lim_{M\to\infty} \dfrac{X_1^{(M)}(t)}{M} = u_1(t) \quad a.s., \\ \lim_{M\to\infty} \dfrac{X_2^{(M)}(t)}{M} = u_2(t) \quad a.s. \end{cases}$$

Proof. We rewrite the solution in the integral form

$$\frac{X_1^{(M)}(t)}{M} = \frac{X_1^{(M)}(0)}{M} + \int_0^{\frac{M-X_1^{(M)}(0)}{M}} \varphi_M(s)ds, \tag{4.1}$$

where the function φ_M is defined by

$$\varphi_M(s) = \chi_{[\sigma_k^{(M)},\infty)}(t) \quad for \quad \frac{k-1}{M} \le s < \frac{k}{M}, \quad 1 \le k \le M. \tag{4.2}$$

We fix any element $\omega \in \Omega$ and $s \in [0, \infty)$ such that

$$\lim_{M\to\infty} \frac{X_1^{(M)}(0)(\omega)}{M} = u_1(0), \tag{4.3}$$

$$\lim_{M\to\infty} \frac{1}{M}\sum_{i=1}^{M}(\tau_i(\omega) - \tau_{i-1}(\omega)) = 1, \tag{4.4}$$

$$\lim_{M\to\infty} \frac{1}{M}\sum_{i=1}^{M}(\tau_i(\omega) - \tau_{i-1}(\omega))^2 = 2, \tag{4.5}$$

$$0 \le s < 1 - u_1(0), \quad 0 < u_1(0) < 1. \tag{4.6}$$

We note that the set of the element $\omega \in \Omega$ satisfying (4.3) - (4.5) has the probability one. In the sequal we shall abbreviate the variable ω.

[Step 1] We claim that

$$\frac{X_1^{(M)}(t)}{M} = \frac{X_1^{(M)}(0)}{M} + \int_0^{1-u_1(0)} \varphi_M(s)ds + o(1) \quad (M \to \infty).$$

By (4.1), we have

$$\frac{X_1^{(M)}(t)}{M} = \begin{cases} \frac{X_1^{(M)}(0)}{M} + \int_0^{1-u_1(0)} \varphi_M(s)ds + \int_{1-u_1(0)}^{\frac{M-X_1^{(M)}(0)}{M}} \varphi_M(s)ds \\ \qquad for\ 1-u_1(0) \le \frac{M-X_1^{(M)}(0)}{M}, \\ \frac{X_1^{(M)}(0)}{M} + \int_0^{1-u_1(0)} \varphi_M(s)ds - \int_{\frac{M-X_1^{(M)}(0)}{M}}^{1-u_1(0)} \varphi_M(s)ds \\ \qquad for\ \frac{M-X_1^{(M)}(0)}{M} \le 1-u_1(0). \end{cases}$$

Since $0 \le \varphi_M(s) \le 1$

$$\frac{X_1^{(M)}(t)}{M} = \frac{X_1^{(M)}(0)}{M} + \int_0^{1-u_1(0)} \varphi_M(s)ds \pm O(|u_1(0) - \frac{X_1^{(M)}(0)}{M}|)$$

Hence, the claim holds.

The convergence of the solution to the deterministic system is now reduced to the integrand $\varphi_M(s)$. For that purpose we shall show the convergence of $\sigma_k^{(M)}$.

We take for each $M > 0$ an integer k_M such that $\frac{k_M-1}{M} \le s < \frac{k_M}{M}$. It is to be noted that $\frac{k_M}{M}$ converges to s as M tends to infinity. We decompose $\sigma_{k_M}^{(M)}$ into

$$\begin{aligned} \sigma_{k_M}^{(M)} &= \textstyle\sum_{1\le i}\Big[\frac{1}{M}(\chi_{\{i\le k_M\}} - \chi_{\{i\le[Ms]\}})\frac{\tau_i-\tau_{i-1}}{\lambda\frac{(X_1^{(M)}(0)+i-1)}{M}(1-\frac{(X_1^{(M)}(0)+i-1)}{M})} \\ &\quad + \frac{1}{M}\chi_{\{i\le[Ms]\}}\Big\{\frac{\tau_i-\tau_{i-1}}{\lambda\frac{(X_1^{(M)}(0)+i-1)}{M}(1-\frac{(X_1^{(M)}(0)+i-1)}{M})} - \frac{\tau_i-\tau_{i-1}}{\lambda(u_1(0)+\frac{i-1}{M})(1-u_1(0)-\frac{i-1}{M})}\Big\} \\ &\quad + \frac{1}{M}\chi_{\{i\le[Ms]\}}\Big\{\frac{\tau_i-\tau_{i-1}}{\lambda(u_1(0)+\frac{i-1}{M})(1-u_1(0)-\frac{i-1}{M})} - \frac{1}{\lambda(u_1(0)+\frac{i-1}{M})(1-u_1(0)-\frac{i-1}{M})}\Big\} \\ &\quad + \frac{1}{M}\chi_{\{i\le[Ms]\}}\frac{1}{\lambda(u_1(0)+\frac{i-1}{M})(1-u_1(0)-\frac{i-1}{M})}\Big] \\ &= \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4. \end{aligned}$$

[Step 2] We claim that $\lim_{M\to\infty} \mathcal{S}_1 = 0$.

By (4.5), there exists a positive constant C_1 such that

$$\frac{1}{M}\sum_{i=\min\{k_M,[Ms]\}+1}^{\max\{k_M,[Ms]\}}(\tau_i-\tau_{i-1})^2<C_1.$$

Moreover, it follows from (4.3) that there is a positive integer M_0 such that for any $M\geq M_0$

$$\frac{u_1(0)}{2}<\frac{X_1^{(M)}(0)}{M}<u_1(0)+\frac{1-u_1(0)-s}{2}. \tag{4.7}$$

Hence,

$$\begin{aligned}|\mathcal{S}_1| &\leq \{\frac{1}{M}\sum_{i=\min\{k_M,[Ms]\}+1}^{\max\{k_M,[Ms]\}}(\tau_i-\tau_{i-1})^2\}^{\frac{1}{2}}\{\frac{1}{M}\sum_{i=\min\{k_M,[Ms]\}+1}^{\max\{k_M,[Ms]\}}\frac{1}{(\lambda\frac{X_1^{(M)}(0)+i-1}{M}\frac{M-(X_1^{(M)}(0)+i-1)}{M})^2}\}^{\frac{1}{2}}\\
&\leq \{C_1\}^{\frac{1}{2}}\{\frac{1}{M}\sum_{i=\min\{Ms,k_M\}+1}^{\max\{Ms,k_M\}}(\frac{1}{\lambda\frac{u_1(0)}{2}\frac{1-u_1(0)-s)}{2}})^2\}^{\frac{1}{2}}\\
&= \{C_1\}^{\frac{1}{2}}\{\frac{|k_M-Ms|}{M}(\frac{1}{\lambda\frac{u_1(0)}{2}\frac{1-u_1(0)-s)}{2}})^2\}^{\frac{1}{2}}.\end{aligned}$$

Therefore it follows from the convergence of $\frac{k_M}{M}\to s$ as $M\to\infty$ that $\lim_{M\to\infty}\mathcal{S}_1=0$.

[Step 3] We claim that $\lim_{M\to\infty}\mathcal{S}_2=0$.

By (4.5), there exists a positive constant C_2 such that

$$\frac{1}{M}\sum_{i=1}^{[Ms]}(\tau_i-\tau_{i-1})^2<C_2.$$

Hence, by using (4.7) of [Step 2], we have the estimate:

$$
\begin{aligned}
|\mathcal{S}_2| \leq & \{\frac{1}{M}(\sum_{i=1}^{[Ms]}(\tau_i - \tau_{i-1})^2) \\
& (\sum_{i=1}^{[Ms]}\frac{1}{M}(\frac{(1-\frac{X_1^{(M)}(0)}{M}-\frac{i-1}{M})-(u_1(0)+\frac{i-1}{M})}{\lambda(\frac{X_1^{(M)}(0)}{M}+\frac{i-1}{M})(1-\frac{X_1^{(M)}(0)}{M}-\frac{i-1}{M})(u_1(0)+\frac{i-1}{M})(1-u_1(0)-\frac{i-1}{M})})^2\}^{\frac{1}{2}} \\
& |\frac{X_1^{(M)}(0)}{M} - u_1(0)| \\
\leq & \{C_2\}^{\frac{1}{2}}\{s(\frac{1}{\lambda\frac{u_1(0)}{2}u_1(0)(1-u_1(0)-s)} \\
& + \frac{1}{\lambda\frac{u_1(0)}{2}\frac{1-u_1(0)-s}{2}(1-u_1(0)-s)})^2\}^{\frac{1}{2}}|\frac{X_1^{(M)}(0)}{M} - u_1(0)|.
\end{aligned}
$$

Therefore, we see from (4.3) that $\lim_{M\to\infty} \mathcal{S}_2 = 0$.

[Step 4] We claim that $\lim_{M\to\infty} \mathcal{S}_3 = 0$.

Now for any arbitrary real number $\epsilon > 0$, we take a natural number N such that $\frac{1}{N} < \frac{\epsilon}{C}$. Here C is a positive constant, which is defined later by (4.8).

Let L be a natural number such that $[Ms]$ divided by L equals N and let r be the remainder: $[Ms] = NL + r$ and $0 \leq r < L$. We note that $M \to \infty$ iff $L \to \infty$. Put

$$
\begin{aligned}
\mathcal{S}_3 &= \frac{1}{M}\sum_{i=1}^{[Ms]} a_{i,M}\xi_i, \\
Z_M &= \frac{L}{M}\sum_{k=1}^{N} a_k(\frac{1}{L}\sum_{i=(k-1)L+1}^{kL}\xi_i),
\end{aligned}
$$

where

$$
\begin{aligned}
a_{i,M} &= \frac{1}{\lambda(u_1(0)+\frac{i-1}{M})(1-u_1(0)-\frac{i-1}{M})}, \\
a_k &= \frac{1}{\lambda(u_1(0)+\frac{s(k-1)}{N})(1-u_1(0)-\frac{s(k-1)}{N})}, \\
\xi_i &= \tau_i - \tau_{i-1} - 1.
\end{aligned}
$$

By Schwarz's inequality, we get

$$
\begin{aligned}
|S_3 - Z_M| \leq & |\frac{1}{M}\sum_{k=1}^{N}\sum_{i=(k-1)L+1}^{kL}(a_{i,M} - a_k)\xi_i| + |\frac{1}{M}\sum_{i=NL+1}^{NL+r} a_{i,M}\xi_i| \\
\leq & (\frac{1}{M}\sum_{k=1}^{N}\sum_{i=(k-1)L+1}^{kL}(\xi_i)^2)^{\frac{1}{2}}(\frac{1}{M}\sum_{k=1}^{N}\sum_{i=(k-1)L+1}^{kL}|a_{i,M} - a_k|^2)^{\frac{1}{2}} \\
& + (\frac{s}{N}\frac{1}{L}\sum_{i=NL+1}^{NL+r}(\xi_i)^2)^{\frac{1}{2}}(\frac{s}{N}\frac{1}{(\lambda u_1(0)(1 - u_1(0) - s))^2})^{\frac{1}{2}}.
\end{aligned}
$$

Since the running suffix i in the region $(k-1)L + 1 \leq i \leq kL$ of the first term of the right-hand side means that

$$
\begin{aligned}
\frac{i-1}{M} - \frac{s(k-1)}{N} &> -s\frac{(k-1)(r+1)}{N^2L} > -s\frac{1}{N}, \\
\frac{i-1}{M} - \frac{s(k-1)}{N} &\leq s\frac{N(L-1) - r(k-1)}{N^2L} < 2s\frac{1}{N},
\end{aligned}
$$

we have the estimate:

$$
\begin{aligned}
& \frac{1}{M}\sum_{k=1}^{N}\sum_{i=(k-1)L+1}^{kL}|a_{i,M} - a_k|^2 \\
& = |\frac{1}{M}\sum_{k=1}^{N}\sum_{i=(k-1)L+1}^{kL}|\frac{(\frac{i-1}{M} - \frac{s(k-1)}{N})\{(u_1(0) + \frac{i-1}{M}) - (1 - u_1(0) - \frac{s(k-1)}{N})\}}{\lambda(u_1(0) + \frac{i-1}{M})(1 - u_1(0) - \frac{i-1}{M})(u_1(0) + \frac{s(k-1)}{N})(1 - u_1(0) - \frac{s(k-1)}{N})}|^2 \\
& \leq \frac{(2s)^2}{MN^2}(\sum_{k=1}^{N}\sum_{i=(k-1)L+1}^{kL}(\frac{1}{\lambda(1 - u_1(0) - \frac{i-1}{M})(u_1(0) + \frac{s(k-1)}{N})(1 - u_1(0) - \frac{s(k-1)}{N})} \\
& \qquad + \frac{1}{\lambda(u_1(0) + \frac{i-1}{M})(1 - u_1(0) - \frac{i-1}{M})(u_1(0) + \frac{s(k-1)}{N})})^2 \\
& \qquad \leq \frac{4s^3}{N^2}(\frac{1}{\lambda u_1(0)(1 - u_1(0) - s)^2} + \frac{1}{\lambda u_1(0)^2(1 - u_1(0) - s)})^2.
\end{aligned}
$$

By (4.5), there exists a positive constant C_3 such that

$$
\max\{\max_{1\leq k\leq N}\{\frac{1}{L}\sum_{i=(k-1)L+1}^{kL}(\tau_i - \tau_{i-1})^2\}, \frac{1}{L}\sum_{i=NL+1}^{NL+r}(\tau_i - \tau_{i-1})^2\} < C_3.
$$

Hence,

$$|\mathcal{S}_3 - Z_M| \le [\{sC_3\}^{\frac{1}{2}}\{2s^3(\frac{1}{\lambda u_1(0)(1-u_1(0)-s)^2} + \frac{1}{u_1(0)^2(1-u_1(0)-s)})^2\}^{\frac{1}{2}}$$
$$+ \{sC_3\}^{\frac{1}{2}}\{s\frac{1}{(\lambda u_1(0)(1-u_1(0)-s))^2}\}^{\frac{1}{2}}]\frac{1}{N}$$
$$= \frac{C}{N} < \epsilon,$$

where

$$C = s\sqrt{C_3}[\sqrt{2}s(\frac{1}{\lambda u_1(0)(1-u_1(0)-s)^2} + \frac{1}{\lambda u_1(0)^2(1-u_1(0)-s)}) + \frac{1}{\lambda u_1(0)(1-u_1(0)-s)}]. \tag{4.8}$$

This fact yields

$$\lim_{M\to\infty} |\mathcal{S}_3 - Z_M| = 0.$$

On the other hand, noting that $0 \le \frac{L}{M} \le \frac{s}{N}$, we see from (4.4) that $\lim_{M\to\infty} Z_M = 0$.

Therfore it follows that $\lim_{M\to\infty} \mathcal{S}_3 = 0$.

[Step 5] It is easy to see that when M tends to infinity, the fourth sum $\mathcal{S}_4$ is convergent to the non-random function $v(s)$ such that

$$v(s) \equiv \frac{1}{\lambda}\int_0^s \frac{1}{(u_1(0)+p)(1-u_1(0)-p)}dp$$
$$= \frac{1}{\lambda}log\frac{(u_1(0)+s)(1-u_1(0))}{u_1(0)(1-u_1(0)-s)}.$$

[Step 6] It follows from Step 1 - Step 5 that $u_1(t) \equiv \lim_{M\to\infty} \frac{X_1^{(M)}(t)}{M}$ exists and it is equal to

$$u_1(0) + \int_0^{1-u_1(0)} \chi_{[v(s),\infty)}(t)ds = \frac{u_1(0)e^{\lambda t}}{u_1(0)e^{\lambda t} + 1 - u_1(0)}.$$

This is a logistic distribution, which coincides with the solution of the deterministic system (3.7).

Consequently we complete the proof of Theorem 4.1. □

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