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Motion of a graph by nonsmooth weighted curvature

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and
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In the memory of Yoshi's first daughter

Abstract. Geometric evolutions of curves represented by graphs are studied when the interface energy is not necessarily smooth. The theory of nonlinear semigroups yields a unique global solution even if the initial curve is not necessarily admissible. Usual evolution law for the crystalline interface energy is justified.

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1. Introduction

We are concerned with a geometric evolution of a curve Γ_t driven by a weighted curvature. We consider

$$V = \frac{-1}{\beta(\vec{n})} \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\frac{\partial \gamma}{\partial p_i}(\vec{n})\right) \quad \text{on} \quad \Gamma_t, \tag{1.1}$$

where $\vec{n}$ represents the unit normal vector (field) of Γ_t and V represents the normal velocity of Γ_t. The function $\gamma = \gamma(p_1, p_2)$ is assumed to be positively homogeneous of degree one. The restriction of γ on the unit circle S^1 is often called the *interface energy (density)*. The function $\beta : S^1 \to \mathbf{R}$ is assumed to be positive; β is called the *kinetic coefficient*. The equation (1.1) is called the *curvature flow equation* if γ and β are isotropic, *i.e.,* γ and β are constant on S^1. In the material science, it is also natural to consider anisotropic interface energy and kinetic coefficient. A slightly general form of (1.1) is derived by the second law of thermodynamics and balances of energy and forces in a series of work by Gurtin. We refer to [G] and references therein.

A fundamental analytic question is to track whole evolution of Γ_t moved by (1.1). There are several ways to attack this problem. See the recent review by

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Taylor, Cahn and Handwerker [TCH] and references therein. Some methods track the evolution after Γ_t experiences singularities even if the surface evolution equation corresponding to (1.1) is considered. In fact, if γ is C^2 (outside the origin) and convex, one can track whole evolution via viscosity formulation provided that β is continuous [CGG] (cf. [ES] for isotropic case.) For the curve evolution equation (1.1) this method is extended to the case that γ is a convexification of smooth one under additional assumptions. See a recent paper of Ohnuma and Sato [OS] and Gurtin, Soner and Souganidis [GSS]; see also [Gi]. This extension applies to nonconvex γ [GSS]. If γ is merely Lipschitz but convex, up to now viscosity formulation seems to be not applicable. A typical example of γ is a *crystalline energy, i.e.,* the Frank diagram of γ

$$\{p \in \mathbf{R}^2; \quad \gamma(p) \leqslant 1\}$$

is a convex polygon. For such nonsmooth γ the evolution law is so far defined only "admissible" curve Γ_t; see [AG] and [T1]. In other words the equation must have diffusion effect on each smooth portion of Γ_t.

Our goal is to formulate (1.1) for general initial shape not necessarily admissible. We restrict ourselves to the case when Γ_t is represented as a graph of a periodic, Lipschitz continuous function, although the same strategy seems to work for other boundary value problems. We adapt the theory of nonlinear semigroups to formulate the problem and then prove the existence of unique global in time solution. Initial data is assumed to be Lipschitz but it is allowed to be non-admissible. However, this method requires that the equation (1.1) has a divergence structure, so it does not apply to surface evolution problem or general curve evolution problem in a standard way. We also show that our solution satisfies evolution law by Taylor [T1, T2] and by Angenent and Gurtin [AG] when γ is a crystalline energy and initial data is admissible. In [AG] the evolution law is derived by an integral identity of the capillary force, while in [T1, T2] it is derived by a gradient flow of the crystalline interface energy. In both work the velocity is assumed to be constant on each 'facet'.

It turns out that our solutions have an approximate property. Namely, if γ is approximated by another one, say $\tilde{\gamma}$, then the approximate solution by $\tilde{\gamma}$ converges to the solution by original γ in a certain sense. This is obtained by applying a stability thorem of Watanabe [W]. In a recent interesting paper [GK] Girão and Kohn studied the approximate rate when smooth γ is approximated by a crystalline energy $\tilde{\gamma}$. Their method is completely different from ours.

2. Subdifferentials

If Γ_t is represented as a graph, the equation (1.1) has a divergence structure. In other words (1.1) is regarded as a gradient flow equation of a functional. We begin by studying functional related to our problem.

Let $L^2(\mathbf{T})$ denote the space of all periodic L^2 function of our variable with period $\omega; \mathbf{T} = \mathbf{R}/\omega\mathbf{Z}$. Suppose that

$$j : \mathbf{R} \to (-\infty, \infty]$$

is convex, lower semicontinuous and proper, *i.e.*, $j \not\equiv +\infty$. Suppose that

$$\lim_{|p|\to\infty} j(p)/|p| = +\infty. \tag{2.1}$$

We consider the functional on the Hilbert space $H = L^2(\mathbf{T})$ defined by

$$\Phi(u) = \begin{cases} \int_0^\omega j(u_x)dx & \text{if} \quad u_x \in L^1(\mathbf{T}) \quad \text{and} \quad j(u_x) \in L^1(\mathbf{T}) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.2}$$

LEMMA 2.1. *The functional Φ is proper, lower semicontinuous and convex on H.*

LEMMA 2.2. *For $u \in H, f \in H$ belongs to the subdifferential $\partial\Phi(u)$ if and only if there is $\eta \in L^1(\mathbf{T})$ such that $\eta(x) \in \partial j(u_x(x))$a.e.$x$ and $f = -\eta_x$.*

Above two results are standard and obtained by adapting the argument in [Br, example 3].

3. Equations with nonsmooth interface energy

Let Γ_t be represented as a graph of a function $u(t,x)$. If $\vec{n}$ is taken upward, then

$$\vec{n} = (-u_x/(1+u_x^2)^{1/2}, \quad (1+u_x^2)^{-1/2}).$$

Using homogeneity of γ and $V = u_t/(1+u_x^2)^{1/2}$, we observe that (1.1) is equivalent to

$$u_t = (g(u_x))_x \tag{3.1}$$

$$\text{with} \quad g(p) = \int_0^p \sqrt{1+q^2}\,\tilde{\beta}(q)^{-1} d\lambda(q), \tag{3.2}$$

$$\lambda(q) = -\partial_{p_1}\gamma(-q,1), \quad \tilde{\beta}(q) = \beta(-q/(1+q^2)^{1/2}, (1+q^2)^{-1/2});$$

see [Gi, §2] for more detail. Since λ may have jumps we should be careful of the meaning of the domain of integration in (3.2); here it represents an open interval $(0,p)$ to fix the idea. If γ is convex, λ is nondecreasing, so that g is nondecreasing.

If γ is Lipschitz, g is not necessarily continuous. Then the meaning of (3.1) is nontrivial even in usual distribution sense or viscosity sense.

Suppose that $g : \mathbf{R} \to \mathbf{R}$ with $g(0) = 0$ is nondecreasing but maybe discontinuous. We formulate initial value problem for (3.1) with periodic, Lipschitz initial data. We take a primitive of g,

$$G(p) = \int_0^p g(q)dq.$$

The function G may not be superlinear *i.e.*,

$$G(p)/|p| \to +\infty \quad \text{as} \quad |p| \to \infty$$

may not hold. For $K > 0$ we take a function $r(p) \in C^\infty(\mathbf{R})$ such that

$$\begin{array}{ll} \text{(i)} & r(p) = 0 \quad \text{for} \quad |p| \leqslant K + 1 \\ \text{(ii)} & r \quad \text{is} \quad \text{convex} \\ \text{(iii)} & r(p)/|p| \to +\infty \quad \text{as} \quad |p| \to \infty. \end{array}$$

It follows that

$$j = G + r$$

is a convex continuous function. By (iii) j satisfies the growth condition (2.1) so Lemmas 2.1 and 2.2 apply to Φ corresponding to j defined by (2.2). Since j depends upon K and r, we often write Φ_{Kr} instead of Φ.

DEFINITION 3.1: Suppose that $g : \mathbf{R} \to \mathbf{R}$ is nondecreasing with $g(0) = 0$. Suppose that $u_0 \in H = L^2(\mathbf{T})$ is Lipschitz. Let K be a constant such that $|u_{0x}| \leqslant K$ on $\mathbf{T}$. We say $u \in C([0, \infty), H)$ solves

$$u_t = (g(u_x))_x \quad \text{with} \quad u(0, x) = u_0(x) \tag{3.3}$$

if u fulfills following properties.
(i) The mapping $x \mapsto u(t, x)$ is Lipschitz and $|u_x| \leqslant K$ for all $t \geqslant 0$, $x \in \mathbf{T}$.
(ii) As an H-valued function u is strongly differentiable for a.e.$t > 0$ and absolutely continuous for $[\delta, \infty)$ for each $\delta > 0$. Moreover, u solves

$$du/dt \in -\partial\Phi_{Kr}(u) \text{ a.e.}x \quad \text{with} \quad u(0, x) = u_0(x). \tag{3.4}$$

PROPOSITION 3.2. *The definition of solutions is independent of the choice of K, r.*

This is because $j(u_x) = G(u_x), \partial j(u_x) = \partial G(u_x)$ for $|u_x| \leqslant K$.

THEOREM 3.3. *Let $u_0 \in L^2(\mathrm{T})$ be a Lipschitz continuous function. Then there is a unique solution of (3.3).*

Proof: Since $\Phi_{K\gamma}$ is a proper, lower semicontinuous, convex function densely defined on $H = L^2(\mathrm{T})$ by Lemma 2.1, the theory of nonlinear semigroups (initiated by Kōmura) guarantees the unique existence of solution $u \in C([0,\infty), H)$ to (3.4); see e.g.[Ba, IV, Theorem 2.1].

It remains to prove (i) of Definition 3.1. We consider an approximate equation

$$u_t = (g_\epsilon(u_x))_x \tag{3.5}$$

by setting

$$g_\epsilon(p) = (j' * \rho_\epsilon)(p) + \epsilon p, \quad (j' = dj/dp),$$

where $\epsilon > 0$ and ρ_ϵ is Friedrichs' mollifier. The equation (3.5) is uniformly parabolic for $|u_x| \leqslant K$. Since we have a priori bound $|u_x| \leqslant K$ for (3.5) by the maximum principle, a standard parabolic theory (see [LUS]) yields a unique classical solution u_ϵ to (3.5) with initial data u_0 such that

$$|u_{\epsilon x}| \leqslant K \quad \text{for all} \quad t \geqslant 0. \tag{3.6}$$

For

$$G_\epsilon(p) = \int_0^p g_\epsilon(q)dq$$

we set a functional $\Phi = \varphi_\epsilon$ defined by (2.2) with $j = G_\epsilon$.
The equation (3.5) is reformulated as an abstract equation

$$du/dt \in -\partial\varphi_\epsilon(u). \tag{3.7}$$

Of course, classical solution u_ϵ to (3.5) is the unique solution of (3.7) with initial data u_0. To obtain $|u_x| \leqslant K$ from (3.6) it suffices to prove

$$u_\epsilon \to u \quad \text{in} \quad C([0,T], H) \tag{3.8}$$

for all $T > 0$. By a stability theorem of Watanabe[W] we obtain the convergence (3.8) provided that $\varphi_\epsilon \to \Phi_{K\gamma}$ in the sense of Mosco. (Note that the form of g_ϵ in (3.5) is not essential to conclude (3.8). If γ and β in (3.1)-(3.2) are approximated by smooth $\gamma_\epsilon, \beta_\epsilon$ so that the corresponding functional φ_ϵ converges to $\Phi_{K\gamma}$, we always obtain (3.8).)

The rest of this section is devoted to the proof of the convergence $\varphi_\epsilon \to \Phi_{K\gamma}$. Note that

$$|G_\epsilon(p) - j(p)| \leqslant C\epsilon(|p| + 1) \tag{3.9}$$

$$G_\epsilon(p) \geqslant C_1|p| - C_2 \tag{3.10}$$

for some positive constants C, C_1 and C_2. Suppose that $u_\epsilon \longrightarrow u$ in H. Then

$$\Phi_{K_\gamma}(u) \leqslant \liminf_{\epsilon\to 0} \varphi_\epsilon(u_\epsilon). \tag{3.11}$$

Indeed, we may assume that the right hand side is finite. By (3.10) this implies that $u_{\epsilon x}$ is bounded in $L^1(\mathrm{T})$. Using (3.9) we observe that

$$|\varphi_\epsilon(u_\epsilon) - \Phi_{K_\gamma}(u_\epsilon)| \leqslant C\epsilon \int_0^\omega (|u_{\epsilon x}| + 1)dx \longrightarrow 0 \quad \text{as} \quad \epsilon \longrightarrow 0. \tag{3.12}$$

Since Φ_{K_γ} is lower semicontinuous, applying (3.12) to

$$\varphi_\epsilon(u_\epsilon) = \Phi_{K_\gamma}(u_\epsilon) + (\varphi_\epsilon(u_\epsilon) - \Phi_{K_\gamma}(u_\epsilon))$$

yields the lower semicontinuity (3.11). We next observe that for each v in the domain of $\partial\Phi_{K_\gamma}$ there is $v_\epsilon \longrightarrow v$ in H such that

$$\lim_{\epsilon\to 0} \varphi_\epsilon(v_\epsilon) = \Phi_{K_\gamma}(v) \tag{3.13}$$

Indeed, we take $v_\epsilon = v$ and apply (3.9) to get (3.13), since $v_x \in L^1(\mathrm{T})$. We have thus proved that φ_ϵ converges to Φ_{K_γ} in the sense of Mosco. ∎

Our Theorem 3.3 yields a unique global solution Γ_t to (1.1) when initial Γ_0 is a graph of a periodic, Lipschitz continuous function. It also asserts that Γ_t stays as a graph of a Lipschitz continuous function.

Since the approximate equation (3.5) has a comparison principle, it is inherited to (3.3).

LEMMA 3.4. *Suppose that u and v are solutions of (3.1) with initial data u_0 and v_0 respectively, where $u_0, v_0 \in L^2(\mathrm{T})$ is Lipschitz. If $u_0 \leqslant v_0$, then $u \leqslant v$ for all $t \geqslant 0$.*

It is likely that solution $u \in C([0,\infty) \times \mathrm{T})$ but we do not touch this problem in this paper.

REMARK 3.5: Our existence theorem applies to any Lipschitz initial data even if it is not admissible in the sense of [AG]. This provides a systematic way to handle non-admissible data; an adhoc approach is given in [T2, 2.4].

4. Speed on admissible facets

We are concerned with evolution driven by crystalline energy. We shall prove that our evolution law (3.4) agrees with usual one for admissible facets appeared in [AG], [T1, T2].

If γ is a crystalline energy, g defined by (3.2) is nondecreasing, piecewise constant function with finitely many jump discontinuities $p_1 < p_2 < \cdots < p_m$. Let $v \in L^2(\mathbf{T})$ be a (continuous) piecewise linear function with finitely many nondifferentiable points. We assume $v_x = p_i$ for some $1 \leqslant i \leqslant m$ at differentiable points. Let $(a,b) \subset \mathbf{R}$ be an interval such that $v_x = p_i$ for some $1 \leqslant i \leqslant m$ and that v is nondifferentiable at a and b. We call the segment of graph v over (a,b) an (admissible) *facet* with slope p_i.

As well known, $A = \partial\Phi_{Kr}$ is a maximal monotone operator in $H = L^2(\mathbf{T})$. We would like to calculate its canonical restriction A^0 at v with $K = \max_i |p_i|$. The value $A^0 v$ is a unique minimizer of

$$\{||w||_H;\ w \in Av\}. \tag{4.1}$$

We often suppress subscripts Kr of Φ.

LEMMA 4.1. *Let $v \in C(\mathbf{T})$ be a piecewise linear function whose graph consists of finitely many admissible facets.*
(i) The set $\partial\Phi(v)$ is non-empty if and only if for $1 \leqslant i \leqslant m$ any facet of slope p_i only touches to facets of slope p_{i-1} or p_{i+1}.
(ii) Suppose that $Av = \partial\Phi(v)$ is non-empty. Suppose that the graph of v over (a,b) is a facet of slope p_i. Then

$$(A^0 v)(x) = -\chi \frac{g(p_i + 0) - g(p_i - 0)}{b - a}, \ x \in (a,b), \tag{4.2}$$

where $g(p_i \pm 0) = \lim_{\epsilon\to 0} g(p_i \pm \epsilon)$. Here $\chi = +1$ if $v_x(a-0) = p_{i-1}$, $v_x(b+0) = p_{i+1}$; $\chi = -1$ if $v_x(a-0) = p_{i+1}$, $v_x(b+0) = p_{i-1}$; $\chi = 0$ otherwise.

Proof: Suppose that $f \in \partial\Phi(v)$. Lemma 2.2 implies that

$$f = -\eta_x \quad \text{with} \quad \eta(x) \in \partial G(v_x(x)) \text{ a.e.} x$$

for some $\eta \in L^1(\mathbf{T})$ since $|v_x| \leqslant K$. Since f is integrable, η is continuous on $\mathbf{T}$. Suppose that the graph of v over (a,b) is a facet of slope p_i. Then for $x \in (a,b)$ we observe that

$$g(p_i - 0) \leqslant \eta(x) \leqslant g(p_i + 0) \tag{4.3}$$

since $\eta(x) \in \partial G(p_i)$. Suppose that the facet over (a,b) touches to a facet over (c,d) with slope p_j other than p_{i-1} or p_{i+1}. Since $p_j \neq p_{i-1}$ and $p_j \neq p_{i+1}$ and since

$$g(p_j - 0) \leqslant \eta(x) \leqslant g(p_j + 0) \quad \text{for} \quad x \in (c,d), \tag{4.4}$$

η becomes discontinuous at a (if $d = a$) or b (if $c = b$). This leads a contradiction to the existence of f. We thus prove a necessary condition that $\partial\Phi(v)$ is non-empty.

Suppose that each facet of v satisfies slope condition in (i). We shall construct a piecewise linear function $\zeta \in C(\mathrm{T})$ which is linear possibly except nondifferential points of v. We assign values of ζ so that

$$\begin{array}{lll}
\zeta(a) = g(p_i - 0), \zeta(b) = g(p_i + 0) & \text{if} & v_x(a-0) = p_{i-1},\ v_x(b+0) = p_{i+1} \\
\zeta(a) = g(p_i + 0), \zeta(b) = g(p_i - 0) & \text{if} & v_x(a-0) = p_{i+1},\ v_x(b+0) = p_{i-1} \\
\zeta(a) = \zeta(b) = g(p_i + 0) & \text{if} & v_x(a-0) = v_x(b+0) = p_{i+1} \\
\zeta(a) = \zeta(b) = g(p_i - 0) & \text{if} & v_x(a-0) = v_x(b+0) = p_{i-1}.
\end{array}$$

Here (a, b) is an arbitrary interval so that the graph of v is a facet with slope p_i over (a, b). The above condition uniquely determines ζ. Since $\zeta(x) \in \partial G(p_i)$ for $x \in (a, b)$ and since ζ_x is piecewise constant, we observe that $-\zeta_x \in \partial\Phi(v)$ which implies that $\partial\Phi(v)$ is non-empty.

It remains to prove that $-\zeta_x$ minimizes (4.1), since $-\zeta_x(x)$ equals the right hand side of (4.2) for $x \in (a, b)$. If $f \in \partial\Phi(v)$, then $f = -\eta_x$ for some $\eta \in C(\mathrm{T})$ satisfying (4.3) and (4.4). Since η is continuous (4.3) and (4.4) imply that $\eta(a) = \zeta(a), \eta(b) = \zeta(b)$ for all possible values of $v_x(a-0)$ and $v_x(b+0)$. Since ζ is linear on $(a, b), \zeta$ minimizes

$$\int_a^b |\eta_x|^2 dx \quad \text{with} \quad \eta(a) = \zeta(a),\ \eta(b) = \zeta(b).$$

We thus observe that $-\zeta_x \in \partial\Phi(v)$ minimizes (4.1). ∎

We next consider (3.4) with $u(0, x) = u_0(x) = v(x)$ and calculate a normal velocity at each facet of v. For $A = \partial\Phi_K$, equations

$$\begin{array}{lll}
du/dt \in -Au & \text{a.e.} t \geqslant 0 & u(0,\cdot) = v \in D(A) \\
d^+u/dt = -A^0 u & t \geqslant 0 & u(0,\cdot) = v \in D(A)
\end{array}$$

are equivalent for absolutely continuous function u with values in H; d^+/dt denotes the right derivative in time; see e.g. [Ba, IV, Theorem 2.2]. Thus

$$\lim_{t\downarrow 0}(u(t,\cdot) - v)/t = -A^0 v.$$

At a facet of slope p_i of the graph v, the upword normal velocity V equals

$$V = (d^+u/dt)(0)/(1 + p_i^2)^{1/2} = -(A^0 v)/(1 + p_i^2)^{1/2}.$$

Since $(b - a)(1 + p_i^2)^{1/2}$ is the length of the facet over (a, b), Lemma 4.1 now yields:

THEOREM 4.2. *Let $v \in C(\mathrm{T})$ be a piecewise linear function whose graph consists of finitely many admissible facets. Then the normal velocity V is of the form*

$$V = \chi(g(p_i + 0) - g(p_i - 0))/L \tag{4.5}$$

for a facet of slope p_i. Here L denotes the length of the facet and $\chi = \pm 1, 0$ as in Lemma 4.1.

We conclude this paper by comparing usual formula of normal velocity at each facets [AG], [T1,T2]. For crystalline energy γ we set

$$f(\theta) = \gamma(\cos\theta, \sin\theta), \quad 0 \leqslant \theta \leqslant 2\pi;$$

this parametrization of interface energy is given in [AG]. Since γ is positively homogeneous, we have

$$f(\theta) + f''(\theta) = \frac{\partial^2 \gamma}{\partial p_1^2}(\cos\theta, \sin\theta)\frac{1}{\sin^2\theta}. \tag{4.6}$$

Indeed,

$$\begin{aligned} f'(\theta) &= -\gamma_1 \sin\theta + \gamma_2 \cos\theta \\ f''(\theta) &= \gamma_{11}\sin^2\theta - 2\gamma_{12}\sin\theta\cos\theta + \gamma_{22}\cos^2\theta \\ &\quad -(\gamma_1\cos\theta + \gamma_2\sin\theta), \end{aligned}$$

where $\gamma_i = \partial\gamma/\partial p_i$, $\gamma_{ij} = \partial^2\gamma/\partial p_i \partial p_j$ evaluated at $(\cos\theta, \sin\theta)$ for $1 \leqslant i,\ j \leqslant 2$. By the homogeneity of γ we have

$$\begin{aligned} \gamma &= \gamma_1\cos\theta + \gamma_2\sin\theta \\ 0 &= \gamma_{11}\cos\theta + \gamma_{12}\sin\theta \\ 0 &= \gamma_{21}\cos\theta + \gamma_{22}\sin\theta. \end{aligned}$$

We thus observe that

$$\begin{aligned} f(\theta) + f''(\theta) &= \gamma_{11}\sin^2\theta - 2\gamma_{12}\sin\theta\cos\theta + \gamma_{22}\cos^2\theta \\ &= (\sin^2\theta + 2\sin\theta\cos\theta/\sin\theta + \cos^2\theta/\sin^2\theta)\gamma_{11} \\ &= \gamma_{11}/\sin^2\theta. \end{aligned}$$

Suppose that $f'(\theta)$ has jump discontinuities at $0 < \theta_1 < \cdots < \theta_m < \pi$ for $\theta \in (0, \pi)$. Then $\gamma_1(-q, 1)$ has a jump discontinuity only at p_i defined by

$$(\cos\theta_i,\ \sin\theta_i) = (-p_i/(1 + p_i^2)^{1/2}, (1 + p_i^2)^{-1/2}) \tag{4.7}$$

so that $p_1 < p_2 < \cdots < p_m$. In other words p_i is a slope of line whose normal equals $(\cos\theta_i,\ \sin\theta_i)$.

Lemma 4.3. *If g is defined by (3.2), then*

$$g(p_i+0)-g(p_i-0)=(f'(\theta_i+0)-f'(\theta_i-0))/\beta(\cos\theta_i,\sin\theta_i),$$

where θ_i and p_i satisfy (4.7). In particular (4.5) is of the form,

$$V=\chi(f'(\theta_i+0)-f'(\theta_i-0))/(L\beta(\cos\theta_i,\ \sin\theta_i)).$$

Proof: By (3.2)

$$\begin{aligned} g(p_i+0)-g(p_i-0) &= \lim_{\epsilon\downarrow 0}\int_{p_i-\epsilon}^{p_i+\epsilon}\tilde{\beta}(q)^{-1}\gamma_{11}(-q/(1+q^2)^{1/2},(1+q^2)^{-1/2})dq \\ &= \beta(\cos\theta_i,\sin\theta_i)^{-1}\lim_{\epsilon\downarrow 0}\int_{\theta(p_i-\epsilon)}^{\theta(p_i+\epsilon)}(\gamma_{11}(\cos\theta,\sin\theta)/\sin^2\theta)d\theta. \end{aligned}$$

Here θ is taken by

$$(\cos\theta,\ \sin\theta)=(-q/(1+q^2)^{1/2},(1+q^2)^{-1/2})$$

so that $q=-1/\tan\theta,\ dq=d\theta/\sin^2\theta$. Applying (4.6) yields

$$g(p_i+0)-g(p_i-0)=\beta(\cos\theta_i,\sin\theta_i)^{-1}(f'(\theta_i+0)-f'(\theta_i-0))$$

since f is continuous. ∎

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