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REMARKS ON QUADRATIC NONLINEAR SCHRÖDINGER EQUATIONS

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REMARKS ON QUADRATIC NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We consider the global Cauchy problem for the nonlinear Schrödinger equations with quadratic nonlinearities. Concerning a class of nonlinearities which ensures the global existence of solutions, we prove that Klainerman's condition and Hayashi's condition are equivalent. An explicit solution to Strauss' equation is also given.

1. Introduction.

Over the last decade a great deal of effort has been put into the Cauchy problem for nonlinear Schrödinger equations of the form

$$\text{(NLS)} \qquad i\partial_t u + \frac{1}{2}\Delta u = F(u, \nabla u, \bar{u}, \nabla \bar{u})$$

where u is a complex valued function of $(t,x) \in \mathbb{R} \times \mathbb{R}^n$, $F(u, \nabla u, \bar{u}, \nabla \bar{u})$ is a local nonlinearity defined pointwisely on $\mathbb{R} \times \mathbb{R}^n$, and F is a complex valued function on $\mathbb{C} \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^n$ with derivatives vanishing at the origin up to the first order [1-11,13-16,18-20,22]. As compared to the special case where the nonlinearity involves no gradient terms, there are fewer results in the general case, even on the basic problems, such as global existence problem for small amplitude solutions. One reason for this situation consists in the difficulty arising from the loss of derivatives in the energy estimates, which is a natural consequence of the presence of gradient terms in the nonlinearity. This leads to the problem of seeking an admissible class of nonlinearities which prevent the loss of derivatives. We should remark here that as for the local existence problem for small amplitude solutions, the problem of loss of derivatives has recently solved in a satisfactory way by Kenig, Ponce & Vega [11]. Actually, their result gives a reason to believe that the local solvability requires no specific structure of the nonlinearities. Meanwhile, there is another restriction on the nonlinearity in connection with the rate of decay of the corresponding free solutions in the uniform norm. In most cases of the global Cauchy problem with small data we need the integrability of the energy norms of nonlinear term, which is reasonably measured in terms of the rate of decay of free solutions. To be specific, for quadratic nonlinearity we could expect that the nonlinear term would be of order $O(|t|^{-n/2})$ in the energy norms as $t \to \pm\infty$, provided that the solution behaves like $O(|t|^{-n/2})$ in the uniform norm and like $O(1)$ in the energy norms. This requires the assumption $n \geq 3$, and at the same time, the critical case becomes $n = 2$. Accordingly the best results available for the quadratic NLS have concerned the case $n \geq 3$ so far. Related results on the scattering problem are discussed in [3, 15, 21, 22].

As described above, for the global existence of small amplitude solutions the nonlinearity is supposed to satisfy two different assumptions. A simple condition proposed by Klainerman [13] (see also [14, 16]) is the following:

(K) Every component of $\partial_{\nabla u} F$ is pure imaginary;

$$|F(u, \nabla u, \bar{u}, \nabla\bar{u})| \leq C(|u| + |\nabla u|)^p \quad \text{with} \quad n(p-1)^2 > 2p. \tag{1.1}$$

Under the condition (K) with (1.1) it is shown in [13, 14, 16] that the Cauchy problem for NLS has a unique global solution in the usual Sobolev space H^m of order $m \geq [n/2] + 3$ with further regularity assumptions on the data, where $[s]$ denotes the largest integer less than or equal to s. The condition (K) avoids the difficulty of derivative loss in order that the highest derivatives in the nonlinearity in hand should be dropped after integration by parts in the energy estimates. The assumption $n(p-1)^2 > 2p$ comes from the integrability condition with respect to the time variable and holds when $p = 2$ with $n \geq 5$ or $p = 3$ with $n \geq 2$, but excludes the case $p = 2$ with $n = 3, 4$. Recently, Hayashi [7] proved an analogous result of global existence in the case $p = 2$ with $n = 3, 4$ under the following condition:

$$\text{(H)} \qquad |\mathrm{Im}(\partial_{\nabla u}F\cdot\nabla\psi,\psi)| \le C(\|\nabla u\|_\infty + \|\nabla^2 u\|_\infty)\|\psi\|_2^2 \quad \text{for all} \quad \psi \in H^1;$$

$$(1.2) \qquad |\partial_u F| + |\partial_{\bar u} F| \le C|\nabla u|,$$

where C is independent of u and $\|\cdot\|_p$ and $(\cdot,\cdot)$ denote the norm in L^p and the scalar product in L^2, respectively. The condition (H) emerges naturally out of the energy estimates, while (1.2) seems to have rather technical reasons. (H) avoids the loss of derivatives by pulling the derivative of ψ back to u, where ψ is supposedly one of the objects to be estimated through the energy estimates, namely, $\Gamma^\alpha u$ with Γ being a differential operator which has a special commutation property with $i\partial_t + \frac{1}{2}\Delta$.

Both (K) and (H) determine the structure of nonlinearity which serves as compensation for the loss of derivatives, though they look much different. The condition (K) is expressed in an algebraic way, while (H) is described in terms of estimates. The first purpose in this paper is to clarify the relation between (K) and (H). We prove:

Theorem 1. *Let F be a quadratic nonlinearity of $(u, \nabla u, \bar u, \nabla \bar u) \in \mathbb{C}\times\mathbb{C}^n\times\mathbb{C}\times\mathbb{C}^n$. Let $u \in H^m$ with $m > n/2+2$. Then (K) and (H) are equivalent.*

The condition $m > n/2+2$ is rather natural since (H) requires that the first and second derivatives of u are bounded.

Our second attention turns to the equation

$$(1.3) \qquad i\partial_t u + \frac{1}{2}\Delta u = \lambda(\nabla u)^2$$

where $\lambda \in \mathbb{C}\setminus\{0\}$ and $(\nabla u)^2 = \sum_{j=1}^n(\partial_j u)^2$. The equation (1.3) has first appeared in Strauss' book [18, p.38] as an example of quadratic NLS which is reduced to a cubic NLS by the method of normal forms of Shatah [17]. Subsequently, Hayashi [6] proved the global existence of small analytic solutions to (1.3) for $n = 3, 4$. We notice that:

(i) The nonlinearity $\lambda(\nabla u)^2$ does not satisfy (K) nor (H).

(ii) For (1.3) the conservation laws of charge and of energy do not hold in any reasonable sense.

Recently, in [8] we proposed the method of gauge transformation which is useful to avoid the loss of derivatives even when suitable energy estimates are no longer available. The method is roughly illustrated as follows. If there exists a function f defined in the $n+1$ dimensional space-time, depending nonlocally on $(u, \nabla u, \bar{u}, \nabla \bar{u})$, such that $\partial_{\nabla u} F = -\nabla f$, then by considering the system of equations for finite unknown functions which are supposedly equal to $(u, e^f \partial^\alpha u : \alpha \neq 0$, finite), the original equation (NLS) is always transformed into a system of equations where no loss of derivatives arises. As a matter of fact, the highest derivatives, $e^f \partial_{\nabla u} F \cdot \nabla \partial^\alpha u$ and $e^f \partial_{\nabla \bar{u}} F \cdot \nabla \partial^\alpha \bar{u}$, coming from the nonlinear term of the equation $(i\partial_t + \frac{1}{2}\triangle) e^f \partial^\alpha u = e^f \partial^\alpha (F(u, \nabla u, \bar{u}, \nabla \bar{u})) + [i\partial_t + \frac{1}{2}\triangle, e^f]\partial^\alpha u$, are controlled in the following way. The term $e^f \partial_{\nabla u} F \cdot \nabla \partial^\alpha u$ is cancelled exactly by $e^f \nabla f \cdot \nabla \partial^\alpha u$ from the last commutator and the influence of the other term $e^f \partial_{\nabla \bar{u}} F \cdot \nabla \partial^\alpha \bar{u}$ is controlled through integration by parts in the space integral of the term multiplied by $\partial^\alpha \bar{u}$, so that one derivative has no effect on the critical terms with highest derivatives through the energy estimates. This method is applicable to NLS [8] as well as to the equations in the KdV hierarchy of Lax type [12]. In [8, 12] one dimensional case is treated, where the potential function f is simply realized as an integral over the interval $(-\infty, x)$ or $(x, +\infty)$. In contrast, we meet some technical obstacles in the higher dimensional case and therefore it would be of considerable interest to develop the method in higher dimensions. With this view Strauss' equation (1.3) provides an instructive example in the attempt. In (1.3) the nonlinearity $\lambda(\nabla v)^2$ has a simple potential function, given by $f = -2\lambda u$. With this choice a direct calculation shows the exact cancellation

$$(i\partial_t + \frac{1}{2}\triangle) e^{-2\lambda u} \nabla u = 0. \tag{1.4}$$

Our analysis of Strauss' equation is based on the exact cancellation (1.4). As was pointed out by Professor Giga, the identity (1.4) is not very surprising if we remember the Hopf-Cole transformation. Since we consider (1.3) in the L^2-setting, u must vanish at infinity and the integration of (1.4) should take the form

$$(i\partial_t + \frac{1}{2}\triangle)(e^{-2\lambda u} - 1) = 0 \tag{1.5}$$

in order that $e^{-2\lambda u} - 1$ should vanish at infinity as well. Consequently, for given initial data, the formal solution u is given by

$$u(t) = -\frac{1}{2\lambda}\log(1 + U(t)(e^{-2\lambda\phi} - 1)) \tag{1.6}$$

where $U(t) = \exp(i\frac{t}{2}\Delta) = \mathscr{F}^{-1}\exp(-i\frac{t}{2}|\xi|^2)\mathscr{F}$ and $\mathscr{F}$ denotes the Fourier transform defined according to the normalization

$$(\mathscr{F}\psi)(\xi) = \hat{\psi}(\xi) = (2\pi)^{-n/2}\int e^{-ix\cdot\xi}\psi(x)dx.$$

For small initial data, (1.6) actually shows the representation formula for the unique global solutions of the Cauchy problem of (1.3).

Theorem 2. *Let $N = [n/2]+1$ and let k be an integer with $k \geq N$. Then for any $\phi \in H^k$ with $||\mathscr{F}(e^{-2\lambda\phi} - 1)||_1 < (2\pi)^{n/2}$, (1.3) has a unique solution $u \in C(\mathbb{R}; H^k) \cap C^1(\mathbb{R}; H^{k-2})$ such that $u(0) = \phi$, and moreover, the solution u is given explicitly by (1.6). If in addition $\phi \in L^1$, then*

$$||u(t)||_\infty = O(|t|^{-n/2}) \quad \text{as} \quad t \to \pm\infty \tag{1.7}$$

and there exists a unique $\psi \in H^k \cap L^1$ such that

$$||u(t) - U(t)\psi||_{H^k} = O(|t|^{-n/2}) \quad \text{as} \quad t \to \pm\infty. \tag{1.8}$$

In fact, ψ is given explicitly by

$$\psi = \frac{1}{2\lambda}(1 - e^{-2\lambda\phi}). \tag{1.9}$$

Remark 1. By the inequalities

$$\begin{aligned}
||\mathscr{F}(e^{-2\lambda\phi} - 1)||_1 &\leq \sum_{j=1}^{\infty}\frac{(2|\lambda|)^j}{j!}||\mathscr{F}(\phi^j)||_1 \\
&= \sum_{j=1}^{\infty}\frac{(2|\lambda|)^j}{j!}(2\pi)^{-\frac{n}{2}(j-1)}||\underbrace{\hat{\phi} * \cdots * \hat{\phi}}_{j-\text{times}}||_1 \\
&\leq \sum_{j=1}^{\infty}\frac{(2|\lambda|)^j}{j!}(2\pi)^{-\frac{n}{2}(j-1)}||\hat{\phi}||_1^j \\
&= (2\pi)^{n/2}(\exp(2|\lambda|(2\pi)^{-n/2}||\hat{\phi}||_1) - 1),
\end{aligned}$$

the assumption $||\mathscr{F}(e^{-2\lambda\phi}-1)||_1 < (2\pi)^{n/2}$ follows from the assumption $||\hat{\phi}||_1 < (2\pi)^{n/2}(\log 2)/(2|\lambda|)$ for instance. Note that $||\hat{\phi}||_1 \leq C||\phi||_{H^N}$ by the Schwarz inequality.

Remark 2. The assumption $||\mathscr{F}(e^{-2\lambda\phi}-1)||_1 < (2\pi)^{n/2}$ is optimal in the sense that there exists $\phi \in H^\infty = \bigcap_{m\geq 0} H^m$ with $||\mathscr{F}(e^{-2\lambda\phi}-1)||_1 = (2\pi)^{n/2}$ such that the corresponding solution cannot be global in time with values in H^N. Indeed, let $\psi(x) = \exp(-|x|^2/2)$ and let $\phi = -\frac{1}{2\lambda}\log(1+U(-t_0)\psi)$ for $t_0 > 1$. Then $||U(-t_0)\psi||_\infty \leq (2\pi t_0)^{-n/2}||\psi||_1 = t_0^{-n/2} < 1$ and therefore in the same way as in the proof of Theorem 2 given below, we find that $\phi \in H^\infty$. Now let $u \in C(\mathbb{R}; H^N) \cap C^1(\mathbb{R}; H^{N-2})$ be a solution of (1.3) with $u(0) = \phi$. In the same way as in the proof of Theorem 2 given below, u satisfies $e^{-2\lambda u}\nabla u \in C(\mathbb{R}; L^2) \cap C^1(\mathbb{R}; H^{-2})$ and $(i\partial_t + \frac{1}{2}\triangle)e^{-2\lambda u}\nabla u = 0$. This leads to $e^{-2\lambda u(t)}\nabla u(t) = U(t)(e^{-2\lambda\phi}\nabla\phi) = -\frac{1}{2\lambda}U(t-t_0)\nabla\psi$ for all $t \in \mathbb{R}$ and hence $e^{-2\lambda u(t)} - 1 = U(t-t_0)\psi$, namely,

$$\exp(-2\lambda u(t,x)) = 1 - (1+i(t-t_0))^{-\frac{n}{2}}\exp(-\frac{|x|^2}{2(1+i(t-t_0))}).$$

This implies $u(t_0) \notin L^\infty$, which contradicts the assumption $u \in C(\mathbb{R}; H^N)$.

As is shown in Theorem 2, (1.3) has solutions which behaves like free solutions for large times. This fact is somewhat exceptional in the case $n \leq 2$ since the nonlinearity is only quadratic. Our method of proof depends on the power series expansion in the logarithmic function in (1.6) and on the estimation of the individual terms of the expansion in suitable norms in order that the resulting majorant power series should converge. Similarly, for small data with exponential fall-off at infinity, the analyticity of the corresponding solutions of (1.3) follows directly from (1.6) by estimating each term of the expansion in suitable equivalent norms in the Bergman and Szegö spaces (see [5, 6]).

As regards the construction of solution of (1.3), we have another problem of seeking solutions u of (1.3) such that (1.8) holds for the prescribed data at $t = \pm\infty$. This is precisely the problem of existence of wave operators. More specifically, we prove:

Theorem 3. *Let $N = [n/2]+1$ and let k be an integer with $k \geq N$. Then for any $\psi \in H^k$ with $||\hat{\psi}||_1 < (2\pi)^{n/2}/(2|\lambda|)$, (1.3) has a unique solution $u \in C(\mathbb{R}; H^k) \cap C^1(\mathbb{R}; H^{k-2})$ such* *that*

$$\|u(t) - U(t)\psi\|_{H^k} \to 0 \quad \text{as} \quad t \to \pm\infty \tag{1.10}$$

In fact, u is given by

$$u(t) = -\frac{1}{2\lambda}\log(1 - 2\lambda U(t)\psi). \tag{1.11}$$

Remark 3. The assumption $\|\hat{\psi}\|_1 < (2\pi)^{n/2}/(2|\lambda|)$ is optimal in the sense that there exists $\psi \in H^\infty$ with $\|\hat{\psi}\|_1 = (2\pi)^{n/2}/(2|\lambda|)$ such that the corresponding solution satisfies (1.10) for any $k \geq N$ while $u \notin C(\mathbb{R}; H^N)$. Indeed, for $\psi(x) = \frac{1}{2\lambda}\exp(-|x|^2/2)$, we have $\|\hat{\psi}\|_1 = (2\pi)^{n/2}/(2|\lambda|)$ and $(U(t)\psi)(x) = \frac{1}{2\lambda}(1+it)^{-n/2}\exp(-\frac{|x|^2}{2(1+it)})$. Let u be defined by (1.11). Then in the same way as in the proof of Theorem 3 given below, u satisfies (1.10). Moreover, $u \in C(\mathbb{R}\setminus\{0\}; H^k) \cap C^1(\mathbb{R}\setminus\{0\}; H^{k-2})$ and u solves (1.3) for $t \in \mathbb{R}\setminus\{0\}$, while $u(0) \notin L^\infty$. Therefore Theorem 3 does not hold when $\|\hat{\psi}\|_1 = (2\pi)^{n/2}/(2|\lambda|)$.

Remark 4. Theorem 3 provides the explicit form of the wave operator $\Omega : \psi \mapsto u(0) = -\frac{1}{2\lambda}\log(1-2\lambda\psi)$ with domain $D = \{\psi \in H^k; \|\hat{\psi}\|_1 < (2\pi)^n/(2|\lambda|)\}$. Clearly D is invariant under $U(t)$. Moreover, (1.11) means the intertwing property

$$V(t)\Omega = \Omega U(t) \quad \text{on} \quad D$$

for any $t \in \mathbb{R}$, where $V(t)$ is the full evolution group associated with (1.3), namely, $V(t)$: $\phi \mapsto u(t) = -\frac{1}{2\lambda}\log(1 + U(t)(e^{-2\lambda\phi} - 1))$.

As we see above, (1.3) has the ordinary wave operator and again, this fact is exceptional in the case $n \leq 2$, where NLS with the usual quadratic nonlinearity does not admit the ordinary wave operators but modified wave operators [3, 15, 21, 22].

2. Proof of Theorem 1.

Assume (K). Let $\psi \in H^1$. Since every component of $\partial_{\nabla u}F$ is pure imaginary, we have $\partial_{\nabla u}F = i\,\mathrm{Im}\,\partial_{\nabla u}F$ and therefore

$$\begin{aligned}\mathrm{Im}(\partial_{\nabla u}F \cdot \nabla\psi, \psi) &= \mathrm{Re}((\mathrm{Im}\,\partial_{\nabla u}F) \cdot \nabla\psi, \psi) \\ &= -\frac{1}{2}(\mathrm{Im}(\mathrm{div}\,\partial_{\nabla u}F), |\psi|^2),\end{aligned}$$

where we have used integration by parts. The condition (H) from the last equality since F is quadratic.

Conversely, assume (H). Choose $\psi \in H^1$ such that $\mathrm{Im}(\partial_1\psi,\psi) \neq 0$ and $\mathrm{Im}(\partial_j\psi,\psi) = 0$ for all j with $2 \leq j \leq n$. For example, take $\psi(x) = \psi(x_1,x') = f(x_1)g(x')$ with $\mathrm{Im}\int_{-\infty}^{\infty}\partial f(x_1)\overline{f(x_1)}dx_1 \neq 0$, $g \in H^1(\mathbb{R};\mathbb{R})\setminus\{0\}$. Now we set $\phi(x) = \psi(\varepsilon^{-1}(x-x_0))$ for $\varepsilon > 0$ and $x_0 \in \mathbb{R}^n$. By making change of variables, we have

$$\begin{aligned}
&||\phi||_2^2 = \varepsilon^n||\psi||_2^2,\\
&(\partial_{\nabla u}F\cdot\nabla\phi,\phi) = \varepsilon^{n-1}\int \partial_{\nabla u}F(u(y),\nabla u(y),\overline{u(y)},\overline{\nabla u(y)})\cdot\nabla\psi(x)\overline{\psi(x)}dx,
\end{aligned}$$

where $y = \varepsilon x + x_0$. Applying (H) to the equalities above, we have

$$\begin{aligned}
&|\mathrm{Im}\int \partial_{\nabla u}F(u(y),\nabla u(y),\overline{u(y)},\overline{\nabla u(y)})\cdot\nabla\psi(x)\overline{\psi(x)}dx|\\
&\qquad\leq c\varepsilon(||\nabla u||_\infty + ||\nabla^2 u||_\infty)||\psi||_2^2.
\end{aligned}$$

Since $u \in H^m$ with $m > n/2+2$, $\partial_{\nabla u}F(u,\nabla u,\bar{u},\nabla\bar{u})$ is bounded and continuous on $\mathbb{R}^n$. Hence, by letting $\varepsilon \to 0$ in the last inequality, we obtain from the dominated convergence theorem that

$$\mathrm{Im}(\partial_{\nabla u}F(u(x_0),\nabla u(x_0),\overline{u(x_0)},\overline{\nabla u(x_0)})\cdot\int\nabla\psi\bar{\psi}dx) = 0,$$

which in turn imlies

$$(\mathrm{Re}\,\partial_{\partial_1 u}F(u(x_0),\nabla u(x_0),\overline{u(x_0)},\overline{\nabla u(x_0)}))\,\mathrm{Im}(\partial_1\psi,\psi) = 0$$

since $\mathrm{Im}(\partial_j\psi,\psi) = 0$ for $2 \leq j \leq n$ and $\mathrm{Re}(\nabla\psi,\psi) = 0$. This proves that $\mathrm{Re}\,\partial_{\partial_1 u}F = 0$ since $\mathrm{Im}(\partial_1\psi,\psi) \neq 0$ and $x_0 \in \mathbb{R}^n$ is arbitrary. Similarly, we obtain $\mathrm{Re}\,\partial_{\partial_j u}F = 0$ for all j, which is precisely (K). Q.E.D.

3. Proof of Theorem 2.

We begin with the following lemma.

Lemma 3.1. *Let $k \geq N$ be an integer. For any $\phi \in H^k$, $e^{-2\lambda\phi} - 1 \in H^k$.*

Proof. We consider the series $\sum_{j=1}^{\infty} \frac{(-2\lambda)^j}{j!}\phi^j$. The series is estimated in the L^2 norm by

$$\begin{aligned} (3.1) \qquad \sum_{j=1}^{\infty} \frac{(2|\lambda|)^j}{j!} \|\phi^j\|_2 &\leq \sum_{j=1}^{\infty} \frac{(2|\lambda|)^j}{j!} \|\phi\|_\infty^{j-1} \|\phi\|_2 \\ &\leq \sum_{j=1}^{\infty} \frac{(2|\lambda|)^j}{j!} C^j \|\phi\|_{H^N}^j \\ &= \exp(2|\lambda| C \|\phi\|_{H^N}) - 1, \end{aligned}$$

so that the series converges absolutely in L^2 and $e^{-2\lambda\phi} - 1 \in L^2$. By the well-known chain rule with respect to the distributional derivatives, $\nabla(e^{-2\lambda\phi} - 1) = -2\lambda e^{-2\lambda\phi}\nabla\phi \in L^2$. Similarly, for any α with $k \geq |\alpha| \geq \max(N, 2)$, $\partial^\alpha(e^{-2\lambda\phi} - 1) \in L^2$ and

$$(3.2) \qquad \partial^\alpha(e^{-2\lambda\phi} - 1) = -2\lambda e^{-2\lambda\phi}\Big(\partial^\alpha\phi + \sum_{j=2}^{|\alpha|} \sum_{\substack{\beta_1+\cdots+\beta_j=\alpha \\ \beta_\ell \neq 0}} C(j, \lambda, \alpha, \{\beta_\ell\}) \prod_{\ell=1}^{j} \partial^{\beta_\ell}\phi\Big),$$

where every term in the summation of the right hand side is in L^2 since the Gagliardo-Nirenberg inequality gives

(3.3)

$$\begin{aligned} \Big\|\prod_{\ell=1}^{j} \partial^{\beta_\ell}\phi\Big\|_2 &\leq \prod_{\ell=1}^{j} \|\partial^{\beta_\ell}\phi\|_{2|\alpha|/|\beta_\ell|} \\ &\leq C \prod_{\ell=1}^{j} \|\nabla^k\phi\|_2^{|\beta_\ell|(2|\alpha|-n)/(|\alpha|(2k-n))} \|\phi\|_\infty^{1-|\beta_\ell|(2|\alpha|-n)/(|\alpha|(2k-n))} \\ &= C\|\nabla^k\phi\|_2^{(2|\alpha|-n)/(2k-n)} \|\phi\|_\infty^{j-(2|\alpha|-n)/(2k-n)}. \end{aligned}$$

Q.E.D.

Lemma 3.2. *Let $k \geq N$ be an integer. Let $\phi \in H^k$ and let $u \in C(\mathbb{R}; H^k) \cap C^1(\mathbb{R}; H^{k-2})$ satisfy (1.3) with $u(0) = \phi$. Then $e^{-2\lambda u} - 1 \in C(\mathbb{R}; H^k) \cap C^1(\mathbb{R}; H^{k-2})$ and*

$$(3.4) \qquad e^{-2\lambda u(t)} - 1 = U(t)(e^{-2\lambda\phi} - 1) \quad \text{for all} \quad t \in \mathbb{R}.$$

Proof. Let $v = e^{-2\lambda u} - 1$ and let $v_m = \sum_{j=1}^{m} \frac{(-2\lambda)^j}{j!} u^j$. We have

$$\begin{aligned} ||v_\ell(t) - v_m(t)||_2 &\le \sum_{j=m+1}^{\ell} \frac{(2|\lambda|)^j}{j!} ||(u(t))^j||_2 \\ &\le \sum_{j=m+1}^{\ell} \frac{(2|\lambda|)^j}{j!} C^j ||u(t)||_{H^N}^j \\ &\to 0 \end{aligned}$$

as ℓ, $m \to \infty$, where the convergence is locally uniformly on $\mathbb{R}$ since $||u(\cdot)||_{H^N}$ is locally bounded on $\mathbb{R}$.

This implies that $v \in C(\mathbb{R}; L^2)$. As in the last lemma, we have from the chain rule that for any $t \in \mathbb{R}$, $\nabla v(t) = -2\lambda e^{-2\lambda u(t)} \nabla u(t) \in L^2$, and therefore $\nabla v \in C(\mathbb{R}; L^2)$ since $u \in C(\mathbb{R}; H^1 \cap L^\infty)$. We have thus proved $\Delta v \in C(\mathbb{R}; H^{-1})$. Another application of the chain rule gives

$$\Delta v = -2\lambda e^{-2\lambda u}(\Delta u - 2\lambda(\nabla u)^2), \tag{3.5}$$

where Δu, $(\nabla u)^2 \in C(\mathbb{R}; H^{N-2})$ since

$$||\nabla w_1 \cdot \nabla w_2||_{H^{N-1}} \le C||w_1||_{H^N}||w_2||_{H^N}. \tag{3.6}$$

Here (3.6) with $N \ge 2$ follows from the Gagliardo-Nirenberg inequality and (3.6) with $N = n = 1$ follows from

$$||fg||_{H^{-1}} \le C||f||_2||g||_2, \tag{3.7}$$

which is proved by the (H^1, H^{-1}) duality and the embedding $H^1 \hookrightarrow L^\infty$. We next prove that $v \in C^1(\mathbb{R}; H^{-1})$ and

$$\partial_t v = -2\lambda e^{-2\lambda u} \partial_t u. \tag{3.8}$$

We compute

(3.9)

$$\begin{aligned} &h^{-1}(v(t+h) - v(t)) + 2\lambda e^{-2\lambda u(t)} \partial_t u(t) \\ &= -2\lambda \int_0^1 (e^{-2\sigma\lambda u(t+h) - 2(1-\sigma)\lambda u(t)} \cdot h^{-1}(u(t+h) - u(t)) - e^{-2\lambda u(t)} \partial_t u(t)) d\sigma \\ &= -2\lambda \int_0^1 e^{-2\sigma\lambda u(t+h) - 2(1-\sigma)\lambda u(t)} d\sigma (h^{-1}(u(t+h) - u(t)) - \partial_t u(t)) \\ &\quad + (2\lambda)^2 \int_0^1 \int_0^1 e^{-2\tau\sigma\lambda u(t+h) - 2(1-\tau\sigma)\lambda u(t)} \sigma d\tau d\sigma (u(t+h) - u(t)) \partial_t u(t). \end{aligned}$$

When $N \geq 2$, we estimate (3.9) in L^2 as

(3.10)
$$\begin{aligned}
&||h^{-1}(v(t+h)-v(t))+2\lambda e^{-2\lambda u(t)}\partial_t u(t)||_2 \\
&\leq 2|\lambda|\exp(2|\lambda|(||u(t+h)||_\infty+||u(t)||_\infty)) \\
&\quad \cdot (||h^{-1}(u(t+h)-u(t))-\partial_t u(t)||_2+|\lambda|||u(t+h)-u(t)||_\infty||\partial_t u(t)||_2) \\
&\to 0
\end{aligned}$$

as $h \to 0$, as required. When $N = n = 1$, in a way a similar to the proof of (3.7) we have

$$||fg||_{H^{-1}} \leq C||f||_{H^1}||g||_{H^{-1}}. \tag{3.11}$$

We use (3.11) to estimate the second term in the right hand side of the last equality of (3.9) in the H^{-1} norm by

$$\begin{aligned}
&C\exp(2|\lambda|(||u(t+h)||_\infty+||u(t)||_\infty)) \\
&\quad \cdot ((||\nabla u(t+h)||_2+||\nabla u(t)||_2)||u(t+h)-u(t)||_\infty+||\nabla(u(t+h)-u(t))||_2)||\partial_t u(t)||_{H^{-1}}
\end{aligned}$$

and we employ the same estimate as (3.10) for the first term so that (3.8) holds in H^{-1} and the continuity in t of the right hand side of (3.8) with values in H^{-1} follows from (3.11). We have thus proved $v \in C(\mathbb{R}; H^1) \cap C^1(\mathbb{R}; H^{-1})$ and moreover, by (3.5), (3.8) and (1.3),

$$i\partial_t v + \frac{1}{2}\Delta v = 0 \tag{3.12}$$

and accordingly, $v(t) = U(t)v(0)$, which is precisely (3.4). Lemma 3.1 and (3.4) then imply $v \in C(\mathbb{R}; H^k) \cap C^1(\mathbb{R}; H^{k-2})$. Q.E.D.

Proof of Theorem 2. We first prove the uniqueness of solutions. Let $\phi \in H^k$ and let u_1 and u_2 be two solutions of (1.3) with $u_j(0) = \phi$, $j = 1, 2$. By (3.4), for every (t, x) $e^{-2\lambda u_1(t,x)} = e^{-2\lambda u_2(t,x)}$ and therefore $u_1(t, x) - u_2(t, x) = \frac{\pi}{\lambda} im(t, x)$ for some integer $m(t, x)$. By the continuity in (t, x) of u_j, we see that m is independent of (t, x), which in turn implies $m = 0$ since $u_j(t) \in L^2$. This proves $u_1 = u_2$, as was to be shown.

We next prove that (1.6) provides a solution in $C(\mathbb{R}; H^k) \cap C^1(\mathbb{R}; H^{k-2})$ of (1.3) under the assumption $||\mathscr{F}(e^{-2\lambda\phi}-1)||_1 < (2\pi)^{n/2}$. By assumption, $\varepsilon \equiv 1-(2\pi)^{-n/2}||\mathscr{F}(e^{-2\lambda\phi}-1)||_1 > 0$ and

$$
\begin{aligned}
(3.13) \qquad & \sup_{t\in\mathbb{R}} ||U(t)(e^{-2\lambda\phi}-1)||_\infty \\
& \leq (2\pi)^{-n/2} \sup_{t\in\mathbb{R}} ||\exp(-i\frac{t}{2}|\cdot|^2)\mathscr{F}(e^{-2\lambda\phi}-1)||_1 \\
& = (2\pi)^{-n/2}||\mathscr{F}(e^{-2\lambda\phi}-1)||_1 = 1-\varepsilon.
\end{aligned}
$$

We consider the series

$$
\frac{1}{2\lambda}\sum_{j=1}^{\infty}\frac{(-1)^j}{j}(U(t)(e^{-2\lambda\phi}-1))^j. \tag{3.14}
$$

By (3.13) and the unitarity of $U(t)$, the series (3.14) converges absolutely in L^2 and the norm is estimated by

$$
\begin{aligned}
& \frac{1}{2|\lambda|}\sum_{j=1}^{\infty}\frac{1}{j}(1-\varepsilon)^{j-1}||e^{-2\lambda\phi}-1||_2 \\
& = \frac{1}{2|\lambda|(1-\varepsilon)}(\log\frac{1}{\varepsilon})||e^{-2\lambda\phi}-1||_2.
\end{aligned}
$$

This proves that the function u defined by the right hand side of (1.6) makes sense and is continuous from $\mathbb{R}$ to L^2. By the chain rule with respect to the distributional derivatives, for any $t\in\mathbb{R}$, $\nabla u(t)\in L^2$ and

$$
\nabla u(t) = \frac{U(t)\nabla\phi}{1+U(t)(e^{-2\lambda\phi}-1)}. \tag{3.15}
$$

The continuity in t of ∇u with values in L^2 follows from (3.15), $||1+U(t)(e^{-2\lambda\phi}-1)||_\infty \geq \varepsilon$, and $U(\cdot)(e^{-2\lambda\phi}-1)\in C(\mathbb{R};L^\infty)$, which follows from the dominated convergence theorem with $\mathscr{F}(e^{-2\lambda\phi}-1)\in L^1$. Similarly, for any α with $k\geq|\alpha|+1\geq N$, $\partial^\alpha\nabla u\in C(\mathbb{R};L^2)$ and

(3.16)

$$
\begin{aligned}
& -2\lambda\partial^\alpha\nabla u \\
& = \frac{\partial^\alpha\nabla w}{1+w} + \sum_{\substack{\beta+\gamma=\alpha \\ \beta\neq 0}}\sum_{j=1}^{|\beta|}\sum_{\substack{\beta_1+\cdots+\beta_j=\beta \\ \beta_\ell\neq 0}}\frac{\alpha!}{\gamma!}\frac{(-1)^{j+1}}{\prod_{\ell=1}^{j}\beta_\ell!}\frac{\partial^\gamma\nabla w}{(1+w)^{j+1}}\prod_{\ell=1}^{j}\partial^{\beta_\ell}w
\end{aligned}
$$

where $w(t) = U(t)(e^{-2\lambda\phi} - 1)$. Here the continuity in t of each term in the summation of the right hand side of (3.16) with values in L^2 follows from the inequality which is obtained by the Gagliardo-Nirenberg inequality

(3.17)
$$\begin{aligned}
&\|\partial^\gamma \nabla \psi_0 \cdot \prod_{\ell=1}^{j} \partial^{\beta_\ell}\psi_\ell\|_2 \\
&\le \|\partial^\gamma \nabla \psi_0\|_{2(|\alpha|+1)/(|\gamma|+1)} \prod_{\ell=1}^{j} \|\partial^{\beta_\ell}\psi_\ell\|_{2(|\alpha|+1)/|\beta_\ell|} \\
&\le C \prod_{\ell=0}^{j} \|\nabla^k \psi_\ell\|_2^{|\beta_\ell|(2|\alpha|+2-n)/((|\alpha|+1)(2k-n))} \|\psi_\ell\|_\infty^{1-|\beta_\ell|(2|\alpha|+2-n)/((|\alpha|+1)(2k-n))},
\end{aligned}$$

where $|\beta_0| = |\gamma| + 1$. In a way similar to the proof of (3.8), we obtain $\partial_t u \in C(\mathbb{R}; H^{-1})$ and

$$-2\lambda \partial_t u = \frac{\partial_t w}{1+w} = \frac{i}{2}\frac{\Delta w}{1+w}. \tag{3.18}$$

As in the derivation of (3.16) from (3.15), (3.18) yields $\partial_t u \in C(\mathbb{R}; H^{k-2})$. This proves $u \in C(\mathbb{R}; H^k) \cap C^1(\mathbb{R}; H^{k-2})$. In the same way as in (3.6) and (3.7), the last fact implies $(\nabla u)^2 \in C(\mathbb{R}; H^{k-2})$. A simple calculation shows $-2\lambda \Delta u = \frac{\Delta w}{1+w} - \frac{(\nabla w)^2}{(1+w)^2} = 4i\lambda \partial_t u - 4\lambda^2 (\Delta u)^2$, which leads to (1.2).

We proceed to (1.7). Estimating (3.11) in the L^∞-norm and using (3.13) and the well-known L^∞-decay estimate of $U(t)$, we obtain

$$\begin{aligned}
\|u(t)\|_\infty &\le \frac{1}{2|\lambda|} \sum_{j=1}^{\infty} \frac{1}{j} \|U(t)(e^{-2\lambda\phi} - 1)\|_\infty^j \\
&\le \frac{1}{2|\lambda|} \sum_{j=1}^{\infty} \frac{1}{j} (1-\varepsilon)^{j-1} (2\pi|t|)^{-n/2} \|e^{-2\lambda\phi} - 1\|_1 \\
&= \frac{\log(1/\varepsilon)}{(2\pi|t|)^{n/2}(1-\varepsilon)} \|\psi\|_1,
\end{aligned} \tag{3.19}$$

where $\psi = \frac{1}{2\lambda}(1 - e^{-2\lambda\phi})$. We note here that $\phi \in L^1 \cap L^\infty$ implies $\psi \in L^1$ by estimation of each term of the expansion of the exponential function of ψ in the L^1 norm.

We finally prove (1.8). In the same way as before,

$$\begin{aligned}
(3.20)\qquad &\|u(t)-U(t)\psi\|_2 \\
&\le \frac{1}{2|\lambda|}\sum_{j=2}^{\infty}\frac{1}{j}\|U(t)(e^{-2\lambda\phi}-1)\|_\infty^{j-1}\|e^{-2\lambda\phi}-1\|_2 \\
&\le \frac{1}{2|\lambda|}\sum_{j=2}^{\infty}\frac{1}{j}(1-\varepsilon)^{j-2}(2\pi|t|)^{-n/2}\|e^{-2\lambda\phi}-1\|_1\|e^{-2\lambda\phi}-1\|_2 \\
&= \frac{2|\lambda|}{(2\pi|t|)^{n/2}}\left(\frac{\log(1/\varepsilon)}{(1-\varepsilon)^2}-1+\varepsilon\right)\|\psi\|_1\|\psi\|_2.
\end{aligned}$$

For any α with $k \ge |\alpha|+1 \ge N$, we obtain from (3.16) and (3.17)

(3.21)

$$\begin{aligned}
&\|\partial^\alpha\nabla(u(t)-U(t)\psi)\|_2 \\
&\le \|(\frac{1}{1+w(t)}-1)\partial^\alpha\nabla U(t)\psi\|_2 \\
&\quad + C\sum_{\substack{\beta+\gamma=\alpha\\ \beta\ne 0}}\sum_{j=1}^{|\beta|}\sum_{\substack{\beta_1+\cdots+\beta_j=\beta\\ \beta_\ell\ne 0}}\varepsilon^{-j-1}\|\partial^\gamma\nabla U(t)\psi\cdot\prod_{\ell=1}^{j}\partial^{\beta_\ell}U(t)\psi\|_2 \\
&\le \varepsilon^{-1}\|w(t)\|_\infty\|\partial^\alpha\nabla U(t)\psi\|_2 \\
&\quad + C\sum_{j=1}^{|\alpha|}\varepsilon^{-j-1}\|\nabla^k U(t)\psi\|_2^{(2|\alpha|+2-n)/(2k-n)}\|U(t)\psi\|_\infty^{j+1-(2|\alpha|+2-n)/(2k-n)} \\
&\le 2|\lambda|\varepsilon^{-1}(2\pi|t|)^{-n/2}\|\psi\|_1\|\partial^\alpha\nabla\psi\|_2 \\
&\quad + C(1+\varepsilon^{-k})\|\nabla^k\psi\|_2^{(2|\alpha|+2-n)/(2k-n)}((2\pi|t|)^{-n/2}\|\psi\|_1+((2\pi|t|)^{-n/2}\|\psi\|_1)^{|\alpha|+1}),
\end{aligned}$$

where we have used $1 \le j \le j+1-(2|\alpha|+2-n)/(2k-n) \le |\alpha|+1$.

This proves (1.8). The uniqueness of ψ follows from (1.8) and the unitarity of $U(t)$ on H^k. Q.E.D.

4. Proof of Theorem 3.

We first prove the uniqueness. Let $u \in C(\mathbb{R};H^k)\cap C^1(\mathbb{R};H^{k-2})$ satisfy (1.3) and (1.10). Let $v = e^{-2\lambda u}-1$. In view of the proof of Lemma 3.2, $v \in C(\mathbb{R};H^1)$ and $v(t)=U(t)v(0)$. By (1.10),

$$\begin{aligned}
(4.1)\qquad \big|\,\|u(t)\|_{H^k}-\|\psi\|_{H^k}\big| &= \big|\,\|u(t)\|_{H^k}-\|U(t)\psi\|_{H^k}\big| \\
&\le \|u(t)-U(t)\psi\|_{H^k} \\
&\to 0
\end{aligned}$$

as $t \to \pm\infty$ and therefore $u \in L^\infty(\mathbb{R}; H^k)$. We have

$$
\begin{aligned}
(4.2)\qquad & ||v(t) - (e^{-2\lambda U(t)\psi} - 1)||_2 \\
&= ||e^{-2\lambda u(t)} - e^{-2\lambda U(t)\psi}||_2 \\
&= ||2\lambda \int_0^1 e^{-2\sigma\lambda u(t) - 2(1-\sigma)\lambda U(t)\psi} d\sigma (u(t) - U(t)\psi)||_2 \\
&\leq 2|\lambda| \exp(2|\lambda|(||u(t)||_\infty + ||U(t)\psi||_\infty))||u(t) - U(t)\psi||_2 \\
&\to 0 \quad \text{as} \quad t \to \pm\infty.
\end{aligned}
$$

Let $\{\psi_j\}$ be a sequence in $L^1 \cap H^k$ such that $\psi_j \to \psi$ in H^k as $j \to \infty$. We have

$$
\begin{aligned}
(4.3)\qquad & ||e^{-2\lambda U(t)\psi_j} - 1 + 2\lambda U(t)\psi_j||_2 \\
&\leq \sum_{m=2}^{\infty} \frac{(2|\lambda|)^m}{m!} ||U(t)\psi_j||_\infty^{m-1} ||\psi_j||_2 \\
&\leq \sum_{m=2}^{\infty} \frac{(2|\lambda|)^m}{m!} (2\pi)^{-n(m-1)/2} ||\hat{\psi}_j||_1^{m-2} \cdot |t|^{-n/2} ||\psi_j||_1 ||\psi_j||_2 \\
&\to 0 \quad \text{as} \quad t \to \pm\infty.
\end{aligned}
$$

As in (4.2),

$$
\begin{aligned}
(4.4)\qquad & ||(e^{-2\lambda U(t)\psi} + 2\lambda U(t)\psi) - (e^{-2\lambda U(t)\psi_j} + 2\lambda U(t)\psi_j)||_2 \\
&\leq ||e^{-2\lambda U(t)\psi} - e^{-2\lambda U(t)\psi_j}||_2 + 2|\lambda| ||\psi - \psi_j||_2 \\
&\leq 2|\lambda| (\exp(2|\lambda|(2\pi)^{-n/2}(||\hat{\psi}||_1 + ||\hat{\psi}_j||_1)) + 1)||\psi - \psi_j||_2.
\end{aligned}
$$

Combining (4.2), (4.3) and (4.4) and letting $t \to \pm\infty$ and then $j \to \infty$, we obtain

(4.5)

$$
\begin{aligned}
& ||v(0) + 2\lambda\psi||_2 = ||U(t)v(0) + 2\lambda U(t)\psi||_2 \\
&\leq ||v(t) - (e^{-2\lambda U(t)\psi} - 1)||_2 + ||(e^{-2\lambda U(t)\psi} + 2\lambda U(t)\psi) - (e^{-2\lambda U(t)\psi_j} + 2\lambda U(t)\psi_j)||_2 \\
&\quad + ||e^{-2\lambda U(t)\psi_j} - 1 + 2\lambda U(t)\psi_j||_2 \\
&\to 0
\end{aligned}
$$

and therefore $v(t) = -2\lambda U(t)\psi$, namely,

$$
(4.6)\qquad e^{-2\lambda u(t)} = 1 - 2\lambda U(t)\psi.
$$

In the same way as in the proof of uniqueness of solutions in Theorem 2, (4.6) proves the required uniqueness. If $||\hat{\psi}||_1 < (2\pi)^{n/2}/(2|\lambda|)$, then

$$\sup_{t\in\mathbb{R}} ||2\lambda U(t)\psi||_\infty \leq 2|\lambda|(2\pi)^{-n/2}||\hat{\psi}||_1 = 1-\delta \tag{4.7}$$

with $\delta = 1 - 2|\lambda|(2\pi)^{-n/2}||\hat{\psi}||_1 > 0$, and hence (4.6) yields (1.11). It remains to prove that u defined by (1.11) satisfies (1.10) and $u \in C(\mathbb{R}; H^k)\cap C^1(\mathbb{R}; H^{k-2})$. In the same way as in the proof of Theorem 2, the last claim holds, so that (1.10) is the main issue. Let $\psi \in H^k$ and let $\{\psi_j\}$ be a sequence in $L^1 \cap H^k$ such that $\psi_j \to \psi$ in H^k as $j \to \infty$ and $\sup_j ||\hat{\psi}_j||_1 \leq (1-\delta/2)(2\pi)^{n/2}/(2|\lambda|)$. Let u_j be the solution associated with ψ_j, namely, $u_j(t) = -\frac{1}{2\lambda}\log(1-2\lambda U(t)\psi_j)$. Similarly as above,

(4.8)

$$\begin{aligned}
||u_j(t) - U(t)\psi_j||_2 &\leq \sum_{m=2}^{\infty} \frac{(2|\lambda|)^{m-1}}{m} ||U(t)\psi_j||_\infty^{m-1}||\psi_j||_2 \\
&\leq \sum_{m=2}^{\infty} \frac{(2|\lambda|(2\pi)^{-n/2})^{m-1}}{m} ||\hat{\psi}_j||_1^{m-2} \cdot |t|^{-n/2}||\psi_j||_1||\psi_j||_2 \\
&\leq \frac{2|\lambda|}{(2\pi)^{n/2}} \sum_{m=2}^{\infty} \frac{1}{m}(1-\frac{\delta}{2})^{m-2} \cdot |t|^{-n/2}||\psi_j||_1||\psi_j||_2 \\
&\to 0 \quad \text{as} \quad t \to \pm\infty.
\end{aligned}$$

Now

(4.9)

$$\begin{aligned}
u(t) - u_j(t) &= U(t)(\psi_j - \psi) \\
&\quad + 2\lambda \int_0^1 (1-\sigma)\left(\frac{(U(t)\psi_j)^2}{(1-2\sigma\lambda U(t)\psi_j)^2} - \frac{(U(t)\psi)^2}{(1-2\sigma\lambda U(t)\psi)^2}\right)d\sigma \\
&= U(t)(\psi_j - \psi) + 2\lambda \int_0^1 \frac{(1-\sigma)U(t)(\psi_j+\psi)}{(1-2\sigma\lambda U(t)\psi_j)^2} d\sigma U(t)(\psi_j - \psi) \\
&\quad + 8\lambda^2 \int_0^1 \frac{(1-\sigma)\sigma(1-2\sigma\lambda U(t)(\psi_j+\psi))}{(1-2\sigma\lambda U(t)\psi_j)^2(1-2\sigma\lambda U(t)\psi)^2} d\sigma (U(t)\psi)^2 U(t)(\psi - \psi_j),
\end{aligned}$$

so that

(4.10)

$$\begin{aligned}
&||u(t) - u_j(t)||_2 \\
&\leq ||\psi_j - \psi||_2 + |\lambda|(2/\delta)^2(2\pi)^{-n/2}(||\hat{\psi}_j||_1 + ||\hat{\psi}||_1)||\psi_j - \psi||_2 \\
&\quad + |\lambda|^2(2/\delta)^4(1 + |\lambda|(2\pi)^{-n/2}(||\hat{\psi}_j||_1 + ||\hat{\psi}||_1))(2\pi)^{-n}||\hat{\psi}||_1^2||\psi_j - \psi||_2.
\end{aligned}$$

Combining (4.8) and (4.10) and letting $t \to \pm\infty$ and then $j \to \infty$, we obtain

$$\begin{aligned}(4.11) \qquad & \|u(t) - U(t)\psi\|_2 \\ & \leq \|u(t) - u_j(t)\|_2 + \|u_j(t) - U(t)\psi_j\|_2 + \|\psi_j - \psi\|_2 \\ & \to 0.\end{aligned}$$

Similarly, the convergence of higher derivatives follows with estimates similar to (3.21). This proves (1.10). Q.E.D.

Remark. After this work has been completed, I am informed of the following two papers:

S.Cohn, Resonance and long time existence for the quadratic semilinear Schrödinger equation, Comm. Pure Appl. Math. 45, 973-1001 (1992),

N.Hayashi, Global and almost global solutions to quadratic nonlinear Schrödinger equations with small initial data, preprint Gunma (1992),

where related problems on the quadratic NLS are discussed, although no part of our results are contained there. I would like to thank Professor Hayashi for that matter.

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